

Weierstraß-Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint

ISSN 0946 – 8633

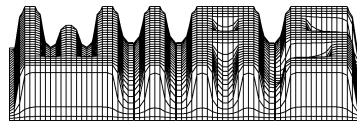
On Thermodynamics of Nonlinear Poroelastic Materials

Krzysztof Wilmański

submitted: 4th December 2002

Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: wilmansk@wias-berlin.de

No. 792
Berlin 2002



2000 *Mathematics Subject Classification.* 74A15, 74E30, 74L05.

Key words and phrases. Continuum thermodynamics, porous media, mixtures.

In memoriam of Prof. Clifford Ambrose Truesdell whose work in continuum mechanics created new standards of research in field theories.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: <http://www.wias-berlin.de/>

In memoriam of Prof. Clifford Ambrose Truesdell whose work in continuum mechanics
created new standards of research in field theories

Abstract

The paper contains a brief presentation of a macroscopical thermodynamic model of poroelastic materials with many fluid components. A particular emphasis is placed on a Lagrangian formulation of the model and, consequently, on a consistent formulation of field equations on the reference configuration of the skeleton (solid phase of the mixture). It is demonstrated that the model possesses an identical structure as that in the pioneering work of C. A. Truesdell on the continuum mixture of fluids. An issue of porosity as an additional microstructural variable is particularly exposed.

1 Introduction

The classical continuum theory of mixtures whose development was started in 1957 by the famous papers of C. A. Truesdell [1] is primarily designed to cover systems of many fluid components. In 1982 R. M. Bowen [2] (see as well: [3]) has extended this classical field on mixtures whose one component is a solid. This has put theories of porous materials on the same footing as mixtures of fluids. During the last twenty years this field of research developed rapidly and in the meantime enhances such systems as suspensions, mixtures of granular materials saturated or not saturated with a fluid and many others.

In spite of this development there are still some controversies concerning a construction of nonlinear models in which large deformations of the skeleton are incorporated. This is related to the fact that in contrast to mixtures of fluids a solid component (skeleton) yields naturally a Lagrangian description of the system. R. M. Bowen was using in his papers a mixed description - Lagrangian for the solid skeleton and Eulerian for the fluid - but such an approach leads to technical difficulties in applications of the model, in formulation of boundary conditions etc. For this reason I have proposed in 1995 a different way of description of two-component porous materials [4]. This may be extended to many components and first results for multicomponent porous materials have been published in the work [5].

In this work we present a full structure of a Lagrangian model of a poroelastic material in which there may be more than one fluid component and the kinematics of the skeleton is formulated in the Lagrangian way.

In Sect. 2 we define the Lagrangian description of multicomponent systems and introduce various kinematical quantities analogous to those appearing in Truesdell's theory of fluid mixtures.

In Sect. 3 we present partial balance equations in the Lagrangian description in their global and local form. It is emphasized that in contrast to such balance equations for single continua they contain convective contributions whose form is objective. We also present a balance equation for the microstructural field of porosity and justify its macroscopic form on phenomenological grounds. This extension of the microstructural model has been proposed in papers [6], [7].

Section 4 contains a discussion of thermodynamic admissibility of constitutive relations for poroelastic materials with ideal fluid components. The whole development is fully macroscopical in contrast to many other works on this subject which are based on the notion of so-called *true (real) densities*. These may be introduced in the present model if needed on any stage of development but they are not necessary for the formulation of the consistent mathematical model. In order to be more specific we limit the attention solely to isotropic systems.

Section 5 is devoted to the specification of some special models which have an important practical bearing. In particular we discuss the simplest model of a two-component poroelastic material.

In Conclusions we indicate advantages of the Lagrangian description for both a theoretical development as well as for numerical evaluations of the boundary value problems.

2 Porous medium as a mixture. Reference configurations, Lagrangian description

The construction of the theory of mixtures of fluids proposed by C. Truesdell [1] is based on the Eulerian description of motion of components. As a continuum model it is based on the assumption that at each point of the space of configurations \mathfrak{R}^3 all components are present simultaneously. Their various contributions are characterized by different concentrations (fractions of partial mass densities to the total mass density) as well as by their own velocity fields.

A model of porous materials requires an extension of this approach. On the one hand side it must account for large deformations of a solid component of the mixture which describes the behaviour of the skeleton of the porous medium. This indicates the necessity of the Lagrangian description which has been in part (solely to the solid component) employed by R. M. Bowen [2]. On the other hand a description of the microstructure must be extended as its properties are described not only by concentrations but also by a volume fraction of voids called porosity. This is particularly visible when the porous material consists only of the solid component, i.e. the mass densities of fluid components are all identically zero. Then the

concentrations are also zero but the microstructure is not trivial. This additional field requires an additional equation and in the above mentioned paper Bowen proposed an evolution equation describing its relaxation properties. An alternative approach has been proposed earlier for granular materials by M. A. Goodman and S. C. Cowin [8]. In their paper the authors proposed a second order equation for a microstructural behaviour. Such an approach related to the so-called principle of self-equilibrated forces has been modified by K. Hutter and B. Svendsen [9] and is applied in the description of avalanches with abrasion [10]. In this paper we rely on a balance equation for porosity introduced in my own works [6], [7].

We consider a porous medium whose channels are filled with a mixture of A fluid components. The model is constructed on a chosen reference configuration \mathcal{B}_0 of the solid component, i.e. all fields are functions of a spatial variable $\mathbf{X} \in \mathcal{B}_0$, and time $t \in \mathcal{T}$. We consider a thermomechanical model in which the governing fields are as follows:

1. ρ^S – mass density of the skeleton in the reference configuration,
2. $\rho^\alpha, \alpha = 1, \dots, A$ – partial mass densities of fluid components referring to the unit volume of the reference configuration of the skeleton,
3. $\dot{\mathbf{x}}^S$ – velocity field of the skeleton,
4. \mathbf{F}^S – deformation gradient of the skeleton,
5. $\dot{\mathbf{x}}^\alpha, \alpha = 1, \dots, A$ – velocity fields of fluid components,
6. θ^S - absolute temperature of the skeleton,
7. $\theta^\alpha, \alpha = 1, \dots, A$ – absolute temperatures of fluid components.
8. n – porosity (the volume fraction of voids).

