Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Global existence result for pair diffusion models

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submitted: November 12th 2002

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Preprint No. 784

Berlin 2002

*Supported by the German Research Foundation (DFG) under grant HU 868/1-2.

2000 *Mathematics Subject Classification.* 35K45, 35K57, 35R05, 35D05, 35B45, 80A30.

Key words and phrases. Reaction-diffusion systems for charged particles, pair diffusion models, global existence, a priori estimates, fixed point theorems.

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Abstract. In this paper we prove a global existence result for pair diffusion models in two dimensions. Such models describe the transport of charged particles in semiconductor heterostructures. The underlying model equations are continuity equations for mobile and immobile species coupled with a nonlinear Poisson equation. The continuity equations for the mobile species are nonlinear parabolic PDEs involving drift, diffusion and reaction terms, the corresponding equations for the immobile species are ODEs containing reaction terms only. Forced by applications to semiconductor technology these equations have to be considered with non-smooth data and kinetic coefficients additionally depending on the state variables.

Our proof is based on regularizations, on a priori estimates which are obtained by energy estimates and Moser iteration as well as on existence results for the regularized problems. These are obtained by applying the Banach Fixed Point Theorem for the equations of the immobile species, and the Schauder Fixed Point Theorem for the equations of the mobile species.

1. The model. Pair diffusion models describe the transport of charged particles (dopant atoms, point defects, dopant-defect pairs) in semiconductors [4, 7]. In [12] we specified a typical mathematical model of this kind which we shall study in this paper, too. We consider m species X_i. The first $l \le m$ species are mobile, the other ones are immobile. We denote by u_i, p_{0i}, $b_i = u_i/p_{0i}$ the density, some reference density, the chemical activity of the i-th species, and by ψ some additional potential. The initial boundary value problem which we are interested in reads as follows:

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} + \nabla\cdot j_i + \sum_{(\alpha,\beta)\in\mathcal{R}^\Omega} (\alpha_i-\beta_i)\,R^\Omega_{\alpha\beta} &= 0 \qquad \text{on } (0,\infty)\times\Omega\,,\\
\nu\cdot j_i - \sum_{(\alpha,\beta)\in\mathcal{R}^\Gamma} (\alpha_i-\beta_i)\,R^\Gamma_{\alpha\beta} &= 0 \qquad \text{on } (0,\infty)\times\Gamma\,,\ i=1,\dots,l\,;\\
\frac{\partial u_i}{\partial t} + \sum_{(\alpha,\beta)\in\mathcal{R}^\Omega} (\alpha_i-\beta_i)\,R^\Omega_{\alpha\beta} &= 0 \qquad \text{on } (0,\infty)\times\Omega\,,\ i=l+1,\dots,m\,;\\
-\nabla\cdot(\varepsilon\nabla\psi) + e(\cdot,\psi) - \sum_{i=1}^m Q_i(\psi)u_i &= f \qquad \text{on } (0,\infty)\times\Omega\,,\\
\nu\cdot(\varepsilon\nabla\psi) &= 0 \qquad \text{on } (0,\infty)\times\Gamma\,;\\
u_i(0) &= U_i \quad \text{on } \Omega\,,\ i=1,\dots,m\,.
\end{aligned}
$$

Here Γ denotes the boundary of the domain $\Omega\subset\mathbb{R}^2$, and ν is the outer unit normal. The transport of the mobile species is governed by the drift-diffusion flux densities

$$j_i = -D_i(\cdot,b,\psi)\,p_{0i}\,(\nabla b_i + Q_i(\psi)\,b_i\nabla\psi)\ ,\ i=1,\dots,l\,,$$

where Q_i denotes the charge number of the i-th species which depends on ψ, and D_i is the diffusivity which depends on the state variables $b=(b_1,\dots,b_m)$ and ψ. All

m continuity equations contain volume source terms generated by mass action type reactions of the form

$$\alpha_1 X_1 + \ldots + \alpha_m X_m \rightleftharpoons \beta_1 X_1 + \ldots + \beta_m X_m\,,\ \ (\alpha,\beta) \in \mathcal{R}^\Omega\,,$$

where α, $\beta \in \mathbb{Z}_+^m$ are the vectors of stoichiometric coefficients, and $\mathcal{R}^\Omega$ describes the set of all reactions under consideration. The corresponding reaction rates $R^\Omega_{\alpha\beta}$ are given by

$$R^\Omega_{\alpha\beta} = k^\Omega_{\alpha\beta}(x, b_1, \ldots, b_m, \psi)\left[\prod_{i=1}^m a_i^{\alpha_i} - \prod_{i=1}^m a_i^{\beta_i}\right]\,,\ a_i = b_i \mathrm{e}^{P_i(\psi)}\,,\ P_i(\psi) = \int_0^\psi Q_i(s)\mathrm{d}s\,.$$

The continuity equations for the mobile species include additional boundary source terms generated by boundary reactions with reaction rates $R^\Gamma_{\alpha\beta}$ given by

$$R^\Gamma_{\alpha\beta} = k^\Gamma_{\alpha\beta}(x, b_1, \ldots, b_l, \psi)\left[\prod_{i=1}^l a_i^{\alpha_i} - \prod_{i=1}^l a_i^{\beta_i}\right]\,,\ \ (\alpha,\beta) \in \mathcal{R}^\Gamma\,.$$

Finally, the nonlinear Poisson equation contains various source terms, namely the fixed charge density f, the charge density e depending on ψ, and the charge density $\sum_{i=1}^m Q_i u_i$ of all particles, ε is the dielectric permittivity.

In heterostructures which we want to include in our considerations the reference densities p_{0i} (and other quantities such as D_i, $k^\Omega_{\alpha\beta}$, $k^\Gamma_{\alpha\beta}$, ε, and e) depend on x, and they may jump when crossing interfaces between different materials. The densities u_i may jump, too, but the chemical activities b_i and the potential ψ remain sufficiently smooth (more precisely, $b_1(t,\cdot), \ldots, b_l(t,\cdot)$, $\psi(t,\cdot)$ belong to $H^1(\Omega)$). In homogeneous structures $p_{0i} = \text{const} > 0$ holds. Then for the mobile species $u_i(t,\cdot) \in H^1(\Omega)$ follows, and the flux densities can be rewritten as

$$j_i = -D_i\left(\nabla u_i + Q_i(\psi)\, u_i \nabla\psi\right)\,,\ \ i = 1, \ldots, l\,.$$

If we know that the chemical activities remain strongly positive, $b_i \geq \text{const} > 0$, then we can reformulate the model equations by using the electrochemical potentials $\zeta_i = \ln a_i = \ln b_i + P_i(\psi)$. For the mobile species $\zeta_i(t,\cdot) \in H^1(\Omega)$ holds, and the kinetic relations are obtained as

$$j_i = -D_i\, u_i\, \nabla\zeta_i\,,\ \ i = 1, \ldots, l\,,$$

$$R^\Omega_{\alpha\beta} = k^\Omega_{\alpha\beta}\left[\mathrm{e}^{\sum_{i=1}^m \alpha_i\zeta_i} - \mathrm{e}^{\sum_{i=1}^m \beta_i\zeta_i}\right]\,,\ \ R^\Gamma_{\alpha\beta} = k^\Gamma_{\alpha\beta}\left[\mathrm{e}^{\sum_{i=1}^l \alpha_i\zeta_i} - \mathrm{e}^{\sum_{i=1}^l \beta_i\zeta_i}\right]\,.$$

If each species has a constant charge number,

$$Q_i(\psi) = q_i = \text{const}\,,\ \ P_i(\psi) = q_i\psi\,,\ \ i = 1, \ldots m\,,$$

then we arrive at a model which we have studied in [8, 9, 10, 11]. There we assumed that all species are mobile, $l = m$, that all diffusivities do not depend on b, and that the initial values U_i are strongly positive. The equations were formulated using the electrochemical potentials as explained above. We proved the global existence and uniqueness of a solution and studied its asymptotic behaviour. The methods developed in the present paper allow us to handle this class of models also in the case that $l < m$, that $D_1, \ldots, D_l$ may depend on b, and that only $U_i \geq 0$ is assumed.

For the pair diffusion models in [4, 7, 12] the charge numbers and their anti-derivatives are given by

$$Q_i(\psi) = \frac{\sum_{k=1}^{k_i} q_{ik} K_{ik} \mathrm{e}^{-q_{ik}\psi}}{\sum_{k=1}^{k_i} K_{ik} \mathrm{e}^{-q_{ik}\psi}}\,,\ P_i(\psi) = \ln \frac{\sum_{k=1}^{k_i} K_{ik}}{\sum_{k=1}^{k_i} K_{ik} \mathrm{e}^{-q_{ik}\psi}} \tag{1.1}$$

where $K_{ik} = \text{const} > 0$, $q_{ik} = \text{const}$. The most important property of the functions Q_i is that $Q_i'(\psi) \le 0$. This property as well as the special structure of the kinetic relations and natural assumptions on the kinetic coefficients ensure that the evolution problem (see $(\mathcal{P})$ later on) as well as needed regularizations of this problem (see $(\mathcal{P}_N)$, $(\mathcal{P}_M)$ later on) have a convex Lyapunov function [12, 15].

It is the aim of the present paper to show that the initial boundary value problem considered here has a global solution in a sense which is precisely defined in Section 2. There also all needed assumptions are given. Section 3 contains the proof of the existence result and related assertions.

Global existence results for simplified versions of the model (for homogeneous two-dimensional structures with smooth boundary, with a special choice of the reactions, and with kinetic coefficients depending only on ψ) were obtained in [20] (all species are mobile, $l = m$) and in [19] (some species can be immobile, $l \le m$). In [1] one may find a local existence result for the same simplified model, but in arbitrary space dimension. A pair diffusion model for uncharged species (then the Poisson equation is dropped) and for homogeneous structures is investigated in [14]. There the case $l < m$ is treated by passing to the limit $D_i \to 0$, $i = l+1, \dots, m$. Several different asymptotic limits for variants of such a model are discussed in [16].

