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Infinitely many elliptic periodic orbits in four-dimensional symplectic maps with a homoclinic tangency

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Abstract

We show that systems having infinitely many coexisting generic 2-elliptic periodic orbits are dense among the four-dimensional symplectic maps with an orbit of homoclinic tangency to a saddle-focus.

Introduction

This paper studies symplectic diffeomorfisms and Hamiltonian systems with homoclinic tangencies. Namely, we speak about the structure of the set of orbits in a small neighborhood of a homoclinic orbit at the points of which the stable and unstable manifolds of a saddle periodic orbit have a quadratic tangency. In a sense, this paper continues a series of papers of the authors where this problem was studied in the framework of general smooth dynamical systems. It was established in [1, 2, 3] that one of the main properties of multidimensional systems with homoclinic tangencies is the *coexistence of periodic trajectories of different topological types* (i.e. with different dimensions of unstable manifolds). This includes the well-known phenomenon [4, 5, 6] of coexistence of hyperbolic sets and stable periodic orbits near homoclinic tangencies. Final criteria for the birth of stable periodic orbits at the bifurcations of a quadratic homoclinic tangency in the case of general dynamical systems were obtained in [1, 2, 7].

When studying analogous problems in the conservative case some peculiarities and differences appear. First, usual conditions of general position often exclude the conservative case, therefore the results obtained in the theory of general systems can rarely be applied to conservative ones. This is true for the systems with homoclinic tangencies as well. Moreover, certain technical difficulties also appear here. The problem is that the study of the behavior near a homoclinic orbit is reduced to the study of the first-return map near some homoclinic point. Usually this map is written in the form $T = T_1 T_0^k$ where T_0 is the Poincaré map near the saddle periodic orbit and T_1 is the map defined by the orbits near the global piece of the homoclinic trajectory. Here, k may take all positive integer values starting with some $\bar{k}$. Since the values of k are not bounded from above, difficulties in the computation of T_0^k appear. In the case of general systems, the use of smooth linearization theorems here may look as the most attractive approach. However, in the conservative dynamics one usually cannot get rid of the resonances of small orders whose presence makes the smooth linearization impossible. Typically, the normal form of a symplectic map in a neighborhood of a saddle fixed point is principally nonlinear. Therefore, for example, the well-known Smale theorem from [8] about the complex structure of the set of orbits in a neighborhood of a transverse homoclinic orbit could not be applied to symplectic diffeomorphisms since the conditions of the theorem included assumptions on the existence of a smooth linearization near the saddle fixed point. These problems completely disappear when one computes the iterations of the local map T_0 in the so-called cross-form ([9, 10]). This was the approach which allowed the second author to solve the Poincaré-Birkhoff problem on the structure of the set of orbits lying near a transverse homoclinic orbit for arbitrary systems, including the Hamiltonian case as well [9]. From the other hand, in the case of conservative systems with homoclinic and heteroclinic tangencies, even a formal removal of the nonlinear terms in the map near a saddle fixed point may be inappropriate: these terms may influence dynamics essentially. Thus, the study of dynamics of Hamiltonian systems with a non-transverse heteroclinic cycle showed [11] that invariants of the Birkhoff-Moser normal form enter formulas for the Ω-moduli (continuous invariants of the local Ω-conjugacy).

The main goal of the present paper is finding conditions under which a symplectic diffeomorfism with a homoclinic tangency to some fixed point has an infinite set of elliptic periodic points. Here we follow Poincaré who proclaimed the question of the existence of stable (elliptic) periodic orbits as one of the main questions of the classical nonlinear dynamics. We must immediately note that among the codimension-1 bifurcations of a homoclinic tangency only two cases are interesting in this connection, a two-dimensional case and a four-dimensional case; moreover, the fixed point must be a saddle-focus in the four-dimensional case. This is connected with the fact that in the other codimension-1 cases there exists an invariant manifold, two- or four-dimensional, which contains the orbit of homoclinic tangency and all the orbits which stay close to it for all, backward and forward iterations of the map [1, 7]. This manifold is saddle which excludes the existence of elliptic points.

The case of two-dimensional symplectic diffeomorfisms with a homoclinic tangency was considered in [12, 13]. It was shown there that generically (namely, if some invariant τ is not an integer) the set of orbits which lie entirely in a small neighborhood of the homoclinic orbit has a nonuniformly hyperbolic structure, hence it does not contain elliptic points (it is well-known, however, that elliptic points appear here indeed when the homoclinic tangency is split [14, 11, 15, 16, 17].)

In the four-dimensional case, when there is an orbit of homoclinic tangency to a saddle-focus fixed point, the birth of elliptic orbits after the tangency is split was established in [18]. In the present paper we investigate the question of the existence of an *infinite set* of elliptic periodic orbits at the moment of tangency itself. Moreover, we consider also the question of the coexistence of periodic trajectories of different topological types. It is known that periodic orbits of four-dimensional symplectic diffeomorphisms, structurally stable in the linear approximation, can be of the three following types :
1) *saddle*, for which one pair of the multipliers lies inside the unit circle and the other pair of multipliers lies outside of it; among the saddle periodic orbits one distinguishes the *saddles*, whose multipliers are real, and the *saddle-foci*, whose multi-

pliers are complex (there are also three types of saddles: $(+,+)$, $(-,-)$ or $(+,-)$ for which, respectively, all the multipliers are positive, negative, or two multipliers are positive and the other two are negative);
2) *saddle-centers* (or 1-elliptic), which have one pair of real multipliers (not equal to 1 in the absolute value) and one pair of complex-conjugate multipliers on the unit circle (one can distinguish two types of saddle-centers: saddle-centers $(+)$ and saddle-centers $(-)$ depending on the sign of the real multipliers);
3) *elliptic* (or 2-elliptic), whose all multipliers $\nu_1, ..., \nu_4$, lie on the unit circle: $\nu_{1,2} = e^{\pm i\omega_1}, \nu_{3,4} = e^{\pm i\omega_2}$, where $0 < \omega_{1,2} < \pi$ and $\omega_1 \neq \omega_2$. In symplectic polar coordinates, the map near an elliptic fixed point may be brought to the following Birkhoff normal form, if there is no strong resonances (i.e., if $\omega_1 \neq \omega_2$, $\omega_1 \neq 2\omega_2$, $\omega_1 \neq 3\omega_2$, $\omega_2 \neq 2\omega_1$, $\omega_2 \neq 3\omega_1$, $\omega_1 + \omega_2 \neq \pi$, $\omega_1 + 2\omega_2 \neq 2\pi$, $2\omega_1 + \omega_2 \neq 2\pi$, $3\omega_1 \pm \omega_2 \neq 2\pi$, $3\omega_2 \pm \omega_1 \neq 2\pi$, $\omega_{1,2} \neq 2\pi/3, \pi/2$):

$$\bar{\rho} = \rho + o(\rho^2), \qquad \bar{\theta} = \theta + \omega + \Omega\rho + o(\rho) \tag{1}$$

where $\rho \in R^2$, $\theta \in T^2$, $\omega = (\omega_1, \omega_2)$ and Ω is a (2×2)-matrix. In the case where Ω is nondegenerate (i.e. $\det \Omega \neq 0$), the corresponding fixed point is called a *generic elliptic point.*

It is obvious that the periodic orbits of the first two types (saddle orbits and saddle-centers) are unstable. In the case of a generic elliptic periodic point the KAM-theory gives the definite positive answer to the question of the eternal stability only in the two-dimensional case. For the four-dimensional case, the KAM-theory gives us that for the majority (in measure) of initial conditions the trajectories never leave a neighborhood of a generic elliptic orbit. However, for the rest of initial conditions one cannot exclude that the corresponding orbits will leave the neighborhood due to the so-called Arnold diffusion. Therefore, when speaking further about the stability of elliptic points we will have in mind the KAM-stability. Of course, to use the KAM-theory one should require a sufficient smoothness of the map.

In this paper we consider C^r-smooth $(r \geq 7)$ symplectic diffeomorfisms with a *saddle-focus* fixed point, whose two-dimensional stable and unstable invariant manifolds have a quadratic tangency at the points of some homoclinic orbit. In the space of C^r-smooth symplectic maps, such diffeomorphisms fill bifurcational surfaces of codimension one. Let $\mathcal{H}$ be such a surface.

Main theorem. *In $\mathcal{H}$ there exists a subset $\mathcal{H}_c$, dense (residual) in the C^r-topology, such that every diffeomorphism from $\mathcal{H}_c$ has 1) an infinite set of generic elliptic periodic orbits; 2) an infinite set of saddle-center periodic orbits; 3) an infinite set of saddle periodic orbits (both saddles and saddle-foci).*

The proof is based on the study of parametric families of diffeomorfisms in $\mathcal{H}$ (i.e. families for which the original homoclinic tangency is not split). It is important to note that we choose, as the parameters, the Ω-moduli, i.e. continuous invariants of the topological conjugacy on the set of orbits lying in a small neighborhood of a homoclinic tangency. By the definition, Ω-moduli are natural governing parameters, because any change in the value of an Ω-modulus leads to a change in the structure of the set of nonwandering orbits, i.e. it leads to bifurcations of periodic, homoclinic, etc., trajectories.

Because of the importance of the Ω-moduli for the bifurcation theory in general, let us elaborate more on this subject. Note, first, that the existence of the Ω-moduli is a characteristic feature of systems with homoclinic tangencies [19, 20, 21, 1, 22]. Namely, such invariants exist in systems with a homoclinic tangency of the so-called third class [4, 23]. In the case of the first two classes one can give a complete description of the set N of trajectories lying in a small neighborhood of an orbit of homoclinic tangency [4]: here N has either a trivial structure (for the systems of the first class), or N admits a complete description in the language of symbolic dynamics (the second class). In the case of the homoclinic tangency of the third class, the set N does not admit a complete description, in general, and its structure changes with any change in the values of the so-called main Ω-modulus

$$\theta \;=\; -\frac{\ln|\lambda|}{\ln|\gamma|}\;,$$

where λ and γ are the leading multipliers of the saddle periodic orbit ($|\lambda| < 1, |\gamma| > 1$). This fact was noticed, first, in the paper [4] for the case of three-dimensional flows (two-dimensional diffeomorfisms). Another effectively computed Ω-modulus is the invariant τ (see [12, 20, 21]), this Ω-modulus is expressed via coefficients of the Poincaré map near the global piece of the homoclinic orbit. In the multidimensional case, when the saddle periodic orbit is a saddle-focus, i.e. when we have for the leading multipliers $\lambda = |\lambda|e^{\pm i\varphi}$ ($\varphi \neq 0, \pi$) and/or $\gamma = |\gamma|e^{\pm i\psi}$ ($\psi \neq 0, \pi$), the corresponding angular arguments φ and ψ are Ω-moduli too [1, 22]. Knowing these moduli helps one to give, in many cases, reasonable answers to the questions concerning the structure and main bifurcations of the set N.

Notably, conservative systems have moduli of local Ω-conjugacy as well. Of course, for a symplectic map, θ is always constant: $\theta = 1$; moreover, $\varphi = \psi$. Therefore, if the leading multipliers are complex (this is possible for symplectic maps starting with the dimension four) the main Ω-modulus is the invariant φ [22, 24]. Other Ω-moduli, analogous to τ, exist here as well. They are certain functions of the coefficients of the Poincaré map near the global piece of the homoclinic orbit — like, for example, invariants α and β (see formula (31) below). In this paper we consider exactly the families where the governing parameters are, along with φ, the Ω-moduli α or β.

The proof or the main theorem consists of two main steps.

As the *first step*, we mainly use the results of our previous paper [18] (collected here in Section 2). Here we consider the problem on the possibility of the birth of (one) elliptic periodic orbit as a result of bifurcations of a homoclinic tangency to a fixed point of the saddle-focus type. Here we consider two-parameter families $F_{\mu\varphi}$ of symplectic diffeomorfisms, which are transverse to $\mathcal{H}$. As the governing parameters, we choose the splitting parameter μ (roughly speaking, it measures the distance to

$\mathcal{H}$) and the angular argument φ. For the original diffeomorfism $F_0 \in \mathcal{H}$, we denote the saddle-focus fixed point as O, and the orbit of homoclinic tangency is denoted as Γ. Let U denote a sufficiently small fixed neighborhood of the set $O \cup \Gamma$. This neighborhood is the union of a small neighborhood U_0 of the point O and a finite number of small neighborhoods of those points of Γ which do not belong to U_0. A periodic or homoclinic to O trajectory, lying entirely in U, is called p-round if it makes exactly p intersection points with every component of the set $U \backslash U_0$. As the first step, we consider bifurcations of the single-round ($p = 1$) periodic orbits in U. As it follows from [18] (see Theorem 1 in [18], or a more general Theorem 1 in Section 2 of the present paper)
in any neighborhood of the point ($\mu = 0, \varphi = \varphi_0$) *in the plane of parameters* (μ, φ) *there exists a region of the parameter values for which the corresponding diffeomorfism* $F_{\mu\varphi}$ *has a single-round periodic orbit in* U*, of any aforehand given type (generic elliptic, saddle-center* $(+)$*, saddle-center* $(-)$*, saddle* $(+,+)$*, saddle* $(+,-)$*, saddle* $(-,-)$ *or saddle-focus).*

Note that the different regions corresponding to the existence of the single-round elliptic periodic orbits do not intersect, in general. This means that the diffeomorphisms close to F_0 cannot, in general, have more than one *single-round* elliptic periodic orbit in a sufficiently small fixed neighborhood of Γ (see Proposition 1 in § 2).

