Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Existence of the Stoneley surface wave at vacuum/porous medium interface: Low-frequency range

Inna Edelman

submitted: 20th November 2002

Weierstrass Institute for
Applied Analysis and Stochastics
Mohrenstr. 39, 10117 Berlin;
E-Mail: edelman@wias-berlin.de

Permanent address:

United Institute of Physics of the Earth
Russian Academy of Sciences
B.Gruzinskaya 10, Moscow, 123810
Russia
E-Mail: edelman@uipe-ras.scgis.ru

No. 789
Berlin 2003

2000 *Mathematics Subject Classification.* 35C20, 74J15, 35L50, 74F10, 35B32, 76M45.

Key words and phrases. Porous medium, interface, surface waves, asymptotics, bifurcation.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Existence and asymptotic behavior of the Stoneley surface wave at vacuum/porous medium interface are investigated in the low frequency range. It is shown that the Stoneley wave possesses a bifurcation in the vicinity of critical wave number k_{cr}. It is proven also that within the k-domain of existence, the Stoneley wave cannot appear for certain values of elastic moduli of the solid phase. Asymptotic formulae for the phase velocity of the Stoneley wave are presented.

Introduction

This paper develops the ideas of the work presented in [1]. Let us remind that paper [1] is devoted to the asymptotic analysis of the surface waves at a free interface of a saturated porous medium in the low frequency range. It was proven the existence of two surface modes: the Stoneley wave and the generalized Rayleigh wave. It was shown that the generalized Rayleigh wave is always possible and propagates almost without attenuation. Unlike high frequency limit, where the Stoneley mode exists always [2,3], at low frequencies it possesses a bifurcation in the vicinity of critical wave number k_{cr} and can appear only if its wave number k is bigger then k_{cr}. It is strongly attenuated mode. Such intricate behavior of the Stoneley wave results from the bifurcation of the Biot slow bulk wave (P2) in the low frequency range [4,5].

The focus of this paper is on the research of domains of existence for the Stoneley surface wave depending on elastic parameter of the porous medium skeleton. We will show that within the k-domain of existence, the Stoneley wave cannot appear for certain values of elastic parameters of the solid phase.

1. Problem Statement

1.1. Mathematical model

Consider two semi-infinite spaces, Ω^- and Ω^+, having a common interface Γ. Let the region Ω^- be occupied by a saturated porous medium and the region Ω^+ be occupied by the vacuum. *In dimensionless variables* the set of balance equations describing a fluid-filled porous medium has the following general form ($x \in \Omega^-,\ t \in [0, T]$) [6,3]:

Mass conservation equations

$$\frac{\partial \rho^F}{\partial t} + \mathrm{div}\big(\rho^F \mathbf{v}^F\big) = 0,$$

$$\frac{\partial \rho^S}{\partial t} + \mathrm{div}(\rho^S \mathbf{v}^S) = 0. \tag{1.1}$$

Here, ρ is the partial mass density, $\mathbf{v}$ is the velocity vector and indices F and S indicate fluid and solid phases, respectively.

Momentum conservation equations

$$\rho^F \left[\frac{\partial}{\partial t} + (v_j^F, \frac{\partial}{\partial x_j})\right] v_i^F - \frac{\partial}{\partial x_j} T_{ij}^F + \Pi(v_i^F - v_i^S) = 0,$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v_j^S, \frac{\partial}{\partial x_j})\right] v_i^S - \frac{\partial}{\partial x_j} T_{ij}^S - \Pi(v_i^F - v_i^S) = 0. \tag{1.2}$$

Here $\mathbf{T}^F$ and $\mathbf{T}^S$ are the partial stress tensors and Π is a positive constant.

Balance equation for the porosity

$$\frac{\partial n}{\partial t} + \big(v_i^S, \frac{\partial}{\partial x_i}\big) n + n_0 \mathrm{div}(\mathbf{v}^F - \mathbf{v}^S) = -(n - n_0), \tag{1.3}$$

where n is the porosity and n_0 is its initial value, assumed to be constant. Stress tensors have the form:

$$\mathbf{T}^F = -p^F \mathbf{1} - \beta(n - n_0)\mathbf{1}, \quad p^F = p_0^F + \kappa(\rho^F - \rho_0^F), \tag{1.4}$$

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S \mathrm{div}\mathbf{u}^S \mathbf{1} + 2\mu^S \mathrm{symgrad}\mathbf{u}^S + \beta(n - n_0)\mathbf{1}. \tag{1.5}$$

