Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

On dynamical properties of diffeomorphisms with homoclinic tangencies

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submitted: 12th December 2002

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No. 795
Berlin 2002

2000 *Mathematics Subject Classification.* 37G25, 37D45, 37C15, 37G30, 34C20, 34C27.

Key words and phrases. Newhouse regions, Henon map, renormalization, strange attractor, chaos, moduli, stable periodic orbits.

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Abstract

We study bifurcations of a homoclinic tangency to a saddle fixed point without non-leading multipliers. We give criteria for the birth of an infinite set of stable periodic orbits, an infinite set of coexisting saddle periodic orbits with different instability indices, non-hyperbolic periodic orbits with more than one multiplier on the unit circle, and an infinite set of stable closed invariant curves (invariant tori). The results are based on the rescaling of the first-return map near the orbit of homoclinic tangency, which is shown to bring the map close to one of four standard quadratic maps, and on the analysis of the bifurcations in these maps.

Poincaré homoclinic orbits, i.e. those which are bi-asymptotic to saddle periodic ones, are one of the most attracting objects of study in the theory of dynamical systems. The reason is that their presence leads to complicated dynamics. Thus, in a neighborhood of a homoclinic orbit corresponding to a transverse intersection of the invariant manifolds of a saddle periodic orbit there exist a countable set of periodic orbits and continuum of non-trivial recurrent orbits [1, 2].

If the system has at least one non-transverse homoclinic orbit, the so-called homoclinic tangency, this implies the existence of an infinite set of regions of structural instability in any neighborhood of the given system: systems with homoclinic tangencies are dense in these regions. This phenomenon was discovered by Newhouse for the case of two-dimensional diffeomorphisms [3]. In the multidimensional case, the Newhouse regions also exist in any neighborhood of any system with a homoclinic tangency, both in the parameter space in finite-parameter families [4] and, naturally, in the space of smooth dynamical systems [4, 5, 6].

From the very beginning we should note that dynamical properties of systems from the Newhouse regions are extremely unusual and complex. Thus, it was established in [7, 8] that a description of dynamics of the systems from the Newhouse regions requires, already in the two-dimensional case, infinitely many invariants (the so-called Ω-moduli [9, 10]). Moreover, systems are dense in the Newhouse regions (in the C^r-topology with any $r \geq 3$) with homoclinic tangencies of an arbitrarily high order and with arbitrarily degenerate periodic orbits [8, 11].

In the present paper we continue the study of dynamics of systems from the Newhouse regions near a diffeomorphism having a saddle fixed point with a homoclinic tangency. We focus only on the so-called basic cases: the two-dimensional one, a three-dimensional case where the fixed point is a saddle-focus with one real multiplier and a pair of complex- conjugate ones, and a four-dimensional case of the saddle-focus which has two pairs of complex-conjugate multipliers. In the general multidimensional case, dynamics near a homoclinic tangency depends mostly on the structure of the set of the so-called leading multipliers of the fixed point. In general position, it is either a pair of real multipliers, or one real multiplier and a pair of complex-conjugate ones, or two pairs of complex-conjugate multipliers. Thus, the diffeomorphisms under consideration are the simplest among all diffeomorphisms with the given set of leading multipliers.

Bifurcations of two-dimensional diffeomorphisms with homoclinic tangencies has been actively studied starting with the paper [12]. Therefore, we pay here our main attention to three- and four-dimensional diffeomorphisms, i.e. to the case of saddle-focus. A fixed point with the multipliers $\lambda e^{\pm i\varphi}$ and γ, where $0 < \lambda < 1$, $0 < \varphi < \pi$ and $|\gamma| > 1$, will be called a saddle-focus (2,1). A point will be called a saddle-focus (1,2) when it has multipliers λ and $\gamma e^{\pm i\psi}$ where $0 < |\lambda| < 1$, $\gamma > 1, 0 < \psi < \pi$. A point is called a saddle-focus (2,2) when it has multipliers $\lambda e^{\pm i\varphi}$ and $\gamma e^{\pm i\psi}$ where $0 < \lambda < 1$, $\gamma > 1$, $0 < \varphi < \pi$, $0 < \psi < \pi$.

In all cases we will assume that the absolute value J of the product of the multipliers does not equal to 1. For definiteness, we will assume $J < 1$ (the case $J > 1$ is reduced to this one if we consider the inverse map).

We show that, similar to the two-dimensional case, *diffeomorphisms with infinitely many stable periodic orbits are dense* in the Newhouse regions at $J < 1$. Earlier, it was known [13, 14, 5] that stable periodic orbits may be born at the bifurcations of a homoclinic tangency provided the unstable manifold of the corresponding fixed point is one-dimensional and the saddle value $\sigma \equiv |\lambda\gamma|$ is less than 1. As our result here shows, neither of these conditions is necessary. From the other hand, we note that the expansion of volumes near the saddle fixed point at $J > 1$ prohibits of stable orbits, both for the system itself and for all close systems [15, 16]. Thus, when $J > 1$, diffeomorphisms with infinitely many completely unstable periodic orbits are dense in the Newhouse regions.

Concerning saddle periodic orbits, we have the following essentially non-twodimensional phenomenon in the case of saddle-foci. We show that under certain conditions *diffeomorphisms which have simultaneously infinitely many coexisting saddle periodic orbits of two or even three different types* (i.e. with different dimensions of the unstable manifolds) are dense in the Newhouse regions. Note that the dimensions of the unstable manifolds of these periodic orbits my even be greater than the dimension of the unstable manifold of the original saddle fixed point. Such phenomenon can be detected in many cases of homoclinic bifurcations: near a homoclinic loop to a saddle-focus [17], near non-transverse heteroclinic cycles [16, 18, 19], near homoclinic tangencies in some cases of codimension 2 [20, 21, 22], and it was explicitly used in the construction of the wild spiral attractor in [23]. In fact, we consider the coexistence of orbits with different numbers of positive Lyapunov exponents as the most general property of multidimensional systems from the Newhouse regions.

The existence of *non-hyperbolic* periodic orbits is another characteristic feature of systems from the Newhouse regions. It is known [12, 14] that bifurcations of a homoclinic tangency are accompanied in the two-dimensional case by the birth of periodic orbits with one multiplier equal to $+1$ or -1. In the present paper we show that in the case of a saddle-focus there may appear periodic orbits with *two* or even *three* multipliers equal to 1 in absolute value, and that diffeomorphisms with such orbits are dense in the corresponding Newhouse regions.

Bifurcations of a periodic orbit with one multiplier equal to $+1$ or -1 are well known: these are the saddle-node bifurcation and the bifurcation of period doubling. In the case of a periodic orbit with two multipliers on the unit circle, $\nu_{1,2} = e^{\pm i\omega}$ for example, its bifurcations can lead to a birth of closed invariant curves. Here, in connection with the problem of coexistence of an infinite number of non-trivial attractors we are specially interested in stable closed curves. Thus, we show that in the case of a homoclinic tangency to a saddle-focus fixed point with $J < 1$ (except for the case of a saddle-focus (2,1) with $|\lambda\gamma| < 1$ when dynamics does not differ much from the case of a saddle) *diffeomorphisms with infinitely many stable invariant curves* are dense in the corresponding Newhouse regions.

In the case of a saddle-focus (2,2) with $\lambda\gamma^2 > 1$ bifurcations of a homoclinic tangency can lead to the birth of periodic orbits with three multipliers on the unit circle. These cases require a separate consideration which we do not conduct in this paper. Note, however, that the normal form in the case of multipliers $(-1, -1, +1)$, for example, is a system of three autonomous differential equations (Morioka-Shimitsu system) which has a Lorenz-like attractor [24]. Therefore, we can expect that diffeomorphisms with infinitely many coexisting strange attractors are dense in the Newhouse regions in the case of a saddle-focus (2,2) with $\lambda\gamma^2 > 1, \lambda\gamma < 1$.

The main results of the paper were announced in [25].

1 Setting the problem and main results

1.1 Main assumptions

Consider a C^r-smooth diffeomorphism f with a saddle fixed point O. Assume that the stable and unstable manifolds $W^s(O)$ and $W^u(O)$ intersect non-transversely at the points of some homoclinic orbit Γ_0.

We assume that the point O does not have non-leading multipliers. Four basic cases appear here: the two-dimensional case, when the multipliers of O are real, two three-dimensional cases, when there are one real multiplier and a pair of complex-conjugate ones, and one four-dimensional case, when the multipliers are complex. Namely, we assume that the following condition holds.

A. The point O belongs to one of the following types:
(1,1) when the multipliers λ and γ of O are real, $|\lambda| < 1$, $|\gamma| > 1$;
(2,1) when O has a pair of complex multipliers $\lambda_{1,2} = \lambda e^{\pm i\varphi}$, where $\lambda \in (0, 1)$, $\varphi \in (0, \pi)$, and one real multiplier γ, where $|\gamma| > 1$;

(1,2) when O has one real multiplier λ, where $|\lambda| < 1$, and a pair of complex multipliers $\gamma_{1,2} = \gamma e^{\pm i\psi}$, where $\gamma > 1$, $\psi \in (0, \pi)$;
(2,2) when O has two pairs of complex multipliers: $\lambda_{1,2} = \lambda e^{\pm i\varphi}$ and $\gamma_{1,2} = \gamma e^{\pm i\psi}$, where $\lambda \in (0, 1)$, $\gamma > 1$, $\varphi, \psi \in (0, \pi)$.

We will call the point O a saddle in the first case and a saddle-focus in the other cases. Let J be the absolute value of the product of the multipliers of O. Assume that f satisfies the following condition.

B. $J < 1$, and $|\lambda\gamma| \neq 1$ in the case (2,1), and $\lambda\gamma^2 \neq 1$ in the case (2,2).

Introduce an integer d_e (we call it "effective dimension") which is defined as follows:

$d_e = 1$ — in the case (1,1), and in the case (2,1) with $|\lambda\gamma| < 1$;
$d_e = 2$ — in the case (2,1) with $|\lambda\gamma| > 1$, in the case (1,2), and in the case (2,2) with $\lambda\gamma^2 < 1$;
$d_e = 3$ — in the case (2,2) with $\lambda\gamma^2 > 1$.

The meaning of the constants J and d_e is quite simple. J is the Jacobian of the map f at the fixed point O. Thus, the diffeomorphism f contracts volumes near O in the case $J < 1$, while it expands volumes if $J > 1$. It is also obvious that if $J < 1$, then the iterations of the map f will exponentially contract any $(d_e + 1)$-dimensional volume near O, while d_e-dimensional volumes can expand.

It is obvious that condition B is not restrictive because the case $J > 1$ reduces to the given one if we consider the inverse map. One should only have in mind that this transition will make the stable manifold unstable, i.e. the case (1,2) becomes (2,1), and vice versa. The definition of the quantity d_e also changes in an obvious way.

Denote as T_0 the restriction of the diffeomorphism f onto a sufficiently small neighborhood U_0 of the fixed point O. We will call T_0 the *local map*. The map T_0 in a small neighborhood of $O(0,0)$ may be written as

$$\bar{x} = Ax + \dots, \quad \bar{y} = By + \dots. \tag{1.1}$$

The eigenvalues of the matrices A and B are the stable (i.e. smaller than 1 in absolute value) and, respectively, unstable (larger than 1 in absolute value) multipliers of O. Thus, if the stable multiplier is real, then $A = \lambda$ and x is a scalar; while if we have a pair of complex stable multipliers, then $x = (x_1, x_2)$ and $A = \lambda \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Analogously, if the unstable multiplier γ is real, then $B = \gamma$ and y is a scalar; and if there is a pair of complex unstable multipliers, then $y = (y_1, y_2)$ and $B = \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$.

The points of intersection of the homoclinic orbit Γ_0 with U_0 belong to the set $W^s \cap W^u$ and converge to O. Countable sets of these points lie in W^s_{loc} and in W^u_{loc}.

Let $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$ be some two points of Γ_0, and let $M^+ = f^{k_0}(M^-)$ for some positive integer k_0. Let Π^+ and Π^- be some sufficiently small neighborhoods of the points M^+ and M^-, lying in U_0. The map $T_1 \equiv f^{k_0} : \Pi^- \to \Pi^+$ will be called the *global map*.

By assumption, $T_1(W^u_{loc})$ is tangent to W^s_{loc} at the point M^+. We will assume that this tangency is *simple*, i.e. the following conditions hold:

C. $T_1(W^u_{loc})$ and W^s_{loc} have a single common tangent vector at the point M^+;

D. the tangency of $T_1 W^u_{loc}$ and W^s_{loc} at the point M^+ is quadratic.

1.2 On bifurcation parameters

Let f be a diffeomorphism with a homoclinic tangency satisfying conditions A–D. Close to f diffeomorphisms which have an orbit of homoclinic tangency close to Γ_0 form a smooth bifurcational surface $\mathcal{H}$ of codimension 1 in the space of C^r-smooth diffeomorphisms with C^r-topology.

In the present paper we consider bifurcations in parametric families f_ε transverse to $\mathcal{H}$ at $\varepsilon = 0$. The minimal number of governing parameters we take equals exactly d_e. As the first parameter we take a parameter μ which estimates the splitting of $W^s(O)$ and $W^u(O)$ near the point M^+ (the exact definition of μ in terms of the coefficients of the Taylor expansion of the global map T_1 see in Section 2, Lemma 5). Formally speaking, μ is a smooth functional defined for diffeomorphisms close to f, such that the bifurcational surface $\mathcal{H}$ is given by the equation $\mu(f) = 0$. The family f_ε is transverse to $\mathcal{H}$ if and only if $\dfrac{\partial}{\partial \varepsilon}(\mu(f_\varepsilon)) \neq 0$ at $\varepsilon = 0$. It is this condition which allows us to take μ as the first component of the vector of parameters ε.