The *second step* is the study of the possibility of the existence of multi-round generic elliptic periodic orbits (as well as the periodic orbits of other types) for the diffeomorfisms in $\mathcal{H}$ close to F_0. Essentially, we consider double-round periodic orbits. We will adhere to the following logic. First, we include F_0 in some one-parameter family F_φ of diffeomorphisms in $\mathcal{H}$. Thus, we take such family of symplectic diffeomorfisms that the homoclinic tangency is not split when the parameter varies, while the angular argument φ of the complex multipliers of the saddle-focus changes monotonically. Since φ is an Ω-modulus, its changes lead to the creation of secondary homoclinic tangencies. Namely, we show that (see Theorems 2 and 2$'$ in Section 3)
under general assumptions, in any sufficiently small interval of values of φ*, the values of* φ *are dense such that the corresponding diffeomorfism* F_φ *has a double-round homoclinic orbit corresponding to a simple quadratic tangency of the invariant manifolds of the saddle-focus fixed point.*

The general assumptions here are the assumption that the original homoclinic tangency of the manifolds $W^s(O)$ and $W^u(O)$ at the points of Γ is simple and quadratic (see the definitions in Section 1), and the inequality

$$\sin(\alpha - \beta) \neq 0. \tag{2}$$

The method of the proof (see the proof of Theorem 2$'$ in § 3) allows to find four different series of the double-round homoclinic tangencies, and the values of φ corresponding to each series are dense in any interval.[1] All these tangencies are simple

[1]Condition (2) is very essential here because the case $\sin(\alpha - \beta) = 0$ is quite special. For example, if $\alpha = \beta$, it may happen that the diffeomorfism F_φ in U does not have, at all, homoclinic orbits other than Γ [24].

and quadratic. Moreover, for fixed φ, the tangencies of the first two series split with non-zero velocities with any change in β, while the tangencies of the third and fourth series split with non-zero velocities with any change in α (see Section 3). Recall that the original homoclinic tangency does not split here, so we may study the bifurcations of the obtained double-round homoclinic tangencies without leaving the bifurcational surface $\mathcal{H}$.

Consider now any two-parameter family $F_{\nu\varphi}$ of diffeomorfisms in $\mathcal{H}$, where the parameter ν is chosen such that

$$\frac{\partial}{\partial \nu}(\alpha, \beta) \neq 0 \tag{3}$$

This ensures that either α or β change monotonically as ν varies. Hence, the homoclinic tangencies of at least two of the above series will always split with a non-zero velocity.

Since the parameter values corresponding to the double-round homoclinic tangencies in $F_{\nu\varphi}$ are dense, and since these tangencies split with non-zero velocities as ν varies, it follows from Theorem 1 that the regions of existence of double-round periodic orbits of any given type are dense in the plane of parameters (ν, φ). Near any point inside any such region, we find (by virtue of Theorem 2′) values of parameters corresponding to some double-round homoclinic tangency, hence (by Theorem 1) we find there a small region for which the system has one more double-round periodic orbit of any aforehand given type. Thus, we obtain that the regions are dense in the parameter plane for the parameter values from which the system has a pair of double-round periodic orbits of arbitrarily chosen types. By repeating the procedure, we obtain that the regions are dense in the parameter plane which correspond to the existence of three double-round periodic orbits of arbitrary types, than four, etc.., and going to the limit proves the main theorem.

In fact, some generalizations of the main theorem are possible (see Theorems 3 and 4 in Section 5). Note also that all the results remain valid both for the case of four-dimensional symplectic diffeomorphisms having a saddle-focus periodic point with a homoclinic tangency and in the case of Hamiltonian systems of three degrees of freedom which have, in some level of the constant value of the Hamiltonian, a saddle-focus periodic orbit with a curve of homoclinic tangency. Concerning this, see more details in § 1.3.

1 Preliminary results: the local and global maps T_0 and T_1

Consider a C^r-smooth ($r \geq 2$) symplectic diffeomorfism F_0 for which the following conditions hold.

A. F_0 has a *saddle-focus* fixed point O with the multipliers $\nu_{1,2} = \lambda_0 e^{\pm i\varphi_0}$, $\nu_{3,4} = \lambda_0^{-1} e^{\pm i\varphi_0}$, where $0 < \lambda_0 < 1, 0 < \varphi_0 < \pi$.

B. The two-dimensional stable and unstable invariant manifolds W^s and W^u of the

point O have a *simple tangency* at the points of some homoclinic orbit Γ. Namely, let $\mathcal{T}_M W$ denote the tangent space to a manifold W at a point $M \in W$. Let M^* be one of the homoclinic points from the orbit Γ. Then we require that:
B.1. $\dim(\mathcal{T}_{M^*} W^s \cap \mathcal{T}_{M^*} W^u) = 1$;
B.2. the tangency of the manifolds W^s and W^u at the point M^* is quadratic.

Conditions B.1, B.2 can be reformulated as a requirement that in some local C^2-coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ near the point M^* the equations of W^s and W^u have the following form

$$W^s = \{\eta_1 = 0, \eta_2 = 0\} \quad \text{and} \quad W^u = \{\xi_2 = 0, \eta_1 = \xi_1^2\}. \tag{4}$$

If one considers one-parameter families which depend smoothly on some parameter ν and which split the given tangency, then C^2-coordinates near M^* can be introduced in such a way that the equations of $W^s(\nu)$ and $W^u(\nu)$ near M^* will take the form

$$W^s = \{\eta_1 = 0, \eta_2 = 0\} \quad \text{and} \quad W^u = \{\xi_2 = 0, \eta_1 = \mu(\nu) + \xi_1^2\}. \tag{5}$$

The quantity $\mu(\nu)$ in (5) is called a splitting parameter for the manifolds W^s and W^u near M^*: at $\mu(\nu) > 0$ the tangency disappears and the manifolds W^s and W^u do not intersect, and at $\mu(\nu) < 0$ two points of a transverse intersection appear. The tangency is said to split generically if $\dfrac{d\mu(0)}{d\nu} \neq 0$.

Let U be a sufficiently small fixed neighborhood of the set $O \cup \Gamma$. It is the union of a small neighborhood U_0 of the point O and a finite number of small neighborhoods of those points of the orbit Γ which do not belong to U_0. A periodic, or a homoclinic to O orbit, entirely lying in U, is called *p-round* if it has exactly p points of intersection with each of the components of the set $U \backslash U_0$. According to this definition, Γ is a single-round homoclinic orbit. Condition B implies that the diffeomorfism F_0 has no other single-round homoclinic orbits in U.

It is obvious that any diffeomorphism close to F_0 has a saddle-focus fixed point $O' \in U_0$ close to O. The diffeomorphisms close to F_0 (in the C^r-topology) may also have a single-round homoclinic to O' orbit Γ' which is close to Γ and which corresponds to a simple tangency of the invariant manifolds of O'. Such diffeomorphisms form a bifurcational surface $\mathcal{H}$ of codimension 1 in the space of four-dimensional symplectic C^r-diffeomorfisms. In the present paper we study dynamical properties and bifurcations of diffeomorfisms from the set $\mathcal{H}$.

As we noticed in the Introduction, our analysis is based on the study of the first-return maps and their iterations. As usual, these maps are represented as compositions of some iteration of the local map T_0 which acts in a small neighborhood of the fixed point O and the global map T_1 defined by the trajectories lying in a small neighborhood of some finite segment of the homoclinic orbit Γ. Below we recall some facts (mostly from [18]) concerning the properties of the local and global maps. Note that along with the diffeomorphism F_0 satisfying conditions A,B, we also consider parametric families[2] F_ε of symplectic C^r-smooth diffeomorfisms which include the diffeomorphism F_0 at $\varepsilon = 0$.

Let the family F_ε be also C^r-smooth with respect to ε. Then the diffeomorphism F_ε has, at all small ε, a saddle-focus fixed point $O \in U_0$ with the multipliers $\lambda(\varepsilon)e^{\pm i\varphi(\varepsilon)}$ and $\lambda^{-1}(\varepsilon)e^{\pm i\varphi(\varepsilon)}$ where $\lambda(0) = \lambda_0$, $\varphi(0) = \varphi_0$. Naturally, the local and global maps will also smoothly depend on ε in this case.

1.1 Properties of the local map T_0

Denote as $T_0(\varepsilon)$ the restriction of the diffeomorphism F_ε onto the neighborhood U_0 of the point O_ε, i.e. $T_0 \equiv F_\varepsilon|_{U_0}$. The map T_0 is called the *local map.* Obviously, one can introduce local coordinates in U_0 such that the point O_ε would be in the origin. Moreover, the following result holds.

Lemma 1 [18] *Let $r \geq 2$. Then there exists $\varepsilon_0 > 0$ and a neighborhood U_0 of O such that for all $\|\varepsilon\| \leq \varepsilon_0$ the local map $T_0(\varepsilon)$ is written in the following form in certain symplectic coordinates in U_0, of class C^r with respect to the phase variables and C^{r-2} with respect to the parameters:*

$$\begin{aligned} \bar{x} &= L(\varepsilon)x + f(x,y,\varepsilon)x, \\ \bar{y} &= L(\varepsilon)^{-\top}y + g(x,y,\varepsilon)y \end{aligned} \tag{6}$$

where

$$L(\varepsilon) = \lambda(\varepsilon)\begin{pmatrix} \cos\varphi(\varepsilon) & -\sin\varphi(\varepsilon) \\ \sin\varphi(\varepsilon) & \cos\varphi(\varepsilon) \end{pmatrix}, \quad L^{-\top}(\varepsilon) = \lambda^{-1}(\varepsilon)\begin{pmatrix} \cos\varphi(\varepsilon) & -\sin\varphi(\varepsilon) \\ \sin\varphi(\varepsilon) & \cos\varphi(\varepsilon) \end{pmatrix}. \tag{7}$$

Here x and y are two-dimensional: $x = (x_1, x_2)$ and $y = (y_1, y_2)$; the C^{r-1}-smooth functions f and g satisfy the following conditions:

$$\begin{aligned} f(0,y,\varepsilon) \equiv 0\ , &\quad g(0,y,\varepsilon) \equiv 0, \\ f(x,0,\varepsilon) \equiv 0\ , &\quad g(x,0,\varepsilon) \equiv 0, \end{aligned} \tag{8}$$

We will use the coordinates of lemma 1 because the iterations of the local map T_0, written in the “cross”-form, will be close in this case to the iterations of a linear map. Namely, denote $(x_k, y_k) = T_0^k(x_0, y_0)$. It is well-known [9, 25, 26] that for a sufficiently small δ, given any $k \geq 0$ and x_0, y_k such that $\|x_0\| \leq \delta/2$, $\|y_k\| \leq \delta/2$, the corresponding segment $(x_j, y_j)_{j=0}^k$ of an orbit of the map T_0 is defined uniquely and all its points lie in the δ-neighborhood of the fixed point $O(0,0)$. Moreover the following result holds true (see [18] and [21, 10]).

[2] we consider here either families transverse to the bifurcational surface $\mathcal{H}$, or families lying within $\mathcal{H}$

Lemma 2 *Let $r \geq 3$ and let identities* (8) *hold.*[3] *Then*

$$\begin{array}{rcl} x_k & = & L(\varepsilon)^k\, x_0 \ + \ k\lambda^{2k} P_k(x_0, y_k, \varepsilon) x_0 \ , \\ y_0 & = & (L(\varepsilon)^\top)^k\, y_k \ + \ k\lambda^{2k} Q_k(x_0, y_k, \varepsilon) y_k \ , \end{array} \tag{9}$$

where the functions P_k and Q_k are uniformly bounded for all k along with all the derivatives up to the order $(r-2)$; the derivatives of the order $(r-1)$ of the right-hand sides of (9) *with respect to (x_0, y_k) tend to zero as $k \to +\infty$.*

1.2 Properties of the global map T_1

In the coordinates of lemma 1 the equations of the two-dimensional manifolds $W^s_{loc}(O(\varepsilon))$ and $W^u_{loc}(O(\varepsilon))$ in U_0 are $y_1 = y_2 = 0$ and $x_1 = x_2 = 0$, respectively. By assumption, the diffeomorfism F_0 has a homoclinic orbit Γ at the points of which the two-dimensional invariant manifolds of the saddle-focus O have a simple tangency, i.e. conditions B.1 and B.2 hold. The points of Γ accumulate to O, so there is an infinite set of the homoclinic points both in $W^s_{loc}(O)$ and in $W^u_{loc}(O)$. Take any pair of these points: $M^+(x^+, 0) \in W^s_{loc}(O) \cap U_0$ and $M^-(0, y^-) \in W^u_{loc}(O) \cap U_0$, where $x^+ = (x_1^+, x_2^+)$, $y^- = (y_1^-, y_2^-)$; here $(x_1^+)^2 + (x_2^+)^2 \neq 0$, $(y_1^-)^2 + (y_2^-)^2 \neq 0$.