Here p^F is the pore pressure; p_0^F and ρ_0^F are the initial values of pore pressure and fluid mass density, respectively; κ is the constant compressibility coefficient of the fluid; β denotes the coupling coefficient of the fluid and solid components; $\mathbf{T}_0^S$ denotes a constant reference value of the partial stress tensor in the skeleton, λ^S and μ^S are the Lamé constants of the skeleton; $\mathbf{u}^S$ is the displacement vector for the solid phase with

$$\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}. \tag{1.6}$$

Let us linearize the system (1.1)-(1.3) about some equilibrium state. The simplest case arises when in the equilibrium state the fields have the following constant values: $\rho^F = \rho_0^F$, $\rho^S = \rho_0^S$, $\mathbf{v}^F = \mathbf{0}$, $\mathbf{v}^S = \mathbf{0}$, and $n = n_0$. After the formal introduction of the displacement vector for the fluid phase $\mathbf{u}^F$ and linearization, the system (1.1)-(1.3) takes the following form:

$$\frac{\partial \rho^F}{\partial t} + r \mathrm{div} \frac{\partial \mathbf{u}^F}{\partial t} = 0, \tag{1.7}$$

$$\frac{\partial \rho^S}{\partial t} + \mathrm{div}\frac{\partial \mathbf{u}^S}{\partial t} = 0, \tag{1.8}$$

$$r\frac{\partial^2 \mathbf{u}^F}{\partial t^2} + \mathrm{grad}(p^F + \beta n) + \Pi\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{1.9}$$

$$\frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \mu^S \Delta \mathbf{u}^S - (\lambda^S + \mu^S)\mathrm{graddiv}\mathbf{u}^S - \beta \mathrm{grad} n - \Pi\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{1.10}$$

$$\frac{\partial n}{\partial t} + n_0 \mathrm{div}\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = -(n - n_0), \tag{1.11}$$

where $r = \rho_0^F/\rho_0^S$.

The general problem of propagation of elastic waves through a bounded space is complicated. We confine ourselves to the consideration of a 2D problem (xy plane). This assumption does not limit the generality for the plane boundary Γ. We investigate surface waves on the interface of a porous medium which occupies the semi-infinite space $y > 0$ (region Ω^-) and is bounded by the vacuum, which fills the semi-infinite space $y < 0$ (region Ω^+).

1.2. Boundary conditions

On the interface $y = 0$, separating the porous medium and the vacuum, the following linearized boundary conditions, which are consequences of the general conditions [2,3], have to be satisfied:

1) the total stress vector must vanish

$$\Big(\frac{\partial u_1^S}{\partial y} + \frac{\partial u_2^S}{\partial x}\Big)|_{y=0} = 0, \tag{1.12}$$

$$\Big(\lambda^S \,\mathrm{div}\,\mathbf{u}^S + 2\mu^S \frac{\partial u_2^S}{\partial y} - \kappa(\rho^F - \rho_0^F)\Big)|_{y=0} = 0, \tag{1.13}$$

2) the relative normal velocity must be equal to zero, i.e. the pores at the interface are completely closed

$$\frac{\partial(u_2^F - u_2^S)}{\partial t}|_{y=0} = 0. \tag{1.14}$$

Our goal is to prove that the boundary value problem (1.7)-(1.14) has solutions in the form of surface waves, i.e. solutions which decrease sufficiently fast as $|y| \to \infty$. For this purpose we will investigate the propagation of a harmonic wave whose frequency is ω, wave number is k, and its amplitude depends on y. The frequency ω is sought as a function of the real wave number $k \in \mathrm{R}^1$. Thus, $\mathrm{Re}(\omega/k)$ defines the phase velocity of waves, while $\mathrm{Im}(\omega)$ defines the attenuation. Below we study the propagation of the surface waves in the low-frequency range. We give attention mainly to the Stoneley surface wave.

2. Surface waves at a free interface of a porous medium

2.1. Construction of solution

Solution in the region Ω^- (porous medium half-space) is sought in the following form [1-3]:

$$\mathbf{u}^F = \mathrm{grad}\varphi^F + \mathrm{rot}\boldsymbol{\Psi}^F, \quad \mathbf{u}^S = \mathrm{grad}\varphi^S + \mathrm{rot}\boldsymbol{\Psi}^S, \tag{2.1}$$

where $\boldsymbol{\Psi}^F = (0, 0, \psi^F)$ and $\boldsymbol{\Psi}^S = (0, 0, \psi^S)$. Consequently, in the explicit form one has

$$u_1^F = \frac{\partial \varphi^F}{\partial x} + \frac{\partial \psi^F}{\partial y}, \quad u_2^F = \frac{\partial \varphi^F}{\partial y} - \frac{\partial \psi^F}{\partial x},$$

$$u_1^S = \frac{\partial \varphi^S}{\partial x} + \frac{\partial \psi^S}{\partial y}, \quad u_2^S = \frac{\partial \varphi^S}{\partial y} - \frac{\partial \psi^S}{\partial x}.$$

Here unknown potentials are sought as

$$\varphi^F = A^F(y)\exp\left(i\left(kx - \omega t\right)\right), \quad \varphi^S = A^S(y)\exp\left(i\left(kx - \omega t\right)\right),$$

$$\psi^F = B^F(y)\exp\left(i\left(kx - \omega t\right)\right), \quad \psi^S = B^S(y)\exp\left(i\left(kx - \omega t\right)\right). \tag{2.2}$$