Further in this work we assume that temperatures of components are the same

$$\theta = \theta^S = \theta^1 = \dots = \theta^A. \quad (1)$$

>From the thermodynamic point of view little has been done for continuum theories of mixtures in which this condition is not satisfied (e.g. [11]). Some semi-kinetic models have been proposed for ionized gases (plasma).

The above fields are related to their Eulerian counterparts in the following way

$$\begin{aligned} \rho_t^S(\mathbf{x}, t) &:= \rho^S(\mathbf{f}^{-1}(\mathbf{x}, t), t) J^{S-1}(\mathbf{f}^{-1}(\mathbf{x}, t), t), \\ \rho_t^\alpha(\mathbf{x}, t) &:= \rho^\alpha(\mathbf{f}^{-1}(\mathbf{x}, t), t) J^{S-1}(\mathbf{f}^{-1}(\mathbf{x}, t), t), \quad \alpha = 1, \dots, A, \\ \mathbf{v}^S(\mathbf{x}, t) &:= \dot{\mathbf{x}}^S(\mathbf{f}^{-1}(\mathbf{x}, t), t), \\ \mathbf{v}^\alpha(\mathbf{x}, t) &:= \dot{\mathbf{x}}^\alpha(\mathbf{f}^{-1}(\mathbf{x}, t), t), \quad \alpha = 1, \dots, A, \\ n(\mathbf{x}, t) &:= n(\mathbf{f}^{-1}(\mathbf{x}, t), t), \end{aligned} \quad (2)$$

where the function of motion of the skeleton

$$\mathbf{x} = \mathbf{f}(\mathbf{X}, t), \quad (3)$$

is assumed to be at least twice continuously differentiable, i.e.

$$\dot{\mathbf{x}}^S = \frac{\partial \mathbf{f}}{\partial t}, \quad \mathbf{F}^S = \text{Grad } \mathbf{f}. \quad (4)$$

Hence the fields $\dot{\mathbf{x}}^S, \mathbf{F}^S$ must satisfy the following integrability conditions

$$\frac{\partial \mathbf{F}^S}{\partial t} = \text{Grad } \dot{\mathbf{x}}^S, \quad \text{Grad } \mathbf{F}^S = (\text{Grad } \mathbf{F}^S)^T. \quad (5)$$

The reference configuration \mathcal{B}_0 is chosen in such a way that it is identical with a configuration at the instant of time $t = t_0$ for which

$$\mathbf{X} \in \mathcal{B}_0 \forall \mathbf{F}^S(\mathbf{X}, t_0) = \mathbf{1}. \quad (6)$$

This choice of reference configuration is convenient for systems in which the solid component forms a skeleton whose topology does not change during the motion. It is the case for modelling of rocks, it may or may not be the case for granular materials, and it is certainly not the case for suspensions of solid particles which appear, for instance, after liquefaction of a granular compact material.

For the above fields field equations follow from general balance equations which we discuss in the next Section.

3 Balance equations

We skip here axiomatic foundations for the integral representation of a general balance law. These may be found in Truesdell's book [12] which is after more than 30 years still the most important reference on this subject.

The general form of this equation for a density $\varphi(\mathbf{X}, t)$, written for an arbitrary domain $\mathcal{P}(t)$ whose motion is described by a velocity field $\mathbf{V}(\mathbf{X}, t)$, is as follows

$$\frac{d}{dt} \int_{\mathcal{P}(t)} \varphi(\mathbf{X}, t) dV = \oint_{\partial \mathcal{P}(t)} \boldsymbol{\Phi}(\mathbf{X}, \mathbf{t}) \cdot \mathbf{N} dS + \int_{\mathcal{P}(t)} \gamma(\mathbf{X}, t) dV, \quad (7)$$

where $\boldsymbol{\Phi}$ is the so-called flux of φ , and γ is its volume supply. The first integral on the right-hand side is evaluated over a closed surface $\partial \mathcal{P}$ of the domain \mathcal{P} and describes the transport through the surface. \mathbf{N} is the field of unit vectors perpendicular to the surface. If we perform the differentiation on the left-hand side and apply the Stokes theorem we obtain

$$\int_{\mathcal{P}(t)} \left[\frac{\partial \varphi}{\partial t} + \text{Div}(\varphi \mathbf{V} - \boldsymbol{\Phi}) - \gamma \right] dV = 0. \quad (8)$$

We apply this relation to partial quantities listed in the previous Section. In order to do so we have to find the kinematics of material domains for each component related to the reference configuration \mathcal{B}_0 . Obviously for domains material with respect to the skeleton we have $\mathbf{V} \equiv \mathbf{0}$. For fluid components we have to use the assumption on the simultaneous appearance of all components in each point of the domain $\mathcal{B}_t := \mathbf{f}(\mathcal{B}_0, t)$ in the configuration space. For the α -component we have then along the trajectory

$$\begin{aligned} \forall_{\mathbf{x} \in \mathcal{B}_t} \forall_{\mathbf{x}' \in \mathcal{N}(\mathbf{x}) \subset \mathcal{B}_t} \mathbf{x}' &= \mathbf{x} + \dot{\mathbf{x}}^\alpha \Delta t + O(\Delta t^2) = \\ &= \mathbf{x} + \mathbf{F}^S [\mathbf{f}^{-1}(\mathbf{x}', t) - \mathbf{f}^{-1}(\mathbf{x}, t)] + \dot{\mathbf{x}}^S \Delta t + O(|\mathbf{x}' - \mathbf{x}|^2), \end{aligned}$$