Let Π^+ and Π^- be some, lying in U_0, neighborhoods of the points M^+ and M^-, respectively. We assume that the neighborhoods Π^+ and Π^- are sufficiently small, so in any case $T_0(\Pi^+) \cap \Pi^+ = \emptyset$ and $T_0^{-1}(\Pi^-) \cap \Pi^- = \emptyset$. By construction, $F_0^{n_0}(M^-) = M^+$ for some positive n_0. Consider the map $T_1 \equiv F_\varepsilon^{n_0} : \Pi^- \to \Pi^+$ which is defined by the orbits of the diffeomorphism F_ε near the global piece of Γ. We will call T_1 the *global map*. By definition, $T_1(M^-) = M^+$ at $\varepsilon = 0$. Denote the coordinates in Π^+ as $(x_0, y_0) = (x_{01}, x_{02}, y_{01}, y_{02})$, and the coordinates in Π^- as $(x_1, y_1) = (x_{11}, x_{12}, y_{11}, y_{12})$. Let us write the Taylor expansion of the global map T_1 at the point $M^-(0, y^-)$ at $\varepsilon = 0$:

$$\bar{x}_0 - x^+ \ = \ a x_1 + b(y_1 - y^-) + ..., \quad \bar{y}_0 \ = \ c x_1 + d(y_1 - y^-) + ..., \tag{10}$$

where the dots stand for the terms of the second order and higher; here a, b, c and d are some (2×2)-matrices. Together, these matrices comprise the following symplectic (4×4)-matrix:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

hence, the following relations hold (see e.g. [18]):

$$\begin{array}{l} 1)\ a^\top c \ = \ c^\top a, \\ 2)\ b^\top d \ = \ d^\top b, \\ 3)\ d^\top a \ - \ b^\top c \ = \ I, \end{array} \tag{11}$$

and

$$\begin{array}{l} 1)\ a\, b^\top \ = \ b\, a^\top, \\ 2)\ c\, d^\top \ = \ d\, c^\top, \\ 3)\ d\, a^\top \ - \ c\, b^\top \ = \ I. \end{array} \tag{12}$$

[3] at $r \geq 3$ identities (8) mean that $f(x, y, \varepsilon) = O(\|x\| \cdot \|y\|)$, $g(x, y, \varepsilon) = O(\|x\| \cdot \|y\|)$

For the matrix which is inverse to S (it is also symplectic) we have the following formula:

$$S^{-1} = \begin{pmatrix} d^\top & -b^\top \\ -c^\top & a^\top \end{pmatrix}. \tag{13}$$

Note that the linear rotation in U_0

$$x_{new} = \mathcal{R}_\theta x, \quad y_{new} = \mathcal{R}_\theta y \ , \tag{14}$$

where

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{15}$$

is a symplectic coordinate transformation. Moreover, this is the unique symplectic transformation which both preserves the form of the matrix L of the linear part of the local map T_0 and keeps identity (8) holding. So we are allowed to make linear rotations in U_0 with arbitrary angles θ, and we will use them in order to simplify the matrix S, i.e. to make zero as many entries of d as possible.

Note that such rotation transforms the matrix d as follows:

$$d_{new} = \mathcal{R}_{-\theta}\, d\, \mathcal{R}_\theta. \tag{16}$$

By condition B.1, the surface $T_1(W^u_{loc})$ is tangent at $\varepsilon = 0$ to the plane W^s_{loc} at the point $M^+(x^+, 0)$ along a single direction. Consider the linear part of T_1 at the point M^-:

$$\bar{x}_0 - x^+ \ = \ ax_1 + b(y_1 - y^-), \ \ \bar{y}_0 \ = \ cx_1 + d(y_1 - y^-). \tag{17}$$

Condition B.1 reads now as follows: the image of the plane $\{x_1 = 0\}$ by this linear map intersects the plane $\{y_0 = 0\}$ at a straight line. In other words, the equation

$$0 \ = \ d(y_1 - y^-) \tag{18}$$

has a one-parameter family of solutions. In this case

$$\det\ d = 0 \quad \text{and} \quad \text{rank}\ d = 1\ , \tag{19}$$

i.e. the rows of the matrix

$$d = \begin{pmatrix} d_{11}\, d_{12} \\ d_{21}\, d_{22} \end{pmatrix}$$

are linearly dependent, but not all its entries are zero. Obviously, one can choose such θ in (16) (and, correspondingly, in (14)), that the matrix d will transform into

$$d_{new} = \begin{pmatrix} 0 & 0 \\ d_{21}\, d_{22} \end{pmatrix}$$

where $d_{21}^2 + d_{22}^2 \neq 0$. We assume that $d_{22} \neq 0$. If this is not the case (i.e. if $d_{22} = 0$ and $d_{21} \neq 0$), then one can take another pair of the homoclinic points, namely $(T_0^{-1}M^-, M^+)$, instead of (M^-, M^+). The new global map will be $T_1' = T_1T_0$, and taking into account that the function g from the formula (6) for T_0 vanishes identically on W^u_{loc} it is easy to see that the corresponding matrix d' will have the form

$$d' \;=\; \lambda^{-1}\cdot\begin{pmatrix}0 & 0\\ d_{21} & 0\end{pmatrix}\cdot\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \;=\; \lambda^{-1}\cdot\begin{pmatrix}0 & 0\\ d_{21}\cos\varphi & -d_{21}\sin\varphi\end{pmatrix}.$$

Since $\sin\varphi \neq 0$, it follows that $d'_{22}(= -d_{21}\sin\varphi) \neq 0$.

Thus, we may assume that the Jacobi matrix S of the global map T_1 computed at the point M^- at $\varepsilon = 0$ has the following form:

$$S = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12}\\ a_{21} & a_{22} & b_{21} & b_{22}\\ c_{11} & c_{12} & 0 & 0\\ c_{21} & c_{22} & d_{21} & d_{22}\end{pmatrix} \tag{20}$$

where $d_{22} \neq 0$. Since S is a symplectic matrix, its entries satisfy the following equalities, according to (11) and (12):

$$\begin{array}{ll} a) & b_{21}d_{22} - b_{22}d_{21} = 0\,,\\ b) & c_{11}d_{21} + c_{12}d_{22} = 0\,,\\ c) & c_{11}\,(b_{12}d_{21} - b_{11}d_{22}) = d_{22}\,,\\ d) & c_{12}\,(b_{12}d_{21} - b_{11}d_{22}) = -d_{21}\,,\\ e) & a_{21}d_{21} - b_{21}c_{21} = 1 + b_{11}c_{11}. \end{array} \tag{21}$$

Since $d_{22} \neq 0$, it follows from (21.c) that

$$c_{11} \neq 0\;,\; b_{11}d_{22} - b_{12}d_{21} \neq 0. \tag{22}$$

Taking into account quadratic terms in the equation for $\bar y_{01}$, we may write the map T_1 as

$$\begin{array}{l} \bar x_{01} - x_1^+ \;=\; a_{11}x_{11} + a_{12}x_{12} + b_{11}(y_{11} - y_1^-) + b_{12}(y_{12} - y_2^-) + \ldots\,,\\ \bar x_{02} - x_2^+ \;=\; a_{21}x_{11} + a_{22}x_{12} + b_{21}(y_{11} - y_1^-) + b_{22}(y_{12} - y_2^-) + \ldots\,,\\ \bar y_{01} \;=\; c_{11}x_{11} + c_{12}x_{12} +\\ \qquad + D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 + \ldots\,,\\ \bar y_{02} \;=\; c_{21}x_{11} + c_{22}x_{12} + d_{21}(y_{11} - y_1^-) + d_{22}(y_{12} - y_2^-) + \ldots\,. \end{array} \tag{23}$$

Since $x_{11} = x_{12} = 0$ on W^u_{loc}, the equation of the surface $T_1W^u_{loc}$ may be written as follows

$$\begin{array}{l} x_{01} - x_1^+ \;=\; b_{11}(y_{11} - y_1^-) + b_{12}(y_{12} - y_2^-) + \ldots\,,\\ x_{02} - x_2^+ \;=\; b_{21}(y_{11} - y_1^-) + b_{22}(y_{12} - y_2^-) + \ldots\,,\\ y_{01} \;=\; D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 + \ldots\,,\\ y_{02} \;=\; d_{21}(y_{11} - y_1^-) + d_{22}(y_{12} - y_2^-) + \ldots\,, \end{array} \tag{24}$$

where $(y_{11} - y_1^-)$ and $(y_{12} - y_2^-)$ are the coordinates on W^u_{loc}. The equation for $T_1W^u_{loc}$ can be written in an explicit form as well. Namely, since $d_{22} \neq 0$, it follows that the last equation of (24) can be resolved with respect to $(y_{12} - y_2^-)$:

$$y_{12} - y_2^- = \frac{1}{d_{22}}y_{02} - \frac{d_{21}}{d_{22}}(y_{11} - y_1^-) + \ldots\,. \tag{25}$$

Plugging (25) into (24), we obtain

$$
\begin{aligned}
x_{01}-x_1^+ &= (b_{11}-b_{12}\frac{d_{21}}{d_{22}})(y_{11}-y_1^-)+\frac{b_{12}}{d_{22}}y_{02}+...,\\
x_{02}-x_2^+ &= (b_{21}-b_{22}\frac{d_{21}}{d_{22}})(y_{11}-y_1^-)+\frac{b_{22}}{d_{22}}y_{02}+...,\\
y_{01} &= \left[D_1-D_2\left(\frac{d_{21}}{d_{22}}\right)+D_3\left(\frac{d_{21}}{d_{22}}\right)^2\right](y_{11}-y_1^-)^2+\tilde{D}_2(y_{11}-y_1^-)y_{02}+\tilde{D}_3y_{02}^2+\dots.
\end{aligned}
\tag{26}
$$

Denote

$$D_0\equiv D_1-D_2\left(\frac{d_{21}}{d_{22}}\right)+D_3\left(\frac{d_{21}}{d_{22}}\right)^2.$$

Since $d_{22}\neq 0$, we deduce from (21) that

$$b_{11}-b_{12}\frac{d_{21}}{d_{22}}=-\frac{1}{c_{11}}\ ,\ \ b_{21}-b_{22}\frac{d_{21}}{d_{22}}=0\ .$$

Now, formulas (26) can be rewritten as

$$
\begin{aligned}
x_{01}-x_1^+ &= -\frac{1}{c_{11}}(y_{11}-y_1^-)+\frac{b_{12}}{d_{22}}y_{02}+O([|y_{02}|+|y_{11}-y_1^-|]^2)\ ,\\
x_{02}-x_2^+ &= \frac{b_{22}}{d_{22}}y_{02}+O([|y_{02}|+|y_{11}-y_1^-|]^2)\ ,\\
y_{01} &= D_0(y_{11}-y_1^-)^2+\tilde{D}_2(y_{11}-y_1^-)y_{02}+\tilde{D}_3y_{02}^2+O([|y_{02}|+|y_{11}-y_1^-|]^3)\ ,
\end{aligned}
$$

or in the following final form

$$
\begin{aligned}
x_{02}-x_2^+ &= \frac{b_{22}}{d_{22}}y_{02}+R_1(x_{01}-x_1^+,y_{02}),\\
y_{01} &= (x_{01}-x_1^+)^2\left[c_{11}^2D_0+R_0(x_{01}-x_1^+)\right]+\\
&\quad +\hat{D}_2\cdot(x_{01}-x_1^+)y_{02}+\hat{D}_3\cdot y_{02}^2+y_{02}\cdot R_2(x_{01}-x_1^+,y_{02}),
\end{aligned}
\tag{27}
$$

where $R_{1,2}=O([|y_{02}|+|x_{01}-x_1^+|]^2)$ and $R_0=O(|x_{01}-x_1^+|)$.