If $d_e \geq 2$, then we need one or two (when $d_e = 3$) more governing parameters, in addition to μ. In this case we require that the family f_ε were transverse at $\varepsilon = 0$ both to the bifurcational surface $\mathcal{H}$ and to the surfaces $\varphi = const$ and/or $\psi = const$, where φ and ψ are the angular arguments of the complex multipliers of O. This transversality condition allows for taking μ, $\varphi - \varphi_0$, $\psi - \psi_0$ as the governing parameters, where φ_0 and ψ_0 are the values of φ and ψ at $\varepsilon = 0$.

Thus, we assume
1) $\varepsilon = \mu$ – in the case (1,1), and in the case (2,1) with $|\lambda\gamma| < 1$;
2) $\varepsilon = (\mu, \varphi - \varphi_0)$ – in the case (2,1) with $|\lambda\gamma| > 1$;
3) $\varepsilon = (\mu, \psi - \psi_0)$ – in the case (1,2), and in the case (2,2) with $\lambda\gamma^2 < 1$;
4) $\varepsilon = (\mu, \varphi - \varphi_0, \psi - \psi_0)$ – in the case (2,2) with $\lambda\gamma^2 > 1$.

Note that φ and ψ are the so-called Ω-moduli – continuous invariants of the topological conjugacy on the set of nonwandering orbits – for the systems with homoclinic tangencies in the case of a saddle-focus. As it was shown in [26, 27], any change in the value of these Ω-moduli (in the class of diffeomorphisms from $\mathcal{H}$, i.e. when the original homoclinic tangency is not split) leads to bifurcations of single-round periodic orbits.[1] This, in particular, explains why having only one governing parameter μ may be insufficient for the analysis of the bifurcations in the cases (2,1), (1,2) and (2,2).

Note that all our results here will hold true for arbitrary families f_ε (e.g. when the number of parameters is larger than d_e) under the only assumption that the above transversality conditions are fulfilled.

One of the general results on the families f_ε is the existence of *Newhouse regions* in these families. First, we recall the following result from [4].

Theorem on Newhouse intervals. *Let f_μ be a one-parameter family of C^r-smooth ($r \geq 3$) diffeomorphisms, transverse to the bifurcational surface $\mathcal{H}$ of diffeomorphisms satisfying conditions A-D.*[2] *Then, in any neighborhood of the point $\mu = 0$ there exist Newhouse intervals such that 1) the values of μ are dense which correspond to the existence of a simple homoclinic tangency to O; 2) the family f_μ is transverse to the corresponding bifurcational surfaces.*

Since the Newhouse regions are open in C^2-topology in the space of dynamical systems, the theorem on Newhouse intervals imply immediately the following result concerning the family f_ε.

Newhouse regions in parametric families. *In the space of parameters ε there exists a sequence of open regions δ_j, converging to $\varepsilon = 0$, such that in δ_j the values of ε are dense which correspond to the existence of an orbit of simple homoclinic tangency to O. Moreover, the family f_ε is transverse to the corresponding bifurcational surfaces.*

1.3 Main results

We will study properties of diffeomorphisms f_ε from the Newhouse regions δ_j. In order to study bifurcations of periodic orbits we will assume sufficient smoothness of f_ε; namely, we assume $r \geq 5$.

First, we discuss the case $d_e = 1$ (recall that we consider one-parameter families with $\varepsilon = \mu$ in this case).

Theorem 1 *In the cases of a saddle (1,1) and a saddle-focus (2,1), when $|\lambda\gamma| < 1$, the following statements hold for the Newhouse intervals δ_j*

[1] We have an analogous situation in the case of a saddle as well, for double-round periodic orbits now. Here any change in the value of the Ω-modulus $\theta = -\ln|\lambda|/\ln|\gamma|$ leads to bifurcations of such orbits [28, 9]. Note that triple-round periodic orbits may, in this case, undergo cusp-bifurcations [29] which correspond to one of the multipliers equal to $+1$ and the first Lyapunov coefficient vanishing at the critical moment.

[2] In [4], instead of condition B, we required only that $\lambda\gamma \neq 1$. Note that our condition B always includes this requirement.

1) the values of μ are dense there, such that the diffeomorphism f_μ has a periodic orbit with a multiplier equal to $+1$;
2) the values of μ are dense such that the diffeomorphism f_μ has a periodic orbit with a multiplier equal to -1;
3) the values of μ are dense (and comprise a residual set) such that the diffeomorphism f_μ has an infinite set of stable periodic orbits.

In essence, items 1 and 2 of this theorem can be found in [12] for the case of a saddle and in [30, 14] for the case of a saddle-focus. Item 3 is known since the paper [13] for the two-dimensional case, the three-dimensional case is considered in [31, 15] (see also [5]). For the sake of completeness, we give the proof of Theorem 1 along with the proofs of the other results listed below.

Further we consider the case $d_e \geq 2$. The main attention here is paid to those properties of the diffeomorphisms f_ε which are new in comparison with the case of a saddle. These are the existence of non-hyperbolic periodic orbits with more than one multiplier on the unit circle (Theorem 2); the coexistence of infinitely many stable closed invariant curves (Theorem 3); the coexistence of infinitely many of (rough) periodic orbits of more than two different types (Theorem 4).

Theorem 2 *In the case $d_e = 2$, i.e. in the cases of a saddle-focus (2,2) with $\lambda\gamma^2 < 1$, a saddle-focus (1,2), and a saddle-focus (2,1) with $|\lambda\gamma| > 1$, in the Newhouse regions δ_j the values of parameters ε are dense such that the corresponding diffeomorphism f_ε has a periodic orbit with any aforehand given pair of multipliers on the unit circle.*

In the case of a saddle-focus (2,2) with $\lambda\gamma^2 > 1$ (i.e. when $d_e = 3$), in the Newhouse regions δ_j the values of ε are dense such that the corresponding diffeomorphism f_ε has a periodic orbit with any aforehand given triplet of multipliers on the unit circle.

Note that we deal here with real diffeomorphisms, therefore we speak in Theorem 2 about such sets of multipliers for which every complex multiplier is accompanied by its conjugate. In particular, we have that in the case of a homoclinic tangency to a saddle-focus with $d_e \geq 2$, in the corresponding Newhouse regions diffeomorphisms with periodic orbits which have a pair of multipliers $e^{\pm i\omega}$ $(0 < \omega < \pi)$ are dense. An analysis of the bifurcations of such periodic orbits, as well as periodic orbits with a pair of multipliers $(-1, -1)$, allows us to establish the following result.

Theorem 3 *Let a C^r-smooth $(r \geq 5)$ diffeomorphism f satisfy conditions A–D. Then, in the case $d_e \geq 2$, in the Newhouse regions δ_j the values of parameters are dense and comprise a residual set for which the diffeomorphism f_ε has infinitely many asymptotically stable closed invariant curves.*

Condition $J < 1$ is essential in this theorem (when $J > 1$, all the orbits are necessarily unstable). In the class of two-dimensional diffeomorphisms with $J \neq 1$ there can be no closed invariant curves near a homoclinic tangency, because we have either contraction (at $J < 1$), or expansion (at $J > 1$) of areas. However, in the case of codimension 2 when $J = 1$ at the moment of homoclinic tangency, the birth of closed invariant curves is possible [20, 22]. Closed invariant curves are also born at the bifurcations of a non-transverse heteroclinic cycle with two saddles when $J < 1$ in one saddle and $J > 1$ in the other saddle. Moreover, it is shown in [18, 19], that near systems with such heteroclinic cycle there exist Newhouse regions where such diffeomorphisms are dense that have simultaneously infinitely many of both stable and completely unstable closed invariant curves.

The next theorem gives us the answer to one of the main problems of the dynamics of systems from the Newhouse regions – on the coexistence of periodic orbits with different numbers of positive Lyapunov exponents.

Theorem 4 *In the Newhouse regions δ_j the values of parameters are dense and comprise a residual set such that the corresponding diffeomorphism f_ε has, simultaneously, an infinite set of stable periodic orbits and, for each d from 1 to d_e, an infinite set of saddle periodic orbits with the dimension of the unstable manifold equal to d.*

Note that here there can be no periodic orbits with the unstable manifolds of the dimension greater than d_e, by virtue of the contraction of $(d_e + 1)$-dimensional volumes [15, 16]. Thus, for example, in the case of a saddle-focus (2,2) with $\lambda\gamma^2 < 1$ we have saddles with one-dimensional and two-dimensional unstable manifolds, while there are no saddles with three-dimensional unstable manifolds here. If we, however, have $\lambda\gamma^2 > 1$, $\lambda\gamma < 1$, then there can simultaneously exist saddles with one-dimensional, two-dimensional and three-dimensional unstable manifolds.

The proof of theorems 1–3 is based on the study of the first-return maps near the orbit of homoclinic tangency. We reduce the study of these maps to the analysis of the following standard quadratic maps:
(i) parabola map — $\bar{y} = M - y^2$ (for the cases of a saddle and a saddle-focus (2,1) with $|\lambda\gamma| < 1$);
(ii) Hénon map — $\bar{x}_1 = y,\ \bar{y} = M - y^2 + Bx_1$ (for the case of a saddle-focus (2,1) with $|\lambda\gamma| > 1$);
(iii) Mira map — $\bar{y}_1 = y_2,\ \bar{y}_2 = M + Cy_2 - y_1^2$ (for the cases of a saddle-focus (1,2) and a saddle-focus(2,2) with $|\lambda\gamma^2| < 1$);
(iv) three-dimensional Hénon map — $\bar{x}_1 = y_1,\ \bar{y}_1 = y_2,\ \bar{y}_2 = M + Cy_2 + Bx_1 - y_1^2$ (for the case of a saddle-focus (2,2) with $\lambda\gamma^2 > 1$).
The linear analysis of the fixed points of these maps is comparatively simple (see Section 4), and it gives us the information necessary for the proof of Theorems 1, 2 and 4. Concerning the stable closed invariant curves of Theorem 3, we derive their existence in the case of saddle-foci (1,2) and (2,2) from a nonlinear bifurcational analysis of maps (iii) and (iv). In the case of a saddle-focus (2,1) with $|\lambda\gamma| > 1$ the problem is that Hénon map (ii) itself has no (asymptotically stable) closed invariant curves. Therefore, in order to prove Theorem 3) in this case we have to deal with the so-called generalized Hénon map (see Lemma 2).

1.4 Rescaling lemma

In the case of diffeomorphisms close to a diffeomorphism with a homoclinic tangency, the first-return maps in a small fixed neighborhood Π^+ of the homoclinic point M^+ are the compositions $T^{(k)} = T_1 T_0^k$, where $k = \bar{k}, \bar{k}+1, ...$, and $\bar{k}$ is sufficiently large. Recall that $T_0 = f_{\varepsilon}|_{U_0}$ where U_0 is some small neighborhood of the fixed point, and $T_1 \equiv f_\varepsilon^{k_0}$ is defined in a small neighborhood Π^- of the homoclinic point M^- and it takes Π^- inside Π^+. Thus, the domain of definition of the map $T^{(k)}$ in Π^+ is the "strip" $\sigma_k^0 = \Pi^+ \cap T_0^{-k}\Pi^-$. The strips σ_k^0 are non-empty for all sufficiently large k (the smaller the size of the neighborhoods Π^+ and Π^-, the large the minimal k is) and they converge to $W^s_{loc} \cap \Pi^+$ as $k \to +\infty$.

The following lemma (the main technical result of the paper) shows that the first-return maps $T^{(k)}$ can be brought, for all large k, to some standard form. Namely, they can be written as maps which are asymptotically close, as $k \to \infty$, to certain one-dimensional, two-dimensional and three-dimensional quadratic maps.

Lemma 1 (Rescaling lemma) *Let f_0 be a C^r-smooth ($r \geq 5$) diffeomorphism satisfying conditions A-D, and let f_ε be a d_e-parameter family transverse to $\mathcal{H}$ at $\varepsilon = 0$. Then, in the space of parameters there exists a sequence of regions Δ_k, converging to $\varepsilon = 0$ as $k \to +\infty$, such that the following holds.*

At $\varepsilon \in \Delta_k$ there exists such transformation of coordinates in σ_k^0 and parameters in Δ_k, C^{r-1}-smooth with respect to the coordinates and C^{r-2}-smooth with respect to the parameters, that brings the first-return map $T^{(k)} : (x,y) \mapsto (\bar{x},\bar{y})$ to one of the following forms:

i) *in the case (1,1) and in the case (2,1) with $\lambda\gamma < 1$ —*

$$\bar{y} = M - y^2 + o(1), \quad \bar{x} = o(1); \tag{1.2}$$

ii) *in the case (2,1) with $\lambda\gamma > 1$ —*

$$\begin{aligned} &\bar{x}_1 = y, \\ &\bar{y} = M - y^2 + Bx_1 + o(1), \qquad \bar{x}_2 = o(1); \end{aligned} \tag{1.3}$$

iii) *in the case (1,2) and in the case (2,2) with $\lambda\gamma^2 < 1$ —*

$$\begin{aligned} &\bar{y}_1 = y_2, \\ &\bar{y}_2 = M + Cy_2 - y_1^2 + o(1), \qquad \bar{x} = o(1); \end{aligned} \tag{1.4}$$

iv) *in the case (2,2) with $\lambda\gamma^2 > 1$ —*

$$\begin{aligned} &\bar{x}_1 = y_1, \\ &\bar{y}_1 = y_2, \\ &\bar{y}_2 = M + Cy_2 + Bx_1 - y_1^2 + o(1), \qquad \bar{x}_2 = o(1). \end{aligned} \tag{1.5}$$

In these coordinates, the domain of definition of the map $T^{(k)}$ is asymptotically large and it covers, in the limit $k \to +\infty$, all finite values of (x, y).