where $\mathcal{N}(\mathbf{x})$ is a neighbourhood of \mathbf{x} . The limit in this relation $\Delta t \rightarrow 0$ yields the following velocity field for material domains of the α -component in the reference configuration of the skeleton

$$\begin{aligned} \forall_{\mathbf{X} \in \mathcal{B}_0} \dot{\mathbf{X}}^\alpha(\mathbf{X}, t) &:= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{f}^{-1}(\mathbf{x}', t) - \mathbf{f}^{-1}(\mathbf{x}, t)}{\Delta t} = \\ &= \mathbf{F}^{S-1}(\mathbf{X}, t) (\dot{\mathbf{x}}^\alpha(\mathbf{X}, t) - \dot{\mathbf{x}}^S(\mathbf{X}, t)). \end{aligned} \quad (9)$$

We call this field the Lagrangian velocity of the α -component.

Assuming that the balance equation (8) for a partial quantity φ^α holds true for any material domain of the α -component we obtain in the standard way the following local form of this equation

$$\frac{\partial \varphi^\alpha}{\partial t} + \text{Div} \left(\varphi^\alpha \dot{\mathbf{X}}^\alpha - \Phi^\alpha \right) = \gamma^\alpha \quad \text{a.e. in } \mathcal{B}_0. \quad (10)$$

Obviously Φ^α denotes the corresponding partial flux, and γ^α is the partial volume supply.

In particular we have

partial mass balance equations

$$\frac{\partial \rho^S}{\partial t} = \hat{\rho}^S, \quad \frac{\partial \rho^\alpha}{\partial t} + \text{Div} \left(\rho^\alpha \dot{\mathbf{X}}^\alpha \right) = \hat{\rho}^\alpha, \quad \alpha = 1, \dots, A, \quad (11)$$

partial momentum balance equations

$$\begin{aligned} \frac{\partial (\rho^S \dot{\mathbf{x}}^S)}{\partial t} &= \text{Div} \mathbf{P}^S + \hat{\mathbf{p}}^S + \rho^S \mathbf{b}^S, \\ \frac{\partial (\rho^\alpha \dot{\mathbf{x}}^\alpha)}{\partial t} + \text{Div} \left(\rho^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{X}}^\alpha \right) &= \text{Div} \mathbf{P}^\alpha + \hat{\mathbf{p}}^\alpha + \rho^\alpha \mathbf{b}^\alpha, \quad \alpha = 1, \dots, A, \end{aligned} \quad (12)$$

partial energy balance equations

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho^S \left(\varepsilon^S + \frac{1}{2} \dot{x}^{S2} \right) \right] &= \text{Div} \left(\mathbf{Q}^S - \mathbf{P}^{ST} \dot{\mathbf{x}}^S \right) + \rho^S \mathbf{b}^S \cdot \dot{\mathbf{x}}^S + \rho^S r^S + \hat{r}^S, \\ \frac{\partial}{\partial t} \left[\rho^\alpha \left(\varepsilon^\alpha + \frac{1}{2} \dot{x}^{\alpha 2} \right) \right] &+ \text{Div} \left[\rho^\alpha \left(\varepsilon^\alpha + \frac{1}{2} \dot{x}^{\alpha 2} \right) \dot{\mathbf{X}}^\alpha \right] = \\ &= \text{Div} \left(\mathbf{Q}^\alpha - \mathbf{P}^{\alpha T} \dot{\mathbf{x}}^\alpha \right) + \rho^\alpha \mathbf{b}^\alpha \cdot \dot{\mathbf{x}}^\alpha + \rho^\alpha r^\alpha + \hat{r}^\alpha, \quad \alpha = 1, \dots, A, \end{aligned} \quad (13)$$

balance equation of porosity

$$\frac{\partial n}{\partial t} = -\text{Div} \mathbf{J} + \hat{n}. \quad (14)$$

In these equations, all functions are defined on the reference configuration \mathcal{B}_0 of the skeleton. In this sense we may call it the *Lagrangian description* even though partial balance equations for fluid components contain convective parts with respect to the corresponding Lagrangian velocities.

The two-point tensors $\mathbf{P}^S, \mathbf{P}^\alpha$ denote the Piola-Kirchhoff partial stress tensors, $\mathbf{b}^S, \mathbf{b}^\alpha$ are partial body forces, $\varepsilon^S, \varepsilon^\alpha$ are partial densities of the internal energy, $\mathbf{Q}^S, \mathbf{Q}^\alpha$ – partial heat fluxes, r^S, r^α are partial energy radiations, \mathbf{J} is the flux of porosity, and all quantities with a hat denote productions.

The balance equation of porosity requires some justification. We have argued in previous works on this subject (e.g. [6], [7]) that the balance equation for n follows from an averaging procedure for a representative elementary volume accounting for geometrical properties of the microstructure. However this argument is not needed if we make an extension of the continuous model of mixtures on the macroscopical phenomenological level. In such a case a new scalar field satisfies in the most general case a balance equation. Second order equations for microstructural variables appearing in some works on this subject indicate that most likely two variables rather than one additional microstructural variable should be introduced and one of them has to be eliminated from the model by substitution of one balance equation in another. The most important question which must be answered in a model with an additional balance law is if such a model can be mathematically well-posed – in particular in relation to additional boundary conditions which may be necessary. The most prominent example for those difficulties appears within the extended thermodynamics (e.g. [16]) where the extension of number of fields and, consequently, an extension of the hierarchy of field equations yields unsolved problems of boundary conditions. Fortunately the above balance equation for porosity specified for two-component poroelastic materials does not require additional boundary conditions – it possesses all properties of an evolution equation. As we shall see further thermodynamic considerations indicate that the flux \mathbf{J} results from the diffusion (relative motion of fluid components with respect to the skeleton), and the source \hat{n} describes relaxation to the thermodynamic equilibrium.