If we introduce now local coordinates near the point M^+ as follows:

$$
\begin{aligned}
&\xi_1=(x_{01}-x_1^+)\sqrt{c_{11}^2D_0+R_0},\ \ \xi_2=(x_{02}-x_2^+)-\frac{b_{22}}{d_{22}}y_{02}-R_1,\\
&\eta_1=y_{01}-\hat{D}_2(x_{01}-x_1^+)y_{02}-\hat{D}_3y_{02}^2-y_{02}R_2,\ \ \eta_2=y_{02},
\end{aligned}
$$

then equations (27) recast as

$$\xi_2\ =\ 0,\ \ \eta_1\ =\ \xi_1^2. \tag{28}$$

Thus, the equations of W^s and W^u in the coordinates $(\xi_1,\xi_2,\eta_1,\eta_2)$ in a small neighborhood of M^+ are the same as (4), provided $D_0\neq 0$. Hence, our condition B.2 of the quadraticity of the homoclinic tangency is equivalent to the requirement

$$D_0\neq 0. \tag{29}$$

We may always assume $D_0>0$ (because the sign of D_0 can always be changed by the coordinate transformation $(x,y)\to(-x,-y)$).

We see that the global map T_1 can be written in the form (23), or it can be written in the following, *cross-form* (with respect to the coordinate y_2):

$$\begin{aligned}
\bar{x}_{01} - x_1^+ &= \tilde{a}_{11}x_{11} + \tilde{a}_{12}x_{12} - \frac{1}{c_{11}}(y_{11} - y_1^-) + \frac{b_{12}}{d_{22}}\bar{y}_{02} + \ldots, \\
\bar{x}_{02} - x_2^+ &= \tilde{a}_{21}x_{11} + \tilde{a}_{22}x_{12} + \frac{b_{22}}{d_{22}}\bar{y}_{02} + \ldots, \\
\bar{y}_{01} &= c_{11}x_{11} + c_{12}x_{12} + \\
&\quad + D_0(y_{11} - y_1^-)^2 + \tilde{D}_2(y_{11} - y_1^-)\bar{y}_{02} + \tilde{D}_3\bar{y}_{02}^2 + \ldots, \\
y_{12} - y_2^- &= -\frac{c_{21}}{d_{22}}x_{11} - \frac{c_{22}}{d_{22}}x_{12} + \frac{1}{d_{22}}\bar{y}_{02} - \frac{d_{21}}{d_{22}}(y_{11} - y_1^-) + \ldots .
\end{aligned} \tag{30}$$

where $\tilde{a}_{ij} = a_{ij} - b_{i2}c_{2j}d_{22}^{-1}$; and the dots stand for the terms of the order two and higher (with respect to the coordinates $x_{11}, x_{12}, y_{11} - y_1^-$ and $\bar{y}_{02}$) in the first, second and fourth equations, and for the terms quadratic in x_{11} and x_{12} and all the terms of the order three and higher in the third equation.

Note that after we brought the map T_0 to the form (6) and the map T_1 to the form (23) (i.e., when there is no linear in y_1 term in the equation for $\bar{y}_{01}$ [4] and $D_0 > 0$), any symplectic coordinate transformation which keeps this form of these maps will have the following form in the restriction onto the stable manifold:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \rho \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

while it will have the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \rho^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

on the unstable manifold, for some $\rho > 0$. It is obvious then that the quantities α, β and A/B, defined by the formulas

$$\begin{aligned}
&A = \sqrt{(y_1^-)^2 + (y_2^-)^2}, \quad \sin\alpha = y_2^-/A, \quad \cos\alpha = y_1^-/A, \\
&B = \sqrt{[(x_1^+)^2 + (x_2^+)^2][c_{11}^2 + c_{12}^2]}, \\
&\sin\beta = \frac{c_{12}x_1^+ - c_{11}x_2^+}{B}, \quad \cos\beta = \frac{c_{11}x_1^+ + c_{12}x_2^+}{B},
\end{aligned} \tag{31}$$

are invariant with respect to such coordinate transformations, hence they are defined by the given diffeomorphism uniquely, for a fixed choice of the pair of homoclinic points. When we choose another pair of homoclinic points, α and β get the same increment, proportional to φ, so the difference $\alpha - \beta$ remains invariant. It is also easy to check that the following formulas are valid for the diffeomorfism F^{-1}:

$$\alpha(F^{-1}) = \beta(F) + \pi, \quad \beta(F^{-1}) = \alpha(F) + \pi, \quad \frac{B(F^{-1})}{A(F^{-1})} = \frac{A(F)}{B(F)}. \tag{32}$$

At non-zero values of the parameter ε, formulas (23) and (30) for the global map are changed as follows. First, all the coefficients (i.e. $x^+, y^-, \ldots, d_{22}$), as well as the

[4] this means that the common tangent vector of $T_1(W^u_{loc})$ and W^s_{loc} at the point M^+ coincides with the x_{01}-axis

terms denoted by the dots, depend now on ε (C^{r-2}-smooth, at least; see details in [18]). It is also easy to see that since $D_0 \neq 0$, the values of $y_1^-(\varepsilon)$ and $y_2^-(\varepsilon)$ can be chosen so that the equation for $\bar{y}_{02}$ would not contain a constant (i.e. zero order) term and, moreover,

$$\det d(\varepsilon) \equiv \det \left.\frac{\partial \bar{y}_0}{\partial y_1}\right|_{(0,0,y_1^-,y_2^-)} = 0$$

(note that the corresponding values of $y_1^-(\varepsilon)$ and $y_2^-(\varepsilon)$ are defined by these conditions uniquely). Next, as we did it at $\varepsilon = 0$, by means of a linear rotation (14) with a small angle $\theta = O(\varepsilon)$ we can always eliminate the linear in $(y_1 - y^-)$ term in the equation for $\bar{y}_{01}$, i.e. $d_{11}(\varepsilon) = d_{21}(\varepsilon) = 0$. Thus, the Jacobi matrix $S(\varepsilon)$ of the global map T_1 at the point $M^-(\varepsilon) = (0, 0, y_1^-, y_2^-)$ will keep its form (20). Since $S(\varepsilon)$ is symplectic, then the equalities (21) remain fulfilled for all small ε.

We see that the main difference with the case $\varepsilon = 0$ is provided by the possible appearance of the non-zero constant term in the equation for $\bar{y}_{01}$. We denote this term as $\mu(\varepsilon)$. So, equation (23) for the map T_1 will take the following form at $\varepsilon \neq 0$:

$$\begin{array}{rcl}
\bar{x}_{01} - x_1^+ &=& a_{11}x_{11} + a_{12}x_{12} + b_{11}(y_{11} - y_1^-) + b_{12}(y_{12} - y_2^-) + \dots\ , \\
\bar{x}_{02} - x_2^+ &=& a_{21}x_{11} + a_{22}x_{12} + b_{21}(y_{11} - y_1^-) + b_{22}(y_{12} - y_2^-) + \dots\ , \\
\bar{y}_{01} &=& \mu + c_{11}x_{11} + c_{12}x_{12} + \\
&& + D_1(y_{11} - y_1^-)^2 + D_2(y_{11} - y_1^-)(y_{12} - y_2^-) + D_3(y_{12} - y_2^-)^2 + \dots\ , \\
\bar{y}_{02} &=& c_{21}x_{11} + c_{22}x_{12} + d_{21}(y_{11} - y_1^-) + d_{22}(y_{12} - y_2^-) + \dots\ ,
\end{array} \tag{33}$$

where all the coefficients, as well as all the terms denoted by the dots, depend C^{r-2}-smoothly on ε. Note that it is easy to see that μ is the splitting parameter for the manifolds $W^s(O)$ and $W^u(O)$, because the equations of $W^s(O)$ and $W^u(O)$ in a small neighborhood of the point M^+ can, in certain local coordinates, be written in the form (5); this is done in the absolutely the same way as we proceeded when deriving (28) from (24).

Since $d_{22} \neq 0$ at all small ε, we can also recast (33) in the cross-form (see (30)):

$$\begin{array}{rcl}
\bar{x}_{01} - x_1^+ &=& \tilde{a}_{11}x_{11} + \tilde{a}_{12}x_{12} - \dfrac{1}{c_{11}}(y_{11} - y_1^-) + \dfrac{b_{12}}{d_{22}}\bar{y}_{02} + \dots\ , \\
\bar{x}_{02} - x_2^+ &=& \tilde{a}_{21}x_{11} + \tilde{a}_{22}x_{12} + \dfrac{b_{22}}{d_{22}}\bar{y}_{02} + \dots\ , \\
\bar{y}_{01} &=& \mu + c_{11}x_{11} + c_{12}x_{12} + \\
&& + D_0(y_{11} - y_1^-)^2 + \tilde{D}_2(y_{11} - y_1^-)\bar{y}_{02} + \tilde{D}_3\bar{y}_{02}^2 + \dots\ , \\
y_{12} - y_2^- &=& -\dfrac{c_{21}}{d_{22}}x_{11} - \dfrac{c_{22}}{d_{22}}x_{12} + \dfrac{1}{d_{22}}\bar{y}_{02} - \dfrac{d_{21}}{d_{22}}(y_{11} - y_1^-) + \dots .
\end{array} \tag{34}$$

1.3 Local and global maps for Hamiltonian flows

Let a Hamiltonian system with three degrees of freedom have a saddle periodic orbit L_0. Let U_0 be a small four-dimensional cross-section to L_0 in the corresponding five-dimensional level of the Hamiltonian. Denote as T_0 the Poincaré map on U_0. Then $L_0 \cap U_0$ will be a fixed point of T_0. We assume that this fixed point is a saddle-focus. Assume also that the stable and unstable manifolds of the orbit L_0 have a simple (quadratic) tangency at the points of some homoclinic curve Γ_0. Then the global map T_1 is defined as the map by trajectories close to Γ_0 which start in a small neighborhood (in U_0) of some point $M^- \in \Gamma_0 \cap U_0 \subset W^u_{loc} \cap U_0$ and end up in a small neighborhood of some point $M^+ \in \Gamma_0 \cap U_0 \subset W^s_{loc} \cap U_0$. The maps T_0 and T_1 preserve the standard symplectic structure, and the statements of Sections 1.1 and 1.2 hold true for these maps as well. Therefore, the results below will hold true both for the case of symplectic maps and for the case of Hamiltonian flows in a fixed level of the Hamiltonian. Note that the value h of the Hamiltonian serves as a natural parameter in the latter case. Here, at small variations of h, the saddle periodic orbit does not disappear, while the homoclinic tangency splits, in general. In that case, it is natural to take h as a splitting parameter μ.

2 On bifurcations of single-round periodic orbits

In this Section we consider bifurcations of single-round periodic orbits in two-parameter, transverse to $\mathcal{H}$, families of diffeomorfisms. Most of the results here are obtained in [18], but we repeat them here because they play a key role in the proof of the main theorem.

Consider a diffeomorfism F_0 satisfying conditions A and B. Embed F_0 in the family $F_{\mu\varphi}$ where μ is the splitting parameter for the pieces $T_1(W^u_{loc})$ and W^s_{loc} of invariant manifolds of the saddle-focus O near the homoclinic point M^+, and φ is the angular argument of the complex multipliers of the saddle-focus. We assume that μ varies in a small neighborhood of $\mu = 0$, and φ varies in a small neighborhood of the point $\varphi_0 \in (0, \pi)$. Now the local map T_0 and the global map T_1 depend smoothly on the parameters $\varepsilon = (\mu, \varphi)$. The map T_0 is given by formula (9), and T_1 is defined by formulas (33) and (34).

The study of single-round periodic orbits of $F_{\mu\varphi}$ is reduced to the study of the fixed points of the maps $T_k \equiv T_1 T_0^k : \sigma_k^0 \to \sigma_k^0$ (i.e. the first-return maps) for all sufficiently large k. Here σ_k^0 is a four-dimensional strip which is the domain of definition of the map T_0^k acting from Π^+ into Π^-. In other words, $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$. Analogously, the image of the strip σ_k^0 by the map T_0^k is the four-dimensional strip $\sigma_k^1 \equiv T_0^k(\sigma_k^0)$, lying in Π^-.

Let the neighborhoods Π^+ and Π^- be defined as follows:

$$\begin{aligned} \Pi^+ &= \{(x_0, y_0) \Big| \, \|x_0 - x^+\| \le \delta_0, \ \|y_0\| \le \delta_0\}, \\ \Pi^- &= \{(x_1, y_1) \Big| \, \|x_1\| \le \delta_0, \ \|y_1 - y^-\| \le \delta_0\} \end{aligned} \tag{35}$$

with some small $\delta_0 > 0$. If the boundary (35) is plugged into the right-hand side of (9), then it is seen immediately that

$$\begin{aligned} \sigma_k^0 \equiv \{(x_0, y_0) \Big| \|x_0 - x^+\| \le \delta_0, \\ | \, y_{01} - \lambda^k A \cos(k\varphi - \alpha) \, | \le \lambda^k \delta_0, \ \ | \, y_{02} + \lambda^k A \sin(k\varphi - \alpha) \, | \le \lambda^k \delta_0\}, \end{aligned} \tag{36}$$

where A and α are given by formulas (31).