Simultaneously,

$$\rho^F - \rho_0^F = A_\rho^F(y)\exp\left(i\left(kx - \omega t\right)\right), \quad \rho^S - \rho_0^S = A_\rho^S(y)\exp\left(i\left(kx - \omega t\right)\right),$$

$$n - n_0 = A_\Delta \exp\left(i\left(kx - \omega t\right)\right). \tag{2.3}$$

Substitution of (2.1) into (1.7)-(1.11) and the following insertion of expressions (2.2), (2.3) result in three equations for the unknown amplitudes $A^F(y)$, $A^S(y)$, and $B^S(y)$

$$\left(c_f^2\left(\frac{d^2}{dy^2} - k^2\right) + \omega^2\right)A_F + \left(\frac{\beta\omega n_0}{r\,(i+\omega)}\left(\frac{d^2}{dy^2} - k^2\right) + \frac{i\Pi\omega}{r}\right)\left(A^F - A^S\right) = 0, \tag{2.4}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \omega^2\right)A^S - \left(\frac{\beta\omega n_0}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\Pi\omega\right)\left(A^F - A^S\right) = 0, \tag{2.5}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \frac{\omega^2}{c_s^2} - \frac{i\Pi\omega^2 r}{c_s^2\left(\omega r + i\Pi\right)}\right)B^S = 0, \tag{2.6}$$

and in four algebraic relations for $B^F(y)$, $A_\Delta(y)$, $A_\rho^S(y)$, and $A_\rho^F(y)$ as follows:

$$B^F = \frac{i\Pi}{\omega r + i\Pi}B^S, \tag{2.7}$$

$$A_\Delta = -\frac{n_0\omega}{i+\omega}(\frac{d^2}{d\,y^2} - k^2)(A^F - A^S), \tag{2.8}$$

$$A^S_\rho = -(\frac{d^2}{d\,y^2} - k^2)A^S, \tag{2.9}$$

$$A^F_\rho = \frac{r\omega^2}{c_f^2}A^F + \frac{1}{c_f^2}\left(\frac{\beta\omega n_0}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\Pi\omega\right)\left(A^F - A^S\right) = 0, \tag{2.10}$$

where

$$c_f = U^F/U^S_{||} < 1, \qquad U^F = \sqrt{\kappa}$$

and

$$c_s = U^S_\perp/U^S_{||} < 1, \qquad U^S_\perp = \sqrt{\mu^S/\rho^S_0}.$$

One can prove [1-3] that a solution to (2.4)-(2.6) exists and has the form

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C_1 \begin{pmatrix} R^F_1 \\ R^S_1 \end{pmatrix} \exp(-\gamma_1\, y) + C_2 \begin{pmatrix} R^F_2 \\ R^S_2 \end{pmatrix} \exp(-\gamma_2\, y), \tag{2.11}$$

$$B^S = C_s \exp(-\gamma_s\, y). \tag{2.12}$$

Here

$$\gamma_s = \sqrt{k^2 - \frac{\omega^2}{c_s^2} + \frac{i\Pi\omega^2 r}{c_s^2\,(\omega r + i\Pi)}}\ , \tag{2.13}$$

functions $\gamma_j,\ j = 1, 2$ are defined from the equation (we consider simplified case when $\beta = 0$)

$$\left(\frac{\gamma_j^2}{k^2} - 1\right)^2 + \frac{\omega}{k}\left(\frac{\omega}{k}\Big(1 + \frac{1}{c_f^2}\Big) + i\frac{\Pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right)\left(\frac{\gamma_j^2}{k^2} - 1\right)$$
$$+\frac{1}{c_f^2}\frac{\omega^3}{k^3}\left(\frac{\omega}{k} + i\frac{\Pi}{k}\Big(1 + \frac{1}{r}\Big)\right) = 0 \tag{2.14}$$

and corresponding eigenvectors are given by

$$(R^F_1,\ R^S_1) = \Big(R^F_1,\ \frac{i\frac{\Pi\omega}{k^2}}{\frac{\gamma_1^2}{k^2} - 1 + \frac{\omega}{k}\Big(\frac{\omega}{k} + i\frac{\Pi}{k}\Big)}R^F_1\Big)$$

$$(R^F_2,\ R^S_2) = \Big(\frac{i\frac{\Pi\omega}{k^2}}{rc_f^2\Big(\frac{\gamma_2^2}{k^2} - 1\Big) + \frac{\omega}{k}\Big(r\frac{\omega}{k} + i\frac{\Pi}{k}\Big)}R^S_2,\ R^S_2\Big). \tag{2.15}$$

The constants C_1, C_2, and C_s are as yet undetermined.