We make an assumption similar to this introduced by C. Truesdell for mixtures of fluids [1] that the bulk productions of mass, momentum, and energy vanish, i.e. the corresponding balance equations reduce to conservation laws. Hence

$$\hat{\rho}^S + \sum_{\alpha=1}^A \hat{\rho}^\alpha = 0, \quad \hat{\mathbf{p}}^S + \sum_{\alpha=1}^A \hat{\mathbf{p}}^\alpha = 0, \quad \hat{r}^S + \sum_{\alpha=1}^A \hat{r}^\alpha = 0. \quad (15)$$

Under these conditions we can introduce bulk quantities which correspond to those introduced by Truesdell for fluid mixtures which satisfy conservation laws of a single component continuum. Due to the fact that we have chosen one of the components - skeleton - as the reference the form of these laws differs from the classical Lagrangian form of conservation equations of a single continuum. Namely by addition of partial mass balance equations (11) we obtain

$$\frac{\partial \rho}{\partial t} + \text{Div} \left(\rho \dot{\mathbf{X}} \right) = 0, \quad \rho := \rho^S + \sum_{\alpha=1}^A \rho^\alpha, \quad \rho \dot{\mathbf{X}} := \sum_{\alpha=1}^A \rho^\alpha \dot{\mathbf{X}}^\alpha. \quad (16)$$

Hence for the single component bulk description we have to identify in relation (8): $\mathbf{V} \equiv \dot{\mathbf{X}}$. This Lagrangian mean velocity takes over the role of the barycentric velocity of the classical mixture theory. However in contrast to the Eulerian description the Lagrangian mean velocity is relative, i.e. similarly to the Lagrangian velocities $\dot{\mathbf{X}}^\alpha$ it is objective. The above definition yields the following conservation laws

momentum

$$\frac{\partial (\rho \dot{\mathbf{x}})}{\partial t} + \text{Div} \left(\rho \dot{\mathbf{x}} \otimes \dot{\mathbf{X}} - \mathbf{P} \right) = \rho \mathbf{b}, \quad (17)$$

$$\rho \dot{\mathbf{x}} := \rho^S \dot{\mathbf{x}}^S + \sum_{\alpha=1}^A \rho^\alpha \dot{\mathbf{x}}^\alpha, \quad \rho \mathbf{b} := \rho^S \mathbf{b}^S + \sum_{\alpha=1}^A \rho^\alpha \mathbf{b}^\alpha,$$

and the bulk Piola-Kirchhoff stress tensor \mathbf{P} is defined by the relation

$$\mathbf{P} := \mathbf{P}_I - \mathbf{F}^S \left\{ \rho^S \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha \left(\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}} \right) \otimes \left(\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}} \right) \right\}, \quad (18)$$

$$\mathbf{P}_I := \mathbf{P}^S + \sum_{\alpha=1}^A \mathbf{P}^\alpha,$$

energy

$$\frac{\partial}{\partial t} \left[\rho \left(\varepsilon + \frac{1}{2} \dot{x}^2 \right) \right] + \text{Div} \left[\rho \left(\varepsilon + \frac{1}{2} \dot{x}^2 \right) \dot{\mathbf{X}} + \mathbf{Q} - \mathbf{P}^T \dot{\mathbf{x}} \right] = \rho \mathbf{b} \cdot \dot{\mathbf{x}} + \rho r, \quad (19)$$

where the bulk internal energy density is defined as follows

$$\rho \varepsilon := \rho \varepsilon_I + \frac{1}{2} \left[\rho^S \mathbf{C}^S \cdot \left(\dot{\mathbf{X}} \otimes \dot{\mathbf{X}} \right) + \sum_{\alpha=1}^A \rho^\alpha \mathbf{C}^S \cdot \left(\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}} \right) \otimes \left(\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}} \right) \right],$$

$$\rho \varepsilon_I := \rho^S \varepsilon^S + \sum_{\alpha=1}^A \rho^\alpha \varepsilon^\alpha, \quad \mathbf{C}^S := \mathbf{F}^{ST} \mathbf{F}^S, \quad (20)$$

and the bulk heat flux has the form

$$\begin{aligned} \mathbf{Q} &= \mathbf{Q}_I + \\ &+ \frac{1}{2} \left[-\rho^S \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \right] \mathbf{C}^S, \\ \mathbf{Q}_I &:= \mathbf{Q}^S + \sum_{\alpha=1}^A \mathbf{Q}^\alpha - \rho^S \varepsilon^S \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha \varepsilon^\alpha (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) + \\ &+ \mathbf{P}^{ST} \mathbf{F}^S \dot{\mathbf{X}} - \sum_{\alpha=1}^A \mathbf{P}^{\alpha T} \mathbf{F}^S (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}), \end{aligned} \quad (21)$$

as well as the radiation

$$\rho r := \rho^S r^S + \sum_{\alpha=1}^A \rho^\alpha r^\alpha - \rho^S \mathbf{b}^S \cdot \mathbf{F}^S \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha \mathbf{b}^\alpha \cdot \mathbf{F}^S (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}). \quad (22)$$

The formal similarity of these relations to the corresponding relations of the fluid mixture theory is obvious. Technical differences are related to the fact that one of the components is solid and, secondly, as the reference we have chosen this solid component rather than a mean barycentric motion of Eulerian description.