Analogously, the image of σ_k^0 by the map T_0^k is the strip σ_k^1 in Π^-, defined by the following inequalities:

$$\sigma_k^1 \equiv \{(x_1, y_1) \big| \mid x_{11} - \lambda^k(\cos k\varphi \cdot x_1^+ - \sin k\varphi \cdot x_2^+) \mid \leq \lambda^k \delta_0, \\ \mid x_{12} - \lambda^k(\cos k\varphi \cdot x_2^+ + \sin k\varphi \cdot x_1^+) \mid \leq \lambda^k \delta_0, \quad \|y_1 - y^-\| \leq \delta_0\}. \tag{37}$$

As $k \to +\infty$, the strips σ_k^1 accumulate on $W^u_{loc}(O)$ and the strips σ_k^0 accumulate on $W^s_{loc}(O)$.

According to (34), the images $T_1(\sigma_k^1)$ of the strips σ_k^1 have a shape of four-dimensional "horse-shoes", winding to the two-dimensional surface $T_1(W^u_{loc}(O)) \cap \Pi^+$. Namely, these horse-shoes are given by the inequalities

$$|x_{02} - x_2^+ - \frac{b_{22}}{d_{22}} y_{02} + R_1(x_{01} - x_1^+, y_{02}) - \\ \lambda^k \left[(\tilde{a}_{21} x_1^+ + \tilde{a}_{22} x_2^+) \cos k\varphi - (\tilde{a}_{21} x_2^+ - \tilde{a}_{22} x_1^+) \sin k\varphi\right] | \leq C \lambda^k \delta_0,$$

$$|y_{01} - \mu - (x_{01} - x_1^+)^2 \left[c_{11}^2 D_0 + R_0[(x_{01} - x_1^+)]\right] + c_{11} \tilde{D}_2 \cdot (x_{01} - x_1^+) y_{02} - \tilde{D}_3 \cdot y_{02}^2 - \\ - y_{02} R_2(x_{01} - x_1^+, y_{02}) - B \lambda^k \cos(k\varphi - \beta)| \leq C \lambda^k \delta_0, \tag{38}$$

where B and β are given by (31), C is some constant, and the functions $R_{0,1,2}$ are estimated as $R_{1,2} = O([|y_{02}| + |x_{01} - x_1^+|]^2)$ and $R_0 = O(|x_{01} - x_1^+|)$ (compare with (27)).

It is clear that the mutual position of the strips and horse-shoes in Π^+ depend in an essential way from the values of the parameters μ and φ. Hence, one can expect that the changes in these parameters can lead to various bifurcations, of periodic and homoclinic orbits in particular. Among the main such bifurcations are bifurcations of single-round periodic orbits, i.e. bifurcations of the fixed points of the maps T_k at large k. In order to study this bifurcations, it is convenient to make, first, a rescaling: the map T_k is written in new coordinates and with new parameters (obtained from the old ones by affine transformations) which are no longer small and may take arbitrary finite values. Namely, we will use the following statement (see [18], lemma 4).

Lemma 3 *For all sufficiently large k, by a smooth transformation of coordinates and parameters, the map T_k can be made asymptotically C^{r-2}-close (as $k \to +\infty$, uniformly in any bounded region of the values of X_1, X_2, Y_1, Y_2 and M_1, M_2) to the following four-dimensional quadratic map:*

$$\bar{X}_2 \;=\; X_1 \;\; \bar{X}_1 \;=\; Y_2, \; \bar{Y}_2 = Y_1, \\ \bar{Y}_1 = M_1(Y_1 + X_1) - X_2 - Y_2^2 + M_2, \tag{39}$$

where

$$M_1 = \lambda^{-k}(d_{22}\cos k\varphi - d_{21}\sin k\varphi + r_k^1),$$
$$M_2 = -\lambda^{-4k} d_{22}^2 D_0(\mu + \frac{1}{\sqrt{d_{21}^2+d_{22}^2}}\lambda^k[x_2^+(c_{12}d_{21} - c_{11}d_{22}) - y_1^- d_{21} - y_2^- d_{22} + r_k^2]), \tag{40}$$

where $r_k^{1,2} \to 0$ *as* $k \to \infty$. *The ranges of values of the new coordinates and parameters cover, in the limit* $k \to \infty$, *all finite values.*

The main (i.e. codimension 1) local bifurcations of symplectic maps are [27] :
1) bifurcations of a fixed point with a double multiplier $(+1)$;
2) bifurcations of a fixed point with a double multiplier (-1);
3) bifurcations of a fixed point with a double complex multiplier on the unit circle (i.e. with a quadruple of multipliers of the form $\nu_{1,2} = \nu_{3,4} = e^{\pm i\omega}$, $\omega \neq 0, \pi$) — the so-called resonance 1:1.

The bifurcation diagram for map (39) is shown in Fig.1. There are five curves there (three bifurcation curves and two auxiliary ones). The curve

$$L^+ : \; M_2 = -(M_1 - 1)^2 \tag{41}$$

corresponds to the bifurcation of the fixed point with a double multiplier $(+1)$. The curve

$$L^- : \; M_2 = (M_1 + 1)(3M_1 - 1) \tag{42}$$

corresponds to the bifurcation of the fixed point with a double multiplier (-1). The curve

$$L^\omega : \; M_2 = \tfrac{1}{8}(1 + \tfrac{M_1^2}{8})(M_1^2 - 16M_1 + 24), \;\; \text{where} \;\; |M_1| < 4 \tag{43}$$

corresponds to the bifurcation of the fixed point with double complex multipliers on the unit circle. Note that the same equation (43) with $|M_1| > 4$ defines the (non-bifurcational) curves L_{d+} and L_{d-} which correspond to the existence of a fixed point with double real multipliers: $\nu_1 = \nu_2, \nu_3 = \nu_4 = \nu_1^{-1}$. The curve L_{d+} lies in the region $M_1 > 4$ and corresponds to positive ν_1, and the curve L_{d-} lies in the region $M_1 < -4$ and corresponds to negative ν_1.

The plane of parameters (M_1, M_2) is divided by the curves L_+, L_-, L_ω and $L_{d\pm}$ into 9 regions D_1 — D_9 (Fig.1). For (M_1, M_2) in the
Region D_1 the map (39) has no fixed points.
For the other regions there exist exactly two fixed points of the following types.
Region D_2. A saddle $(+,-)$ (i.e. it has a pair of positive and a pair of negative real multipliers) and a saddle-center $(-)$ (i.e. a fixed point which has a pair of complex-conjugate multipliers on the unit circle and a pair of real negative multipliers).
Region D_3. A saddle $(+,-)$ and an elliptic point.
Region D_4. A saddle-center $(+)$ and an elliptic point.
Region D_5. A saddle-center $(+)$ and a saddle-focus.
Region D_6. A saddle $(+,+)$ and a saddle-center $(+)$.
Region D_7. A saddle $(+,-)$ and a saddle $(-,-)$.

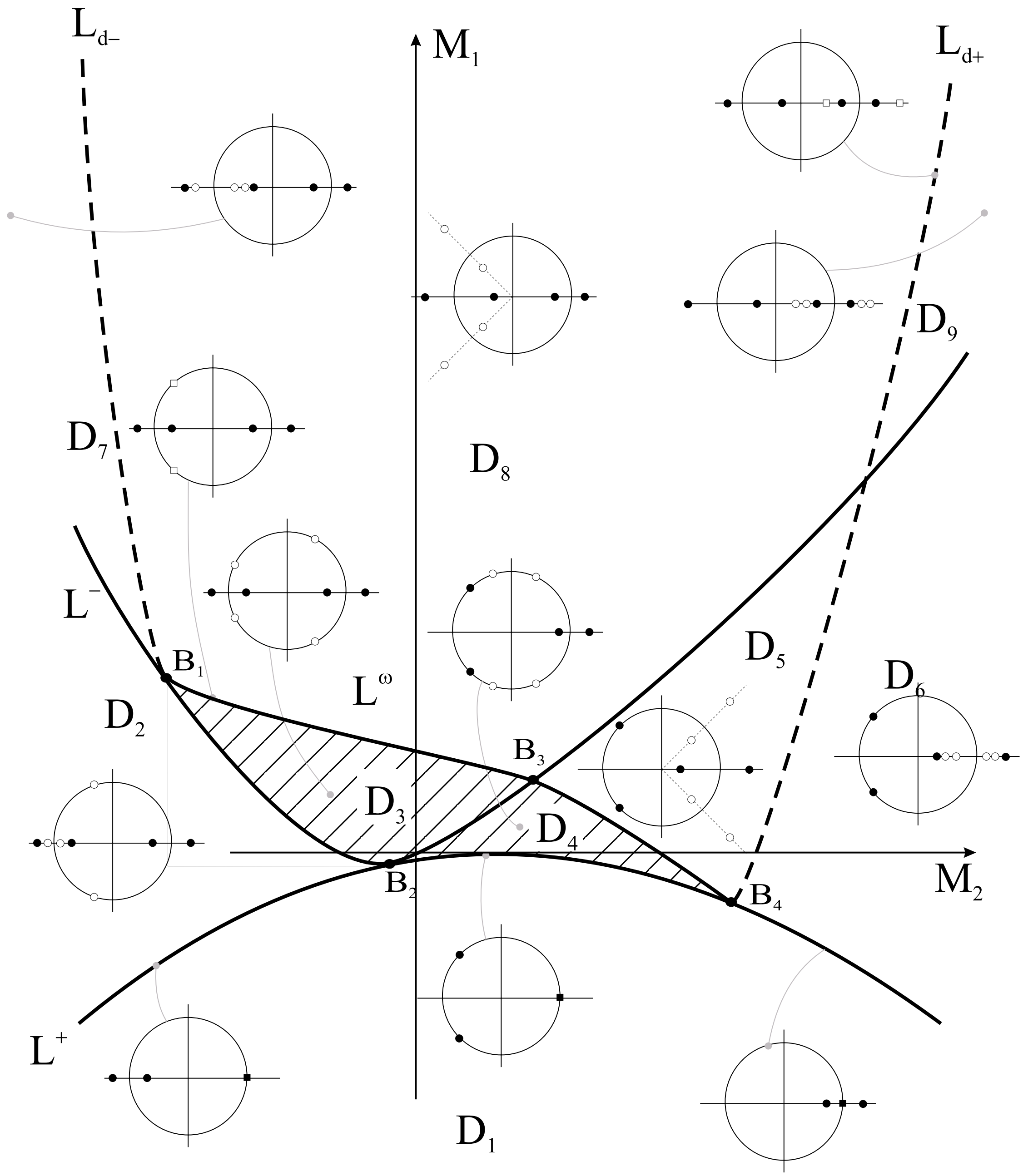

Figure 1: The bifurcation diagram for the fixed points of the map (39). Bifurcation curves L^+, L^- and L^φ which correspond to the existence of fixed points with double multipliers $+1$, -1 and $e^{\pm i\omega}$ respectively, and the auxiliary (non-bifurcational) curves L_{d+} and L_{d-} which correspond to saddles with double real multipliers, resp. positive and negative, divide the plane of parameters (M_1, M_2) into 9 regions. In region D_1, the map (39) has no fixed points. In regions D_2 — D_9 there exist exactly two fixed points. The type of the corresponding fixed points is indicated by showing the position of their multipliers with respect to the unit circle: bold points denote the multipliers of one of the points and the circles denote the multipliers of the other point; boxes correspond to double multipliers.

Region D_8. A saddle $(+,-)$ and a saddle-focus.
Region D_9. A saddle $(+,-)$ and a saddle $(+,+)$.

Note also that there are four codimension-2 points on the bifurcation diagram:
the point B_1 – map (39) has a fixed point with the multipliers $(-1,-1,-1,-1)$;
the point B_2 – the map has a fixed point with the multipliers $(-1,-1,+1,+1)$;
the point B_4 – the map has a fixed point with the multipliers $(+1,+1,+1,+1)$;
the point B_3 – one fixed point of map (39) has a double multiplier (-1) and the other fixed point has double complex multipliers on the unit circle.

Note that for the parameter values in the regions D_3 and D_4 (dashed curvilinear triangle in Fig.1) map (39) has an elliptic fixed point. It was shown in [18] that the elliptic fixed point of (39) is generic for almost all parameter values from the region $D_3 \cup D_4$: exclusive parameter values, if such exist, lie on some set of finitely many curves.