2. Notation, assumptions and main result.

2.1. Notation. The notation of function spaces corresponds to that in [17]. By $\mathbb{Z}_+^n$, $\mathbb{R}_+^n$, L_+^p, H_+^1 we denote the cones of non-negative elements. If $u \in \mathbb{R}^n$ then $u \ge 0$ $(u > 0)$ means $u_i \ge 0\,\forall i$ $(u_i > 0\,\forall i)$. For the scalar product in $\mathbb{R}^n$ we use a centered dot. If u, $v \in \mathbb{R}^n$ then $uv = \{u_i v_i\}_{i=1,\dots,n}$ and u/v is to be understood analogously. Finally, if $u \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{Z}_+^n$ then u^α means the product $\prod_{i=1}^n u_i^{\alpha_i}$. In our estimates positive constants, which depend at most on the data of our problem, are denoted by c. Some auxiliary results which are relevant for the paper are collected in the appendix.

2.2. Assumptions. Let us summarize all needed assumptions which we suppose to be fulfilled up to the end of the paper:

$$\begin{cases} \Omega \subset \mathbb{R}^2 \text{ is a bounded Lipschitzian domain}\,,\ \Gamma = \partial\Omega\,, \\ m, l \in \mathbb{N}\,,\ 1 \le l \le m\,,\ U \in L_+^\infty(\Omega, \mathbb{R}^m)\,,\ f \in L^2(\Omega)\,; \end{cases} \tag{2.1}$$

$$\begin{cases} \text{for } i = 1, \dots, m\ : \\ Q_i \in C^1(\mathbb{R})\,,\ |Q_i(\psi)| \le c\,,\ Q_i'(\psi) \le 0\,,\ P_i(\psi) = \displaystyle\int_0^\psi Q_i(s)\,\mathrm{d}s\,, \\ p_i(x,\psi) = p_{0i}(x)\,\mathrm{e}^{-P_i(\psi)}\,,\ x \in \Omega\,,\ \psi \in \mathbb{R} \\ \text{where } p_{0i} \in L^\infty(\Omega)\,,\ \operatorname{ess\,inf}_{x\in\Omega} p_{0i}(x) \ge \epsilon_0 > 0\,; \end{cases} \tag{2.2}$$

$$
(2.3)\quad \begin{cases}
\varepsilon \in L^\infty(\Omega)\,,\ \operatorname{ess\,inf}_{x\in\Omega}\varepsilon(x) > 0\,, \\
e\colon \Omega\times\mathbb{R}\to\mathbb{R} \text{ satisfies the Carathéodory conditions,} \\
e(x,\cdot) \text{ is locally Lipschitz continuous uniformly w.r.t. } x\,, \\
|e(x,\psi)| \le c\,e^{c|\psi|} \text{ f.a.a. } x\in\Omega\,,\ \forall\psi\in\mathbb{R} \text{ with some } c>0\,, \\
e(x,\psi)-e(x,\overline{\psi}) \ge e_0(x)\,(\psi-\overline{\psi}) \text{ f.a.a. } x\in\Omega\,,\ \forall\psi,\overline{\psi}\in\mathbb{R} \text{ with } \psi\ge\overline{\psi} \\
\text{where } e_0\in L^\infty_+(\Omega)\,,\ \|e_0\|_{L^1}>0\,;
\end{cases}
$$

$$
(2.4)\quad \begin{cases}
\mathcal{R}^\Omega \subset \mathbb{Z}^m_+\times\mathbb{Z}^m_+\,,\ m_\Omega = m\,, \\
\mathcal{R}^\Gamma \subset \left\{(\alpha,\beta)\in\mathbb{Z}^m_+\times\mathbb{Z}^m_+ \colon \alpha_i=\beta_i=0\,,\ i=l+1,\dots,m\right\}\,,\ m_\Gamma = l\,, \\
\text{for } \Sigma=\Omega,\,\Gamma\,,\ (\alpha,\beta)\in\mathcal{R}^\Sigma\ : \\
R^\Sigma_{\alpha\beta}(x,b,\psi) = k^\Sigma_{\alpha\beta}(x,b,\psi)\,\left(a^\alpha - a^\beta\right)\,,\ x\in\Sigma\,,\ b\in\mathbb{R}^{m_\Sigma}_+\,,\ \psi\in\mathbb{R} \\
\text{where } a_i = b_i \mathrm{e}^{P_i(\psi)}\,,\ i=1,\dots,m_\Sigma\,, \\
k^\Sigma_{\alpha\beta}\colon \Sigma\times\mathbb{R}^{m_\Sigma}_+\times\mathbb{R}\to\mathbb{R}_+ \text{ satisfies the Carathéodory conditions,} \\
k^\Sigma_{\alpha\beta}(x,\cdot,\cdot) \text{ is locally Lipschitz continuous uniformly w.r.t. } x\,, \\
\text{for all } R>0 \text{ there exists } c_R>0 \text{ such that} \\
k^\Sigma_{\alpha\beta}(x,b,\psi) \le c_R \text{ f.a.a. } x\in\Sigma\,,\ \forall b\in\mathbb{R}^{m_\Sigma}_+\,,\ \forall\psi\in[-R,R]\,;
\end{cases}
$$

$$
(2.5)\quad \begin{cases}
\displaystyle\sum_{i=l+1}^m \alpha_i \cdot \sum_{i=l+1}^m \beta_i = 0 \quad \forall(\alpha,\beta)\in\mathcal{R}^\Omega\,, \\
\text{there exists } c>0 \text{ such that for } a\in\mathbb{R}^m_+\ : \\
\displaystyle\max_{k=1,\dots,m} \left\{(a^\alpha-a^\beta)(\beta_k-\alpha_k)\right\} \le c\sum_{k=1}^m a_k^2 + c \quad \forall(\alpha,\beta)\in\mathcal{R}^\Omega\,, \\
\displaystyle\max_{k=1,\dots,l} \left\{(a^\alpha-a^\beta)(\beta_k-\alpha_k)\right\} \le c\sum_{k=1}^l a_k + c \quad \forall(\alpha,\beta)\in\mathcal{R}^\Gamma\,;
\end{cases}
$$

$$
(2.6)\quad \begin{cases}
\text{for } i=1,\dots,l\ : \\
D_i\colon \Omega\times\mathbb{R}^m_+\times\mathbb{R}\to\mathbb{R}_+ \text{ satisfies the Carathéodory conditions,} \\
D_i(x,b,\psi) \ge c_{2.6} > 0 \text{ f.a.a. } x\in\Omega\,,\ \forall b\in\mathbb{R}^m_+\,,\ \forall\psi\in\mathbb{R}\,, \\
\text{for all } R>0 \text{ there exists } c_R>0 \text{ such that} \\
D_i(x,b,\psi) \le c_R \text{ f.a.a. } x\in\Omega\,,\ \forall b\in\mathbb{R}^m_+\,,\ \forall\psi\in[-R,R]\,;
\end{cases}
$$

$$
(2.7)\quad \begin{cases}
\text{for } i = l+1,\dots,m \ : \\
\text{there is a reaction of the form} \\
R^{\Omega}_{\alpha_{(i)}\beta_{(i)}}(x,b,\psi) = k^{\Omega}_{\alpha_{(i)}\beta_{(i)}}(x,b,\psi)\left[\prod_{k=1}^{l} a_k^{\alpha_{(i)k}} - a_i^2\right] \\
\text{where for all } R>0 \text{ there exists } c_R>0 \text{ such that} \\
k^{\Omega}_{\alpha_{(i)}\beta_{(i)}}(x,b,\psi) \ge c_R \text{ f.a.a. } x\in\Omega\,,\ \forall b\in\mathbb{R}^m_+\,,\ \forall\psi\in[-R,R]\,.
\end{cases}
$$

2.3. Formulation of the problem. We use the function spaces

$$Y = L^2(\Omega,\mathbb{R}^m)\,,\ X = \left\{b\in Y\colon b_i\in H^1(\Omega)\,,\ i=1,\dots,l\right\}$$

and define operators $B\colon Y\to Y$ and

$$A, R\colon [X\cap L^\infty_+(\Omega,\mathbb{R}^m)]\times[H^1(\Omega)\cap L^\infty(\Omega)]\to X^*\,,\ E\colon H^1(\Omega)\times Y\to H^1(\Omega)^*$$

by the relations

$$(Bb,\overline{b})_Y = \int_\Omega \sum_{i=1}^m p_{0i}\, b_i\, \overline{b}_i \,\mathrm{d}x\,,\quad \overline{b}\in Y\,,$$

$$\langle A(b,\psi),\overline{b}\rangle_X = \int_\Omega \sum_{i=1}^l D_i(\cdot,b,\psi)p_{0i}(\nabla b_i + b_iQ_i(\psi)\nabla\psi)\cdot\nabla\overline{b}_i\,\mathrm{d}x\,,\quad \overline{b}\in X\,,$$

$$\begin{aligned}\langle R(b,\psi),\overline{b}\rangle_X = &\int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}^\Omega} R^\Omega_{\alpha\beta}(\cdot,b_1,\dots,b_m,\psi)\sum_{i=1}^m(\beta_i-\alpha_i)\,\overline{b}_i\,\mathrm{d}x \\ &+\int_\Gamma \sum_{(\alpha,\beta)\in\mathcal{R}^\Gamma} R^\Gamma_{\alpha\beta}(\cdot,b_1,\dots,b_l,\psi)\sum_{i=1}^l(\beta_i-\alpha_i)\,\overline{b}_i\,\mathrm{d}\Gamma\,,\quad \overline{b}\in X\,,\end{aligned}$$

$$\langle E(\psi,u),\overline{\psi}\rangle_{H^1} = \int_\Omega\left\{\varepsilon\nabla\psi\cdot\nabla\overline{\psi} + e(\cdot,\psi)\overline{\psi} - \sum_{i=1}^m u_iQ_i(\psi)\overline{\psi} - f\overline{\psi}\right\}\mathrm{d}x\,,\quad \overline{\psi}\in H^1(\Omega)\,.$$

The precise formulation of the initial boundary value problem considered in Section 1 reads as follows:

$$
(\mathcal{P})\quad \begin{cases}
u'(t) + A(b(t),\psi(t)) = R(b(t),\psi(t))\,, \\
E(\psi(t),u(t)) = 0\,,\ u(t) = Bb(t) \text{ f.a.a. } t>0\,,\ u(0) = U\,, \\
u\in H^1_{\mathrm{loc}}(\mathbb{R}_+,X^*)\cap L^2_{\mathrm{loc}}(\mathbb{R}_+,Y)\,,\ b\in L^2_{\mathrm{loc}}(\mathbb{R}_+,X)\cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+,L^\infty_+(\Omega,\mathbb{R}^m))\,, \\
\psi\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^1(\Omega))\cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+,L^\infty(\Omega))\,.
\end{cases}
$$

Remark 2.1. Let (u,b,ψ) be a solution of $(\mathcal{P})$. Lemma 4.1 ii), iii) ensure that u, $b\in C(\mathbb{R}_+,Y)$. Furthermore one easily obtains that u, $b\in C(\mathbb{R}_+,(L^\infty(\Omega,\mathbb{R}^m),w^*))$, and $\psi\in C(\mathbb{R}_+,H^1(\Omega))$, see Lemma 3.1, too. These properties imply that the relations

$$(2.8)\qquad \begin{aligned} &E(\psi(t),u(t)) = 0 \text{ in } H^1(\Omega)^*\,, \\ &u(t) = p_0\,b(t) \text{ in } L^\infty(\Omega,\mathbb{R}^m)\,,\ u(t),\, b(t)\ge 0 \text{ a.e. on } \Omega\end{aligned}$$

are fulfilled for all $t\in\mathbb{R}_+$.