4 Field equations and thermodynamic admissibility for isotropic materials

Thermodynamics of mixtures of fluids needed more than 10 years since the publication of Truesdell's paper [1] to start to develop. The pioneering work of I. Müller [13] contains the most fundamental extension of the Clausius-Duhem inequality which has been used as a condition for thermodynamic admissibility of various single component models. It is the assumption that the heat flux and the entropy flux are not related to each other by a classical universal Fourier relation: $\mathbf{h} = \frac{\mathbf{q}}{\theta}$. The review of basic results for mixtures following from this extension can be found in the book [14].

The formal thermodynamic construction of a continuous model proceeds as follows. We need field equations for the following fields

$$\mathcal{F} := \{ \rho^S, \rho^\alpha, \mathbf{F}^S, \dot{\mathbf{x}}^S, \dot{\mathbf{x}}^\alpha, \theta, n \}, \quad \alpha = 1, \dots, A. \quad (23)$$

They follow from the balance equations (11), (5), (12), (19), (14). However, in order to transform these equations into field equations we have to perform the so-called *closure*. Namely the following quantities

$$\mathcal{R} := \{ \hat{\rho}^\alpha, \mathbf{P}^S, \mathbf{P}^\alpha, \hat{\mathbf{p}}^\alpha, \varepsilon_I, \mathbf{Q}_I, \mathbf{J}, \hat{n} \}, \quad \alpha = 1, \dots, A, \quad (24)$$

must be specified in terms of fields and their derivatives in order to close the system. This is the *constitutive problem* defining materials contributing to the mixture. The mass and momentum sources for the skeleton do not appear in the above list because, according to (15), they are not independent. Let us remark that in many cases of practical bearing additional constitutive relations may have the form of evolution equations. For instance this is the case when the skeleton has some plastic properties, or when mass sources result from chemical reactions or adsorption/desorption processes. We do not consider such problems in this work and limit further our attention to the so-called *poroelastic* materials. Then the set of *constitutive variables* is as follows

$$\begin{aligned} \mathcal{C} &:= \{ \rho^S, \rho^\alpha, \mathbf{F}^S, \dot{\mathbf{X}}^\alpha, \theta, \mathbf{G}, n, \mathbf{N} \}, \quad \alpha = 1, \dots, A, \\ \mathbf{G} &:= \text{Grad } \theta, \quad \mathbf{N} := \text{Grad } n. \end{aligned} \quad (25)$$

Usually this set is still much too complicated for the full thermodynamic analysis and one considers simpler models. For example in the case of a *simple two-component isotropic model of isothermal processes without mass exchange* scalar constitutive functions depend on the following set of constitutive variables

$$\mathcal{C}_{simple} := \{ \rho^F, I, II, III, IV, V, VI, n \}, \quad (26)$$

where the six invariants I, \dots, VI are defined as follows

$$\begin{aligned} I &:= \text{tr} \mathbf{C}^S, \quad II := \frac{1}{2} (I^2 - \text{tr} \mathbf{C}^{S2}), \quad III := \det \mathbf{C}^S, \\ IV &:= \dot{\mathbf{X}}^F \cdot \dot{\mathbf{X}}^F, \quad V := \dot{\mathbf{X}}^F \cdot \mathbf{C}^S \dot{\mathbf{X}}^F, \quad VI := \dot{\mathbf{X}}^F \cdot \mathbf{C}^{S2} \dot{\mathbf{X}}^F, \end{aligned} \quad (27)$$

with $\dot{\mathbf{X}}^F$ being the Lagrangian velocity of the single fluid component: $\alpha = F$. We present some results for such a model further in this paper.

The fundamental assumption of a continuous modelling has the form of the following constitutive relation

$$\mathcal{R} = \mathcal{R}(\mathcal{C}), \quad (28)$$

where the mapping is assumed to be at least once continuously differentiable.

The constitutive functions (28) are said to be *thermodynamically admissible* if any solution of field equations satisfies identically the following *entropy inequality*

$$\frac{\partial(\rho\eta)}{\partial t} + \text{Div} \left(\rho\eta\dot{\mathbf{X}} + \mathbf{H} \right) \geq 0, \quad \eta = \eta(\mathcal{C}), \quad \mathbf{H} = \mathbf{H}(\mathcal{C}). \quad (29)$$

This is the Lagrangian form of the second law of thermodynamics proposed by I. Müller for mixtures.

As shown in 1973 by I-Shih Liu (e.g. see [15]) the limitation to solutions of field equations can be eliminated from the above formulation by means of Lagrange multipliers. The equivalent form of the second law is then as follows. For all fields the following inequality must be fulfilled identically

$$\begin{aligned}
& \frac{\partial(\rho\eta)}{\partial t} + \text{Div} \left(\rho\eta\dot{\mathbf{X}} + \mathbf{H} \right) - \Lambda^S \left(\frac{\partial\rho^S}{\partial t} - \hat{\rho}^S \right) - \\
& - \sum_{\alpha=1}^A \Lambda^\alpha \left(\frac{\partial\rho^\alpha}{\partial t} + \text{Div} \left(\rho^\alpha\dot{\mathbf{X}}^\alpha \right) - \hat{\rho}^\alpha \right) - \\
& - \boldsymbol{\lambda}^S \cdot \left(\rho^S \frac{\partial\dot{\mathbf{x}}^S}{\partial t} - \text{Div} \mathbf{P}^S - \hat{\mathbf{p}}^S + \hat{\rho}^S\dot{\mathbf{x}}^S \right) - \\
& - \sum_{\alpha=1}^A \boldsymbol{\lambda}^\alpha \cdot \left(\rho^\alpha \left(\frac{\partial\dot{\mathbf{x}}^\alpha}{\partial t} + \dot{\mathbf{X}}^\alpha \cdot \text{Grad} \dot{\mathbf{x}}^\alpha \right) - \text{Div} \mathbf{P}^\alpha - \hat{\mathbf{p}}^\alpha + \hat{\rho}^\alpha\dot{\mathbf{x}}^\alpha \right) - \\
& - \boldsymbol{\Lambda}^{\mathbf{F}} \cdot \left(\frac{\partial\mathbf{F}^S}{\partial t} - \text{Grad} \dot{\mathbf{x}}^S \right) - \Lambda^n \left(\frac{\partial n}{\partial t} + \text{Div} \mathbf{J} - \hat{n} \right) - \\
& - \Lambda^\varepsilon \left(\frac{\partial\rho\varepsilon}{\partial t} + \text{Div} \left(\rho\varepsilon\dot{\mathbf{X}} + \mathbf{Q} - \mathbf{P}^T\dot{\mathbf{x}} \right) \right) \geq 0, \tag{30}
\end{aligned}$$