The rescaled parameters M, B and C are expressed via the original parameters μ, φ and ψ as follows:

$$\begin{gathered} M = M_0 \gamma^{2nk}(\mu + O(|\lambda|^k + |\gamma|^{-k})), \\ B = B_0 \left(\lambda \gamma^n\right)^k \cos(k\varphi + \alpha_k(\varepsilon)), \quad C = C_0 \gamma^k \cos(k\psi + \beta_k(\varepsilon)), \end{gathered} \tag{1.6}$$

where $n = \dim W^u(O)$, the constants M_0, B_0, C_0 are non-zero, and the functions α_k and β_k are uniformly bounded for all k, along with the derivatives. Here, when ε runs the region Δ_k the values of M, B and C run asymptotically large regions which cover, in the limit $k \to +\infty$, all finite values.

Here we denote as $o(1)$ some functions (of the rescaled coordinates and the parameters M, B, C) which tend to zero as $k \to \infty$ along with all the derivatives up to the order $(r-2)$ with respect to the coordinates and $(r-3)$ with respect to the parameters, uniformly on any bounded subset of the space (x, y, M, B, C). Note also that in the case of saddle-foci the regions Δ_k, corresponding to finite values of B and C, may consist of many connected components (by virtue of the periodic dependence of B and C on φ and ψ respectively).

In case (ii) of Lemma 1 we need a more accurate account of the asymptotically small terms in the map (1.3), which leads us to the following result.

Lemma 2 *In the case (2,1) with $\lambda\gamma > 1$, when $\varepsilon = (\mu, \varphi - \varphi_0) \in \Delta_k$ and when the corresponding value of B is bounded away from zero, the map $T^{(k)}$ in the form* (1.3) *has a two-dimensional attracting invariant C^{r-2}-smooth manifold $\mathcal{M}_k^s \subset \sigma_k^0$, which is the graph of a function x_2 vs. (x_1, y) such that $x_2 = o(1)$ as $k \to \infty$. The map $T^{(k)}\big|_{\mathcal{M}_k^s}$ has the form*

$$\begin{aligned} \bar{x}_1 &= y, \\ \bar{y} &= M - y^2 + Bx_1 + \frac{2J_1}{B}(\lambda^2\gamma)^k \left(x_1 y + o(1)\right), \end{aligned} \tag{1.7}$$

where $J_1 \neq 0$ is some constant (namely, J_1 is the Jacobian of the global map T_1, taken at the homoclinic point M^- at $\varepsilon = 0$).

The maps of the form (1.7) are called generalized Hénon maps. They were introduced in [20, 22] where it was shown, in particular, that they undergo a non-degenerate Andronov-Hopf bifurcation and have a stable closed invariant curve for the values of parameters (M, B) from some open regions (see Section 4).

The paper is organized as follows. In Section 2, appropriate formulas are obtained for the local and global maps $T_0(\varepsilon)$ and $T_1(\varepsilon)$. In Section 3 the first-return maps are studied and Lemmas 1 and 2 are proved. In Section 4 the analysis of the maps (1.2)–(1.5) and (1.7) is conducted and Theorems 1–4 are proved.

2 Properties of the local and global maps

In order to study the first return maps $T^{(k)} = T_1 T_0^k$ at all large k and small ε, we will need appropriate formulas for the maps T_0 and T_1. Here, naturally, the main attention is paid to the form of the local map $T_0(\varepsilon)$. This map, at all small parameter values, has a fixed point O_ε which we assume to be in the origin of coordinates. By choosing the coordinate axes appropriately we may write the map $T_0(\varepsilon)$ in the form (1.1). Moreover, by a C^r-smooth transformation of coordinates we may straighten the local stable and unstable manifolds of O_ε. This brings T_0 to the following form:

$$\bar{x} = A(\varepsilon)x + p(x, y, \varepsilon), \quad \bar{y} = B(\varepsilon)y + q(x, y, \varepsilon), \tag{2.1}$$

where the C^r-smooth functions p and q vanish at the origin along with the first derivatives; moreover, $p(0, y, \varepsilon) \equiv 0$, $q(x, 0, \varepsilon) \equiv 0$. In this case $W^s_{loc} = \{y = 0, v = 0\}$, $W^u_{loc} = \{x = 0, y = 0\}$. Note that the straightening alone of the manifolds W^s_{loc} and W^u_{loc} is not sufficient for our purposes. In essence, the problem is that the right-hand sides of (2.1) contain too many non-resonant terms. However, with the help of some additional coordinate transformation a significant portion of these terms can be killed. Namely, the following lemma holds.

Lemma 3 *Let $r \geq 3$. For all sufficiently small ε, one can introduce C^{r-1}-coordinates (x, u, y, v) in U_0, which are C^{r-2} with respect to parameters, such that the map $T_0(\varepsilon)$ is written in these coordinates as*

$$\bar{x} \;=\; A(\varepsilon)x + P(x, y, \varepsilon)x, \quad \bar{y} \;=\; B(\varepsilon)y + Q(x, y, \varepsilon)y, \tag{2.2}$$

where

$$P(0, y, \varepsilon) = P(x, 0, \varepsilon) \equiv 0, \quad Q(x, 0, \varepsilon) = Q(0, y, \varepsilon) \equiv 0. \tag{2.3}$$

The main advantage here is that in the coordinates of Lemma 3 the map $T_0^k : U_0 \to U_0$, when written in the so-called "cross-form" is linear to the leading order for all sufficiently large k. Namely, let $T_0(\varepsilon)$ be in the form (2.2), and let identities (2.3) hold. Let (x_i, y_i), $i = 0, ..., k$, be points in U_0 such that $(x_i, y_i) = T_0(x_{i-1}, y_{i-1})$.

Lemma 4 *For all sufficiently large k and for all sufficiently small ε, the map $T_0^k(\varepsilon) \;:\; (x_0, y_0) \;\to\; (x_k, y_k)$ can be written in the following form:*

$$x_k \;-\; A_1^k(\varepsilon)x_0 \;=\; \hat{\lambda}^k \xi_k(x_0, y_k, \varepsilon), \quad y_0 \;-\; B_1^{-k}(\varepsilon)y_k \;=\; \hat{\gamma}^{-k}\eta_k(x_0, y_k, \varepsilon), \tag{2.4}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are some constants such that $0 < \hat{\lambda} < |\lambda|$, $\hat{\gamma} > |\gamma|$; the functions ξ_k and η_k are uniformly bounded for all k, along with all the derivatives with respect to the coordinates and parameters up to the order $(r - 2)$.

The proof of Lemmas 3 and 4 can be found, for different cases, in [9, 10, 32].

Concerning the global map $T_1(\varepsilon)$, we will also find now a convenient form for it, using conditions C and D of the quadraticity of the homoclinic tangency. Recall also that the condition of the transversality of the family f_ε to the bifurcational surface $\mathcal{H}$ means that among the parameters ε we can select the parameter μ which measures the splitting of the invariant manifolds of O near the chosen homoclinic point M^+. In this case, the global map $T_1(\varepsilon)$ can be written in the form described in the following lemma.

Lemma 5 *The coordinates defined in Lemma 3 can be introduced in U_0 in such a way that the global map $T_1(\varepsilon)$ will have the following form for all small ε:*

— in the case (1,1) (here $x \in R^1$, $y \in R^1$)

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y - y^-) + \dots, \\ \bar{y} &= \mu + cx + D_0(y - y^-)^2 + \dots, \end{aligned} \tag{2.5}$$

— in the case (2,1) (here $x \in R^2$, $y \in R^1$)

$$\begin{aligned} \bar{x} - x^+ &= ax + \binom{b_0}{0}(y - y^-) + \dots, \\ \bar{y} &= \mu + c_1x_1 + c_2x_2 + D_0(y - y^-)^2 + \dots, \end{aligned} \tag{2.6}$$

— in the case (1,2) (here $x \in R^1$, $y \in R^2$)

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= \mu + cx + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots, \end{aligned} \tag{2.7}$$

— in the case (2,2) (here $x \in R^2$, $y \in R^2$)

$$\begin{aligned} \bar{x} - x^+ &= ax + \binom{b_0}{0}(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= \mu + c_1x_1 + c_2x_2 + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + e_1x_1 + e_2x_2 + \dots, \end{aligned} \tag{2.8}$$

where $b_0 \neq 0$, $c \neq 0$, $D_0 \neq 0$, $d_2 \neq 0$, $x^+ \neq 0$, $y^- \neq 0$.

Note that formulas (2.7) and (2.8) represent the global map T_1 in the cross-form with respect to the coordinate y_2, i.e. the right-hand sides are functions of (x, y_1) and $\bar{y}_2$.

In essence, formulas (2.5)–(2.8) are Taylor expansions with an appropriately chosen $y^-(\varepsilon)$; the dots stand for nonlinear terms (except for the quadratic term which is written explicitly). Note also that the coefficients $a, \dots, e_2$, as well as x^+ and y^-, and the terms denoted by dots, depend on the parameters ε. The corresponding class of smoothness with respect to ε is here C^{r-3}: in the coordinates of Lemma 3 the map T_1, along with its first derivative with respect to (x, y), is C^{r-2}-smooth with respect to ε (see [32]), therefore the coefficient $D_0(\varepsilon)$ of the quadratic term is C^{r-3}-smooth.

Proof of Lemma 5. Let the coordinates of Lemma 3 be introduced in U_0. Let $x^+ \neq 0$ and $y^- \neq 0$ be the coordinates of the homoclinic points $M^+ \in W^s_{loc}$ and $M^- \in W^u_{loc}$, i.e. $M^+ = M^+(x^+, 0)$ and $M^- = M^-(0, y^-)$. Since $T_1 M^- = M^+$ at $\varepsilon = 0$, the map $T_1(\varepsilon)$ can be written in the following form for all small ε:

$$\begin{aligned} \bar{x} - x^+(\varepsilon) \ &= \ \hat{a}x + \hat{b}(y - y^-(\varepsilon)) + ..., \\ \bar{y} \ &= \ y^+(\varepsilon) + \hat{c}x + \hat{d}(y - y^-(\varepsilon)) + ..., \end{aligned} \tag{2.9}$$

where the dots stand for the nonlinear terms, all the coefficients depend on ε, and $y^+(0) = 0$. Moreover,

$$det \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \neq 0. \tag{2.10}$$

Let us find restrictions on the coefficients in (2.9) imposed by condition C. It means that the manifold $T_1 W^u_{loc}$ has, at $\varepsilon = 0$, exactly one common tangent vector with W^s_{loc} at the point M^+. Since the equation of W^u_{loc} is $x = 0$, and the equation of W^s_{loc} is $\bar{y} = 0$, it follows from (2.9) that the intersection of tangent spaces of $T_1 W^u_{loc}$ and W^s_{loc} at the point M^+ is one-dimensional if and only if the equation $\hat{d}(y - y^-) = 0$ has a one-parameter family of solutions at $\varepsilon = 0$. Thus, in the case where $y \in R^1$ and $\hat{d}$ is a scalar (i.e. in the cases (1,1) and (2,1)) we have

$$\hat{d} \ = \ 0 \quad \text{at} \quad \varepsilon = 0. \tag{2.11}$$

If $y \in R^2$ (the cases (1,2) and (2,2)), then $\hat{d}$ is a (2×2)-matrix, and we have

$$\det \hat{d} \ = \ 0 \quad \text{and} \quad \text{rank}\, \hat{d} = 1 \quad \text{at} \quad \varepsilon = 0. \tag{2.12}$$

In case $y \in R^1$, the second equation of (2.9) can be written as

$$\bar{y} \ = \ y^+(\varepsilon) + \hat{c}x + D_0(y - y^-)^2 + \dots, \tag{2.13}$$

where we wrote explicitly the linear terms and one second order term. Since $x = 0$ on W^u_{loc}, we obtain that the equation of $T_1 W^u_{loc}$ is

$$\begin{cases} \bar{x} - x^+(\varepsilon) = \hat{b}(y - y^-) + \dots, \\ \bar{y} \ = \ y^+(\varepsilon) + D_0(y - y^-)^2 + \dots. \end{cases} \tag{2.14}$$

This is a parabola-like curve, parametrized by the coordinate y on W^u_{loc}. Obviously, condition D of the quadraticity of the tangency of this curve to the plane W^u_{loc} : $\{\bar{y} = 0\}$ at $\varepsilon = 0$ means that $D_0 \neq 0$. Note that the right-hand side of (2.13) does not contain the term linear in $(y - y^-)$: since $D_0 \neq 0$, this term can be killed at all small ε by means of an appropriate choice of $y^-(\varepsilon)$.

Thus, in the case (1,1) we indeed obtain formula (2.5) for the map T_1, where $b_0 = \hat{b}, c = \hat{c}$, and $b_0 c \neq 0$ by virtue of (2.10) and (2.12). Note also that we may put

$\mu = y^+(\varepsilon)$ in (2.5) because $y^+(\varepsilon)$ measures the splitting of the manifolds $W^s(O)$ and $W^u(O)$ near the homoclinic point M^+ (see (2.14)).

In the case (2,1), in order to obtain formula (2.6), we make a linear rotation in the x-plane (this coordinate transformation, obviously, does not destroy identities (2.3)) so that the vector $b = (\hat{b}_1, \hat{b}_2)$ transforms into $(b_0, 0)$ where $b_0 = \sqrt{\hat{b}_1^2 + \hat{b}_2^2} \neq 0$. It is easy to see that this is achieved by means of the rotation $x \mapsto R_\omega x$ where $\omega =$ arctan $(-\hat{b}_2/\hat{b}_1)$. Note that this gives us

$$c_1 = \frac{\hat{b}_1\hat{c}_1 - \hat{b}_2\hat{c}_2}{b_0}, \quad c_2 = \frac{\hat{b}_2\hat{c}_1 + \hat{b}_1\hat{c}_2}{b_0},$$

so $c_1^2 + c_2^2 \neq 0$ by virtue of (2.10).