2.4. Main result. Now we formulate the main result of the paper.

THEOREM 2.2. *There exists a solution of* $(\mathcal{P})$.

Remark 2.3. In [12] we found some further results concerning properties of solutions of $(\mathcal{P})$. Uniqueness was obtained using some restrictive assumptions on the diffusivities (see [12, Lemma 7.2]). Global estimates as well as asymptotic properties of solutions were derived, too (see [12, Theorems 4.1, 6.1]). Here additional assumptions on the underlying reaction system were needed.

3. Proofs.

3.1. The nonlinear Poisson equation. We start with some results concerning the Poisson equation which we need in the sequel.

LEMMA 3.1. *For any $u \in Y$ there exists a unique solution $\psi \in H^1(\Omega)$ of the equation $E(\psi, u^+) = 0$. Moreover, there are an exponent $q > 2$, a positive constant c and a monotonously increasing function $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$(3.1)\quad \|\psi - \overline{\psi}\|_{H^1} \le c\,\|u - \overline{u}\|_Y \quad \forall u, \overline{u} \in Y\,,\ E(\psi, u^+) = E(\overline{\psi}, \overline{u}^+) = 0\,,$$

$$(3.2)\quad \|\psi\|_{L^\infty} \le c\left\{1 + \sum_{i=1}^m \|u_i^+ \ln u_i^+\|_{L^1} + d(\|\psi\|_{H^1})\right\} \quad \forall u \in Y\,,\ E(\psi, u^+) = 0\,,$$

$$(3.3)\quad \|\psi\|_{W^{1,q}} \le c\left\{1 + \sum_{i=1}^m \|u_i\|_{L^{2q/(2+q)}} + d(\|\psi\|_{H^1})\right\} \quad \forall u \in Y\,,\ E(\psi, u^+) = 0\,,$$

$$(3.4)\quad \|\psi\|_{H^1} \le c(1 + \|u\|_Y) \quad \forall u \in Y\,,\ E(\psi, u^+) = 0\,.$$

Finally, let $S = [0, T]$, $T > 0$. Then for every $u \in L^2(S, Y)$ there exists a unique $\psi \in L^2(S, H^1(\Omega))$ such that

$$E(\psi(t), u^+(t)) = 0 \text{ f.a.a. } t \in S\,.$$

If $u \in C(S, Y)$ then $\psi \in C(S, H^1(\Omega))$ follows and the last equation holds for all $t \in S$.

Proof. For the first existence result and the estimates (3.1), (3.2) we refer to [15, Lemma 1]. The estimate (3.3) is a consequence of Gröger's regularity result for elliptic equations [13, Theorem 1] and of Trudinger's imbedding theorem [23]. Moreover, let ψ_0 be the (unique) solution of $E(\psi_0, 0) = 0$. According to (3.1) we have $\|\psi - \psi_0\|_{H^1} \le c\|u\|_Y$ if $u \in Y$ and $E(\psi, u^+) = 0$. Thus (3.4) follows. The last assertions result from the pointwise existence result and (3.1). ∎

3.2. First regularized problem $(\mathcal{P}_N)$. In order to prove Theorem 2.2 we shall consider two regularized problems which are defined on an arbitrary given interval $S = [0, T]$. First we introduce a problem $(\mathcal{P}_N)$ as follows. Let $N \in \mathbb{R}$, $N > 0$, be given and let $\rho_N\colon \mathbb{R}^{m+1} \to [0, 1]$ be a Lipschitz continuous function with

$$\rho_N(y) = \begin{cases} 0 & \text{if} \quad |y|_\infty \ge N\,, \\ 1 & \text{if} \quad |y|_\infty \le N/2 \end{cases}\,, \quad |y|_\infty = \max\{|y_1|, \dots, |y_{m+1}|\}\,.$$

We define the functions $r_i^\Sigma\colon \Sigma \times \mathbb{R}_+^{m_\Sigma} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m_\Sigma$, $\Sigma = \Omega$, Γ, by

$$r_i^\Omega(x, b, \psi) = \rho_N(b, \psi) \sum_{(\alpha,\, \beta) \in \mathcal{R}^\Omega} R_{\alpha\beta}^\Omega(x, b, \psi)(\beta_i - \alpha_i)\,,$$

$$r_i^\Gamma(x,b_1,\dots,b_l,\psi)=\rho_N(b_1,\dots,b_l,0,\dots,0,\psi)\sum_{(\alpha,\,\beta)\in\mathcal{R}^\Gamma}R^\Gamma_{\alpha\beta}(x,b_1,\dots,b_l,\psi)(\beta_i-\alpha_i)\,.$$

These functions satisfy the Carathéodory conditions, and the functions $r_i^\Sigma(x,\cdot,\cdot)$ are uniformly Lipschitz continuous since $R^\Sigma_{\alpha\beta}(x,\cdot,\cdot)$ are uniformly locally Lipschitz continuous and ρ_N is a Lipschitz continuous function with compact support. Further important properties of these functions are

$$|r_i^\Sigma(x,b,\psi)|\le c(N)\quad \text{f.a.a. } x\in\Sigma\,,\ \forall (b,\psi)\in\mathbb{R}_+^{m_\Sigma}\times\mathbb{R}\,,\ i=1,\dots,m_\Sigma\,, \tag{3.5}$$

$$\sum_{i=1}^{m_\Sigma} r_i^\Sigma(x,b,\psi)\ (\ln b_i+P_i(\psi))\le 0\quad \text{f.a.a. } x\in\Sigma\,,\ \forall (b,\psi)\in\mathbb{R}_+^{m_\Sigma}\times\mathbb{R}\,,\ b>0\,. \tag{3.6}$$

We define the operator $R_N\colon X_+\times H^1(\Omega)\to X^*$ by

$$\begin{aligned}\langle R_N(b,\psi),\overline{b}\rangle_X=&\int_\Omega\sum_{i=1}^m r_i^\Omega(\cdot,b_1,\dots,b_m,\psi)\,\overline{b}_i\,\mathrm{d}x\\&+\int_\Gamma\sum_{i=1}^l r_i^\Gamma(\cdot,b_1,\dots,b_l,\psi)\,\overline{b}_i\,\mathrm{d}\Gamma\,,\ \overline{b}\in X\,.\end{aligned}$$

Now our first regularized problem is formulated as follows:

$$(\mathcal{P}_N)\qquad\left\{\begin{array}{c} u'(t)+A(b(t),\psi(t))=R_N(b(t),\psi(t))\,,\\ E(\psi(t),u(t))=0\,,\ u(t)=Bb(t)\quad \text{f.a.a. } t\in S\,,\ u(0)=U\,,\\ u\in H^1(S,X^*)\cap L^2(S,Y)\,,\ b\in L^2(S,X)\cap L^\infty(S,L_+^\infty(\Omega,\mathbb{R}^m))\,,\\ \psi\in L^2(S,H^1(\Omega))\cap L^\infty(S,L^\infty(\Omega))\,.\end{array}\right.$$

3.3. Energy estimates for solutions of $(\mathcal{P}_N)$. We summarize some results which can be obtained as in [12, 15]. Let $\widetilde{F}_1,\widetilde{F}_2:Y\to\mathbb{R}$ be given by

$$\widetilde{F}_1(u)=\int_\Omega\left\{\frac{\varepsilon}{2}|\nabla\psi|^2+g(\cdot,\psi)+\sum_{i=1}^m u_i(P_i(\psi)-Q_i(\psi)\psi)\right\}\mathrm{d}x\,,\ u\in Y_+\,, \tag{3.7}$$

where $g(\cdot,\psi)=e(\cdot,\psi)\psi-\int_0^\psi e(\cdot,z)\,\mathrm{d}z$ and $\psi\in H^1(\Omega)\cap L^\infty(\Omega)$ is the unique solution of the Poisson equation $E(\psi,u)=0$,

$$\widetilde{F}_2(u)=\int_\Omega\sum_{i=1}^m\left\{u_i\left[\ln\frac{u_i}{p_{0i}}-1\right]+p_{0i}\right\}\mathrm{d}x\,,\ u\in Y_+\,, \tag{3.8}$$

$$\widetilde{F}_1(u)=+\infty\,,\ \widetilde{F}_2(u)=+\infty\,,\ u\in Y\setminus Y_+\,.$$

Finally, we define the functionals

$$F_k=(\widetilde{F}_k^*|_X)^*:X^*\to\overline{\mathbb{R}}\,,\ k=1,2\,,\ F=F_1+F_2:X^*\to\overline{\mathbb{R}}\,. \tag{3.9}$$

The value $F(u)$ represents the free energy of the state $u\in X^*$.

LEMMA 3.2. *The functional $F = F_1 + F_2 : X^* \to \overline{\mathbb{R}}$ is proper, convex and lower semi-continuous. For $u \in Y_+$ it can be evaluated according to* (3.7), (3.8).