where the Lagrange multipliers $\Lambda := \{\Lambda^S, \Lambda^\alpha, \boldsymbol{\lambda}^S, \boldsymbol{\lambda}^\alpha, \boldsymbol{\Lambda}^{\mathbf{F}}, \Lambda^n, \Lambda^\varepsilon\}$ are functions of constitutive variables \mathcal{C} .

The exploitation of the inequality is now standard. Applying the chain rule we separate a linear part which must vanish. This yields relations for multipliers and some restrictions of constitutive relations. The remaining nonlinear part of the inequality defines the *dissipation* in the system. We skip here a discussion of fully general restrictions of constitutive relations. These can be found in the paper [5] and in the book [15]. We present their particular cases further in this work. However it is worthwhile to expose the structure of the dissipation for constitutive variables \mathcal{C} in which we leave out the dependence on \mathbf{G} and \mathbf{N} . After some calculations we obtain the following so-called residual inequality

$$\begin{aligned}
\mathcal{D} & := \sum_{\alpha=1}^A (\Lambda^\alpha - \Lambda^S) \hat{\rho}^\alpha + \Lambda^n \hat{n} + \\
& + \sum_{\alpha=1}^A (\boldsymbol{\lambda}^\alpha - \boldsymbol{\lambda}^S) \cdot (\hat{\mathbf{p}}^\alpha - \hat{\rho}^\alpha\dot{\mathbf{x}}^\alpha) - \boldsymbol{\lambda}^S \cdot \mathbf{F}^S \sum_{\alpha=1}^A \hat{\rho}^\alpha\dot{\mathbf{X}}^\alpha \geq 0, \tag{31}
\end{aligned}$$

where the multipliers are given by the relations

$$\Lambda^S = \rho^S \left(\frac{\partial\eta}{\partial\rho^S} - \Lambda^\varepsilon \frac{\partial\varepsilon}{\partial\rho^S} \right), \quad \Lambda^\alpha = \rho^\alpha \left(\frac{\partial\eta}{\partial\rho^\alpha} - \Lambda^\varepsilon \frac{\partial\varepsilon}{\partial\rho^\alpha} \right), \quad \Lambda^\varepsilon = \left(\frac{\partial\eta}{\partial\theta} \right) \left(\frac{\partial\varepsilon}{\partial\theta} \right)^{-1},$$

$$\begin{aligned}\rho^S \boldsymbol{\lambda}^S &= -\rho \mathbf{F}^{S-T} \sum_{\alpha=1}^A \left(\frac{\partial \eta}{\partial \dot{\mathbf{X}}^\alpha} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \dot{\mathbf{X}}^\alpha} \right), & \rho^\alpha \boldsymbol{\lambda}^\alpha &= \rho \mathbf{F}^{S-T} \left(\frac{\partial \eta}{\partial \dot{\mathbf{X}}^\alpha} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial \dot{\mathbf{X}}^\alpha} \right), \\ \Lambda^n &= \rho \left(\frac{\partial \eta}{\partial n} - \Lambda^\varepsilon \frac{\partial \varepsilon}{\partial n} \right).\end{aligned}\tag{32}$$

The first contribution to the dissipation function \mathcal{D} (31) describes the dissipation due to the *mass exchange* between components. The second contribution is the dissipation due to the *relaxation of porosity* to its equilibrium value, say n_E . Finally the last contribution is the dissipation due to the *relative motion* of components. It is known from the classical theory of mixtures that momentum sources are objective solely in the combination with mass sources. This property is also present in the model for poroelastic materials and, consequently, the second line in the definition of \mathcal{D} should be considered as a whole. There is no contribution of dissipation due to the *heat conduction* because we have left out the dependence on the temperature gradient \mathbf{G} . The lack of dependence on the gradient of porosity \mathbf{N} does not lead to any simplifications in the dissipation.

The thermodynamic equilibrium state is defined by the requirement that $\mathcal{D} = 0$ in this state. It means that mass, momentum and porosity sources vanish in this state, and simultaneously the dissipation function \mathcal{D} reaches the minimum.

The second law of thermodynamics does not specify constitutive relations for sources but it limits their form by the residual inequality. This statement can be made more specific by the assumption that deviations from the thermodynamic equilibrium are small. Then the dissipation becomes a quadratic function of nonequilibrium variables. We present further the results of this simplification.