Recall that in the coordinates of lemma 3 the map T_k is sufficiently close to (39) for large k. Therefore, the structure of the bifurcation diagram for the fixed points of the map T_k is the same as of the above described bifurcation diagram for the map (39). Thus, returning to the original parameters (μ,φ) by formulas (40), we obtain the following statement.

Theorem 1 *Let $F_{\mu\varphi}$ be a two-parameter family of diffeomorphisms, which includes, at $\mu=0, \varphi=\varphi_0$, the diffeomorphism F_0 satisfying conditions A and B. In the plane of parameters (μ,φ) there exist eight infinite sequences of regions Δ_k^l , $l=2,\ldots,9$, which accumulate at the point $(\mu,\varphi)=(0,\varphi_0)$, such that the diffeomorfism $F_{\mu\varphi}$ has, at $(\mu,\varphi)\in\Delta_k^l$, two single-round periodic orbits of the same type as the fixed points of map (39) with (M_1,M_2) from the region D_l. If $r\geq 7$ (where r is the smoothness of the map $F_{\mu\varphi}$), then the regions Δ_k^3 and Δ_k^4 consist of a finite number of open regions such that $F_{\mu\varphi}$ has a generic elliptic single-round periodic orbit when (μ,φ) belongs to these regions.*

Note that systems in $\mathcal{H}$ (i.e. with $\mu=0$) correspond, in general, to large (of order λ^{-4k}) values of M_1, while map (39)) can have an elliptic fixed point only for a bounded interval of M_1. Thus, by an immediate computation, one can prove the following

Proposition. *If $r_0 \equiv x_2^+(c_{12}d_{21}-c_{11}d_{22})-y_1^-d_{21}-y_2^-d_{22}\neq 0$ for the diffeomorphism F_0, then for a sufficiently small and fixed neighborhood of the homoclinic orbit Γ we have that*
1) neither F_0, nor close to it diffeomorfisms from $\mathcal{H}$ can have single-round elliptic periodic orbits;
2) no diffeomorphism close to F_0 can have more than one single-round elliptic periodic orbit.

This fact is the main reason why we further consider double-round periodic orbits (for the diffeomorphisms from $\mathcal{H}$). We prefer to do it not immediately, rather by means of the analysis of double round homoclinic tangencies.

3 Secondary homoclinic tangencies for the diffeomorphisms from $\mathcal{H}$

The goal of this Section is to establish the following result.

Theorem 2 *In the set of four-dimensional symplectic diffeomorfisms satisfying conditions A and B the diffeomorphisms are dense which, along with the original orbit of homoclinic tangency Γ, have a double-round homoclinic orbit which corresponds to a simple tangency of the invariant manifolds of the saddle-focus O.*

Theorem 2 follows immediately from Theorem 2′ which we formulate below for one-parameter families of diffeomorphisms for which the original tangency is not split. Therefore, we will focus on the proof of Theorem 2′.

We will consider one-parameter families of diffeomorphisms in $\mathcal{H}$. Namely, let F_0 be a symplectic diffeomorphism satisfying conditions A and B, and let the following inequalities hold:

$$\alpha \neq \beta \ \text{ and } \ \alpha \neq \beta \pm \pi,$$

where α and β are the angles from (31). Embed F_0 in a one-parameter family F_φ of the diffeomorphisms in $\mathcal{H}$ (i.e. the original homoclinic tangency is not split). Let φ vary in any interval $I = (\varphi_0 - \varepsilon_1, \varphi_0 + \varepsilon_1) \subset (0, \pi)$.

Theorem 2′. *In I the values of φ are dense such that the corresponding diffeomorfism F_φ has a double-round homoclinic orbit which corresponds to a simple tangency of the invariant manifolds of the saddle-focus O_φ.*

Proof. Since $x_1 = 0$ on W^u_{loc}, we obtain from (30) that the equation of the two-dimensional surface $T_1(W^u_{loc}) \cap \Pi^+$ can be written in an implicit form as follows:

$$\begin{aligned}
x_{01} - x_1^+ &= -\frac{1}{c_{11}}(y_{11} - y_1^-) + \frac{b_{12}}{d_{22}} y_{02} + O\left[(|y_{02}| + |y_{11} - y_1^-|)^2\right], \\
x_{02} - x_2^+ &= \frac{b_{22}}{d_{22}} y_{02} + O\left[(|y_{02}| + |y_{11} - y_1^-|)^2\right], \\
y_{01} &= D_0(y_{11} - y_1^-)^2 + \tilde{D}_2(y_{11} - y_1^-)y_{02} + \tilde{D}_3 y_{02}^2 + O\left[(|y_{02}| + |y_{11} - y_1^-|)^3\right],
\end{aligned} \tag{44}$$

where $(y_{11} - y_1^-)$ is the coordinate on $W^u_{loc} \cap \Pi^-$, i.e. it runs the interval $|y_{11} - y_1^-| \leq \delta_0$.

The strip $\sigma_k^0 \subset \Pi^+$ — the domain of definition of the map $T_0^k : \Pi^+ \to \Pi^-$ — is defined by inequalities (36). Hence, the intersection $T_1(W^u_{loc}) \cap \sigma_k^0$ is defined by system (44) with the condition that the coordinates y_{01} and y_{02} in (44) satisfy the inequalities

$$|y_{01} - \lambda^k A \cos(k\varphi - \alpha)| \leq \lambda^k \delta_0, \ |y_{02} + \lambda^k A \sin(k\varphi - \alpha)| \leq \lambda^k \delta_0, \tag{45}$$

where the coefficients A and α are defined by formula (31).

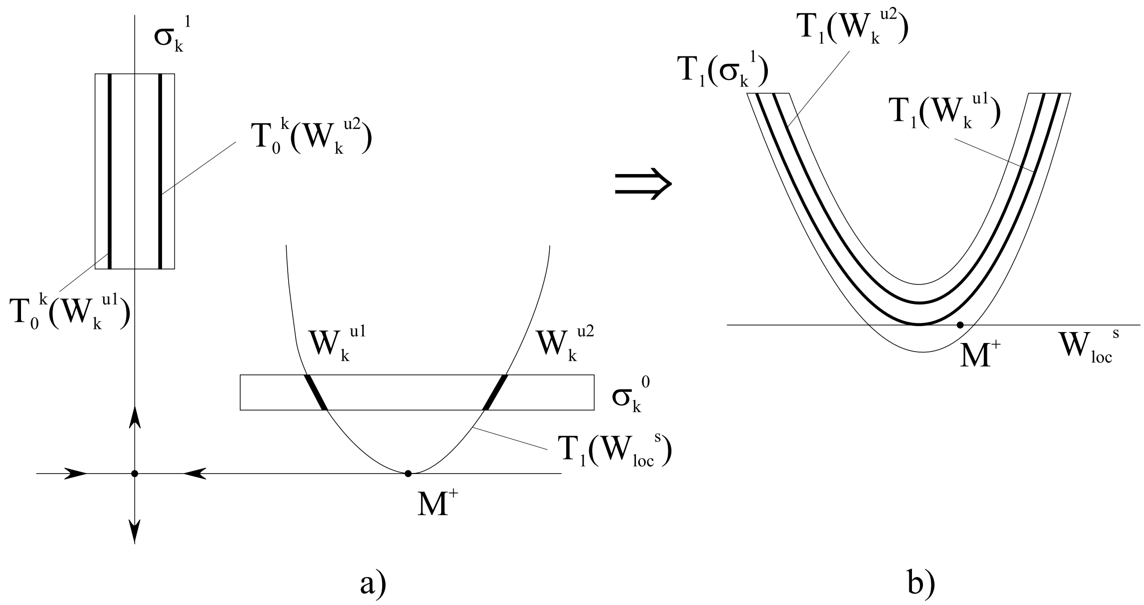

Figure 2: Schematic illustration to theorem $2'$: formation of double-round homoclinic tangencies of the first type. The intersection of $T_1(W^u_{loc}(O))$ with σ^0_k consists of two connected components W^{u1}_k and W^{u2}_k (Fig.2a), and the intersection of the horseshoe $T_1(\sigma^1_k)$ with the piece W^s_{loc} of the stable manifold of O consists of one component (Fig.2b). In this case either $T_1T^k_0(W^{u1}_k)$, or $T_1T^k_0(W^{u2}_k)$ may have a tangency with $W^s_{loc}(O)$.

It follows from the third equation in (44) that if k is sufficiently large and if the inequality

$$A\cos(k\varphi-\alpha)<-\delta_1, \tag{46}$$

hold with some fixed $\delta_1>\delta_0$, then the surface $T_1(W^u_{loc})$ does not intersect the strip σ^0_k: when (46) holds, the left-hand side and right-hand side of the third equation in (44) have different signs (recall that $D_0>0$). On the contrary, if k satisfies the inequality

$$A\cos(k\varphi-\alpha)>\delta_1, \tag{47}$$

the surface $T_1(W^u_{loc})$ intersects the strip σ^0_k twice; we denote the two connected components of the intersection as W^{u1}_k and W^{u2}_k (Fig.2a). Each of this components can be defined by an explicit expression. Namely, the third equation of (44) can be resolved with respect to y_{11} in the following way:

$$y_{11}-y_1^-=-\frac{\tilde D_2}{2D_0}y_{02}(1+\ldots)+(-1)^{l+1}\frac{1}{\sqrt{D_0}}\sqrt{y_{01}-(\tilde D_3-\tilde D_2^2/4D_0)y_{02}^2+\ldots}\equiv\Phi_l(y_{01},y_{02}), \tag{48}$$

$l=1,2$. Since we assume that inequality (47) holds, we have that the expression under the square root sign is, by virtue of (45), positive for sufficiently large k. Hence, the function $\Phi_l(y_{01},y_{02})$ is smooth (of the same smoothness as the right-hand side in (44), i.e. it is a least C^{r-2} with respect to all variables and parameters). Let us fix $l=1$, for definiteness, i.e. we will deal with the component W^{u1}_k. Correspondingly, we omit the index “1”, assuming now $W_k^{u1} \equiv W_k^u$, $\Phi_1 \equiv \Phi$, etc.. For the component W_k^{u2}, all the constructions remains the same.

We have from (44) and (48) that the surface W_k^u is given by the following system of equations:

$$\begin{aligned} x_{01} - x_1^+ &= -\frac{1}{c_{11}}\Phi(y_{01}, y_{02}) + \frac{b_{12}}{d_{22}} y_{02} + O\left[(|y_{02}| + |\Phi(y_{01}, y_{02})|)^2\right], \\ x_{02} - x_2^+ &= \frac{b_{22}}{d_{22}} y_{02} + O\left[(|y_{02}| + |\Phi(y_{01}, y_{02})|)^2\right], \end{aligned} \tag{49}$$

where the coordinates y_{01} and y_{02} run the region given by the inequalities (45).

The map T_0^k takes the surface W_k^u into the two-dimensional surface $T_0^k(W_k^u)$, lying in the vertical strip $\sigma_k^1 \subset \Pi^-$. Let us show that $T_0^k(W_k^u)$ is given by the equation

$$\begin{aligned} \bar{x}_{11} &= \lambda^k \cos k\varphi \cdot x_1^+ - \lambda^k \sin k\varphi \cdot x_2^+ + \lambda^k \phi_k^1(\bar{y}_{11}, \bar{y}_{12}), \\ \bar{x}_{12} &= \lambda^k \cos k\varphi \cdot x_2^+ + \lambda^k \sin k\varphi \cdot x_1^+ + \lambda^k \phi_k^2(\bar{y}_{11}, \bar{y}_{12}), \end{aligned} \tag{50}$$

where $(\bar{x}_1, \bar{y}_1)$ are the coordinates on Π^- (i.e. $\|\bar{y}_1 - y^-\| \le \delta_0$, in particular) and the functions $\phi_k^{1,2}$ are small along with their derivatives up to the second order. Indeed, by (9) we have the following relation for $T_0^k(W_k^u)$:

$$\begin{aligned} \bar{x}_{11} &= \lambda^k \cos k\varphi \cdot x_{01} - \lambda^k \sin k\varphi \cdot x_{02} + k\lambda^{2k} P_k^1(x_0, \bar{y}_1) x_0, \\ \bar{x}_{12} &= \lambda^k \cos k\varphi \cdot x_{02} + \lambda^k \sin k\varphi \cdot x_{01} + k\lambda^{2k} P_k^2(x_0, \bar{y}_1) x_0, \end{aligned} \tag{51}$$

where (x_{01}, x_{02}) are defined by formula (49) as functions of the coordinates (y_{01}, y_{02}) which, in turn, are defined as follows:

$$\begin{aligned} y_{01} &= \lambda^k \cos k\varphi \cdot \bar{y}_{11} + \lambda^k \sin k\varphi \cdot \bar{y}_{12} + k\lambda^{2k} Q_k^1(x_0, \bar{y}_1) \bar{y}_1, \\ y_{02} &= \lambda^k \cos k\varphi \cdot \bar{y}_{12} - \lambda^k \sin k\varphi \cdot \bar{y}_{11} + k\lambda^{2k} Q_k^2(x_0, \bar{y}_1) \bar{y}_1, \end{aligned} \tag{52}$$

and $\|\bar{y}_1 - y^-\| \le \varepsilon_0$. It is now seen immediately, that the equation for $T_0^k(W_k^u)$ can be written in the form (50) indeed, where

$$\phi_k^{1,2}(y_{11}, y_{12}) = O(|\Phi(y_{01}, y_{02})| + k\lambda^k), \tag{53}$$

and y_{01} and y_{02} satisfy (52). From (53), using (47),(48) and (52), we obtain the following estimate:

$$\|\phi_k\| \le C_1 k\lambda|^k + \sqrt{\frac{1}{D_0}} \cdot \sqrt{\frac{\delta_1}{2}\lambda^k - C_2\lambda^{2k}} \le C_3\sqrt{\delta_1}\lambda^{k/2}, \tag{54}$$

where C_1, C_2, C_3 are some positive constants, and δ_1 is taken from (47). Next, we have

$$\begin{aligned} \left\| \frac{\partial \phi_k^l}{\partial \bar{y}_{11}, \partial \bar{y}_{12}} \right\| &\le \frac{1}{2\sqrt{D_0}} \cdot (y_{01} - (\tilde{D}_2 - \tilde{D}_3^2/4D_0)y_{02}^2 + ...)^{-1/2} \|\frac{\partial y_0}{\partial \bar{y}_1}\| \le \\ &\le C_4\lambda^k \left(\lambda^k \cdot \frac{\delta_1}{2}\right)^{-1/2} \le C_5\delta_1^{-1/2}\lambda^{k/2}. \end{aligned} \tag{55}$$

Analogously, we obtain the inequality

$$\left\| \frac{\partial^2 \phi_k^l}{\partial y_{11}^p \partial y_{12}^{2-p}} \right\| \leq C_6 \delta_1^{-3/2} \lambda^{k/2}, \tag{56}$$

where $p \in \{0, 1, 2\}$.

These three inequalities and (50) imply that
for those k for which inequality (47) holds, the surfaces $T_0^k(W_k^u)$ accumulate, as $k \to \infty$, on $(W_{loc}^u \cap \Pi^-) = \{x_{11} = 0, x_{12} = 0\}$ in the C^2-topology.

Thus, it follows that for all sufficiently large k satisfying (47), when the surface $T_1[T_0^k(W_k^u)]$ is tangent to the surface $W_{loc}^s \cap \Pi^+$, this tangency will satisfy conditions B.1 and B.2, as the original homoclinic tangency of $T_1 W_{loc}^u$ and W_{loc}^s does. Let us prove that the tangency of such type exists indeed for a dense set of values of φ.

According to (30) and (50), the equation of $T_1[T_0^k(W_k^u)]$ can be written in the following form:

$$\begin{aligned}
\bar{\bar{x}}_{01} - x_1^+ &= \lambda^k \xi_{k1}^*(\varphi) - \frac{1}{c_{11}}(\bar{y}_{11} - y_1^-) + \bar{\bar{y}}_{02}\frac{b_{12}}{d_{22}} + \\
&\quad + O[\lambda^k(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)^2], \\
\bar{\bar{x}}_{02} - x_2^+ &= \lambda^k \xi_{k2}^*(\varphi) + \frac{b_{22}}{d_{22}}\bar{\bar{y}}_{02} + O[\lambda^k(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)^2], \\
\bar{\bar{y}}_{01} &= M_k(\varphi) + D_0(\bar{y}_{11} - y_1^-)^2 + \tilde{D}_2(\bar{y}_{11} - y_1^-)\bar{\bar{y}}_{02} + \tilde{D}_3\bar{\bar{y}}_{02}^2 + \\
&\quad + O[\lambda^k(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)(|\bar{\bar{y}}_{02}| + |\bar{y}_{11} - y_1^-|)^3],
\end{aligned} \tag{57}$$

where ξ_{k1}^* and ξ_{k2}^* are constant (independent of the coordinates) terms, uniformly bounded at all k. The constant term in the third equation of (57) is written as

$$M_k(\varphi) = B\lambda^k \cos(k\varphi - \beta) + O(\lambda^{3k/2}), \tag{58}$$

where the coefficients B and β are given by (31). Note that formulas (57) and (58) hold true only for those k for which (47) holds with δ_1 bounded away from zero, because the constant factors in the $O(\cdot)$-terms may depend on δ_1 (see (54)-(56)).

Let us now write system (57) in an explicit form, resolving the first equation with respect to $(y_{11} - y_1^-)$. We obtain the following equation for the surface $T_1[T_0^k(W_k^u)]$ (to simplify notation, we remove bars from x and y):

$$\begin{aligned}
x_{02} - x_2^+ &= O(\lambda^k) + O(y_{02}), \\
y_{01} &= \hat{M}_k(\varphi) + D_0 c_{11}^2 (x_{01} - x_1^+ - \lambda^k \hat{\xi}_k)^2 (1 + O[x_{01} - x_1^+ - \lambda^k \hat{\xi}_k]) + O(y_{02}),
\end{aligned} \tag{59}$$

where $\hat{\xi}_k$ is some bounded coefficient (close to ξ_{k1}^*), and $\hat{M}_k(\varphi)$ is the new constant term in the equation for y_{01}, which still satisfies (58) (by this reason, we will further denote this term simply as $M_k(\varphi)$).

It is obvious that one may introduce local coordinates

$$\begin{aligned}
&\xi_1 = x_{01} - x_1^+ - O(\lambda^k) - O(y_{02}), \ \xi_2 = (x_{02} - x_2^+ - \lambda^k \hat{\xi}_k)(1 + \ldots), \\
&\eta_1 = y_{01} - O(y_{02}), \ \eta_2 = y_{02}
\end{aligned} \tag{60}$$

in a neighborhood of the point

$$(x_{01} = x_1^+ + \lambda^k \hat{\xi}_k,\ x_{02} = x_2^+,\ y_{01} = 0,\ y_{02} = 0),$$

such that system (59) will recast as

$$\xi_2 = 0,\ \eta_1 = M_k(\varphi) + D_0 c_{11}^2 \xi_1^2. \tag{61}$$

This equation defined the piece $T_1 T_0^k(W_k^u)$ of the unstable manifold of O in the coordinates (60). In the same coordinates, the piece $W_{loc}^s \cap \Pi^+$ of the stable manifold of O has the form $\eta_1 = \eta_2 = 0$ (this corresponds to $y_{01} = y_{02} = 0$ in the old coordinates).

Since $D_0 \neq 0$ and $c_{11} \neq 0$, the two-dimensional surfaces $T_1 T_0^k(W_k^u)$ and $W_{loc}^s \cap \Pi^+$ have a quadratic tangency at the point $(\xi, \eta) = 0$ at $M_k(\varphi) = 0$; they do not have intersections at $M_k(\varphi) > 0$, and have two intersection points at $M_k(\varphi) < 0$. It is obvious that the quantity $M_k(\varphi)$ serves as a splitting parameter for the given tangency.

Thus, the diffeomorphism F_φ has a double-round homoclinic orbit corresponding to a simple tangency of W^u and W^s if $M_k(\varphi) = 0$. By (58), this condition can be written as

$$\varphi = \varphi_{kj}^{\pm} = \pm\frac{\pi}{2k} + \frac{\beta}{k} + 2\pi\frac{j}{k} + O(\lambda^{k/2}), \tag{62}$$

where j runs arbitrary integer values. By construction, the sought homoclinic tangency corresponds to those values of $\varphi = \varphi_{kj}^{\pm}$ which satisfy inequality (47). When φ is given by (62), the inequality (47) reduces to

$$\pm A \sin(\alpha - \beta) > \delta_1 + O(\lambda^{k/2}).$$

It is clear that the corresponding $\delta_1 > \delta_0$, independent of k, exists always, provided $\sin(\alpha - \beta) \neq 0$. Thus, the sought homoclinic tangency corresponds to the values φ_{kj}^+ of φ in the case $\sin(\alpha - \beta) > 0$ and to the values φ_{kj}^- in the case $\sin(\alpha - \beta) < 0$. Since both the sequences φ_{kj}^+ and φ_{kj}^- are dense in any interval, this completes the proof of theorem $2'$.

Analogous computations for the second component W_k^{u2} of the intersection $T_1(W_{loc}^u) \cap \sigma_k^0$ (see (48)) show that the homoclinic tangency of the surface $T_1 T_0^k(W_k^{u2})$ with W_{loc}^s exists for values of φ which also satisfy the asymptotic relations (62). Hence, if $\sin(\alpha - \beta) \neq 0$, there exist two sequences $\varphi = \varphi_{kj}^{(1)}$ and $\varphi = \varphi_{kj}^{(2)}$ corresponding to a simple double-round homoclinic tangency. Recall that

$$\varphi_{kj}^{(1,2)} = \begin{cases} \dfrac{\pi}{2k} + \dfrac{\beta}{k} + 2\pi\dfrac{j}{k} + O(\lambda^{k/2}) \ \text{ when } \ \sin(\alpha - \beta) > 0, \\ -\dfrac{\pi}{2k} + \dfrac{\beta}{k} + 2\pi\dfrac{j}{k} + O(\lambda^{k/2}) \ \text{ when } \ \sin(\alpha - \beta) < 0, \end{cases} \tag{63}$$

where j and k run arbitrary integer numbers such that $\varphi_{kj}^{(1,2)} \in I$ (we must also require that k is sufficiently large and positive).

Note that it follows from formula (63) that there exist, in fact, two more series of homoclinic tangencies. Namely, the corresponding values of φ are given by the formula

$$\varphi = \tilde{\varphi}_{kj}^{(1,2)} \; = \left\{ \begin{array}{l} \dfrac{\pi}{2k} + \dfrac{\alpha}{k} + 2\pi\dfrac{j}{k} + O(|\lambda|^{k/2}) \;\; \text{when} \;\; \sin(\alpha-\beta) < 0, \\ -\dfrac{\pi}{2k} + \dfrac{\alpha}{k} + 2\pi\dfrac{j}{k} + O(|\lambda|^{k/2}) \;\; \text{when} \;\; \sin(\alpha-\beta) > 0, \end{array} \right. \tag{64}$$

where j and k run the integer values such that $\tilde{\varphi}_{kj}^{(1,2)} \in I$ (both the sequences $\tilde{\varphi}_{kj}^1$ and $\tilde{\varphi}_{kj}^2$ are dense in I).

Indeed, as it follows from (32), formula (63) transforms into (64) if one considers the diffeomorphism F^{-1} instead of F.

Thus, we have two different cases: $\sin(\alpha-\beta) > 0$ and $\sin(\alpha-\beta) < 0$. In both cases, for one-parameter families $F_\varphi \subset \mathcal{H}$ there are 4 sequences of the values of the parameter φ, corresponding to the homoclinic tangencies. The sequences are given by formulas (63) and (64).

Note that we have two types of the homoclinic tangencies here. The tangencies of the first type correspond to φ given by (63), and the tangencies of the second type correspond to φ from (64). The difference is clearly seen in Figs.2 and 3 where we show, schematically, in a two-dimensional projection, the behavior of the stable and unstable manifolds of the saddle-focus. In the first case (Fig.2), we obtain the double-round homoclinic tangency as a result of the following mutual position of the invariant manifolds, strips and horse-shoes: the intersection of the piece $T_1(W^u_{loc})$ of the unstable manifold of O with the strip σ_k^0 is "regular", it consists of two connected components W_k^{u1} and W_k^{u2}, while the intersection of the horse-shoe $T_1(\sigma_k^1)$ with the piece W^s_{loc} of the stable manifold of O is "irregular" and consists of one component. Therefore, either $T_1T_0^k(W_k^{u1})$, or $T_1T_0^k(W_k^{u2})$ can have a tangency with W^s_{loc} in this case, which corresponds to the tangency of the first type. It is exactly the type of tangency, existence of which we established in the proof of the theorem. In the second case (Fig.3), we have the following geometry: the intersection of the piece $T_1(W^u_{loc})$ of the unstable manifold of O with the strip σ_k^0 is irregular and consists of one component $\tilde{W}_k^u$, while the intersection of the horse-shoe $T_1(\sigma_k^1)$ with the piece W^s_{loc}) of the stable manifold of O is regular and consists of two segments. The double-round homoclinic tangencies of the second type correspond here to the tangency of the surface $T_1T_0^k(\tilde{W}_k^u)$ either with $\tilde{W}^s_{k1}$ (Fig.3a), or with $\tilde{W}^s_{k2}$ (Fig.3b). Formally speaking, the proof of the existence of the homoclinic tangencies of the second type and of their quadraticity for the diffeomorphism F requires a method different from what we used for the tangencies of the first type. We, however, noticed that the tangency of the second type becomes the tangency of the first type for the diffeomorphism F^{-1}, so the denseness in $\mathcal{H}$ of the diffeomorfisms with the double-round homoclinic tangencies of the second type followed automatically, by virtue of the natural symmetry between symplectic maps F and F^{-1} (see formula (32)).