For the proof see [12, Lemma 3.2]. Next, we introduce the functional

$$D : \left\{ u \in L^\infty_+(\Omega, \mathbb{R}^m) : \sqrt{a_i} \in H^1(\Omega)\,,\ a_i = u_i/p_i(\psi)\,,\ i = 1, \dots, l\,,\ E(\psi, u) = 0 \right\} \to \mathbb{R}$$

by the formula

$$D(u) = c_{2.6} \int_\Omega \sum_{i=1}^{l} p_i(\cdot, \psi) |\nabla \sqrt{a_i}|^2 \,\mathrm{d}x \tag{3.10}$$

with $c_{2.6}$ from assumption (2.6). This functional is a non-negative lower estimate for the dissipation rate of problem $(\mathcal{P}_N)$ (and of problem $(\mathcal{P})$, too) where the contributions arising from the reactions have been omitted in view of (3.6). Again using the properties (3.6) and following the ideas in [15, Section 5]) and [12] (see also the proof of Lemma 3.15) we obtain

LEMMA 3.3. *Along any solution (u, b, ψ) of $(\mathcal{P}_N)$ the free energy $F(u)$ remains bounded from above and decreases monotonously, more precisely*

$$F(u(t_2)) + \int_{t_1}^{t_2} D(u(t)) \,\mathrm{d}t \le F(u(t_1)) \le F(U)\,, \quad 0 \le t_1 \le t_2 \le T$$

holds. Moreover, there exist constants c, $c_{3.11} > 0$ depending only on the data but not on N and T such that

$$\begin{aligned} &\sum_{i=1}^{m} \|u_i \ln u_i\|_{L^\infty(S, L^1(\Omega))} \le c\,, \quad \|u\|_{L^\infty(S, L^1(\Omega, \mathbb{R}^m))} \le c\,, \\ &\|\psi\|_{L^\infty(S, H^1(\Omega))} \le c\,, \quad \|\psi\|_{L^\infty(S, L^\infty(\Omega))}\,,\ \|\psi\|_{L^\infty(S, L^\infty(\Gamma))} \le c_{3.11} \end{aligned} \tag{3.11}$$

for any solution of $(\mathcal{P}_N)$.

Remark 3.4. Note that the last two estimates of Lemma 3.3 together with the properties (2.4) and (2.6) ensure the existence of constants c, $\tilde{\epsilon}$, $\epsilon > 0$ such that

$$\begin{aligned} &k^{\Sigma}_{\alpha\beta}(\cdot, b_1, \dots, b_{m_\Sigma}, \psi) \le c \text{ a.e. in } S \times \Sigma\,,\ (\alpha, \beta) \in \mathcal{R}^{\Sigma}\,,\ \Sigma = \Omega, \Gamma\,, \\ &\tilde{\epsilon} \le 2k^{\Omega}_{\alpha(i)\beta(i)}(\cdot, b, \psi)\, \mathrm{e}^{P_i(\psi)} \text{ a.e. in } S \times \Omega\,,\ i = l+1, \dots, m\,, \\ &\epsilon \le D_i(\cdot, b, \psi) p_{0i} \le c \text{ a.e. in } S \times \Omega\,,\ i = 1, \dots, l\,, \end{aligned}$$

for any solution (u, b, ψ) of $(\mathcal{P}_N)$.

3.4. Further a priori estimates for solutions of $(\mathcal{P}_N)$. The constants in the estimates of this subsection will depend on T. Therefore it is not possible to use these results to obtain global (w.r.t. time) bounds for solutions of $(\mathcal{P})$. Such global bounds are derived in [12] by a modified method.

LEMMA 3.5. *There is a constant $c_{3.12} > 0$ not depending on N such that*

$$\|b_i(t)\|_{L^2} \le c_{3.12} \quad \forall t \in S\,,\ i = 1, \dots, m\,, \tag{3.12}$$

for any solution (u, b, ψ) of $(\mathcal{P}_N)$.

Proof. Let (u, b, ψ) be a solution of $(\mathcal{P}_N)$.

1. Choosing q as in Lemma 3.1 we obtain by Lemma 3.1 and Lemma 3.3 that

$$\|\psi(t)\|_{W^{1,q}} \le c \left(1 + \sum_{i=1}^{m} \|b_i(t)\|_{L^{2q/(2+q)}} \right) \text{ f.a.a. } t \in S\,. \tag{3.13}$$

2. We take into account the assumptions (2.5) and (2.7) concerning the order of the source terms of the reactions and the presence of reactions with quadratic sink terms for the immobile species, respectively. Since $\|\psi\|_{L^\infty(S,L^\infty(\Sigma))} \le c_{3.11}$, $\Sigma = \Omega, \Gamma$, and $|\rho_N(b,\psi)| \le 1$ we find that

$$\int_\Omega \sum_{i=1}^m r_i^\Omega(\cdot, b, \psi)\, b_i \,\mathrm{d}x$$
$$\le \int_\Omega \rho_N(b,\psi) \sum_{k=l+1}^m \left\{ c \sum_{i=1}^l (b_i^3 + b_i^2 b_k + b_i b_k^2 + b_k^2 + 1) - \tilde{\epsilon}\, b_k^3 \right\} \mathrm{d}x \le c \sum_{i=1}^l \|b_i\|_{L^3}^3 + c\,,$$
$$\int_\Gamma \sum_{i=1}^l r_i^\Gamma(\cdot, b_1, \dots, b_l, \psi)\, b_i \,\mathrm{d}\Gamma \le c \sum_{i=1}^l \|b_i\|_{L^2(\Gamma)}^2 + c\,.$$

3. Testing the evolution equation in $(\mathcal{P}_N)$ with $2b$, and using the estimates from step 2 as well as (4.1), (4.3) and Young's inequality we obtain

$$\sum_{i=1}^m \left(\epsilon_0 \|b_i(t)\|_{L^2}^2 - c\|U_i\|_{L^2}^2 \right)$$
$$\le \int_0^t \sum_{i=1}^l \left\{ -2\epsilon \|b_i\|_{H^1}^2 + c(\|b_i\|_{L^r}\|\psi\|_{W^{1,q}}\|b_i\|_{H^1} + \|b_i\|_{L^3}^3 + \|b_i\|_{L^2(\Gamma)}^2 + 1) \right\} \mathrm{d}s$$
$$\le \int_0^t \sum_{i=1}^l \left\{ -\epsilon \|b_i\|_{H^1}^2 + \overline{c}(\|b_i\|_{L^r}\|\psi\|_{W^{1,q}}\|b_i\|_{H^1} + \|b_i\|_{L^2}^4 + 1) \right\} \mathrm{d}s \quad \forall t \in S$$

where $r = 2q/(q-2)$. Using the estimate $\|b_k\|_{L^{2q/(2+q)}} \le \|b_k\|_{L^1}^{(r-2)/r} \|b_k\|_{L^2}^{2/r}$ as well as Lemma 3.3 and (3.13), (4.3) we calculate

$$\overline{c}\|b_i\|_{L^r}\|\psi\|_{W^{1,q}}\|u_i\|_{H^1} \le c\|b_i\|_{L^2}^{2/r} \left(1 + \sum_{k=1}^m \|b_k\|_{L^2}^{2/r} \right) \|b_i\|_{H^1}^{2(r-1)/r}$$
$$\le \epsilon \|b_i\|_{H^1}^2 + c\|b_i\|_{L^2}^2 \sum_{k=1}^m \|b_k\|_{L^2}^2 + c\,.$$

Therefore we can continue the first estimate in step 3 as

$$\sum_{k=1}^m \|b_k(t)\|_{L^2}^2 \le c \int_0^t \sum_{k=1}^m \sum_{i=1}^l \|b_i\|_{L^2}^2 \|b_k\|_{L^2}^2 \,\mathrm{d}s + c \quad \forall t \in S\,.$$

Let i, $1 \le i \le l$, be fixed. By Lemma 3.3 and (3.10) we find that

$$\|\nabla\sqrt{a_i}\|_{L^2(S,L^2)} \le c\,, \ \|u_i\|_{L^\infty(S,L^1)} \le c\,, \ \|\psi\|_{L^\infty(S,L^\infty)} \le c$$

and $\|\sqrt{a_i}\|_{L^2(S,H^1)} \le c$, $\|\sqrt{a_i}\|_{L^\infty(S,L^2)} \le c$. Thus interpolation yields $\|\sqrt{a_i}\|_{L^4(S,L^4)} \le c$ and $\|b_i\|_{L^2(S,L^2)} \le c$. A special form of Gronwall's lemma (cf. [24, p. 14, 15]) leads to the desired result. □

Again, let q be chosen as in Lemma 3.1. Since $2q/(2+q) < 2$ we obtain from (3.12), (3.13) the estimate $\|\psi\|_{L^\infty(S,W^{1,q})} \le c_q$. We define

$$\kappa = c_q^{2r} + 1 \text{ where } r = 2q/(q-2)\,,\ q \text{ as in Lemma 3.1.} \tag{3.14}$$

Lemma 3.6. *There is a constant $c_{3.15} \geq 1$ not depending on N such that*

$$\|b_i(t)\|_{L^4} \leq c_{3.15} \quad \forall t \in S\,,\ i = 1,\dots,m\,, \tag{3.15}$$

for any solution (u,b,ψ) of $(\mathcal{P}_N)$.

Proof. Let (u,b,ψ) be a solution of $(\mathcal{P}_N)$. We use the test function $4(b_1^3,\dots,b_m^3)$. Arguing similar as in step 2 of the proof of Lemma 3.5 we find that

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^m r_i^\Omega(\cdot,b,\psi)\, b_i^3 \,\mathrm{d}x \\
&\leq \int_\Omega \rho_N(b,\psi) \sum_{k=l+1}^m \Big\{ c\sum_{i=1}^l \big((b_i^2+1)b_k^3 + (b_k^2+1)b_i^3 + b_i^5\big) - \tilde\epsilon b_k^5 \Big\} \mathrm{d}x \leq c\sum_{i=1}^l \|b_i\|_{L^5}^5 + c\,, \\
&\int_\Gamma \sum_{i=1}^l r_i^\Gamma(\cdot,b_1,\dots,b_l,\psi)\, b_i^3 \,\mathrm{d}\Gamma \leq c\sum_{i=1}^l \|b_i\|_{L^4(\Gamma)}^4 + c\,.
\end{aligned}$$

Therefore we obtain for all $t \in S$

$$\begin{aligned}
&\sum_{i=1}^m \big(\epsilon_0 \|b_i(t)\|_{L^4}^4 - c\|U_i\|_{L^4}^4\big) \\
&\leq \int_0^t \sum_{i=1}^l \Big\{ -2\epsilon \|b_i^2\|_{H^1}^2 + c\Big(\|\nabla\psi\|_{L^q} \|\nabla(b_i^2)\|_{L^2} \|b_i^2\|_{L^r} + \|b_i\|_{L^5}^5 + \|b_i\|_{L^4(\Gamma)}^4 + 1\Big)\Big\} \,\mathrm{d}s\,.
\end{aligned}$$