It should be emphasized that we are interested in the solutions in the form of surface waves, i.e. in the solutions which attenuate with y. Thus, solution (2.11), (2.12) bounded in y requires that:

$$\operatorname{Re}\gamma_s > 0, \qquad \operatorname{Re}\gamma_j > 0,\ j = 1,2. \tag{2.16}$$

The latter means that frequencies of all surface waves, which propagate at the free interface of a porous medium, should satisfy conditions (2.16).

As is earlier proven [1-3], functions $\gamma_j,\ j = 1,2$ have different structure in low and high frequency ranges. In this paper we investigate in more detail the low frequency range, i.e. long waves. Asymptotic expansion of equation (2.14) with respect to small wave number k results in two solutions:

$$\tilde{\gamma}_1 = \sqrt{1 - i\tilde{\omega}\frac{1}{\tilde{k}}\Big(1 + \frac{1}{rc_f^2}\Big) - \frac{\tilde{\omega}^2}{c_f^2}\frac{c_f^2 + \frac{1}{rc_f^2}}{1 + \frac{1}{rc_f^2}}} + O\Big(\sqrt{\tilde{k}}\Big) \tag{2.17}$$

and

$$\tilde{\gamma}_2 = \sqrt{1 - \tilde{\omega}^2\frac{1+r}{1+rc_f^2}} + O\Big(\sqrt{\tilde{k}}\Big), \tag{2.18}$$

where $\tilde{\gamma}_j = \gamma_j/k,\ j = 1,2,\ \ \tilde{\omega} = \omega/k,\ \ \tilde{k} = k/\Pi$.

Obviously, leading terms in expansions (2.17), (2.18) have different orders. Let us remind that the Biot slow wave (P2) does not propagate if its wave number k is less then critical value [1,4,5]

$$k_{cr} \approx c_f\Big(1 + \frac{1}{2rc_f^2}\Big)\Pi\,. \tag{2.19}$$

It is fully attenuated mode. If $k > k_{cr}$ then P2 wave begins to emerge. This fact is also illustrated by equation (2.14), since it reduces to the second order if $k = 0$. Consequently, solutions (2.17), (2.18) are of different order and we conclude that P1 and P2 waves have various hierarchy in the domain $D_{cr} = \{k \mid\ k \le k_{cr}\}$ [7,1]. Thus, terms of different order appear in the solution (2.11). However, for $y \ge 0$ the first term in (2.11) is negligible except in a narrow region, or layer, near $y = 0$. In this layer, which is roughly of width k_{cr}, the term $\exp\big(-\gamma_1 y\big)$ drops rapidly from its value of 1 at $y = 0$ to nearly zero. We say that solution (2.11) has a layer of rapid transition at $y = 0$. This layer is called a boundary layer (it occurs at a boundary point of the region considered).

However, expansion (2.17) is not uniformly valid throughout the k-domain of interest. Namely, (2.17) fails in a vicinity of the bifurcation point k_{cr} [1]. The procedure of the construction of suitable expansion for γ_1 in a neighborhood of k_{cr} will be discussed in detail in subsequent sections.

2.2. Dispersion equation

By substituting (2.11),(2.12) into the boundary conditions (1.12)-(1.14) one obtains the following system of equations for unknown constants $C_1,\ C_2,\ C_s$:

$$\tilde{\gamma}_1 C_1 R_1^S + \tilde{\gamma}_2 C_2 R_2^S + \frac{i}{2}\Big(\tilde{\gamma}_s^2 + 1\Big) C_s = 0, \tag{2.20}$$

$$\begin{aligned}(\tilde{\gamma}_1^2 - 1) C_1 R_1^S + (\tilde{\gamma}_2^2 - 1) C_2 R_2^S + 2c_s^2 (C_1 R_1^S + C_2 R_2^S) \\ +2ic_s^2 \tilde{\gamma}_s C_s - \Big(\tilde{\omega}^2 r + i\tilde{\omega}\frac{1}{\tilde{k}}\Big)(C_1 R_1^F + C_2 R_2^F) + i\tilde{\omega}\frac{1}{\tilde{k}}\Big(C_1 R_1^S + C_2 R_2^S\Big) = 0,\end{aligned} \tag{2.21}$$

$$\tilde{\gamma}_1 C_1 (R_1^F - R_1^S) + \tilde{\gamma}_2 C_2 (R_2^F - R_2^S) - iC_s\Big(1 - \frac{i}{\tilde{k}\tilde{\omega} r + i}\Big) = 0. \tag{2.22}$$

Requesting that the determinant of this system must vanish yields the dispersion equation for the definition of the frequencies of the surface waves. It takes different form depending on wave number [1]. Next we consider two cases: 1) the region $k \le k_{cr}$, where the Biot bulk wave does not propagate and 2) the small neighborhood of the critical point k_{cr}, where $k > k_{cr}$ and all three bulk waves, appearing in an unbounded porous medium, exist.