Consider now the case where $y \in R^2$ (i.e. the cases (1,2) and (2,2)). Equations for $\bar{y}$ from (2.9) will have the following form:

$$\begin{aligned} \bar{y}_1 &= y_1^+(\varepsilon) + \hat{c}_1 x + \hat{d}_{11}(y_1 - y_1^-) + \hat{d}_{12}(y_2 - y_2^-) + \ldots, \\ \bar{y}_2 &= y_2^+(\varepsilon) + \hat{c}_2 x + \hat{d}_{21}(y_1 - y_1^-) + \hat{d}_{22}(y_2 - y_2^-) + \ldots. \end{aligned} \tag{2.15}$$

Note that the rotation in the y-plane does not change the form of equations (2.15), but the coefficients may change. At $\varepsilon = 0$, since det $\hat{d} = 0$, we may rotate the y-coordinates so that the following equalities will be fulfilled:

$$\hat{d}_{11} = 0, \quad \hat{d}_{12} = 0. \tag{2.16}$$

Without loss of generality we will assume that these equalities hold at $\varepsilon = 0$ from the very beginning. Since rank $\hat{d} = 1$ at $\varepsilon = 0$, it follows that at least one of the coefficients $\hat{d}_{21}$ or $\hat{d}_{22}$ is non-zero. Assume that

$$\hat{d}_{22} \neq 0. \tag{2.17}$$

If this is not the case (i.e. if $\hat{d}_{22} = 0$ and, hence, $\hat{d}_{21} \neq 0$), we will take another homoclinic point, namely, the point $T_0^{-1}(M^-)$, and we will consider it as the new point M^-. For the new global map ($T_{1new} = T_1 T_0$), the new matrix $\hat{d}$ will be written as

$$\hat{d}_{new} = \hat{d} \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

By (2.16),

$$\hat{d}_{new} = \begin{pmatrix} 0 & 0 \\ \hat{d}_{21}\cos\varphi + \hat{d}_{22}\sin\varphi & -\hat{d}_{21}\sin\varphi + \hat{d}_{22}\cos\varphi \end{pmatrix}.$$

Thus, if $\hat{d}_{22} = 0$, then by choosing the new homoclinic point we will indeed obtain (2.17) (since $\hat{d}_{21} \neq 0$ and $\sin\varphi > 0$).

Let us now take into account quadratic terms as well. Then the equation for $\bar{y}_1$ from (2.15) will take the following form at $\varepsilon = 0$:

$$\bar{y}_1 = c_1 x + D_1(y_1 - y_1^-)^2 + D_2(y_1 - y_1^-)(y_2 - y_2^-) + D_3(y_2 - y_2^-)^2 + \ldots. \tag{2.18}$$

Since $\hat{d}_{22} \neq 0$, the second equation in (2.15) can be resolved with respect to $(y_2 - y_2^-)$. Correspondingly, we have at $\varepsilon = 0$:

$$y_2 - y_2^- \;=\; d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots, \tag{2.19}$$

where $d_1 = -\hat{d}_{21}/\hat{d}_{22}$, $d_2 = \hat{d}_{22}^{-1}$. By plugging (2.19) in (2.18), we obtain

$$\bar{y}_1 \;=\; c_1 x + D_0(y_1 - y_1^-)^2 + \tilde{D}_1(y_1 - y_1^-)\bar{y}_2 + \tilde{D}_2\bar{y}_2^2 + \dots, \tag{2.20}$$

where

$$D_0 \equiv D_1 + d_1 D_2 + d_1^2 D_3, \tag{2.21}$$

and $\tilde{D}_{1,2}$ are some coefficients. Thus the map T_1 is written in the following cross-form at $\varepsilon = 0$:

$$\begin{aligned} \bar{x} - x^+ &= ax + b_0(y_1 - y_1^-) + b_1\bar{y}_2 + \dots, \\ \bar{y}_1 &= cx + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + ex + \dots \end{aligned} \tag{2.22}$$

in the case (1,2), and

$$\begin{aligned} \bar{x}_1 - x_1^+ &= a_{11}x_1 + a_{12}x_2 + b_0(y_1 - y_1^-) + b_{11}\bar{y}_2 + \dots, \\ \bar{x}_2 - x_2^+ &= a_{21}x_1 + a_{22}x_2 + b_{12}\bar{y}_2 + \dots, \\ \bar{y}_1 &= c_1x_1 + c_2x_2 + D_0(y_1 - y_1^-)^2 + \dots, \\ y_2 - y_2^- &= d_1(y_1 - y_1^-) + d_2\bar{y}_2 + e_1x_1 + e_2x_2 + \dots \end{aligned} \tag{2.23}$$

in the case (2,2), with some new coefficients a, b, c, d, e (in the case (2,2) we make the coefficient of $(y_1 - y_1^-)$ in the equation for $\bar{x}_2$ equal to zero by means of an appropriate rotation in the x-coordinates, in the same way as in the case (2,1)). Condition (2.10) recasts now as

$$det\ \frac{\partial(\bar{x}, \bar{y}_1)}{\partial(x, y_1)} \neq 0, \tag{2.24}$$

which gives us, in both cases, $b_0 \neq 0$, $c \neq 0$.

Since $x = 0$ on W^u_{loc}, it follows from (2.22), (2.23) that $T_1 W^u_{loc}$ is given by the following equations near the point M^+:

$$y_1 \;=\; \frac{D_0}{b_0^2}(x - x^+)^2 + \dots \tag{2.25}$$

in the case (1,2), and

$$\begin{aligned} x_2 - x_2^+ &= b_{12}y_2 + \dots, \\ y_1 &= \frac{D_0}{b_0^2}(x_1 - x_1^+)^2 + \dots \end{aligned} \tag{2.26}$$

in the case (2,2). In any case it is obvious that condition D of the quadraticity of the tangency of this surface with $W^s_{loc} : y = 0$ is equivalent to the condition $D_0 \neq 0$.

At $\varepsilon \neq 0$ the map T_1 is still given by the equations (2.22) and (2.23): since $D_0(\varepsilon) \neq 0$ at all small ε, we may always choose $y_1^-(\varepsilon)$ and $y_2^-(\varepsilon)$ and rotate additionally the

y-coordinates so that the coefficients $d_{11}(\varepsilon)$ and $d_{12}(\varepsilon)$ will vanish identically for all small ε. The only difference with the case $\varepsilon = 0$ is that a non-zero constant term $y_1^+(\varepsilon)$ appears in the equation for $\bar{y}_1$. As before, the condition of transversality of the family f_ε to the bifurcational surface $\mathcal{H}$ allows us to assume $y_1^+(\varepsilon) = \mu$. This finishes the proof of Lemma 5.

3 Proof of rescaling lemmas

In this Section we study the first-return maps

$$T^{(k)}(\varepsilon) \equiv T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$$

for all sufficiently large $k \ : k = \bar{k}, \bar{k}+1, \ldots,$ and small ε, $\|\varepsilon\| \leq \varepsilon_0$. We will use formula (2.4) from Lemma 4 for the map $T_0^k : \sigma_k^0 \to \sigma_k^1$, with $(x_0, y_0) \in \Pi^+, (x_k, y_k) \in \Pi^-$. For the global map $T_1(\varepsilon)$ we will use the corresponding formulas from Lemma 5. According to Lemma 4, for all small x_0, y_k and any sufficiently large k the corresponding coordinates x_k, y_0 are defined uniquely. Therefore, we may use (x_0, y_k) as the coordinates in σ_k^0; the coordinate y_0 is computed by formula $y_0 = B_1^{-k}(\varepsilon) y_k + \hat{\gamma}^{-k} \eta_k(x_0, y_k, \varepsilon)$ (see Lemma 4). Note that the size of the strip σ_k^0 in the new coordinates (x_0, y_k) is bounded away from zero in all directions, for all k. Thus, if we define the neighborhoods Π^+ and Π^- as $\{\|x - x^+\| \leq \rho_0, \|y\| \leq \rho_0\}$ and $\{\|x\| \leq \rho_0, \|y - y^-\| \leq \rho_0\}$ respectively, where ρ_0 is a small positive constant, then each strip σ_k^0 is defined as $\{\|x_0 - x^+\| \leq \rho_0, \|y_k - y^-\| \leq \rho_0\}$.

3.1 First-return maps in the case (1,1)

Here, the coordinates x and y are one-dimensional, $A = \lambda, B = \gamma$. By (2.4) and (2.5), the first-return map $T^{(k)} \equiv T_1 T_0^k$ takes the following form for all sufficiently large k and small ε:

$$\begin{aligned} \bar{x}_0 - x^+(\varepsilon) \ &= \ a\lambda^k x_0 + b_0(y_k - y^-) + O\left((y_k - y^-)^2 + |\lambda|^k |x_0||y_k - y^-| + \hat{\lambda}^k |x_0|\right), \\ \gamma^{-k}\bar{y}_k + \hat{\gamma}^{-k} O(|\bar{x}_0| + |\bar{y}_k|) \ &= \ \mu + c x_0 \lambda^k + D_0 (y_k - y^-)^2 + \\ &\quad + O\left((y_k - y^-)^3 + |\lambda|^k |x_0||y_k - y^-| + \hat{\lambda}^k |x_0|\right), \end{aligned} \tag{3.1}$$

where $b_0 \neq 0$, $c \neq 0$, $D_0 \neq 0$. Note that we, hereafter, choose $\hat{\lambda}$ sufficiently close to $|\lambda|$ (it is always less than $|\lambda|$, of course), so that $\hat{\lambda} > \lambda^2$, in particular.

Let us shift the origin of the coordinates:

$$x = x_0 - x^+(\varepsilon) + \tilde{\nu}_k^1, \ y = y_k - y^- + \tilde{\nu}_k^2,$$

in such a way that the first equation of (3.1) would not contain constant terms (i.e. those which depend only on ε), and the second equation would not contain the linear in y term. This can always be achieved by a proper choice of $\tilde{\nu}_k^j = O(\lambda^k + \hat{\gamma}^{-k})$. As a result, system (3.1) is rewritten as

$$\begin{aligned} \bar{x} &= O(|\lambda|^k|x| + |y|), \\ \bar{y} + (\hat{\gamma}/\gamma)^{-k} O(|\bar{x}| + |\bar{y}|) &= \gamma^k M_1 + \tilde{D}_0 \gamma^k y^2 + \gamma^k O(|y|^3 + |\lambda|^k|x|), \end{aligned} \tag{3.2}$$

where

$$M_1 \equiv \mu - \gamma_1^{-k} y^-(1 + \ldots) + c\lambda_1^k x^+(1 + \ldots),$$

and $\tilde{D}_0 = D_0(1 + \beta_k)$, where $\beta_k = O(\lambda^k + \hat{\gamma}^{-k})$ is a small quantity.

Let us now scale the coordinates as follows:

$$x = -\frac{\gamma^{-k}}{\rho^k} x_{new}, \; y = -\frac{1}{\tilde{D}_0} \gamma^{-k} y_{new}, \tag{3.3}$$

where ρ is a number from the interval

$$\max \left\{ |\lambda\gamma|, |\gamma|^{-1} \right\} < \rho < 1. \tag{3.4}$$

Since $|\lambda\gamma| < 1$ and $|\gamma| > 1$, such ρ exist indeed, and the scaling factors in (3.3) are asymptotically small as $k \to \infty$. Hence, since the size of the strip σ_k^0 in the coordinates (x_0, y_k) is bounded away from zero, the range of values of the rescaled coordinates (x, y) becomes unboundedly large as k grows.

In the new coordinates, system (3.2) recasts as

$$\begin{aligned} \bar{x} &= O(\rho^k|y| + |\lambda|^k|x|), \\ \bar{y} + (\hat{\gamma}/\gamma)^{-k} O(\rho^{-k}|\bar{x}| + |\bar{y}|) &= -\tilde{D}_0 \gamma^{2k} M_1 - y^2 + O\left(|\gamma|^{-k}|y|^3 + \frac{|\lambda\gamma|^k}{\rho^k}|x| \right). \end{aligned} \tag{3.5}$$

Now, by virtue of (3.4), taking into account that $|\lambda\gamma| < 1$, $\hat{\lambda} < |\lambda|$, system (3.5) is immediately brought to the sought form (1.2), where we put

$$M = -\tilde{D}_0 \gamma_1^{2k} [\mu - \gamma_1^{-k} y^-(1 + \ldots) + c\lambda_1^k x^+(1 + \ldots)]. \tag{3.6}$$

Note that the parameter M, as well as the coordinates (x, y), may now take arbitrary finite values at large k.