We apply the trace inequality (4.1), Gagliardo–Nirenberg's inequality (4.3), (3.14) and Young's inequality,

$$\begin{aligned}
\epsilon_0 \sum_{i=1}^m \|b_i(t)\|_{L^4}^4 \leq& \int_0^t \sum_{i=1}^l \Big\{ -\frac{\epsilon}{2}\|b_i^2\|_{H^1}^2 + c\Big(\|\psi\|_{W^{1,q}} \|b_i^2\|_{L^1}^{1/r} \|b_i^2\|_{H^1}^{2-1/r} \\
&\qquad + \|b_i^2\|_{L^1} \|b_i^2\|_{H^1}^{3/2} + \|b_i^2\|_{L^1}^{1/2} \|b_i^2\|_{H^1}^{3/2} + 1\Big)\Big\} \,\mathrm{d}s + c \\
\leq&\ c\int_0^t \sum_{i=1}^l \big(\kappa\|b_i^2\|_{L^1}^2 + \|b_i^2\|_{L^1}^4 + \|b_i^2\|_{L^1}^2 + 1\big) \,\mathrm{d}s + c \quad \forall t \in S\,,
\end{aligned}$$

and the assertion follows from Lemma 3.5. □

Theorem 3.7. *There exists a constant $c_{3.16} > 0$ not depending on N such that*

$$\begin{aligned}
&\|b_i(t)\|_{L^\infty} \leq c_{3.16} \quad \forall t \in S, \quad i = 1,\dots,m\,, \\
&\|b_i\|_{L^\infty(S,L^\infty(\Gamma))} \leq c_{3.16}\,, \quad i = 1,\dots,l\,,
\end{aligned} \tag{3.16}$$

for any solution (u,b,ψ) of $(\mathcal{P}_N)$.

Proof. The proof will be done in two steps. Firstly, by Moser iteration we establish global upper bounds for the mobile species. Then, using these bounds we derive global upper bounds for the immobile species. Let (u,b,ψ) be a solution of $(\mathcal{P}_N)$. Let $K = \max\{1, \max_{i=1,\dots,m} \|U_i/p_{0i}\|_{L^\infty}\}$ and define $z_i = (b_i - K)^+$, $i = 1,\dots,m$.
1. *Bounds for the mobile species.* Let $p \geq 8$. We use $p(z_1^{p-1},\dots,z_l^{p-1},0,\dots,0)$ as test function and define $w_i = z_i^{p/2}$, $i = 1,\dots,l$. At first let us remark that

$$\sum_{i=1}^l r_i^\Omega(\cdot,b,\psi)\, z_i^{p-1} \leq c\sum_{i=1}^l \sum_{k=1}^m (b_k^2+1) z_i^{p-1} \leq c\sum_{i=1}^l \Big(z_i^{p+1} + \sum_{k=l+1}^m z_i^{p-1} z_k^2 \Big) + c\,.$$

With Lemma 3.6 and Hölder's inequality we can estimate

$$\int_\Omega z_i^{p-1} z_k^2 \,\mathrm{d}x \leq \|z_i\|_{L^{2(p-1)}}^{p-1} \|z_k\|_{L^4}^2 \leq c_{3.15}^2 \|w_i\|_{L^{4(p-1)/p}}^{2(p-1)/p} .$$

Therefore we obtain for all $t \in S$

$$\epsilon_0 \sum_{i=1}^l \|w_i(t)\|_{L^2}^2 \leq \int_0^t \sum_{i=1}^l \Big\{ -2\epsilon \|w_i\|_{H^1}^2 + cp\Big(\|\nabla\psi\|_{L^q} \|\nabla w_i\|_{L^2} (\|w_i\|_{L^r} + 1) \\ + \|w_i\|_{L^{2(p+1)/p}}^{2(p+1)/p} + c_{3.15}^2 \|w_i\|_{L^{4(p-1)/p}}^{2(p-1)/p} + \|w_i\|_{L^2(\Gamma)}^2 + 1 \Big) \Big\} \,\mathrm{d}s .$$

We apply for $k = 1$ and $\widetilde{p} = r$, $\widetilde{p} = 2(p+1)/p$ and $\widetilde{p} = 4(p-1)/p$, respectively, Gagliardo–Nirenberg's inequality (4.3) and continue

$$\begin{aligned}
&\epsilon_0 \sum_{i=1}^l \|w_i(t)\|_{L^2}^2 \\
&\leq \int_0^t \sum_{i=1}^l \Big\{ -\epsilon \|w_i\|_{H^1}^2 + cp^{2r} (\|\psi\|_{W^{1,q}}^{2r} + 1)(\|w_i\|_{L^1}^2 + 1) + cp\Big(\|w_i\|_{H^1}^{(p+2)/p} \|w_i\|_{L^1} \\
&\qquad + c_{3.15}^2 \|w_i\|_{H^1}^{(3p-4)/2p} \|w_i\|_{L^1}^{1/2} + \|w_i\|_{H^1}^{3/2} \|w_i\|_{L^1}^{1/2} + 1 \Big) \Big\} \,\mathrm{d}s \\
&\leq \int_0^t \sum_{i=1}^l c \Big\{ p^{2r} \kappa (\|w_i\|_{L^1}^2 + 1) + p^4 \|w_i\|_{L^1}^{2p/(p-2)} + p^4 c_{3.15}^8 \|w_i\|_{L^1}^{2p/(p+4)} + p^4 \|w_i\|_{L^1}^2 \Big\} \,\mathrm{d}s \\
&\leq cp^{2r} \left(\kappa + c_{3.15}^8\right) \int_0^t \sum_{i=1}^l \left(\|w_i\|_{L^1}^{2p/(p-2)} + 1 \right) \mathrm{d}s \\
&\leq cp^{2r} \left(\kappa + c_{3.15}^8\right) T \sum_{i=1}^l \left(\sup_{s\in S} \|z_i(s)\|_{L^{p/2}}^{p^2/(p-2)} + 1 \right) .
\end{aligned}$$

Therefore the iteration formula

$$\sum_{i=1}^l \|z_i(t)\|_{L^p}^p + 1 \leq \overline{c}\, p^{2r} \left(\kappa + c_{3.15}^8\right) T \left(\sum_{i=1}^l \sup_{s\in S} \|z_i(s)\|_{L^{p/2}}^{p/2} + 1 \right)^{2p/(p-2)} \qquad \forall t \in S$$

results where $\overline{c} > 1$ depends only on the data, and κ, r, $c_{3.15}$ are defined in (3.14) and Lemma 3.6. Now we set $p = 2^k$, $k \in \mathbb{N}$, $k \geq 3$. The iteration formula yields

$$\gamma_k \leq \left(2^{4r} \left(\kappa + c_{3.15}^8\right) T\, \overline{c}\, \gamma_2\right)^{c_0 2^k} , \quad \gamma_k = \sum_{i=1}^l \sup_{s\in S} \|z_i(s)\|_{L^{2^k}}^{2^k} + 1 , \quad c_0 = \prod_{j=1}^\infty \frac{2^j}{2^j - 1} .$$

Passing to the limit $k \to \infty$ we obtain

$$\sum_{i=1}^l \|z_i(t)\|_{L^\infty} \leq \sqrt{l} \left(2^{4r} \left(\kappa + c_{3.15}^8\right) T\, \overline{c} \left(\sum_{i=1}^l \sup_{s\in S} \|z_i(s)\|_{L^4}^4 + 1 \right) \right)^{c_0} \qquad \forall t \in S .$$

With Lemma 3.6 and (4.2) the desired estimates for b_i, $i = 1, \ldots, l$, are verified.

2. *Bounds for the immobile species.* Now, let $p \geq 2$. We use the test function $p(0,\dots,0,z_{l+1}^{p-1},\dots,z_m^{p-1})$. Taking into account the assumptions (2.7), (2.5), the estimates for b_i, $i=1,\dots,l$, obtained in step 1 as well as the inequalities $b_k \geq z_k \geq 0$ we find that

$$\begin{aligned}\sum_{k=l+1}^m r_k^\Omega(\cdot,b,\psi)\, z_k^{p-1} &\leq c\sum_{i=1}^l \sum_{k=l+1}^m (b_i^2+b_i+1) z_k^{p-1} - \tilde\epsilon \sum_{k=l+1}^m z_k^{p+1} \\ &\leq \tilde c \sum_{k=l+1}^m z_k^{p-1} - \tilde\epsilon \sum_{k=l+1}^m z_k^{p+1} \leq (m-l)\frac{\tilde c^{(p+1)/2}}{\tilde\epsilon^{(p-1)/2}}\,.\end{aligned}$$

The last estimate follows from Young's inequality. Therefore we obtain

$$\epsilon_0 \sum_{k=l+1}^m \|z_k(t)\|_{L^p}^p \leq pT|\Omega|(m-l)\,\frac{\tilde c^{(p+1)/2}}{\tilde\epsilon^{(p-1)/2}} \quad \forall t\in S\,.$$

And consequently,

$$\|z_k(t)\|_{L^p} \leq (pT|\Omega|(m-l)/\epsilon_0)^{1/p}\ \frac{\tilde c^{(p+1)/2p}}{\tilde\epsilon^{(p-1)/2p}} \leq c_\infty\ \forall t\in S\,,\ k=l+1,\dots,m\,.$$

Passing to the limit $p\to\infty$ we get $\|z_k(t)\|_{L^\infty} \leq c_\infty$ for all $t\in S$, $k=l+1,\dots,m$, which leads to the desired estimates for b_k, $k=l+1,\dots,m$. ∎

3.5. Second regularized problem ($\mathcal{P}_M$). We prove the solvability of $(\mathcal{P}_N)$ for fixed $N>0$ by means of a second regularization $(\mathcal{P}_M)$.