2.3. The generalized Rayleigh surface wave

For wave numbers $k \le k_{cr}$ dispersion equation has the form [1]:

$$\Big(-\tilde{\omega}_0^2 \frac{1+r}{1+rc_f^2} + 2c_s^2 - \tilde{\omega}_0^2 r\Big)\Big(1 - \tilde{\omega}_0^2 \frac{1+r}{2c_s^2}\Big) - 2c_s^2 \sqrt{1 - \tilde{\omega}_0^2 \frac{1+r}{1+rc_f^2}}\sqrt{1 - \tilde{\omega}_0^2 \frac{1+r}{2c_s^2}} = 0, \tag{2.23}$$

where $\tilde{\omega}_0$ is the leading term of the asymptotic expansion for $\tilde{\omega}$ (see [1] for the details). It has been mentioned already, that the Biot slow wave does not propagate with small wave numbers $k \le k_{cr}$. Consequently, in this region wave properties of a porous medium are very similar to those of an elastic solid and, as a result, only one surface wave can appear at the free interface of a porous material. Indeed, one can prove that dispersion equation (2.23) has a unique root, corresponding to the generalized Rayleigh surface wave. Evidently, if $r = \rho_0^F/\rho_0^S \to 0$ (limit passage to elastic medium), than equation (2.23) is degenerated into the classical Rayleigh equation:

$$\mathcal{P}_R(\tilde{\omega}) = \left(2 - \frac{\tilde{\omega}^2}{c_s^2}\right)^2 - 4\sqrt{1 - \tilde{\omega}^2}\sqrt{1 - \tilde{\omega}^2/c_s^2}. \tag{2.24}$$

Let us consider r to be a small parameter $\varepsilon \equiv r$ that is indeed fulfilled by virtue of physical meaning: $r < 1$. Asymptotic expansion for the root of (2.23) has the structure:

$$\tilde{\omega}_0 = c_R + \varepsilon\Omega_1 + \dots , \tag{2.25}$$

where c_R is the speed of the classical Rayleigh wave in an elastic half-space and Ω_1 obeys the equation:

$$\Omega_1 \frac{\mathcal{P}_R(\tilde{\omega})}{d\tilde{\omega}}|_{\tilde{\omega}=c_R} = \left(1 - \frac{c_R^2}{2c_s^2}\right)\left(3c_R^2 - 2c_s^2 c_f^2\right)$$

$$+\sqrt{1-c_R^2}\sqrt{1-c_R^2/c_s^2}\left(1 - \frac{c_R^2}{c_s^2 - c_R^2} - \frac{c_R^2 - c_f^2}{1-c_R^2}\right). \tag{2.26}$$

As is easily checked, provisos (2.16) are fulfilled for (2.25). Thereby, the solution (2.11),(2.12) has a form of a surface wave. As it follows from (2.25), the generalized Rayleigh wave propagates almost without attenuation in the region $k \leq k_{cr}$. These results are consistent with those obtained for the classical Biot model [8,9].

2.4. The Stoneley surface wave

Next consider in more detail the surface modes, which can appear in a small neighborhood of the bifurcation point k_{cr}, where $k > k_{cr}$ and P2 bulk wave is propagatory.

Requesting that the determinant of the system (2.20)-(2.22) must vanish yields the following dispersion equation, which holds true for any k:

$$\tilde{\gamma}_1\left(i\mathcal{R}_1 + \frac{1}{2}(\tilde{\gamma}_s^2+1)(1-i\mathcal{R}_1)\frac{r\tilde{\omega}+i\frac{\Pi}{k}}{r\tilde{\omega}}\right)\left((\tilde{\gamma}_2^2-1)+2c_s^2\right.$$

$$\left.-2c_s^2\tilde{\gamma}_s\tilde{\gamma}_2(1-i\mathcal{R}_2)\frac{r\tilde{\omega}+i\frac{\Pi}{k}}{r\tilde{\omega}} - \tilde{\omega}\left(r\tilde{\omega}+i\frac{\Pi}{k}\right)i\mathcal{R}_2 + i\tilde{\omega}\frac{\Pi}{k}\right)$$

$$-\tilde{\gamma}_2\left(1 - \frac{1}{2}(\tilde{\gamma}_s^2+1)(1-i\mathcal{R}_2)\frac{r\tilde{\omega}+i\frac{\Pi}{k}}{r\tilde{\omega}}\right)\left((\tilde{\gamma}_1^2-1)i\mathcal{R}_1 + 2c_s^2 i\mathcal{R}_1\right.$$

$$\left.+2c_s^2\tilde{\gamma}_s\tilde{\gamma}_1(1-i\mathcal{R}_1)\frac{r\tilde{\omega}+i\frac{\Pi}{k}}{r\tilde{\omega}} - \tilde{\omega}\left(r\tilde{\omega}+i\frac{\Pi}{k}\right) - \tilde{\omega}\mathcal{R}_1\frac{\Pi}{k}\right) = 0. \tag{2.27}$$