3.2 First-return maps in the case (2,1)

Here $x = (x_1, x_2)$ is two-dimensional, y is one-dimensional, and

$$A \equiv \lambda \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \; B \equiv \gamma.$$

By (2.4) and (2.6), the first-return map $T^{(k)} \equiv T_1 T_0^k$ is written in the following form for all large k and small ε:

$$
\begin{aligned}
\bar{x}_{01} - x_1^+(\varepsilon) &= \lambda^k A_{11}(k\varphi)x_{01} + \lambda^k A_{12}(k\varphi)x_{02} + b_0(y_k - y^-) + \\
&\qquad + O\left((y_k - y^-)^2 + \lambda^k \|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right), \\
\bar{x}_{02} - x_2^+(\varepsilon) &= \lambda^k A_{21}(k\varphi)x_{01} + \lambda^k A_{22}(k\varphi)x_{02} + \\
&\qquad + O\left((y_k - y^-)^2 + \lambda^k \|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right), \\
\gamma^{-k}\bar{y}_k + \hat{\gamma}^{-k}\eta_k(\bar{x}_0, \bar{y}_k, \varepsilon) &= \mu + D_0(y_k - y^-)^2 + \\
&\quad + \lambda^k\left[x_{01}(c_1\cos k\varphi + c_2 \sin k\varphi)x_{01} + (c_2\cos k\varphi - c_1\sin k\varphi)x_{02}\right] + \\
&\qquad + O\left((y_k - y^-)^3 + \lambda^k \|x_0\||y_k - y^-| + \hat{\lambda}^k\|x_0\|\right),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
A_{11}(k\varphi) &= a_{11}\cos k\varphi - a_{12}\sin k\varphi, & A_{12}(k\varphi) &= a_{12}\cos k\varphi + a_{11}\sin k\varphi, \\
A_{21}(k\varphi) &= a_{21}\cos k\varphi + a_{22}\sin k\varphi, & A_{22}(k\varphi) &= a_{22}\cos k\varphi - a_{21}\sin k\varphi.
\end{aligned}
\tag{3.8}
$$

Let us shift the origin of coordinates:

$$
x_1 = x_{01} - x_1^+(\varepsilon) + \tilde{\nu}_k^1, \quad x_2 = x_{02} - x_2^+(\varepsilon) + \tilde{\nu}_k^2, \quad y = y_k - y^-(\varepsilon) + \tilde{\nu}_k^3.
$$

We do it in such a way that the first and second equations of (3.7) will not contain constant terms and the third equation will not contain the linear in y term (here $\tilde{\nu}_k^i = O(\lambda^k + \hat{\gamma}^{-k})$). Let us also do the following: in all terms in the left-hand side of the third equation of (3.7) which do not depend on $\bar{y}$ we change $\bar{x}_{01}$ and $\bar{x}_{02}$ to their expressions from the first and second equation. Then, system (3.7) recasts as

$$
\begin{aligned}
\bar{x}_1 &= \lambda^k A_{11}(k\varphi)x_1 + \lambda^k A_{12}(k\varphi)x_2 + b_0 y + O(y^2 + \lambda^k|y| + \hat{\lambda}^k\|x\|), \\
\bar{x}_2 &= \lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + O(y^2 + \lambda^k|y| + \hat{\lambda}^k\|x\|), \\
\bar{y} + (\hat{\gamma}/\gamma)^{-k}O(|\bar{y}|) &= \gamma^k M_1 + \tilde{D}_0\gamma^k y^2 + \lambda^k\gamma^k\Big[(c_1\cos k\varphi + c_2\sin k\varphi)x_1 + \\
&\quad +(c_2\cos k\varphi - c_1\sin k\varphi)x_2 + \left((\hat{\lambda}/\lambda)^k + \hat{\gamma}^{-k}\right)O(\|x\|)\Big] + \gamma^k O\left(|y|^3 + \lambda^k\|x\||y|\right),
\end{aligned}
\tag{3.9}
$$

where

$$
M_1 \equiv \mu - \gamma^{-k}y^-(1 + \ldots) + C_0\lambda^k\left(\cos(k\varphi + \vartheta_1) + \ldots\right),
$$

and

$$
C_0 = \sqrt{(c_1^2 + c_2^2)({x_1^+}^2 + {x_2^+}^2)}, \quad \sin\vartheta_1 = \frac{c_1 x_2^+ - c_2 x_1^+}{C_0}, \quad \cos\vartheta_1 = \frac{c_1 x_1^+ + c_2 x_2^+}{C_0};
\tag{3.10}
$$

$\tilde{D}_0 = D_0(1 + \beta_k)$ where $\beta_k = O(\lambda^k + \hat{\gamma}^{-k})$ is some small coefficient.

Consider the case $|\lambda\gamma| < 1$ first. Like in the case (1,1), we scale the coordinates in the following way:

$$
x = \frac{\gamma^{-k}}{\rho^k}x_{new}, \; y = -\tilde{D}_0^{-1}\gamma^{-k}y_{new},
$$

where ρ is a number from the interval (3.4). In the new coordinates, system (3.9) takes the form

$$\begin{aligned}
\bar{x}_1 &= \rho^k O(y) + \lambda^k O(x) \\
\bar{x}_2 &= \rho^k \gamma^{-k} O(y^2) + \lambda^k O(x), \\
\bar{y} + (\hat{\gamma}/\gamma)^{-k} O(\bar{y}) &= M - y^2 + \frac{\lambda^k \gamma^k}{\rho^k} O(x) + |\gamma|^{-k} O(y^3),
\end{aligned} \tag{3.11}$$

where

$$M \equiv -\tilde{D}_0 \gamma^{2k} \left[\mu - \gamma^{-k} y^-(1 + \ldots) + \lambda^k (C_0 \cos(k\varphi + \vartheta_1) + \ldots) \right]. \tag{3.12}$$

By virtue of (3.4), it is obvious that after we resolve the last equation with respect to $\bar{y}$ the map (3.11) is immediately brought to the sought form (1.2).

Consider now the case $|\lambda\gamma| > 1$ (and $|\lambda^2\gamma| < 1$ here, as before). We scale the coordinates in (3.9) in the following way:

$$x_1 = -(b_0 \tilde{D}_0^{-1}) \gamma^{-k} x_{1new}, \ x_2 = -\rho^k (b_0 \tilde{D}_0^{-1}) \gamma^{-k} x_{2new}, \ y = -\tilde{D}_0^{-1} \gamma^{-k} y_{new},$$

where ρ is some constant from the interval

$$|\gamma|^{-1} < \lambda < \rho < |\lambda\gamma|^{-1}, \tag{3.13}$$

which is non-empty because

$$1 > \frac{1}{|\lambda\gamma|} = \frac{\lambda}{|\lambda^2\gamma|} > \lambda$$

(recall that $|\lambda^2\gamma| < 1$).

In the new coordinates, system (3.9) takes the form

$$\begin{aligned}
\bar{x}_1 &= y + \lambda^k O(\|x\| + |y|), \\
\bar{x}_2 &= \rho^{-k} \lambda^k A_{21}(k\varphi) x_1 + \lambda^k A_{22}(k\varphi) x_2 + \rho^{-k} \lambda^k O(y) + \rho^{-k} \hat{\lambda}^k O(x), \\
\bar{y} + (\hat{\gamma}/\gamma)^{-k} O(\bar{y}) &= M - y^2 + \\
&\quad + \lambda^k \gamma^k b_0 \left\{ (c_1 \cos k\varphi + c_2 \sin k\varphi + \nu_k^1) x_1 + \rho^k (c_2 \cos k\varphi - c_1 \cos k\varphi + \nu_k^2) x_2 \right\} + \\
&\quad + O\left(|\gamma|^{-k} |y|^3 + \lambda^k \|x\| |y| + (\hat{\lambda}^k + \lambda^k \hat{\gamma}^{-k}) \|x\|^2 \right),
\end{aligned} \tag{3.14}$$

where the parameter M again satisfies formula (3.12), and ν_k^1, ν_k^2 are some small coefficients, $\nu_k^{1,2} = O\left((\hat{\lambda}/\lambda)^k + \hat{\gamma}^{-k} \right)$. Let us resolve the third equation of (3.14) with respect to $\bar{y}$. The right-hand sides will keep their form, but a coefficient of order $1 + O\left((\hat{\gamma}/\gamma)^{-k} \right)$ will appear in front of the $(-y^2)$ term in the third equation. We can make this coefficient equal to 1 again by additional rescaling the coordinate

y: $y_{new} = y(1 + \hat{\beta}_k)$, where $\hat{\beta}_k = O\left((\hat{\gamma}/\gamma)^{-k}\right)$ is some small quantity. After that, system (3.14) will take the form

$$
\begin{aligned}
\bar{x}_1 &= y + \lambda^k O(\|x\| + |y|), \\
\bar{x}_2 &= \rho^{-k}\lambda^k A_{21}(k\varphi)x_1 + \lambda^k A_{22}(k\varphi)x_2 + \rho^{-k}\lambda^k O(y) + \rho^{-k}\hat{\lambda}^k O(\|x\|), \\
\bar{y} &= M - y^2 + \\
&\quad + \lambda^k\gamma^k b_0 \left\{(c_1 \cos k\varphi + c_2 \sin k\varphi + \nu_k^1)x_1 + \rho^k(c_2 \cos k\varphi - c_1 \cos k\varphi + \nu_k^2)x_2\right\} + \\
&\qquad + O\left(|\gamma|^{-k}|y|^3 + \lambda^k\|x\||y| + \hat{\lambda}^k\|x\|^2\right),
\end{aligned}
\tag{3.15}
$$

where the new coefficients M and $\nu_k^{1,2}$ differ from the old ones by small quantities of order $O\left((\hat{\gamma}/\gamma)^{-k}\right)$; the coefficient b_0 is kept unchanged.

Since $|\lambda\gamma| > 1$, the coefficient

$$
B_k(\varphi) \equiv b_0\lambda^k\gamma^k(c_1 \cos k\varphi + c_2 \sin k\varphi + \nu_k^1) \tag{3.16}
$$

from the third equation of (3.15) is no longer small. Nevertheless, since $c \neq 0$ (see Lemma 5), $B_k(\varphi)$ may take arbitrary finite values for sufficiently large k, when the parameter φ varies near those values where $c_1 \cos k\varphi + c_2 \sin k\varphi = 0$, i.e. near

$$
\varphi = -\frac{1}{k}\arctan\left(\frac{c_1}{c_2}\right) + \pi\frac{j}{k}, \quad j \in Z. \tag{3.17}
$$

Note that the values (3.17) of the angle φ for all possible k and j fill the interval $(0, \pi)$ densely.

We denote $B = B_k(\varphi)$, stressing that B is one more governing parameter, along with M. Note that M takes arbitrary finite values when μ varies near $\mu_k^0 = \gamma_1^{-k}y^- - C_0\lambda^k \cos(k\varphi_0 + \vartheta_1)$ (see formula (3.12)).

Introduce a new coordinate $y_{new} = y + \lambda^k O(\|x\| + |y|)$ so that we would have $\bar{x}_1 = y$. Then, by virtue of (3.13), the map (3.15) takes the sought form (1.3).

3.3 Proof of Lemma 2

Here we continue to study the case (2,1) with $|\lambda\gamma| > 1$ and $|\lambda^2\gamma| < 1$. Assume that $B \neq 0$ in (3.15). Since $\lambda^k\gamma^k\rho^k \to 0$ as $k \to \infty$, we may introduce a new coordinate

$$
x_{1new} = x_1 + \frac{1}{B}b_0\lambda^k\gamma^k\rho^k(c_2 \cos k\varphi - c_1 \cos k\varphi + \nu_k^2)x_2.
$$

Map (3.15) then takes the form

$$\begin{aligned} \bar{x}_1 &= y + \frac{b_0 A_{21}(k\varphi)}{B}(c_2\cos k\varphi - c_1 \sin k\varphi + \nu_k^2)\lambda^{2k}\gamma^k x_1 + O(\lambda^k), \\ \bar{x}_2 &= O\left(\frac{\lambda^k}{\rho^k}\right), \\ \bar{y} &= M - y^2 + Bx_1 + O(\lambda^k). \end{aligned} \tag{3.18}$$

Note that this map is exponentially contracting in the x_2-direction (with the contraction coefficient of order $O(\lambda^k\rho^{-k})$), while in those regions of the phase space where there is a contraction in the x_1- and y-directions the corresponding contraction coefficient is bounded away from zero at $B \neq 0$. Thus, theorem 4.4 of [32] implies that for any $Q, R > 0$, for all sufficiently large k, map (3.18) in the region $\|(x,y)\| \le Q$ has, at $\|(M,B)\| \le R$, $|B| > 1/R$, a C^{r-2}-smooth, asymptotically stable, invariant non-local center manifold $\mathcal{M}_k^c$ of the form $x_2(x_1, y, M, B) = O(\lambda^k\rho^{-k})$. The map (3.18) on $\mathcal{M}_k^c$ is written as follows:

$$\begin{aligned} \bar{x}_1 &= y + \frac{b_0 A_{21}(k\varphi)}{B}(c_2\cos k\varphi - c_1 \sin k\varphi + \nu_k^2)\lambda^{2k}\gamma^k x_1 + O(\lambda^k), \\ \bar{y} &= M - y^2 + Bx_1 + O(\lambda^k). \end{aligned} \tag{3.19}$$

In the region where B is uniformly bounded, $|B| < Q$, we found from (3.16) that $c_1 \cos k\varphi + c_2 \sin k\varphi = O(\lambda^{-k}\gamma^{-k})$. Since $|\lambda\gamma| > 1$, this gives us

$$c_2 \cos k\varphi - c_1 \sin k\varphi = \pm\sqrt{c_1^2 + c_2^2} + \dots,$$

where the dots stand for the terms tending to zero as $k \to \infty$. Also, we have (see (3.8))

$$A_{21}(k\varphi) = a_{21}\cos k\varphi + a_{22}\sin k\varphi = \pm\frac{a_{21}c_2 - a_{22}c_1}{\sqrt{c_1^2 + c_2^2}} + \dots .$$

Thus,

$$b_0 A_{21}(k\varphi)(c_2\cos k\varphi - c_1 \sin k\varphi + \nu_k^2) = b_0(a_{21}c_2 - a_{22}c_1) + \dots .$$

It is easy to see from (2.6) that the constant $J_1 = b_0(a_{21}c_2 - a_{22}c_1)$ is the Jacobian of the global map T_1, taken at the point $(x = 0, y_1 = y^-)$ at $\varepsilon = 0$. Note also that $\lambda^{2k}\gamma^k$ constitutes the main part of the Jacobian of the local map T_0^k. Denote

$$J_k = J_1\lambda^{2k}\gamma^k.$$

Map (3.19) may be recast as

$$\bar{x}_1 = y + \frac{J_k}{B}x_1 + o(J_k), \quad \bar{y} = M - y^2 + Bx_1 + O(\lambda^k). \tag{3.20}$$

Let us make one more coordinate transformation

$$x_{1new} = x_1, \; y_{new} = y + \frac{J_k}{B}x_1 + o(J_k) \equiv \bar{x}_1.$$

Map (3.20) will take the form

$$\bar{x}_1 \; = \; y, \quad \bar{y} \; = \; M - y^2 + Bx_1 + \frac{J_k}{B} y + \frac{2J_k}{B} x_1 y + o(J_k). \tag{3.21}$$

The following additional shifts of the coordinate y and the parameter M:

$$y_{new} = y - \frac{J_k}{2B} \; , \; M_{new} = M - \frac{J_k^2}{4B^2},$$

brings map (3.21) to the form (1.7). Lemma 2 is proven.