Let $M \geq M^* = \max\{N+1, \max_{i=1,\dots,m}\|U_i/p_{0i}\|_{L^\infty}\}$. We denote by σ_M the projection from $\mathbb{R}$ onto $[-M,M]$,

$$\sigma_M(y) = \operatorname{sign}(y)\min\{|y|,M\}\,,\ y\in\mathbb{R}\,.$$

Moreover, we introduce the functions

$$D_{iM}(x,b,\psi) = D_i(x,b^+,\sigma_M(\psi))\,,\ i=1,\dots,l\,,\ x\in\Omega\,,\ b\in\mathbb{R}^m\,,\ \psi\in\mathbb{R}\,,$$

define the operator $A_M\colon X\times H^1(\Omega)\to X^*$,

$$\langle A_M(b,\psi),\overline{b}\rangle_X = \int_\Omega \sum_{i=1}^l D_{iM}(\cdot,b,\psi)p_{0i}(\nabla b_i + [\sigma_M(b_i)]^+ Q_i(\psi)\nabla\psi)\cdot\nabla\overline{b}_i\,\mathrm{d}x\,,\ \overline{b}\in X\,,$$

and consider the following problem:

$$(\mathcal{P}_M)\qquad \left\{\begin{array}{c} u'(t) + A_M(b(t),\psi(t)) = R_N(b^+(t),\psi(t))\,, \\ E(\psi(t),u^+(t)) = 0\,,\quad u(t) = Bb(t)\quad \text{f.a.a. } t\in S\,,\quad u(0)=U\,, \\ u\in H^1(S,X^*)\cap L^2(S,Y)\,,\ b\in L^2(S,X)\,,\ \psi\in L^2(S,H^1(\Omega))\,. \end{array}\right.$$

Let us remark that we have u, $b\in C(S,Y)$, $\psi\in C(S,H^1(\Omega))$ for solutions of $(\mathcal{P}_M)$.

3.6. Existence result for $(\mathcal{P}_M)$. First we derive an equivalent formulation of $(\mathcal{P}_M)$. We write b in the form $b=(v,w)$, $v=(b_1,\dots,b_l)$, $w=(b_{l+1},\dots,b_m)$ and introduce the spaces

$$Y^l=L^2(\Omega,\mathbb{R}^l)\,,\ Y^{m-l}=L^2(\Omega,\mathbb{R}^{m-l})\,,\ X^l=H^1(\Omega,\mathbb{R}^l)\,,$$

and the operators $B_v\colon L^2(S,Y^l)\to L^2(S,Y^l)$, $B_w\colon L^2(S,Y^{m-l})\to L^2(S,Y^{m-l})$ by

$$\langle (B_v v)(t),\overline{v}\rangle_{Y^l}=\int_\Omega\sum_{i=1}^{l}p_{0i}\,v_i(t)\,\overline{v}_i\,\mathrm{d}x\,,\quad \overline{v}\in Y^l\,,$$

$$\langle (B_w w)(t),\overline{w}\rangle_{Y^{m-l}}=\int_\Omega\sum_{i=1}^{m-l}p_{0(l+i)}\,w_i(t)\,\overline{w}_i\,\mathrm{d}x,\quad \overline{w}\in Y^{m-l}\,,\quad t\in S\,.$$

Moreover, we define the operators

$$\begin{aligned}
&R_v\colon L^2(S,X^l)\times L^2(S,Y^{m-l})\times L^2(S,H^1(\Omega))\to L^2(S,X^{l*})\,,\\
&R_w\colon L^2(S,X^l)\times L^2(S,Y^{m-l})\times L^2(S,H^1(\Omega))\to L^2(S,Y^{m-l})\,,\\
&A_v\colon L^2(S,X^l)\times L^2(S,X^l)\times L^2(S,Y^{m-l})\times L^2(S,H^1(\Omega))\to L^2(S,X^{l*})\,,\\
&A_v^0\colon L^2(S,X^l)\times L^2(S,Y^{m-l})\times L^2(S,H^1(\Omega))\to L^2(S,X^{l*})
\end{aligned}$$

as follows:

$$\langle R_v(v,w,\psi),\overline{v}\rangle_{L^2(S,X^l)}=\int_S\langle R_N(v^+,w^+,\psi),(\overline{v},0)\rangle_X\,\mathrm{d}s\,,\quad \overline{v}\in L^2(S,X^l)\,,$$

$$\langle R_w(v,w,\psi),\overline{w}\rangle_{L^2(S,Y^{m-l})}=\int_S\langle R_N(v^+,w^+,\psi),(0,\overline{w})\rangle_X\,\mathrm{d}s\,,\quad \overline{w}\in L^2(S,Y^{m-l})\,,$$

$$\langle A_v(v;\widehat{v},w,\psi),\overline{v}\rangle_{L^2(S,X^l)}=\int_S\int_\Omega\sum_{i=1}^{l}(D_{iM}(\cdot,\widehat{v},w,\psi)\,p_{0i}\,\nabla v_i\cdot\nabla\overline{v}_i+v_i\overline{v}_i)\,\mathrm{d}x\,\mathrm{d}s\,,$$

$$\begin{aligned}\langle A_v^0(v,w,\psi),\overline{v}\rangle_{L^2(S,X^l)}=\int_S\int_\Omega\sum_{i=1}^{l}&(D_{iM}(\cdot,v,w,\psi)\,p_{0i}\,[\sigma_M(v_i)]^+Q_i(\psi)\nabla\psi\cdot\nabla\overline{v}_i\\
&-v_i\overline{v}_i)\,\mathrm{d}x\,\mathrm{d}s\,,\quad \overline{v}\in L^2(S,X^l)\,.\end{aligned}$$

For any given $v\in L^2(S,Y^l)$, $w\in L^2(S,Y^{m-l})$ we have that $(B_v v,B_w w)\in L^2(S,Y)$ and by Lemma 3.1 we find a unique $\psi\in L^2(S,H^1(\Omega))\cap L^\infty(S,L^\infty(\Omega))$ such that

$$E(\psi(t),((B_v v)^+(t),(B_w w)^+(t)))=0\quad \text{f.a.a. } t\in S\,.$$

We denote by $\mathcal{T}_\psi\colon L^2(S,Y^l)\times L^2(S,Y^{m-l})\to L^2(S,H^1(\Omega))$ the corresponding solution operator, $\psi=\mathcal{T}_\psi(v,w)$. Problem $(\mathcal{P}_M)$ is obviously equivalent to the following system of equations:

$$\tag{3.17}\begin{gathered}(B_v v)'+A_v(v;v,w,\mathcal{T}_\psi(v,w))=R_v(v,w,\mathcal{T}_\psi(v,w))-A_v^0(v,w,\mathcal{T}_\psi(v,w))\,,\\
(B_v v)(0)=U_v=(U_1,\dots,U_l)\,,\quad v\in W^l\end{gathered}$$

where $W^l=\big\{v\in L^2(S,X^l)\colon(B_v v)'\in L^2(S,X^{l*})\big\}\subset C(S,Y^l)$,

$$\tag{3.18}\begin{gathered}(B_w w)'=R_w(v,w,\mathcal{T}_\psi(v,w))\,,\\
(B_w w)(0)=U_w=(U_{l+1},\dots,U_m)\,,\quad B_w w\in H^1(S,Y^{m-l})\,.\end{gathered}$$

Let us shortly outline how these equations will be solved. We start with some fixed $\widehat{v} \in W^l$. First we solve the initial value problem

$$(3.19)\quad (B_w w)' = R_w(\widehat{v}, w, \mathcal{T}_\psi(\widehat{v}, w)),\ (B_w w)(0) = U_w,\ B_w w \in H^1(S, Y^{m-l}).$$

This problem has a unique solution $w = \mathcal{T}_w \widehat{v}$ (see Lemma 3.8). Next we solve the initial boundary value problem

$$(3.20)\quad \begin{aligned} (B_v v)' + A_v(v; \widehat{v}, \mathcal{T}_w\widehat{v}, \mathcal{T}_\psi(\widehat{v}, \mathcal{T}_w\widehat{v})) &= R_v(\widehat{v}, \mathcal{T}_w\widehat{v}, \mathcal{T}_\psi(\widehat{v}, \mathcal{T}_w\widehat{v})) \\ &\quad - A_v^0(\widehat{v}, \mathcal{T}_w\widehat{v}, \mathcal{T}_\psi(\widehat{v}, \mathcal{T}_w\widehat{v})), \\ (B_v v)(0) &= U_v,\ v \in W^l. \end{aligned}$$

Also this problem has a unique solution $v = \mathcal{Q}\widehat{v}$ (see Lemma 4.2). The operator $\mathcal{Q}$ is completely continuous (see Lemma 3.10). Using Schauder's Fixed Point Theorem we obtain a fixed point v of $\mathcal{Q}$ (see Lemma 3.11). Then $(v, \mathcal{T}_w v)$ is a solution of (3.17), (3.18).

Now we give the detailed proofs. The constants c in the estimates of this subsection can depend on M, N (and on T).

LEMMA 3.8. *For any $\widehat{v} \in W^l$ there exists a unique solution w of problem* (3.19), *and w belongs to $H^1(S, Y^{m-l}) \subset C(S, Y^{m-l})$.*

Proof. Let $\widehat{v} \in W^l$ be fixed. The initial value problem (3.19) is obviously equivalent to the initial value problem

$$(3.21)\qquad w' + G\,w = 0, \quad w(0) = (B_w)^{-1}U_w, \quad w \in H^1(S, Y^{m-l})$$

where $G: L^2(S, Y^{m-l}) \to L^2(S, Y^{m-l})$ is defined by

$$G\,w = -(B_w)^{-1}[R_w(\hat{v}, w, \mathcal{T}_\psi(\widehat{v}, w))],\ w \in L^2(S, Y^{m-l}).$$

For $\widehat{v} \in L^2(S, Y^l)$, $w \in L^2(S, Y^{m-l})$ we have $\mathcal{T}_\psi(\widehat{v}, w) \in L^2(S, H^1(\Omega))$ and

$$\|\mathcal{T}_\psi(\widehat{v}, w^1) - \mathcal{T}_\psi(\widehat{v}, w^2)\|_{L^2(S,H^1)} \le c\|w^1 - w^2\|_{L^2(S,Y^{m-l})} \quad \forall w^1,\, w^2 \in L^2(S, Y^{m-l}).$$

Since the functions $r_i^\Omega(x, \cdot, \cdot)$ are uniformly Lipschitz continuous the estimate

$$\|G\,w^1 - G\,w^2\|_{L^2(S,Y^{m-l})} \le L\|w^1 - w^2\|_{L^2(S,Y^{m-l})} \quad \forall w^1,\, w^2 \in L^2(S, Y^{m-l})$$

follows, and [6, Chap. V, Theorem 1.3] ensures the existence of a unique solution of (3.21). In principle, this result was obtained by means of the Banach Fixed Point Theorem. □

We denote by $\mathcal{T}_w: W^l \to H^1(S, Y^{m-l})$ the operator which assigns to $\widehat{v}$ the solution w of (3.19).