Here

$$\mathcal{R}_1 = \frac{\frac{\Pi}{k}\tilde{\omega}}{\tilde{\gamma}_1{}^2 - 1 + \tilde{\omega}(\tilde{\omega}+i\frac{\Pi}{k})}, \quad \mathcal{R}_2 = \frac{\frac{\Pi}{k}\tilde{\omega}}{rc_f^2(\tilde{\gamma}_2{}^2-1)+\tilde{\omega}(r\tilde{\omega}+i\frac{\Pi}{k})}. \tag{2.28}$$

To choose appropriate expansions for the wave number k in a neighborhood of k_{cr} and for the frequency ω let us call to mind the construction of corresponding expansions for the Biot bulk wave. It has been proven (we refer to [1,4] for details) that P2 mode begins to emerge if its wave number $k > k_{cr}$ (see (2.19)). For any small parameter ϵ and wave number

$$k = k_{cr}\big(1 + \epsilon^2 k_2\big) + \Pi O(\epsilon^3) \tag{2.29}$$

asymptotic expansion for P2 wave frequency ω_{P2} has the form:

$$\omega_{P2} = -i\,\Pi\,\Omega_{cr} + \epsilon\omega_1 + \Pi O(\epsilon^2), \qquad \omega_1 = 2k_{cr}\sqrt{k_2/\mathcal{A}} \tag{2.30}$$

where

$$\Omega_{cr} \approx 1/(2\,r) + 2c_f^2(1 + 3rc_f^2 - 2c_f^2),$$

$$\mathcal{A} = \frac{1+c_f^2}{c_f^2} + \frac{1-c_f^2}{c_f^2 g(\Omega_{cr})\sqrt{g(\Omega_{cr})}}\Big(- r^3(1-c_f^2)^3\Omega_{cr}^3 + 3r^2(1-c_f^2)^2(1-rc_f^2)\Omega_{cr}^2$$

$$-3r(1-c_f^2)(1+r^2c_f^4)\Omega_{cr} + (1-rc_f^2)(1+rc_f^2)^2\Big) > 0,$$

$$g(\Omega) = \Omega^2 r^2(1-c_f^2)^2 - 2r\Omega(1-c_f^2)(1-rc_f^2) + (1+rc_f^2)^2$$

and $k_2 = O(1)$ with respect to small parameter ϵ.

Similar to (2.29), (2.30), assume now that for the surface waves in a vicinity of the critical point k_{cr}

$$k = \frac{\Pi}{2rc_f}\Big(1 + \varepsilon^2 k_2 + \cdots\Big) \tag{2.31}$$

and

$$\tilde{\omega} = -i\varepsilon + \tilde{\omega}_1\varepsilon^2 + \cdots . \tag{2.32}$$

In expansions (2.31), (2.32) we have to set a small parameter $\varepsilon \equiv c_f$ (this choice is dictated by the limit problem). Radicals (2.17) and (2.13) remain to be valid for any k. Thus, one obtains

$$\tilde{\gamma}_2 = 1 + \frac{1}{2}\Big(1 + 2\,r\Big)\varepsilon^2 + O(\varepsilon^3) \tag{2.33}$$

and

$$\tilde{\gamma}_s = 1 + \frac{1}{2c_s^2}\Big(1 - 2r\Big)\varepsilon^2 + O(\varepsilon^3). \tag{2.34}$$

However, as it was mentioned above, expansion (2.17), which corresponds to P2 wave, fails in a neighborhood of the bifurcation point k_{cr}. Solution for $\tilde{\gamma}_1$ in a vicinity of k_{cr} is sought in the following form:

$$\tilde{\gamma}_1 = \Gamma_1\varepsilon^2 + \Gamma_2\varepsilon^3 + \cdots \tag{2.35}$$

By substituting (2.31)-(2.35) into (2.14) and (2.27), from the lowest approximations one obtains the expressions for Γ_1 and $\tilde{\omega}_1$ as follows:

$$\Gamma_1 = \frac{r(4c_s^2 - 1)\Big(2(1+2\,r) - \frac{1}{2c_s^2}(1-2r)\Big)}{(1+2\,r)(1-c_s^2)} \tag{2.36}$$

and

$$\tilde{\omega}_1^2 = 2\Big(k_2 - 2r\Big). \tag{2.37}$$

Finally, for wave number (2.31) dispersion equation (2.27) has the root:

$$\tilde{\omega}_{St} = -i\varepsilon + \sqrt{2\Big(k_2 - 2r\Big)}\varepsilon^2 + O(\varepsilon^3), \tag{2.38}$$

which defines the Stoneley surface wave. Obviously, real solution for $\tilde{\omega}_1$ exists if expression in right-hand side of (2.37) is positive. Therefore, similar to the Biot bulk wave, the Stoneley surface mode has bifurcation behavior in a neighborhood of the bifurcation point

$$k_{cr} \approx \frac{\Pi}{2rc_f}\Big(1 + c_f^2 k_2\Big), \tag{2.39}$$

where

$$k_2 = 2r. \tag{2.40}$$

Thus, if $k_2 \leq 2r$, i.e. $k \leq k_{cr}$, than the Stoneley wave does not propagate; it is fully attenuated mode. Otherwise, if $k_2 > 2r$, i.e. $k > k_{cr}$, it begins to emerge with velocity very close to the speed of P2 wave. Unlike high frequency limit (see [2,3]), the Stoneley surface mode is strongly attenuated at low frequencies (leaky mode).