3.4 First-return maps in the case (1,2)

Here x is one-dimensional and $y = (y_1, y_2)$ is two-dimensional;

$$A \equiv \lambda, \quad B \equiv \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

By (2.4) and (2.7), the first-return map $T^{(k)} \equiv T_1 T_0^k$ is written in the following form for all sufficiently large k and all small ε:

$$\begin{aligned} \bar{x}_0 - x^+ \; = \; & a\lambda^k x_0 + b_0(y_{k1} - y_1^-) + b_1\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k2} + \sin k\psi \cdot \bar{y}_{k1}) + \\ & + O\left((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\right), \\ \gamma^{-k}(\cos k\psi \cdot \bar{y}_{k1} - \sin k\psi \cdot \bar{y}_{k2}) \; = \; & \mu + c\lambda^k x_0 + D_0(y_{k1} - y_1^-)^2 + \\ & + O\left((y_{k1} - y_1^-)^3 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\right), \\ y_{k2} - y_2^- \; = \; & e\lambda^k x_0 + d_1(y_{k1} - y_1^-) + d_2\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k2} + \sin k\psi \cdot \bar{y}_{k1}) + \\ & + O\left((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(|\lambda|^k|x_0| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k|x_0| + \hat{\gamma}^{-k}(|\bar{x}_0| + \|\bar{y}_k\|)\right), \end{aligned} \tag{3.22}$$

where, we recall, $0 < \hat{\lambda} < |\lambda|$, $\hat{\gamma} > \gamma$; moreover, we assume that $\hat{\lambda}$ and $\hat{\gamma}$ are sufficiently close to $|\lambda|$ and γ respectively.

Let us shift the origin of coordinates:

$$x_{new} = x_0 - x^+(\varepsilon) + \tilde{\nu}_k^1, \;\; y_{1new} = y_{k1} - y_1^-(\varepsilon) + \tilde{\nu}_k^2, \;\; y_{2new} = y_{k2} - y_2^-(\varepsilon) + \tilde{\nu}_k^3,$$

so that the first and third equations of (3.22) would not contain the constant (i.e. depending only on ε) terms, and the second equation would not contain the linear in y_1 term. Here we have $\tilde{\nu}_k^i(\varepsilon) = O(\gamma^{-k})$. If, in addition, we resolve the first equation with respect to $\bar{x}$ and plug the corresponding expression in the right-hand side of the other equations, the system (3.22) will recast as

$$
\begin{aligned}
\bar{x} &= O\left(|y_1| + |\lambda|^k|x| + \gamma^{-k}\|\bar{y}\|\right),\\
\gamma^{-k}\left\{\cos k\psi\cdot\bar{y}_1 - \sin k\psi\cdot\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|)\right\} &= M_1 + c\lambda^k\, x + \tilde{D}_0\, y_1^2 + \\
&\quad +O\left(|y_1|^3 + |\lambda|^k|x||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1|\right),\\
y_2 - e\lambda^k\, x - \tilde{d}_1 y_1 &= d_2\gamma^{-k}\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + \nu_k^2)\bar{y}_1\} + \\
&\quad +O\left(y_1^2 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\lambda}^k|x| + \hat{\gamma}^{-k}\|\bar{y}\|^2\right),
\end{aligned}
\tag{3.23}
$$

where $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$, and the coefficients $\tilde{D}_0$ and $\tilde{d}_1$ differ from, respectively, D_0 and d_1 by some small quantities of order $O(\gamma^{-k})$. We also denote here

$$
M_1 \equiv \mu - \gamma^{-k}E_0\ \cos(k\psi - \vartheta_2 + ...) + c\lambda^k(x^+ + ...),
\tag{3.24}
$$

where

$$
E_0 = \sqrt{(y_1^-)^2 + (y_2^-)^2},\ \ \cos\vartheta_2 = y_1^-/E_0,\ \sin\vartheta_2 = y_2^-/E_0.
\tag{3.25}
$$

Make one more coordinate transformation:

$$
x_{new} = x,\ \ y_{1new} = y_1,\ \ y_{2new} = y_2 - \tilde{d}_1 y_1.
$$

Map (3.23) will take the form

$$
\begin{aligned}
\bar{x} &= O\left(|y_1| + |\lambda|^k|x| + \gamma^{-k}\|\bar{y}\|\right),\\
&(\cos k\psi - d_1\sin k\psi)\bar{y}_1 - \sin k\psi\cdot\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\|\bar{y}\|) = \\
&\quad = M_1\gamma^k + c\lambda^k\gamma^k x + \tilde{D}_0\gamma^k y_1^2 + O\left(\gamma^k|y_1|^3 + |\lambda|^k\gamma^k|x||y_1| + \hat{\lambda}^k\gamma^k|x| + \|\bar{y}\||y_1|\right),\\
y_2 - e\lambda^k x &= d_2\gamma^{-k}\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^2)\bar{y}_1\} + \\
&\quad +O\left(y_1^2 + |\lambda|^k|x||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\gamma}^{-k}\|\bar{y}\|^2\right),
\end{aligned}
\tag{3.26}
$$

with some new coefficients $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$.

Introduce new coordinates y_1 and y_2:

$$
\begin{aligned}
y_{1new} &= (\cos k\psi + \nu_k^1)y_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^2)y_1,\\
y_{2new} &= \frac{1}{d_2}\gamma^k\left(y_2 - e\lambda^k x)\right).
\end{aligned}
\tag{3.27}
$$

The old coordinates are expressed via the new ones by the formulas

$$
\begin{aligned}
y_2 &= d_2\gamma^{-k}y_{2new} + e\lambda^k x,\\
y_1 &= \frac{1}{s_0}y_{1new} - \frac{1}{s_0}(\cos k\psi + \nu_k^1)(d_2\gamma^{-k}y_{2new} + e\lambda^k x),
\end{aligned}
\tag{3.28}
$$

where

$$
s_0 \equiv s_0(k\psi) = \sin k\psi + d_1\cos k\psi + \nu_k^2.
\tag{3.29}
$$

We will consider only those ψ for which $s_0 \neq 0$. Then the coordinate transformation (3.27) is non-degenerate, and (3.26) is rewritten in the following form in the new coordinates:

$$\begin{aligned}
\bar{x} &= O(|y_1| + |\lambda|^k|x| + \gamma^{-k}(|y_2| + |\bar{y}|)), \\
\gamma^k\bar{y}_1(\cos k\psi - d_1\sin k\psi + \nu_k^3) - d_2\bar{y}_2 + (\hat{\gamma}/\gamma)^{-k}O(\bar{y}_2) + |\lambda\gamma|^k O(\bar{x}) &= \\
= \gamma^{2k}s_0M_1 + cs_0\lambda^k\gamma^{2k}x + \tilde{D}_0(s_0)^{-1}\gamma^{2k}y_1^2 + \\
+\gamma^{2k}O\left(y_1^3 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\|\|y\| + \hat{\lambda}^k|x| + \gamma^{-k}\|y\|^2 + \hat{\gamma}^{-k}\|\bar{y}\|^2\right), \\
y_2 = \bar{y}_1 + \gamma^k O\left(y_1^2 + |\lambda|^k|x||y_1| + \gamma^{-k}\|\bar{y}\|\|y\| + \gamma^{-k}\|\bar{y}\|^2 + \hat{\lambda}^k|x|\right),
\end{aligned} \tag{3.30}$$

where $\nu_k^3 = O(\hat{\gamma}^{-k}\gamma^k)$ is some small coefficient.

We now scale the coordinates:

$$x = \rho^k\gamma^{-2k}\, x_{new}, \quad y_1 = \frac{d_2s_0}{\tilde{D}_0}\gamma^{-2k}y_{1new}, \quad y_2 = \frac{d_2s_0}{\tilde{D}_0}\gamma^{-2k}y_{2new}, \tag{3.31}$$

where

$$1 < \rho < \frac{1}{|\lambda|\gamma^2}$$

(recall that $|\lambda\gamma^2| < 1$ by assumption, and that we also assume that s_0 is bounded away from zero).

After the scaling (3.31), the map (3.30) takes the form

$$\begin{aligned}
\bar{x}_1 &= \phi_k^1(x, y, \bar{y}), \\
\frac{1}{d_2}\gamma^k\bar{y}_1(\cos k\psi - d_1\sin k\psi + \nu_k^3) - \bar{y}_2 &= \tilde{M} + y_1^2 + \phi_k^2(x, y, \bar{y}), \\
y_2 &= \bar{y}_1 + \phi_k^3(x, y, \bar{y}),
\end{aligned} \tag{3.32}$$

where

$$\tilde{M} = \gamma^{4k}\frac{D_0}{d_2^2}\left[\mu - \gamma^{-k}(y_1^-\cos k\psi - y_2^-\sin k\psi + ...) + c\lambda^k(x^+ + ...)\right], \tag{3.33}$$

and $\phi_k^l = o(1)$ as $k \to \infty$.

Note that the trigonometric coefficient

$$C(k\psi) \equiv \frac{1}{d_2}\gamma^k(\cos k\psi - d_1\sin k\psi + \nu_k^3)$$

from (3.32) may be bounded for large k if only $\cos k\psi - d_1\sin k\psi$ is close to zero, i.e. for those values of ψ which are close to

$$\psi = \frac{1}{k}\ \text{arctan}\ \left(\frac{1}{d_1}\right) + \pi\frac{j}{k}, \quad j \in Z. \tag{3.34}$$

The coefficient s_0 from (3.29) is indeed bounded away from zero for such ψ: $s_0^2 = 1 + d_1^2 + \ldots$.

Note that the values (3.34) of the angle ψ are dense in $(0, \pi)$. This means that given any $Q > 0$, in any neighborhood of any point $\psi_0 \in (0, \pi)$ there exist intervals (of size $\sim Q\gamma^{-k}$) such that when ψ runs any of them the coefficient $C(k\psi)$ runs all the values from the interval $[-Q, Q]$.

In the region of the values of ψ where C is finite, we may resolve system (3.32) with respect to $\bar{y}$. The map $T^{(k)}$ will take the form

$$\begin{aligned} \bar{x}_1 &= \tilde{\phi}_k^1(x, y, M, C), \\ \bar{y}_2 &= M - y_1^2 + Cy_2 + \tilde{\phi}_k^2(x, y, M, C), \\ \bar{y}_1 &= y_2 + \tilde{\phi}_k^3(x, y, M, C), \end{aligned} \tag{3.35}$$

where $M = -\tilde{M}, C = C(k\psi)$, and $\tilde{\phi}_k = o(1)$. By putting $y_{2new} = y_2 + \tilde{\phi}_k^3$, we obtain exactly the map (1.4) from Lemma 1.

3.5 First-return maps in the case (2,2)

Here $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two-dimensional,

$$A \equiv \lambda \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad \text{and} \quad B \equiv \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

By (2.4) and (2.8), the first-return map $T^{(k)} \equiv T_1 T_0^k$ is written in the following form for all sufficiently large k and all small ε:

$$\begin{aligned} \bar{x}_{01} - x_1^+ &= b_0(y_{k1} - y_1^-) + b_{11}\gamma^{-k}(\cos k\psi\; \bar{y}_{k2} + \sin k\psi\; \bar{y}_{k1}) + \\ &\quad + O\left((y_{k1} - y_1^-)^2 + \gamma^{-k}|y_{k1} - y_1^-|\|\bar{y}_k\| + \lambda^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right), \\ \bar{x}_{02} - x_2^+ &= b_{12}\gamma^{-k}(\cos k\psi\; \bar{y}_{k2} + \sin k\psi\; \bar{y}_{k1}) + \\ &\quad + O\left((y_{k1} - y_1^-)^2 + \gamma^{-k}|y_{k1} - y_1^-|\|\bar{y}_k\| + \lambda^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right), \\ \gamma^{-k}(\cos k\psi \cdot \bar{y}_{k1} &- \sin k\psi \cdot \bar{y}_{k2}) = \\ &= \mu + \lambda^k C_1(k\varphi)x_{01} + \lambda^k C_2(k\varphi)x_{02} + D_0(y_{k1} - y_1^-)^2 + \\ &+ O\left((y_{k1} - y_1^-)^3 + |y_{k1} - y_1^-|(\lambda^k\|x_0\| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right), \\ y_{k2} - y_2^- &= \lambda^k E_1(k\varphi)x_{01} + \lambda^k E_2(k\varphi)x_{02} + d_1(y_{k1} - y_1^-) + \\ &\quad + d_2\gamma^{-k}(\cos k\psi \cdot \bar{y}_{k2} + \sin k\psi \cdot \bar{y}_{k1}) + \\ &+ O\left((y_{k1} - y_1^-)^2 + |y_{k1} - y_1^-|(\lambda^k\|x_0\| + \gamma^{-k}\|\bar{y}_k\|) + \hat{\lambda}^k\|x_0\| + \hat{\gamma}^{-k}(\|\bar{x}_0\| + \|\bar{y}_k\|)\right), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} C_1 = c_1\cos k\varphi + c_2\sin k\varphi,\;\; C_2 = c_2\cos k\varphi - c_1\sin k\varphi, \\ E_1 = e_1\cos k\varphi + e_2\sin k\varphi,\;\; E_2 = e_2\cos k\varphi - e_1\sin k\varphi, \end{aligned} \tag{3.37}$$

and, as before, $0 < \hat{\lambda} < \lambda,\;\; \hat{\gamma} > \gamma$.