LEMMA 3.9. *There exists a constant $c > 0$ such that the following estimates hold:*

$$\begin{aligned} \|\mathcal{T}_w\widehat{v}^1 - \mathcal{T}_w\widehat{v}^2\|_{C(S,Y^{m-l})} &\le c\|\widehat{v}^1 - \widehat{v}^2\|_{L^2(S,Y^l)} \quad \forall\, \widehat{v}^1, \widehat{v}^2 \in W^l, \\ \|\mathcal{T}_w\widehat{v}\|_{C(S,Y^{m-l})} &\le c \quad \forall\, \widehat{v} \in W^l. \end{aligned}$$

Proof. Let $w^k = \mathcal{T}_w\widehat{v}^k$, $k = 1, 2$. Using the test function $\overline{w} = w^1 - w^2$ for the corresponding problems (3.19), the Lipschitz continuity of r_i^Ω, Hölder's inequality and

Lemma 3.1 we find that

$$
\begin{aligned}
&\|\overline{w}(t)\|^2_{Y^{m-l}} \le c\|(B_w(\overline{w}(t))\|^2_{Y^{m-l}} \\
&\le c\int_0^t \left(\|\overline{w}\|_{Y^{m-l}} + \|\widehat{v}^1-\widehat{v}^2\|_{Y^l} + \|\mathcal{T}_\psi(\widehat{v}^1,w^1)-\mathcal{T}_\psi(\widehat{v}^2,w^2)\|_{H^1}\right)\|\overline{w}\|_{Y^{m-l}}\,\mathrm{d}s \\
&\le c\int_0^t \left(\|\overline{w}\|^2_{Y^{m-l}} + \|\widehat{v}^1-\widehat{v}^2\|^2_{Y^l}\right)\,\mathrm{d}s\,.
\end{aligned}
$$

Gronwall's Lemma leads to the first assertion. Next, testing (3.19) with $w=\mathcal{T}_w\widehat{v}$ and using (3.5) the estimate

$$
\|w(t)\|^2_{Y^{m-l}} \le c + c\int_0^t \|w(s)\|_{Y^{m-l}}\,\mathrm{d}s \le c + \int_0^t \|w(s)\|^2_{Y^{m-l}} \quad \forall t\in S
$$

follows where c does not depend on $\widehat{v}$. Again applying Gronwall's Lemma the second assertion is obtained. $\square$

Next we conclude that for given $\widehat{v}\in W^l$ the initial boundary value problem (3.20) has a unique solution. This follows from Lemma 4.2 since B_v and A_v are diagonal and the right hand side belongs to $L^2(S,X^{l*})$. We denote by $\mathcal{Q}\colon W^l\to W^l$ the operator which assigns to $\widehat{v}$ the solution v of (3.20).

LEMMA 3.10. *The mapping $\mathcal{Q}$ is completely continuous.*

Proof. Let $\{\widehat{v}_n\}\subset W^l$ be bounded. Because of Lemma 4.1 v) we may assume that there exists an element $\widehat{v}\in W^l$ such that $\widehat{v}_n\to\widehat{v}$ in $L^2(S,Y^l)$ as well as in $L^2(S,L^2(\Gamma,\mathbb{R}^l))$. Let

$$
v_n=\mathcal{Q}\widehat{v}_n\,,\ v=\mathcal{Q}\widehat{v}\,,\ w_n=\mathcal{T}_w\widehat{v}_n\,,\ w=\mathcal{T}_w\widehat{v}\,,\ \psi_n=\mathcal{T}_\psi(\widehat{v}_n,w_n)\,,\ \psi=\mathcal{T}_\psi(\widehat{v},w)\,.
$$

From Lemma 3.9 and Lemma 3.1 it follows that

$$
w_n\to w \text{ in } L^2(S,Y^{m-l})\,,\ \psi_n\to\psi \text{ in } L^2(S,H^1)\,.
$$

Using the test function v_n-v we obtain

$$
\begin{aligned}
&\frac{\epsilon_0}{2}\|(v_n-v)(t)\|^2_{Y^l} + \int_0^t \epsilon\|v_n-v\|^2_{X^l}\,\mathrm{d}s \\
&\le \int_0^t c\Bigg\{ \left(\|\widehat{v}_n-\widehat{v}\|_{L^2(\Gamma,\mathbb{R}^l)} + \|\psi_n-\psi\|_{L^2(\Gamma)}\right)\|v_n-v\|_{L^2(\Gamma,\mathbb{R}^l)} \\
&\qquad + \left(\|\widehat{v}_n-\widehat{v}\|_{Y^l} + \|w_n-w\|_{Y^{m-l}} + \|\psi_n-\psi\|_{L^2}\right)\|v_n-v\|_{Y^l} \\
&\qquad + \int_\Omega \sum_{i=1}^l \Big\{ \left(|\nabla v_i|+|\nabla\psi|\right)|D_{iM}(\cdot,\widehat{v}_n,w_n,\psi_n)-D_{iM}(\cdot,\widehat{v},w,\psi)|\,|\nabla(v_{ni}-v_i)| \\
&\qquad\quad + \left(|Q_i(\psi_n)-Q_i(\psi)| + |[\sigma_M(\widehat{v}_{ni})]^+ - [\sigma_M(\widehat{v}_i)]^+|\right)|\nabla\psi|\,|\nabla(v_{ni}-v_i)| \\
&\qquad\quad + |\nabla(\psi_n-\psi)|\,|\nabla(v_{ni}-v_i)|\Big\}\,\mathrm{d}x\Bigg\}\,\mathrm{d}s \quad \forall t\in S\,.
\end{aligned}
$$

Applying Hölder's inequality and Lemma 3.9 we arrive at

$$
\begin{aligned}
&\|v_n - v\|^2_{L^2(S,X^l)} \\
&\le c\|v_n - v\|_{L^2(S,X^l)} \Big\{ \|\widehat{v}_n - \widehat{v}\|_{L^2(S,Y^l)} + \|\widehat{v}_n - \widehat{v}\|_{L^2(S,L^2(\Gamma,\mathbb{R}^l))} + \|\psi_n - \psi\|_{L^2(S,H^1)} \\
&+ \sum_{i=1}^{l} \Big\{ \Big[\int_0^T \int_\Omega |D_{iM}(\cdot,\widehat{v}_n,w_n,\psi_n) - D_{iM}(\cdot,\widehat{v},w,\psi)|^2\, |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |D_{iM}(\cdot,\widehat{v}_n,w_n,\psi_n) - D_{iM}(\cdot,\widehat{v},w,\psi)|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |Q_i(\psi_n) - Q_i(\psi)|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |[\sigma_M(\widehat{v}_{ni})]^+ - [\sigma_M(\widehat{v}_i)]^+|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \Big\} \Big\}.
\end{aligned}
$$

Properties of superposition operators ensure that the last four bracket terms tend to zero if $n \to \infty$. Thus in summary we find that $v_n \to v$ in $L^2(S,X^l)$. Next we obtain

$$
\begin{aligned}
\|(B_v v_n)' - (B_v v)'\|_{L^2(S,X^{l*})} \le{}& \|R_v(\widehat{v}_n,w_n,\psi_n) - R_v(\widehat{v},w,\psi)\|_{L^2(S,X^{l*})} \\
&+ \|A_v(v_n;\widehat{v}_n,w_n,\psi_n) - A_v(v;\widehat{v},w,\psi)\|_{L^2(S,X^{l*})} \\
&+ \|A_v^0(\widehat{v}_n,w_n,\psi_n) - A_v^0(\widehat{v},w,\psi)\|_{L^2(S,X^{l*})}
\end{aligned}
$$

$$
\begin{aligned}
&\le c \Big\{ \|v_n - v\|_{L^2(S,X^l)} + \|\widehat{v}_n - \widehat{v}\|_{L^2(S,Y^l)} + \|\widehat{v}_n - \widehat{v}\|_{L^2(S,L^2(\Gamma,\mathbb{R}^l))} \\
&+ \|w_n - w\|_{L^2(S,Y^{m-l})} + \|\psi_n - \psi\|_{L^2(S,H^1)} \\
&+ \sum_{i=1}^{l} \Big\{ \Big[\int_0^T \int_\Omega |D_{iM}(\cdot,\widehat{v}_n,w_n,\psi_n) - D_{iM}(\cdot,\widehat{v},w,\psi)|^2\, |\nabla v_i|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |D_{iM}(\cdot,\widehat{v}_n,w_n,\psi_n) - D_{iM}(\cdot,\widehat{v},w,\psi)|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |Q_i(\psi_n) - Q_i(\psi)|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \\
&\qquad + \Big[\int_0^T \int_\Omega |[\sigma_M(\widehat{v}_{ni})]^+ - [\sigma_M(\widehat{v}_i)]^+|^2\, |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}s \Big]^{1/2} \Big\} \Big\} \to 0 \text{ for } n \to \infty
\end{aligned}
$$

and we arrive at $v_n \to v$ in W^l. The continuity of the operator $\mathcal{Q}$ can be shown by similar arguments. ∎

LEMMA 3.11. *The mapping $\mathcal{Q}$ has a fixed point.*

Proof. Let $\widehat{v} \in W^l$, $\psi = \mathcal{T}_\psi(\widehat{v}, \mathcal{T}_w \widehat{v})$ and $v = \mathcal{Q}\widehat{v}$. We use v as test function for (3.20), take into account the properties of D_{iM}, Q_i, apply Lemma 3.1, (4.1), Lemma 3.9, the boundedness of r_i^Σ and Young's inequality. Thus we obtain

$$
\begin{aligned}
&\epsilon_0\|v(t)\|_{Y^l}^2 - c\|(U_1,\dots,U_l)\|_{Y^l}^2 + 2\epsilon\int_0^t \|v\|_{X^l}^2\,\mathrm{d}s\\
&\le c\int_0^t \left(1+\|v\|_{Y^l}^2+\|\widehat{v}\|_{Y^l}^2+\|\psi\|_{H^1}\|v\|_{X^l}+\|v\|_{L^2(\Gamma,\mathbb{R}^l)}^2\right)\mathrm{d}s \qquad (3.22)\\
&\le \int_0^t \left(\epsilon\|v\|_{X^l}^2 + c\left(1+\|v\|_{Y^l}^2+\|\widehat{v}\|_{Y^l}^2\right)\right)\mathrm{d}s \quad \forall t\in S\,.
\end{aligned}
$$