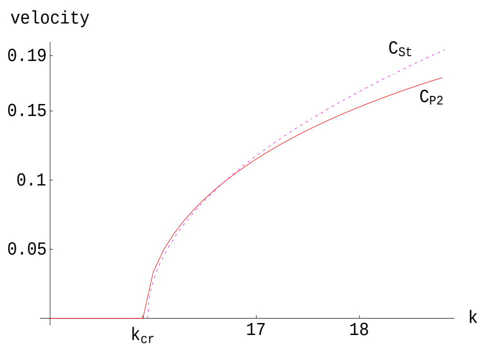

Figure 1: Velocity of the P2 wave (solid line) and velocity of the Stoneley surface wave (dashed line): $r = 0.1$ and $c_f = 0.32$ (water-saturated sandstone [5,10])

Fig.1 shows the calculated velocities of the Biot slow bulk wave and of the Stoneley surface wave in water-saturated sandstone as functions of the wave number. One sees that in the vicinity of k_{cr} velocity of the Stoneley wave $c_{St} = \mathrm{Re}\,\tilde{\omega}_{St}$ is very close but somewhat less than the speed c_{P2} of the P2 wave. Further deviation of

c_{St} from c_{P2} results from the fact that only one term of asymptotic expansion to c_{St} was taken into account (see (2.38)).

Next, let us verify whether solution (2.11), (2.12) indeed has a form of a surface wave, i.e. whether conditions (2.16) hold true for (2.38), (2.31). It is easily seen, that $\operatorname{Re}\gamma_s > 0$ and $\operatorname{Re}\gamma_2 > 0$. However, $\operatorname{Re}\gamma_1$ is not always positive. Last statement follows from the analysis of the leading term in expansion for γ_1 (see (2.35),(2.36)). Coefficient Γ_1 is a single-valued function of parameters r (the ratio of the initial density of the fluid to that of the solid phase) and c_s (the ratio of the shear wave velocity to the speed of the longitudinal wave in an unbounded elastic medium). In Fig.2 the plots of Γ_1 are evaluated for $r = 0.1$ (water-saturated sandstone [5,10]) and for $r = 0.06$ (oil-saturated sandstone [5,10]). From the left plot ($r = 0.1$) it is evident, that $\Gamma_1 < 0$ and, consequently, $\operatorname{Re}\gamma_1 < 0$, if $0.41 < c_s < 0.5$. Thereby, solution (2.11),(2.12) is not a surface wave. Hence we conclude that if elastic moduli of the solid skeleton of a porous medium are such that the ratio of the speeds of the shear and longitudinal waves $c_s \in (0.41, 0.5)$, then at low frequencies the Stoneley surface wave cannot appear at the free interface of this porous medium. The case r=0.06 gives similar result. Inspection of the right plot in Fig.2 shows that if elastic moduli of the solid phase are such that $c_s \in (0.44, 0.5)$, then at low frequencies the Stoneley surface wave does not exist at the free interface of this porous medium. One can observe, that domain of non-existence of the Stoneley surface mode is getting smaller as r decreases. It is interesting to note also, that upper limit for values of c_s, for which this surface wave cannot appear, is always 0.5, while lower limit is not fixed and depends on r.

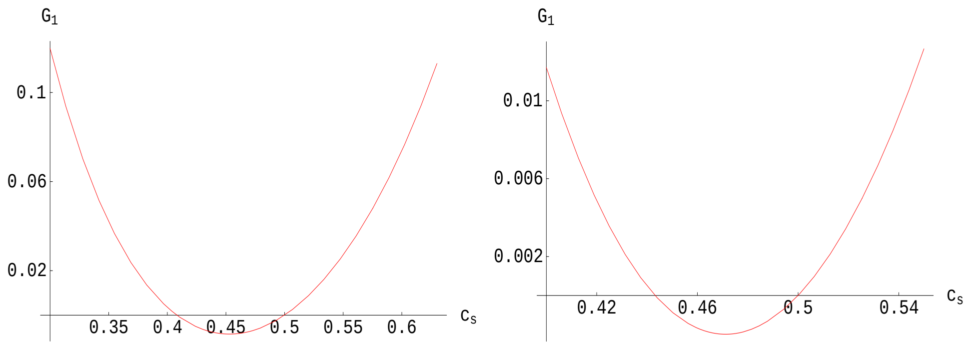

Figure 2: Coefficient Γ_1 as a function of c_s: $r = 0.1$ (left) and $r = 0.06$ (right)

Comment. It is well known that for all elastic materials the ratio c_s of the velocity of the shear wave to that of the lonfitudinal wave varies as: $0 < c_s < \frac{1}{\sqrt{2}} \approx 0.7$ [11].