Introduce new coordinates (a shift of the origin):

$$x_{1new} = x_1 - x_1^+(\varepsilon) + \tilde{\nu}_k^1, \quad x_{2new} = x_2 - x_2^+(\varepsilon) + \tilde{\nu}_k^2,$$
$$y_{1new} = y_{k1} - y_1^- + \tilde{\nu}_k^3, \quad y_{2new} = y_{k2} - y_2^- + \tilde{\nu}_k^4.$$

Here the small shifts $\tilde{\nu}_k^i(\varepsilon)$ (of order $O(|\gamma|^{-k})$) are chosen in such a way that the first, second and fourth equations of (3.36) will not contain the constant terms, and the third equation will not contain the linear in y_1 term. We will also resolve the first and second equation with respect to $\bar{x}$ and plug the obtained expressions into the right-hand side of the third and fourth equations. As a result, system (3.36) takes the following form:

$$\begin{aligned}
\bar{x}_1 &= b_0 y_1 + O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + |\bar{y}|)\right), \\
\bar{x}_2 &= O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + |\bar{y}|)\right), \\
\gamma^{-k}&\left\{\cos k\psi \cdot \bar{y}_1 - \sin k\psi \cdot \bar{y}_2 + (\hat{\gamma}/\gamma)^{-k} O(\|\bar{y}\|)\right\} = \\
&= M_1 + C_1\lambda^k x_1 + C_2\lambda^k x_2 + \tilde{D}_0\, y_1^2 + \\
&\quad + O\left(|y_1|^3 + \lambda^k\|x\||y_1| + \hat{\lambda}^k\|x\| + \gamma^{-k}\|\bar{y}\||y_1|\right), \\
y_2 - \tilde{d}_1 y_1 - E_1\lambda^k x_1 - E_2\lambda^k x_2 &= d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^1)\bar{y}_2 + (\sin k\psi + \nu_k^2)\bar{y}_1\right\} + \\
&\quad + O\left(y_1^2 + \lambda^k\|x\||y_1| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\lambda}^k\|x\| + \hat{\gamma}^{-k}\|\bar{y}\|^2\right),
\end{aligned} \tag{3.38}$$

where $\nu_k^{1,2} = O(\hat{\gamma}^{-k}\gamma^k)$, and the coefficients $\tilde{D}_0$ and $\tilde{d}_1$ differ from, respectively, D_0 and d_1 by some small quantities of order $O(\gamma^{-k})$. We also denote

$$M_1 \equiv \mu - \gamma^{-k}E_0\cos(k\psi + \vartheta_2 + \ldots) + \lambda^k C_0\ \cos(k\varphi - \vartheta_1 + \ldots), \tag{3.39}$$

see formulas (3.25) and (3.10).

Introduce a new coordinate y_2 by formula $y_{2new} = y_2 - \tilde{d}_1 y_1$. Map (3.38) will take the following form:

$$\begin{aligned}
\bar{x}_1 &= b_0 y_1 + O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\
\bar{x}_2 &= O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\
(\cos k\psi - \tilde{d}_1\sin k\psi)\bar{y}_1 &- \sin k\psi \cdot \bar{y}_2 + (\hat{\gamma}/\gamma)^{-k} O(\|\bar{y}\|) = \\
&= M_1\gamma^k + C_1\lambda^k\gamma^k x_1 + C_2\lambda^k\gamma^k x_2 + \tilde{D}_0\gamma^k y_1^2 + \\
&\quad + O\left(\gamma^k|y_1|^3 + \lambda^k\gamma^k\|x\||y_1| + \hat{\lambda}^k\gamma^k\|x\| + \|\bar{y}\||y_1|\right), \\
y_2 - \lambda^k E_1 x_1 - \lambda^k E_2 x_2 &= d_2\gamma^{-k}\left\{(\cos k\psi + \nu_k^3)\bar{y}_2 + (\sin k\psi + d_1\cos k\psi + \nu_k^4)\bar{y}_1\right\} + \\
&\quad + O\left(y_1^2 + \lambda^k\|x\||y_1| + \hat{\lambda}^k|x| + \gamma^{-k}\|\bar{y}\||y_1| + \hat{\gamma}^{-k}\|\bar{y}\|^2\right),
\end{aligned} \tag{3.40}$$

where $\nu_k^{3,4} = O(\hat{\gamma}^{-k}\gamma^k)$. Introduce new coordinates y:

$$\begin{aligned} y_{1new} &= (\cos k\psi + \nu_k^3)y_2 + (\sin k\psi + d_1 \cos k\psi + \nu_k^4)y_1, \\ y_{2new} &= \gamma^k \frac{1}{d_2}\left(y_2 - E_1\lambda^k x_1 - E_2\lambda^k x_2\right). \end{aligned} \tag{3.41}$$

For the old coordinates (y_1, y_2) we have

$$\begin{aligned} y_2 &= \gamma^{-k} d_2 y_{2new} + E_1\lambda^k x_1 + E_2\lambda^k x_2, \\ y_1 &= \frac{1}{s_0}y_{1new} - \frac{d_1}{s_0}(d_2\cos k\psi + \nu_k^3)(\gamma^{-k}y_{2new} + E_1\lambda^k x_1 + E_2\lambda^k x_2), \end{aligned} \tag{3.42}$$

where

$$s_0 \equiv s_0(k\psi) = \sin k\psi + d_1\cos k\psi + \nu_k^4. \tag{3.43}$$

We will consider only such ψ for which s_0 is uniformly bounded away from zero. In this case we may rewrite (3.40) as follows:

$$\begin{aligned} \bar{x}_1 &= \frac{b_0}{s_0}y_1 + O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\ \bar{x}_2 &= O\left(y_1^2 + \lambda^k\|x\| + \gamma^{-k}(\|y\| + \|\bar{y}\|)\right), \\ \frac{1}{d_2}\gamma^k\bar{y}_1(\cos k\psi - d_1\sin k\psi + \nu_k^5 + O(\bar{y}_1)) &- \bar{y}_2(1 + \nu_k^6 + O(\bar{y}_2)) + (\lambda\gamma)^k O(\bar{x}) = \\ &= \gamma^{2k}s_0M_1 + \tilde{D}_0(s_0)^{-1}\gamma^{2k}y_1^2 + \tilde{C}_1 s_0\lambda^k\gamma^{2k}x_1 + \tilde{C}_2 s_0\lambda^k\gamma^{2k}x_2 + \\ &\quad + \gamma^{2k}O\left(|y_1|^3 + \lambda^k\|x\||y_1| + \gamma^{-k}(\|\bar{y}\|\|y\| + \|y\|^2) + \hat{\lambda}^k\|x\|^2 + \hat{\gamma}^{-k}\|\bar{y}\|^2\right), \\ y_2 &= \bar{y}_1 + O\left(\gamma^k\|y\|^2 + \lambda^k\gamma^k\|x\|\|y\| + \|\bar{y}\|\|y\| + \|\bar{y}\|^2 + \hat{\lambda}^k\gamma^k\|x\|\right), \end{aligned} \tag{3.44}$$

where $\nu_k^{5,6} = O(\hat{\gamma}^{-k}\gamma^k)$, and the coefficients $\tilde{C}_1$ and $\tilde{C}_2$ differ from, respectively, C_1 and C_2 by quantities of order $O(\hat{\lambda}^k\lambda^{-k})$.

Consider, first, the case $\lambda\gamma^2 < 1$. Scale the coordinates in (3.44) as follows:

$$\begin{aligned} x_1 &= \rho^{-k}\frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}x_{1new}, \qquad & x_2 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}x_{2new}, \\ y_1 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}y_{1new}, \qquad & y_2 &= \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k}y_{2new}, \end{aligned} \tag{3.45}$$

where ρ is a number such that $\lambda\gamma^2 < \rho < 1$.

Since the scaling coefficients in (3.45) are asymptotically small, the range of values of the new coordinates (x, y) will grow with the increase of k, and it will cover all finite values in the limit $k \to \infty$. This allows us to assume that our map is defined in the region $\|(x_{new}, y_{new})\| \leq Q$ for some $Q > 0$, and this constant Q can be taken as large as we want. After the scaling, map (3.44) may be written in the form

$$\bar{x}_1 = \rho^k O(y_1) + \gamma^{-k} O(\|(x,y,\bar{y})\|), \qquad \bar{x}_2 = \gamma^{-k} O(\|(x,y,\bar{y})\|),$$

$$\frac{1}{d_2}\gamma^k C(k\psi)\bar{y}_1 - \bar{y}_2 = \tilde{M} + y_1^2 + \left(\frac{\lambda^k\gamma^{2k}}{\rho^k} + \gamma^{-k}\right) O(\|(x,y,\bar{y})\|), \tag{3.46}$$

$$y_2 = \bar{y}_1 + \gamma^{-k} O(\|(x,y,\bar{y})\|).$$

where

$$\tilde{M} = \gamma^{4k}\frac{\tilde{D}_0}{d_2^2} M_1, \tag{3.47}$$

M_1 satisfies formula (3.39), and

$$C(k\psi) = \cos k\psi - d_1 \sin k\psi + \nu_k^5. \tag{3.48}$$

Note that the coefficients $\tilde{M}$ and $C = d_2^{-1} C(k\psi)\gamma^k$ may take arbitrary finite values as $k \to +\infty$, for appropriately chosen values of the original parameters μ and ψ.

Note also that C may stay uniformly bounded only when $\cos k\psi - d_1 \sin k\psi$ is asymptotically close to zero, i.e. when ψ is close to the values given by formula (3.34). As we mentioned there, the value s_0 from (3.43) is uniformly bounded away from zero for such ψ: $|s_0| = \sqrt{1 + d_1^2}(1 + ...)$. Further, only such ψ are considered.

As a result, for any bounded region of the values of (x, y, M, C), map (3.44) may be written in the form

$$\begin{aligned} \bar{x}_1 &= o(1), \\ \bar{x}_2 &= o(1), \\ \bar{y}_1 &= y_2 + o(1), \\ \bar{y}_2 &= -\tilde{M} + Cy_2 - y_1^2 + o(1), \end{aligned} \tag{3.49}$$

where we denote as $o(1)$ functions of all coordinates and parameters which tend to zero as $k \to \infty$, uniformly in any bounded region of values of (x, y, M, C), along with all the derivatives up to the order $(r-2)$ with respect to the coordinates and $(r-3)$ with respect to the parameters. If we put $M = -\tilde{M}$ and $y_{2new} = y_2 + o(1)$ in (3.49), then we immediately arrive at the sought map (1.4).

Consider now the case $\lambda\gamma^2 > 1$ (and $\lambda\gamma < 1$, as before). Let us make the following scaling in (3.44):

$$x_1 = \frac{d_2 b_0}{\tilde{D}_0}\gamma^{-2k} x_{1new}, \quad x_2 = q^k \gamma^{-2k} x_{2new}, \quad y_1 = \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{1new}, \quad y_2 = \frac{d_2 s_0}{\tilde{D}_0}\gamma^{-2k} y_{2new}, \tag{3.50}$$

where q is a number from the interval $q \in (\gamma^{-1}, (\lambda\gamma^2)^{-1})$. This interval is non-empty and lies in $(0,1)$, since

$$1 > \frac{1}{\lambda\gamma^2} = \frac{\gamma^{-1}}{\lambda\gamma} > \gamma^{-1}.$$

The map (3.44) takes the following form in coordinates (3.50):

$$
\begin{aligned}
\bar{x}_1 &= y_1 + \gamma^{-k} O(\|(x,y,\bar{y})\|), \qquad \bar{x}_2 = \frac{\gamma^{-k}}{q^k} O(\|(x,y,\bar{y})\|), \\
\frac{1}{d_2}\gamma^k C(k\psi)\bar{y}_1 - \bar{y}_2 &= M + y_1^2 + \frac{b_0}{d_2}\lambda^k\gamma^{2k}\left(c_{11}\cos k\varphi + c_{12}\sin\varphi + l_k\right)x_1 + \\
&\quad + \lambda^k\gamma^{2k}q^k O(x_2) + \gamma^{-k}O(\|(x,y,\bar{y})\|), \\
y_2 &= \bar{y}_1 + \gamma^{-k}O(\|(x,y,\bar{y})\|),
\end{aligned}
\tag{3.51}
$$

where $l_k = O((\hat{\lambda}/\lambda)^k)$ is some small coefficient, and M and $C(k\psi)$ satisfy formulas(3.47) and (3.48) above.

In comparison with (3.46), in map (3.51) there is one more independent parameter, along with M and $C = \gamma^k C(k\psi)$. It is the parameter

$$
B = B(k\varphi) \equiv \frac{b_0}{d_2}\lambda^k\gamma^{2k}(c_{11}\cos k\varphi + c_{12}\sin\varphi + l_k).
$$

Since $\lambda\gamma^2 > 1$, the coefficient $B(k\varphi)$ is no longer small (as it was in the case $\lambda\gamma^2 < 1$), and it may take arbitrary finite values when φ varies, provided k is large enough. Bounded values of B correspond to the values of φ close to

$$
\varphi = -\frac{1}{k}\arctan\left(\frac{c_1}{c_2}\right) + \pi\frac{j}{k}, \qquad j \in Z. \tag{3.52}
$$

In the region of bounded values of (x, y, M, B, C) the map (3.51) may be written in the form

$$
\begin{aligned}
\bar{x}_1 &= y_1 + o(1), \\
\bar{x}_2 &= o(1), \\
\bar{y}_2 &= -M - Bx_1 + Cy_1 - y_1^2 + o(1), \quad , \\
\bar{y}_1 &= y_2 + o(1).
\end{aligned}
$$

After changing the signs of M and B, this map is easily brought to the form (1.5).

Thus, rescaling lemma is proven.

4 Proof of main theorems

The proof of theorems 1–4 is based on the rescaling lemmas. They allow us to make comparatively simple analysis of the first-return maps $T^{(k)}(\varepsilon)$, using their closeness, at $\varepsilon \in \Delta_k$, to the standard quadratic maps in the rescaled coordinates. It is convenient for us to prove Theorem 2 first, then Theorem 4 (here we need only a linear analysis of the fixed points of the first-return maps); after that we prove Theorem 3. We derive Theorem 1 in the process of the proof of Theorems 2 and 4.