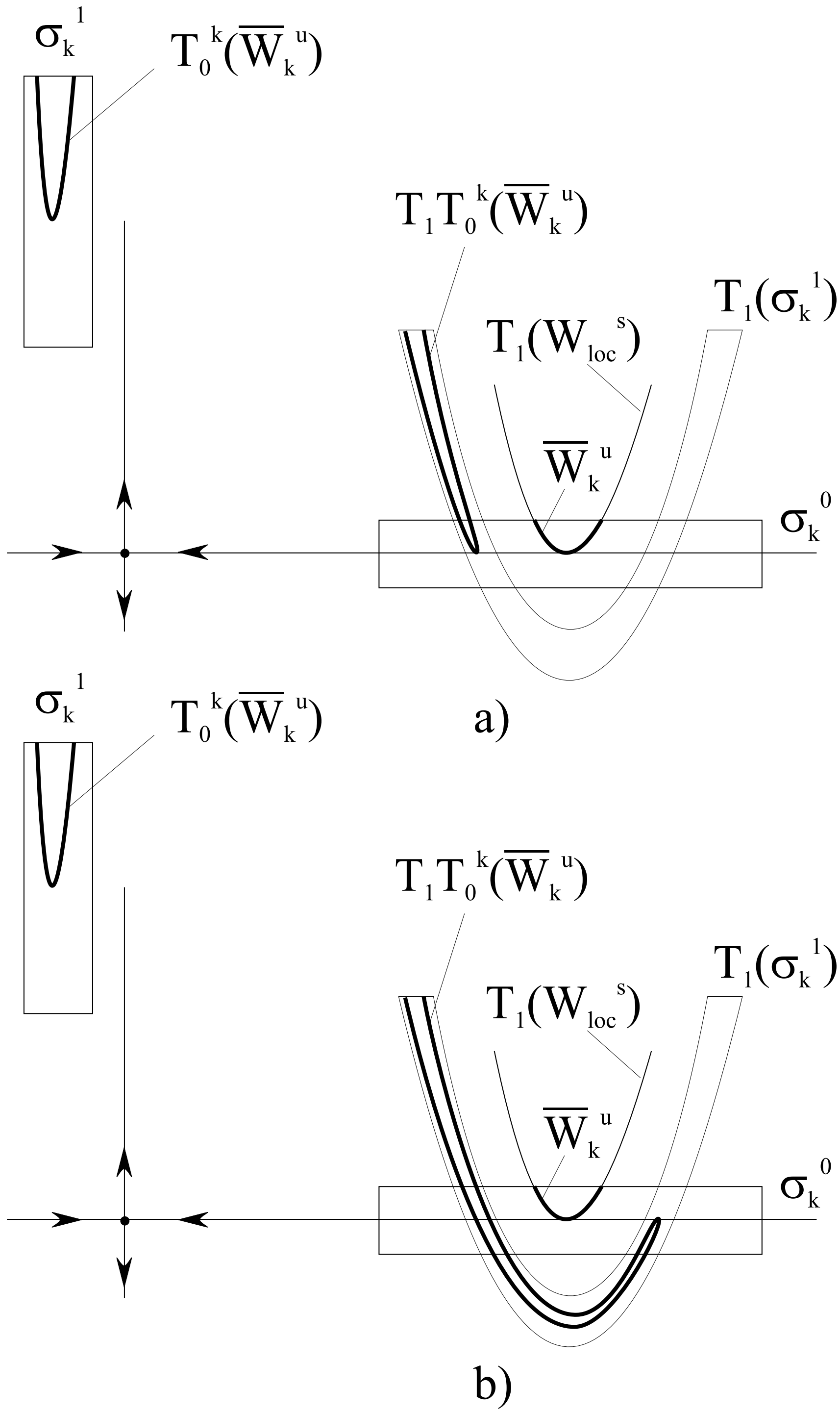

Figure 3: Schematic illustration of the formation of double-round homoclinic tangencies of the second type. The intersection of the piece $T_1(W^u_{loc})$ with σ^0_k consists of one connected component $\tilde{W}^u_k$, while the intersection of $T_1(\sigma^1_k)$ with $W^s_{loc})$ consists of two segments $\tilde{W}^s_{k1}$ and $\tilde{W}^s_{k2}$. Double-round homoclinic tangencies of the second type correspond to the tangency of the surface $T_1T_0^k(\tilde{W}^u_k)$ with either $\tilde{W}^s_{k1}$ (Fig.3a), or $\tilde{W}^s_{k2}$ (Fig.3b).

4 Proof of the main theorem, and some corollaries

Note, first, that it is essential that in proving theorem $2'$, we not only showed that diffeomorphisms with double-round homoclinic tangencies are dense but we have also determined in which families in $\mathcal{H}$ these tangencies are split in a generic way. Namely, the splitting parameter here is either (see formulas (61),(58)) the quantity

$$M_k(\varphi) \sim \lambda^k \cos(k\varphi - \beta) + O(\lambda^{3k/2})$$

for the homoclinic tangencies of the first type which correspond to the values of φ given by (63), or the symmetric quantity

$$\tilde{M}_k(\varphi) \sim \lambda^k \cos(k\varphi - \alpha) + O(\lambda^{3k/2})$$

for the homoclinic tangencies of the second type which correspond to the values of φ from (64). Hence, in any two-parameter family $F_{\nu\varphi}$ of diffeomorfisms from $\mathcal{H}$, for which inequality (3) holds (i.e. either $\alpha'_\nu \neq 0$, or $\beta'_\nu \neq 0$), the tangencies of at least of one of these two types will split generically (for all sufficiently large k) at arbitrarily small changes of ν with the value of φ fixed.

It means that theorem 1 is applied to these families. Thus, applying theorem 1 to a double-round homoclinic tangency gives us the existence of (eight) regions in the plane of parameters (ν, φ) near the point corresponding to the given secondary tangency, for which the diffeomorphism have periodic orbits of different types. These periodic orbits are double-round in U (they are single-round with respect to a small neighborhood of the secondary homoclinic tangency).

We see that it is natural in the analysis of homoclinic bifurcations in the class of diffeomorphisms from $\mathcal{H}$ to take φ and β, or φ and α as the governing parameters. For definiteness, we will consider below two-parameter families of diffeomorphisms from $\mathcal{H}$ parametrized by φ and β.

Let F_0 be a symplectic diffeomorfism satisfying conditions A and B, and let the inequalities $\alpha \neq \beta$ and $\alpha \neq \beta \pm \pi$ hold. Consider a two-parameter family $F_{\varphi,\beta}$ of diffeomorphisms from $\mathcal{H}$ such that the range of values of φ contains an interval $I_0 = (\varphi_0 - \nu_0, \varphi_0 + \nu_0)$ and the range of values of β contains an interval $B_0 = (\beta_0 - \nu_1, \beta_0 + \nu_1)$, where ν_0 and ν_1 are sufficiently small. Denote $J = I_0 \times B_0$. Let us show that in J there exists a dense subset $\tilde{J}$ such that if $(\varphi, \beta) \in \tilde{J}$, then the diffeomorphism $F_{\varphi,\beta}$ has, simultaneously, infinitely many of (double-round) periodic orbits of all generic types (saddle, saddle-center, elliptic). This will give us the main theorem immediately.

Take any point $P_1 \in J$. Let Δ_0 be its small neighborhood in J. By theorem $2'$, in Δ_0 there exists a point $\tilde{P}_1 = (\varphi_0^1, \beta_0^1)$ such that the diffeomorphism $F_{\varphi_0^1,\beta_0^1}$ has a double-round orbit of homoclinic tangency Γ_{i_0}. By theorem 1, the point $\tilde{P}_1$ is a limit of a sequence of regions $\Delta_k^l(\tilde{P}_1)$ in the plane of parameters φ and β, which correspond to the existence of double-round periodic orbits of the same type as the fixed points in the parameter regions D_l , $l = 2, \ldots, 9$, for the map (39). Consider a region $\Delta_k^2(\tilde{P}_1) \subset \Delta_0$. Here, the diffeomorfism $F_{\varphi,\beta}$ has two periodic orbits of the type saddle $(+,-)$ and saddle-center $(-)$. In the region Δ_k^2 we again find a point $\tilde{P}_1^{(2)}$ such that the corresponding diffeomorfism $F_{\varphi,\beta}$ has a double-round orbit of homoclinic tangency Γ_{i_1}. Near this point we find a region $\Delta_{k_1}^3(\tilde{P}_1^{(2)}) \subset \Delta_k^2(\tilde{P}_1) \subset \Delta_0$ such that, when the parameters belong to this region, the diffeomorfism $F_{\varphi,\beta}$ has, along with the previously constructed saddle $(+,-)$ and saddle-center $(-)$ periodic orbits, two new periodic orbits — a new saddle $(+,-)$ and an elliptic orbit. Inside $\Delta_{k_1}^3$, we find again a point $\tilde{P}_1^{(3)}$ corresponding to a new double-round homoclinic tangency, and a region $\Delta_{k_2}^4(\tilde{P}_1^{(3)})$, and so on, until we construct a region $\Delta_{k_7}^9(\tilde{P}_1^{(8)})$. At $(\varphi,\beta) \in \Delta_{k_7}^9$ the diffeomorfism $F_{\varphi,\beta}$ has, by construction, double-round periodic orbits of "all types". Now we repeat this procedure infinitely many times. We obtain a sequence

$$\Delta_{k_7}^9 \supset \Delta_{k_7^1}^9 \supset \ldots \supset \Delta_{k_7^n}^9 \supset \ldots$$

of nested regions. For a point of intersection of these regions, $P^* = (\varphi^*, \beta^*) \in \Delta_0$, we have that the diffeomorfism F_{φ^*,β^*} has infinitely many of double-round periodic orbits of all types. Since the original point P_1 were taken arbitrarily, and the point P^* is found in an arbitrarily small neighborhood of P_1, this completes the proof of the theorem.

Let us now discuss some generalizations of the main theorem. Recall that theorem 1 from § 2 is proven by means of the analysis of the fixed points of the first-return maps near the orbit of homoclinic tangency. The main result here is lemma 3 which says that the first-return map is sufficiently close, in a specially chosen coordinates (X_1, X_2, Y_1, Y_2), to the quadratic map (39), where the coordinates (X,Y) and the parameters M_2 and M_1 can take arbitrary finite values (essentially, these are the small parameters which govern the splitting of the tangency and the changes in the values of φ, divided to some small factors so that the range of their values becomes large, see formula (40)). We call the quadruple of non-zero complex numbers $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *symplectic* if it is invariant (up to a permutation) both with respect to the inversion $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1})$ and the complex conjugation $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*)$. It is well-known that the multipliers of a periodic point of a real four-dimensional symplectic map always form a symplectic quadruple. It is easy to check that when the parameters M_1 and M_2 vary, the multipliers of the fixed points of the map (39) may run all symplectic quadruples. Hence, due to the closeness of the first-return maps to the map (39), we immediately obtain the following result (an analog of theorem 1).

Theorem 3 *Let $F_{\mu\varphi}$ be a two-parameter family of diffeomorphisms which include, at $\mu = 0, \varphi = \varphi_0$, a diffeomorfism F_0 satisfying conditions A and B. Let μ be a splitting parameter for the invariant manifolds of the saddle-focus near a point of homoclinic tangency, and φ be an angular argument of the multipliers of the saddle-focus. Given any symplectic quadruple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, in any neighborhood of the point $\mu = 0, \varphi = \varphi_0$ in the plane of parameters (μ, φ) there exist parameter values* *for which the corresponding diffeomorphism has, in a small neighborhood of the orbit of homoclinic tangency, a single-round periodic orbit with exactly the multipliers* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Based on this result and on theorem $2'$, using exactly the same construction with nested domains as we used in the proof of the main theorem we obtain the following generalization of the latter.

Theorem 4 *For any two-parameter family* $F_{\nu\varphi}$ *of diffeomorfisms in* $\mathcal{H}$*, satisfying* (2),(3)*, the values of parameters are dense such that the corresponding diffeomorphism has an infinite set of double-round periodic orbits the sets of multipliers of which form a dense subset in the set of all symplectic quadruples.*

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