Above, we have considered only the crudest approximation to the Stoneley surface wave. Certainly, one can construct next terms in the expansion (2.35) so as to confirm the statement on the domains of non-existence of the Stoneley mode.

Continuing the calculations, one finds

$$\Gamma_2 = 2ir\tilde{\omega}_1 \frac{1 - r - 4r^2 + 8c_s^4(1+2r)^2 - 2c_s^2(3+4r)}{(1-c_s^2)(1+2r)^2 c_s^2}. \tag{2.41}$$

Thus, above-made conclusions on the Stoneley wave remain valid.

It should be noted that the Stoneley surface wave has been registered experimentally in the fluid-filled porous natural rocks with closed surface pores at ultrasonic frequencies [12]. Also the existence of this surface mode has been shown numerically [13] for all values of the frame moduli on the base of the high frequency limit of the Biot model. These experimental and numerical results are in good agreement with our analytical predictions [3]. However, the transition region between low and frequency ranges, i.e. the vicinity of the critical point k_{cr}, has never been investigated. As it follows from the preceding analysis, in the transition domain the Stoneley mode exists not for all values of the skeleton moduli.

3. Conclusions

The results presented in the paper concern existence and asymptotic properties of the Stoneley surface wave at the free interface of a fluid-filled porous medium at low frequencies. Asymptotic expansion (2.38) to the frequency of the Stoneley wave shows that, similar to the Biot slow bulk wave, the surface mode has bifurcation behavior in a vicinity of the critical wave number k_{cr}. If wave number of the Stoneley mode $k \leq k_{cr}$ then it is fully attenuated. If $k > k_{cr}$ then it begins to emerge with phase velocity very close to the velocity of P2 wave. Unlike high frequency range, the Stoneley wave is strongly attenuated at low frequencies. It is a leaky mode. It was proven also that within the k-domain of existence, the Stoneley wave cannot appear for certain values of elastic moduli of the solid phase. As it follows from (2.36), (2.16), the domain of non-existence of the surface mode is related to the ratio of the velocities of the shear and longitudinal waves in a skeleton.

Acknowledgments

Support by the Alexander von Humboldt Foundation is gratefully acknowledged.

References

[1] Edelman, I.: 2002, Asymptotic analysis of surface waves at vacuum/porous medium interface: Low-frequency range, *WIAS, Preprint No.745.*

[2] Edelman, I.: (2001), Waves on boundaries of porous media, *Physics Doklady* **46**, 517-521.

[3] Edelman, I., Wilmanski, K.: 2002, Asymptotic analysis of surface waves at vacuum/porous medium and liquid/porous medium interfaces, *Continuum Mech. Thermodyn.* **14(1)**, 25-44.

[4] Edelman, I.: 2002, On the bifurcation of the Biot slow wave in a porous medium, *WIAS, Preprint No.738.*

[5] Edelman, I.: 2002, On the velocity of the Biot slow wave in a porous medium: Uniform asymptotic expansion, *WIAS, Preprint No.775.*

[6] Wilmanski, K.: 1995, Lagrangean model of two-phase porous material, *J. Non-Equilib. Thermodyn.* **20**, 50–77.

[7] Whitham, G.B.: 1974, *Linear and nonlinear waves*, John Wiley & Sons, New York.

[8] Deresiewicz, H.: 1962, The effect of boundaries on wave propagation in a liquid-filled porous solid. IV. Surface waves in a half-space, *Bull. Seism. Soc. Am.* **52(3)**, 627-638.

[9] Biot, M.A.: 1956, Theory of propagation of elastic waves in a fluid - saturated porous solids I. Low frequency range & II. High frequency range, *J. Acoust. Soc. Am.* **28**, 168-186.

[10] Bourbie, T., Coussy, O., Zinszner, B.: 1987, *Acoustics of porous media*, Editions Technip, Paris.

[11] Landau, L.D., Lifshitz, E.M.: 1970, *Theory of elasticity*, Pergamon, New-York.

[12] Adler, L., Nagy, P.B.: 1994, Measurements of acoustic surface waves on fluid-filled porous rocks, *J. Geophys. Res. Solid Earth* **99(B9)**, 17,863-17,869.

[13] Feng, S., Johnson, D.L.: 1983, High-frequency Acoustic Properties of a Fluid/Porous Solid Interface. I. New Surface Mode & II. The 2D Reflection Green's Function, *J. Acoust. Soc. Am.* **74(3)**, 906-924.