4.1 Proof of Theorem 2 and items 1, 2 of Theorem 1

First we analyze fixed points of the first-return maps (1.2)–(1.5) and (1.7), in order to find the values of the parameters M, B, C for which these maps have fixed points with the multipliers on the unit circle.

Map (1.2).
Consider the one-dimensional parabola map

$$\bar{y} \;=\; M - y^2.$$

Let $\nu_1 \neq 0$ be the multiplier of some its fixed point. The coordinate y of this fixed point satisfy equations $M = y + y^2$ and $2y = -\nu_1$. Thus, we have that the parabola map has a fixed point with the multiplier ν_1 at

$$M \;=\; \frac{\nu_1^2}{4} - \frac{\nu_1}{2}. \tag{4.1}$$

Since map (1.2) is close to the parabola map along with a sufficient number of derivatives, it must also have a fixed point with a multiplier equal to ν_1 at the value of $M = M_k(\nu_1)$ which is asymptotically close, as $k \to \infty$, to the value (4.1). The other multipliers of the fixed point (one multiplier in the case (1,1) and two multipliers in the case (2,1)) are always less than 1 in the absolute value – they tend to zero as $k \to \infty$.

Map (1.3).
Consider Hénon map (the limit map for (1.3)):

$$\bar{x} = y, \;\; \bar{y} \;=\; M + Bx - y^2.$$

Let ν_1 and ν_2 be the multipliers of some its fixed point (they are either both real, or they comprise a complex-conjugate pair; we also assume $\nu_1\nu_2 \neq 0$). The coordinates $x = y$ of the fixed point satisfy the equation $M = y(1-B) + y^2$. The characteristic equation is $\nu^2 + 2y\nu - B = 0$. It is easy to find that

$$B(\nu_1, \nu_2) \;=\; -\nu_1\nu_2, \;\; M(\nu_1, \nu_2) \;=\; \frac{\nu_1 + \nu_2}{4}(\nu_1 + \nu_2 - 2\nu_1\nu_2 - 2). \tag{4.2}$$

It is clear that the map (1.3) will also have a fixed point with the given multipliers ν_1 and ν_2, at the values of M and B which are asymptotically close to those given by formula (4.2). The third multiplier is always less than 1 in the absolute value (it tends to zero as $k \to \infty$).

Map (1.4).
Consider the map (limit for (1.4)):

$$\bar{y}_1 = y_2, \;\; \bar{y}_2 \;=\; M + Cy_2 - y_1^2.$$

Let ν_1 be ν_2 the multipliers of some its fixed point (again, they are either both real, or they comprise a complex-conjugate pair; and we assume again that $\nu_1\nu_2 \neq 0$). The coordinates $y_1 = y_2 = y$ of the fixed point satisfy the equation $M = y(1-C) + y^2$; the characteristic equation is $\nu^2 - C\nu + 2y = 0$. One can easily find

$$C = \nu_1 + \nu_2, \quad M = \frac{\nu_1\nu_2}{2}(1-C) + \frac{(\nu_1\nu_2)^2}{4}. \tag{4.3}$$

Map (1.4) will also have a fixed point with the given multipliers ν_1 and ν_2, at M and C which are asymptotically close to those given by formula (4.3). The other multipliers (the third one in the case (1,2), and the third and fourth multipliers in the case (2,2)) are always less than 1 in the absolute value.

Map (1.5).
Consider the three-dimensional map (limit for (1.5)):

$$\bar{x} = y_1, \ \bar{y}_1 = y_2, \ \bar{y}_2 = M + Bx + Cy_2 - y_1^2. \tag{4.4}$$

Let ν_1, ν_2, ν_3 be the multipliers (all non-zero) of some its fixed point (either all three of them are real, or one multiplier is real and the other two comprise a complex-conjugate pair). The coordinates $x = y_1 = y_2$ of the fixed point satisfy the equation $M = x(1-B-C) + x^2$, and the characteristic equation is $-\nu^3 + C\nu^2 - 2x\nu + B = 0$. This gives

$$\begin{aligned} B &= \nu_1\nu_2\nu_3, \quad C = \nu_1 + \nu_2 + \nu_3, \\ M &= (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)(1-B-C) + \frac{(\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)^2}{4}. \end{aligned} \tag{4.5}$$

The original map (1.5) will also have a fixed point with the given multipliers ν_1, ν_2, ν_3, at the values of M, B and C which are asymptotically close to those given by formula (4.5). The fourth multiplier of this point is, at large k, always less than 1 in the absolute value.

Thus, given any set $\{\nu_1, \ldots, \nu_{d_e}\}$ of d_e multipliers (where $d_e = 1$ in the case of map (1.2), $d_e = 2$ for the maps (1.3) and (1.4), and $d_e = 3$ for the map (1.5)), each of the maps (1.2)-(1.5) has the values of parameters $M = M_k$, $B = B_k$, $C = C_k$ for which there exists a fixed point, d_e multipliers of which are equal exactly to $\nu_1, \ldots, \nu_{d_e}$. Note that the corresponding values M_k, B_k, C_k are uniformly bounded for all large k. According to (1.6), we have for the corresponding values of the original parameters $(\mu, \varphi, \psi) = (\mu_k, \varphi_k, \psi_k)$ that, first, $\mu_k \to 0$ as $k \to +\infty$, and that if $d_e \geq 2$, then there is always a subsequence (φ_k, ψ_k) which converges to (φ_0, ψ_0) where φ_0 and ψ_0 are the values of the angular arguments of the complex multipliers of the fixed point O for the diffeomorphism f_0. Thus, we obtain the following

Corollary 1 *Given any set of multipliers $\{\nu_1, \ldots, \nu_{d_e}\}$ there exists a sequence $\varepsilon_k \to 0$ of the values of parameters ε such that the diffeomorphism f_ε has, at $\varepsilon = \varepsilon_k$, a single-round periodic orbit whose d_e multipliers are equal exactly to $\nu_1, \ldots, \nu_{d_e}$, and the rest of multipliers lies strictly inside the unit circle.*

Theorem 2 follows from this statement immediately. Indeed, in the Newhouse region δ_j, near any $\varepsilon \in \delta_j$ there exist the values of parameters corresponding to homoclinic tangencies to O, for which conditions A-D hold. As we just established it, arbitrarily small perturbations within the same family f_ε give periodic orbits (single-round with respect to these secondary homoclinic tangencies) with any given set of d_e multipliers on the unit circle, in a complete agreement with Theorem 2.

In the case of Theorem 1 we have $d_e = 1$, i.e. we may speak here about periodic orbits with one multiplier equal to $\nu_1 = +1$ or $\nu_1 = -1$. Thus, in this case, Corollary 1 gives us items 1 and 2 of Theorem 1 for the Newhouse intervals δ_j.

4.2 Proof of Theorem 4 and item 3 of Theorem 1

Here we again use Corollary 1, now for hyperbolic periodic orbits, whose multipliers $\nu_1, ..., \nu_{d_e}$ does not equal to 1 in the absolute value. We have here $d_e + 1$ different types of orbits, according to the number of multipliers outside the unit circle: 0, 1, ... , or, maximum, d_e. The first case corresponds to a stable periodic orbit.

Recall that arbitrarily close to any parameter value from the Newhouse regions δ_j there exists a value of ε for which the point O has an orbit of a simple homoclinic tangency. According to Corollary 1, arbitrarily close to this value of ε there is a parameter value for which f_ε has a hyperbolic periodic orbit with exactly d multipliers outside the unit circle, for any given $d = 0, ..., d_e$. This orbit exists in some region in the space of parameters. Repeating the arguments, inside this region we find a smaller region which corresponds to the existence of one more hyperbolic periodic orbit with d multipliers outside the unit circle, with the same d, or with any other d from 0 to d_e, etc.. By repeating this procedure infinitely many times for every $d = 0, ..., d_e$, we obtain a sequence of nested domains such that the values of ε from the intersection of these domains correspond to the existence of infinitely many periodic orbits with all possible numbers from 0 to d_e of multipliers outside the unit circle. By construction, the obtained set of values of ε is an intersection of a countable number of open and dense in δ_j sets, i.e. it is a residual set. The theorem is proven.

4.3 Proof of Theorem 3

Like in the proof of Theorem 4, it is enough to prove that the first-return maps $T^{(k)}$ have, for some region of parameters (M, B), (M, C) or (M, C, B), a stable closed invariant curve. An infinite set of coexisting closed invariant curves is obtained by means of the construction with nested domains, as in Theorem 4.

Consider, first, the cases of a saddle-focus (1,2) and a saddle-focus (2,2) with $\lambda\gamma^2 < 1$. By Lemma 1, the map $T^{(k)}$ is brought to the following form in this case:

$$\begin{aligned} &\bar{x} = o(1), \\ &\bar{y}_1 = y_2, \quad \bar{y}_2 = M + Cy_2 - y_1^2 + o(1). \end{aligned} \tag{4.6}$$

The limit map

$$\bar{y}_1 = y_2, \quad \bar{y}_2 = M + Cy_2 - y_1^2 \tag{4.7}$$

has a fixed point with multipliers $\nu_{1,2} = e^{\pm i\omega}$ for the values of (M, C) on the following curve (see formula (4.3)): $L : \{M = \frac{3}{4} - \frac{1}{2}C,\ C = 2\cos\omega\}$ (i.e. $|C| < 2$). At $\omega \neq \pi/2, 2\pi/3$ the stability of the closed invariant curve which is born at the bifurcations of such fixed point is determined by the sign of the first Lyapunov coefficient (see e.g. [32, 33]). Recall that the Lyapunov coefficient G_1 is the coefficient of a cubic term in the normal form of the map near the fixed point, written in polar coordinates (ρ, θ): $\bar\rho = \rho + G_1\rho^3 + o(\rho^3), \bar\theta = \theta + \omega + O(r^2)$. It is not hard to compute for the map (4.7) that here $G_1 = -1 - \dfrac{1}{2(1-\cos\omega)}$, i.e. the Lyapunov coefficient is always negative for this map. Since G_1 is a coefficient of the cubic term, the Lyapunov coefficient would remain negative for all maps which are C^3-close to (4.7).

Consider now the map (4.6). For all sufficiently large k, it also has a curve in the parameter plane close to the curve L, which corresponds to the existence of a fixed point with two multipliers equal to $e^{\pm i\omega}$ (the rest of multipliers lies inside the unit circle). We denote this curve as L_k. On the center manifold, the map (4.6) is C^{r-2}-close to (4.7). Since $r \geq 5$, we have that the corresponding Lyapunov coefficient is negative for the map (4.6), hence a stable closed invariant curve is born when the parameters cross the curve L_k, and it exists for the values of parameters from a certain open region, as required.

In the case of a saddle-focus (2,2) with $\lambda\gamma^2 > 1$, the first-return map $T^{(k)}$ is brought to the form

$$\begin{aligned} &\bar x_2 = o(1), \\ &\bar x_1 = y_1, \quad \bar y_1 = y_2, \quad \bar y_2 = M + Cy_2 + Bx_1 - y_1^2 + o(1). \end{aligned} \tag{4.8}$$

Here, for small B, the fixed point with the multipliers $\nu_{1,2} = e^{\pm i\omega}$, $\nu_3 = B + o(1)$, $\nu_4 = o(1)$ has a negative Lyapunov coefficient too. It follows immediately from the fact that the map (4.8) at $B = 0$ degenerates, as $k \to +\infty$, to the map (4.7) in the coordinates y_1 and y_2, and the negativity of the Lyapunov coefficient for the latter map has been already established. Thus, in this case we also have that the first-return map has a stable closed invariant curve for some open region of parameter values, for all sufficiently large k.

In the case of a saddle-focus (2,1) with $d_e = 2$, i.e. at $\lambda\gamma > 1$, in order to find stable closed invariant curves, we will use the form of the first-return maps obtained in Lemma 2. It is the so-called generalized Hénon map

$$\begin{aligned} &\bar x_1 = y, \\ &\bar y = M - y^2 + Bx_1 + Q_k x_1 y + o(Q_k), \end{aligned} \tag{4.9}$$

where $Q_k \neq 0$ and $Q_k \to 0$ as $k \to +\infty$. Such maps were studied in [20, 22] where it was shown, in particular, that the maps of type (4.9) have, for all k large enough, a stable closed invariant curve for the values of parameters (M, B) from some open region. Namely, such region emanates from the point $(M = M_k^*, B = B_k^*)$ where $M_k^* = 3 - Q_k + o(Q_k)$, $B_k^* = -1 + Q_k/2 + o(Q_k)$, for which map (4.9) has a fixed point with the multipliers $(-1, -1)$ (in [22] it was shown that this point is non-degenerate and corresponds to the "soft" case, i.e. "the case $s = -1$", of the 1:2 resonance in terminology of [33], so the known results about the birth of closed invariant curves from this point are applied to map (4.9)).

Thus, in all cases with $d_e \geq 2$, for the family f_ε there exists an infinite sequence of regions $\tilde{\Delta}_k \subset \Delta_k$, which converge to $\varepsilon = 0$ as $k \to \infty$, such that at $\varepsilon \in \tilde{\Delta}_k$ the diffeomorphism f_ε has a stable closed invariant curve. In order to obtain an infinite set of closed invariant curves for a dense set of parameter values from the Newhouse regions δ_j, it remains to use the construction with the nested domains from the proof of Theorem 4.

This work was supported by the grants No. 02-01-00273 and No. 01-01-00975 of RFBR, grant No. 2000-221 INTAS, grant DFG No. 436 RUS, scientific program "Russian Universities", project No. 1905, and by Portugal-Russian fellowship program. The second author acknowledges support of the Alexander von Humboldt foundation.

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