5 Some special cases

Let us begin with a rather formal simplification of the multicomponent model which indicates a possible structure of energy, entropy and porosity fluxes. We assume that the intrinsic parts of the internal energy ε_I and the entropy η are independent of relative velocities $\dot{\mathbf{X}}^\alpha$. This assumption is motivated by the fact that scalar functions for isotropic materials must be at least quadratic in their dependence on vector arguments. For small deviations from the thermodynamic equilibrium such a dependence on $\dot{\mathbf{X}}^\alpha$ can be left out. If so then relations (32)_{4,5} for the multipliers become quite explicit and we obtain

$$\boldsymbol{\lambda}^S = \Lambda^\varepsilon \mathbf{F}^S \dot{\mathbf{X}}, \quad \boldsymbol{\lambda}^\alpha = \Lambda^\varepsilon \mathbf{F}^S \left(\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}} \right).\tag{33}$$

Then restrictions following from the second law which we are not quoting in this paper yield the following general form of fluxes for processes in isotropic materials

with a small deviation from the thermodynamic equilibrium

$$\begin{aligned}
\mathbf{Q} &= \sum_{\alpha=1}^A (Q_0^\alpha \mathbf{1} + Q_1^\alpha \mathbf{C}^S + Q_2^\alpha \mathbf{C}^{S2}) \dot{\mathbf{X}}^\alpha, \\
\mathbf{H} &= \sum_{\alpha=1}^A (H_0^\alpha \mathbf{1} + H_1^\alpha \mathbf{C}^S + H_2^\alpha \mathbf{C}^{S2}) \dot{\mathbf{X}}^\alpha, \\
\mathbf{J} &= \sum_{\alpha=1}^A (J_0^\alpha \mathbf{1} + J_1^\alpha \mathbf{C}^S + J_2^\alpha \mathbf{C}^{S2}) \dot{\mathbf{X}}^\alpha,
\end{aligned} \tag{34}$$

where the scalar coefficients $Q_0^\alpha, \dots, J_2^\alpha$ are solely functions of equilibrium variables

$$\mathcal{C}_{equil} = \{I, II, III, \rho^\alpha, \theta, n_E\}. \tag{35}$$

Particularly the last result is important because it allows to specify the equilibrium porosity. Namely the balance equation of porosity (14) reduces in this case as follows

$$\frac{\partial n_E}{\partial t} = 0 \implies \rho^S \frac{\partial n_E}{\partial \rho^S} + \sum_{\alpha=1}^A \rho^\alpha \frac{\partial n_E}{\partial \rho^\alpha} = 0, \tag{36}$$

which is the partial differential equation for n_E . It shows that n_E can be left out in the list (35) because it is not independent from the other variables.

In the simple case of two components the solution of the differential equation (36) has the form

$$n_E = n_E \left(\frac{\rho^F}{\rho^S} \right). \tag{37}$$

The above simplification of the dependence on relative velocities and the structure of the dissipation function indicate as well the following structure of momentum and porosity sources

$$\hat{\mathbf{p}}^\alpha - \hat{\rho}^\alpha \dot{\mathbf{x}}^\alpha = \pi^\alpha \mathbf{F}^S \dot{\mathbf{X}}^\alpha, \quad \hat{n} = -\frac{n - n_E}{\tau}, \quad \pi^\alpha, \tau > 0, \tag{38}$$

where parameters π^α, τ may depend on equilibrium variables.

We proceed to present the model for an important special case of the two-component poroelastic material. This models the so-called saturated porous materials whose components on the macroscopic level are the elastic skeleton and the ideal fluid. The thermodynamic admissibility following from the second law of thermodynamics leads for isothermal processes without mass exchange to the following constitutive relations (e.g. [17])

partial Cauchy stresses

$$\mathbf{T}^S := J^{S-1} \mathbf{P}^S \mathbf{F}^{ST} = \mathfrak{N}_0 \mathbf{1} + \mathfrak{N}_1 \mathbf{B}^S + \mathfrak{N}_2 \mathbf{B}^{S2} - \theta \Lambda^n (n - n_E) \mathbf{1},$$

$$\mathbf{T}^F = - (p^F - \theta \Lambda^n (n - n_E)) \mathbf{1}, \quad \mathbf{B}^S := \mathbf{F}^S \mathbf{F}^{ST}, \quad (39)$$

where

$$\begin{aligned} \aleph_\Gamma &= \aleph_\Gamma (I, II, III, \theta), \quad \Gamma = 0, 1, 2, \quad p^F = p^F (\rho^F, \theta), \\ \Lambda^n &= \Lambda^n (I, II, III, \rho^F, \theta). \end{aligned} \quad (39)$$

porosity flux and momentum source

$$\mathbf{J} = n_E \dot{\mathbf{X}}^F, \quad \hat{\mathbf{p}} = \pi \mathbf{F}^S \dot{\mathbf{X}}^F, \quad \pi = \pi (III, \rho^F, \theta). \quad (40)$$

Consequently the model is analogous to the model of simple mixtures of fluids (e.g. [14]) in which interactions of components reduce to momentum sources and, what is characteristic for poroelastic materials, to nonequilibrium changes of porosity.

We conclude these considerations with a few remarks concerning boundary conditions. Very little has been done for the case of models with more than two components. Therefore we limit the attention solely to this last case.

The natural condition on the boundary $\partial\mathcal{B}_0$ is the condition for the total loading. If we denote by \mathbf{t}_{ext} the vector of force density on this surface which is controlled from the external world then it must be taken over by the total stress vector, i.e.

$$\mathbf{t}_{ext} = \mathbf{PN}|_{\partial\mathcal{B}_0}, \quad (41)$$

provided the interface $\partial\mathcal{B}_0$ does not possess any intrinsic structure of its own. This may not be fulfilled by many porous materials which, for instance, may possess a surface tension on contact surfaces.

In addition to this dynamical condition we have to formulate a kinematical condition depending on a relative motion of components. The tangential component of this vectorial condition has been intensively investigated and the early results of G. S. Beavers and D. D. Joseph [18] have been confirmed. In the case of ideal fluid components this condition reduces to the following one

$$\dot{\mathbf{X}}^F - \dot{\mathbf{X}}^F \cdot \mathbf{NN}|_{\partial\mathcal{B}_0} = 0. \quad (42)$$

The remaining normal component must be determined from investigations of a boundary layer which is created by fluid components flowing out of the porous material through a permeable boundary. A phenomenological model of this flow has been proposed by H. Deresiewicz and R. Skalak [19] and not much has been modified in this condition even though some questions seem to be still open. For two porous materials in contact through the permeable interface $\partial\mathcal{B}_0$ this condition has the form

$$\rho^F \dot{\mathbf{X}}^F \cdot \mathbf{N} + \alpha_0 \left[\left[\frac{p^F}{n} \right] \right] \Big|_{\partial\mathcal{B}_0} = 0, \quad (43)$$

where the double brackets denote a jump, α_0 is a phenomenological coefficient of surface permeability, and the quantity in the brackets describes the difference of the pore pressure on both sides of the interface. It is a kind of a *driving force* for the flow of the fluid through the surface.

It remains the problem of a boundary condition for the porosity. Note that the equation of porosity does not contain a divergence of porosity. Consequently it is a heterogeneous evolution equation rather than a real balance equation. For this reason it does not require any boundary condition at all. This may not be the case if we rely on the model proposed by Goodmann and Cowin in which the equation for the microstructural variable does contain spatial derivatives.

6 Conclusions

The general framework of a nonlinear model of poroelastic materials reminds very much this designed by C. A. Truesdell for mixtures of fluids. The Lagrangian formulation of the present model is solely a technical issue which enables to incorporate large deformations of the solid component but it does not change anything in the "philosophy" of the construction of the model.

A new element grows only from the fact that we have to incorporate an additional microstructural parameter into the model. The model presented in this work contains only one such parameter – the porosity. However the experience with soil and rock mechanics, mechanics of snow and glaciers indicates that the number of those parameters must be larger in many problems of practical bearing. For example, it may be tortuosity, double porosity, anisotropy of microstructure, plastic deformation of the skeleton etc. In such cases the model must be extended even further but the fundamental elements of the theory of mixtures would remain in such extensions.

Finally let us remark that the linear version of the model has been extensively investigated and it seems to work very well, particularly in applications to acoustics of porous materials. Nonlinear problems of poroelastic materials are being solved usually by means of numerical methods for which the Lagrangian formulation is particularly useful. In such a description a mesh of finite elements or finite volumes does not have to be changed in time to follow the motion of fluid components. Analytical results are very rare (e.g. [20]) because very little is known about the form of constitutive relations for large deformations of the skeleton.

References

- [1] Truesdell, C. A.: Sulle basi della termomeccanica, *Acad. Naz. dei Lincei, Rend. della Classe di Scienze Fisiche, Matematiche e Naturali*, **22**(8), 33-38, 158-166 (1957).

- [2] Bowen, R. M.: Compressible Porous Media Models by Use of the Theory of Mixtures, *Int. J. Engn. Sci.*, **20**(6), 697-735 (1982).
- [3] Truesdell, C. A.: *Rational Thermodynamics*, second edition, Springer, N. Y. (1985).
- [4] Wilmanski, K.: Lagrangean Model of Two-phase Porous Material, *J. Non-Equilib. Thermodyn.*, **20**, 50-77 (1995).
- [5] Wilmanski, K.: Toward an Extended Thermodynamics of Porous and Granular Materials, in: G. Ioos, O. Guès, A. Nouri (eds.), *Trends in Applications of Mathematics to Mechanics*, Chapman&Hall/CRC, 147-160 (2000).
- [6] Wilmanski, K.: Porous Media at Finite Strains - the New Model With the Balance Equation of Porosity, *Arch. Mech.*, **48**(4), 591-628 (1996).
- [7] Wilmanski, K.: A Thermodynamic Model of Compressible Porous Materials with the Balance Equation of Porosity, *Transport in Porous Media*, **32**, 21-47 (1998).
- [8] Goodman, M. A., Cowin, S. C.: A Continuum Theory For Granular Materials, *Arch. Rat. Mech. Anal.*, **48**, 249-266 (1972).
- [9] Svendsen, B., Hutter, K.: On the Thermodynamics of a Mixture of Isotropic Materials With Constraints, *Int. J. Engn. Sci.*, **33** 2021-2054 (1995).
- [10] Kirchner, N. P.: Thermodynamically Consistent Modelling of Abrasive Granular Materials. I. Non-equilibrium Theory, *Proc. R. Soc. Lond. (A)*, **458**, 2153-2176 (2002).
- [11] Dunwoody, N. T., Müller, I.: Thermodynamic Theory of Two Chemically Reacting Ideal Gases with Different Temperatures, *Arch. Rat. Mech. Anal.*, **29** (1968).
- [12] Truesdell, C.: *A First Course in Rational Continuum Mechanics*, The Johns Hopkins University, Baltimore, Maryland, 1972.
- [13] Müller, I.: A Thermodynamic Theory of Mixtures of Fluids, *Arch. Rat. Mech. Anal.*, **28**, (1968).
- [14] Müller, I.: *Thermodynamics*, Pitman, N.Y. (1985).
- [15] Wilmanski, K.: *Thermomechanics of Continua*, Springer, Heidelberg (1998).
- [16] Müller, I., Ruggeri, T.: *Rational Extended Thermodynamics*, Springer, N.Y. (1998).
- [17] Wilmanski, K.: Mass Exchange, Diffusion and Large Deformations of Poroe-lastic Materials, in: G. Capriz, V. N. Ghionna, P. Giovine (eds.), *Modeling and Mechanics of Granular and Porous Materials*, Birkhäuser, 213-244 (2002).

- [18] Beavers, G. S., Joseph, D. D.: Boundary Conditions at a Naturally Permeable Wall, *J. Fluid Mech.*, **30**(19, 197-207 (1967).
- [19] Deresiewicz, H., Skalak, R.: On Uniqueness in Dynamic Poroelasticity, *Bull. Seismol. Soc. Am.*, **53**, 783-788 (1963).
- [20] Albers, B., Wilmanski, K.: An Axisymmetric Steady-State Flow Through a Poroelastic Medium Under Large Deformations, *Arch. Appl. Mech.*, **69**, 121-132 (1999).