Therefore we find a constant $\overline{c}>0$ such that for all $k>0$

$$
\begin{aligned}
&e^{-kt}\left(\|v(t)\|_{Y^l}^2+\int_0^t\|v\|_{X^l}^2\,\mathrm{d}s\right)\\
&\le \overline{c}+\overline{c}e^{-kt}\int_0^t\left\{\left\{\|v\|_{Y^l}^2+\|\widehat{v}\|_{Y^l}^2+\int_0^s\left(\|v\|_{X^l}^2+\|\widehat{v}\|_{X^l}^2\right)\mathrm{d}\tau\right\}e^{-ks}e^{ks}\right\}\mathrm{d}s\\
&\le \overline{c}+\overline{c}e^{-kt}\sup_{s\in S}\left\{\left\{\|v(s)\|_{Y^l}^2+\|\widehat{v}(s)\|_{Y^l}^2+\int_0^s\left(\|v\|_{X^l}^2+\|\widehat{v}\|_{X^l}^2\right)\mathrm{d}\tau\right\}e^{-ks}\right\}\frac{e^{kt}-1}{k}\,.
\end{aligned}
$$

Choosing now $k\ge 3\overline{c}$ we obtain

$$
\begin{aligned}
&\sup_{t\in S} e^{-kt}\left(\|v(t)\|_{Y^l}^2+\int_0^t\|v(s)\|_{X^l}^2\,\mathrm{d}s\right)\\
&\le \frac{3}{2}\overline{c}+\frac{1}{2}\sup_{t\in S}\left\{e^{-kt}\left(\|\widehat{v}(t)\|_{Y^l}^2+\int_0^t\|\widehat{v}(s)\|_{X^l}^2\,\mathrm{d}s\right)\right\}\,.
\end{aligned}
$$

Again using Lemma 3.1 and Lemma 3.9 we estimate

$$
\begin{aligned}
&\|(B_v v)'\|_{L^2(S,X^{l*})}\\
&=\sup_{\|\overline{v}\|_{L^2(S,X^l)}\le 1}\langle R_v(\widehat{v},\mathcal{T}_w\widehat{v},\psi)-A_v(v;\widehat{v},\mathcal{T}_w\widehat{v},\psi)-A_v^0(\widehat{v},\mathcal{T}_w\widehat{v},\psi),\overline{v}\rangle_{L^2(S,X^l)}\\
&\le c\left(\|v\|_{L^2(S,X^l)}+\|\psi\|_{L^2(S,H^1)}+1\right)\le c\left(\|v\|_{L^2(S,X^l)}+\|\widehat{v}\|_{L^2(S,Y^l)}+1\right)\\
&\le \widetilde{c}\left(\|v\|_{L^2(S,X^l)}+\left[\sup_{t\in S}\left\{e^{-kt}\left(\|\widehat{v}(t)\|_{Y^l}^2+\int_0^t\|\widehat{v}(s)\|_{X^l}^2\,\mathrm{d}s\right)\right\}e^{kT}\right]^{1/2}+1\right)\,.
\end{aligned}
$$

Now we define the set

$$
\begin{aligned}
\mathcal{B}=\Bigg\{v\in W^l\colon &\sup_{t\in S}\left\{e^{-kt}\left(\|v(t)\|_{Y^l}^2+\int_0^t\|v\|_{X^l}^2\,\mathrm{d}s\right)\right\}\le 3\overline{c}\,,\\
&\|(B_v v)'\|_{L^2(S,X^{l*})}\le\widetilde{c}\left(2\sqrt{3\overline{c}e^{kT}}+1\right)\Bigg\}\,.
\end{aligned}
$$

This set is a non-empty, bounded, closed and convex subset of W^l with the property that $\mathcal{Q}(\mathcal{B})\subset\mathcal{B}$. Since the mapping $\mathcal{Q}$ is completely continuous the assertion follows from the Schauder Fixed Point Theorem. ∎

THEOREM 3.12. *There exists a solution* (u, b, ψ) *of* $(\mathcal{P}_M)$.

Proof. Because of Lemma 3.11 there exists a solution v of the problem

$$(B_v v)' + A_v(v; v, \mathcal{T}_w v, \mathcal{T}_\psi(v, \mathcal{T}_w v)) = R_v(v, \mathcal{T}_w v, \mathcal{T}_\psi(v, \mathcal{T}_w v)) - A_v^0(v, \mathcal{T}_w v, \mathcal{T}_\psi(v, \mathcal{T}_w v)),$$

$$(B_v v)(0) = (U_1, \dots, U_l)\,, \quad v \in W^l\,.$$

We set $w = \mathcal{T}_w v \in H^1(S, Y^{m-l})$. Then the pair (v, w) fulfils the equations (3.17) and (3.18) which represent an equivalent formulation of problem $(\mathcal{P}_M)$. □

3.7. Energy estimates for solutions of $(\mathcal{P}_M)$. First, we proof

LEMMA 3.13. *For any solution* (u, b, ψ) *of* $(\mathcal{P}_M)$ *and for every* $t \in S$ *the inequalities* $b(t)\,, u(t) \geq 0$ *a.e. on* Ω *are fulfilled.*

Proof. Let (u, b, ψ) be a solution of $(\mathcal{P}_M)$. We test the evolution equation with the function $-b^-$. Taking into account that

$$(\nabla b_i + [\sigma_M(b_i)]^+ Q_i(\psi) \nabla \psi) \cdot \nabla b_i^- \leq 0\,, \quad i = 1, \dots, l\,,$$

$$-r_i^\Sigma(\cdot, b_1^+, \dots, b_{m_\Sigma}^+, \psi)\, b_i^- \leq 0\,, \quad i = 1, \dots, m_\Sigma\,,\ \Sigma = \Omega, \Gamma\,,$$

we find that $\|b^-(t)\|_Y^2 \leq 0$ for all $t \in S$. □

Next, we introduce a regularized free energy functional F_M which is adapted to the regularizations in problem $(\mathcal{P}_M)$. We define the function

$$l_M(y) = \begin{cases} \ln y & \text{if} \quad 0 < y \leq M\,, \\ \ln M - 1 + \dfrac{y}{M} & \text{if} \quad y > M\,, \end{cases}$$

and the functional $\widetilde{F}_{M2} : Y \to \overline{\mathbb{R}}$ by

$$\widetilde{F}_{M2}(u) = \begin{cases} \displaystyle\int_\Omega \sum_{i=1}^m \int_{p_{0i}}^{u_i} l_M(z/p_{0i})\, \mathrm{d}z\, \mathrm{d}x & \text{if} \quad u \in Y_+\,, \\ +\infty & \text{if} \quad u \in Y \setminus Y_+\,. \end{cases} \tag{3.23}$$

Moreover, we set

$$F_{M2} = (\widetilde{F}^*_{M2}|_X)^* : X^* \to \overline{\mathbb{R}}, \quad F_M = F_1 + F_{M2} : X^* \to \overline{\mathbb{R}}$$

where F_1 was introduced in Subsection 3.3. Since the function l_M has the same essential properties as the function ln which occurs in the definition of the functional F_2 similar arguments as in [12] give the following results.

LEMMA 3.14. *The functional* $F_M = F_1 + F_{M2} : X^* \to \overline{\mathbb{R}}$ *is proper, convex and lower semi-continuous. For* $u \in Y_+$ *it can be evaluated according to* (3.7), (3.23). *The restriction* $F_M|_{Y_+}$ *is continuous. If* $u \in Y_+$ *then* $P(\psi) \in \partial F_1(u)$ *where* ψ *is the solution of* $E(\psi, u) = 0$. *If* $u \in Y$, $u \geq \delta > 0$ *and* $u/p_0 \in X$ *then* $l_M(u/p_0) \in \partial F_{M2}(u)$ *where* $l_M(b)$ *denotes the vector* $\{l_M(b_i)\}_{i=1,\dots,m}$.

By the definition of l_M the inequality

$$\int_{p_{0i}}^{y} l_{Mi}(z/p_{0i})\, \mathrm{d}z \geq y \ln \frac{y}{p_{0i}} - y + p_{0i} \quad \text{a.e. on } \Omega\,,\ \forall y \in \mathbb{R}_+$$

holds. Therefore it follows that

$$F_M(u) \geq c_1 \left\{ \|\psi\|^2_{H^1} + \sum_{i=1}^m \|u_i \ln u_i\|_{L^1} \right\} - c_2\,,\ u \in Y_+\,. \tag{3.24}$$

LEMMA 3.15. *Along any solution (u, b, ψ) of $(\mathcal{P}_M)$ the regularized free energy $F_M(u)$ remains bounded by its initial value and decreases monotonously,*

$$F_M(u(t_2)) \leq F_M(u(t_1)) \leq F_M(U) = F(U)\,, \quad 0 \leq t_1 \leq t_2 \leq T\,.$$

Moreover, there exist positive constants c, $c_{3.25}$, $c_{3.26}$ not depending on M, such that

$$\sum_{i=1}^m \|u_i \ln u_i\|_{L^\infty(S,L^1(\Omega))} \leq c\,, \quad \|u\|_{L^\infty(S,L^1(\Omega,\mathbb{R}^m))} \leq c\,,$$

$$\sum_{i=1}^m \|b_i \ln b_i\|_{L^\infty(S,L^1(\Omega))} \leq c_{3.25}\,, \tag{3.25}$$

$$\|\psi\|_{L^\infty(S,H^1(\Omega))} \leq c\,, \quad \|\psi\|_{L^\infty(S,L^\infty(\Omega))}\,,\ \|\psi\|_{L^\infty(S,L^\infty(\Gamma))} \leq c_{3.26} \tag{3.26}$$

for any solution of $(\mathcal{P}_M)$.

Proof. Let (u, b, ψ) be a solution of $(\mathcal{P}_M)$.
1. We know that $u \in H^1(S, X^*)$, $\psi \in L^2(S, H^1(\Omega))$, $P(\psi) \in L^2(S, X)$, $\nabla P(\psi) = Q(\psi)\nabla\psi$. By Lemma 3.14 we find that $P(\psi(t)) \in \partial F_1(u(t))$ f.a.a. $t \in S$. Thus, the function $t \mapsto F_1(u(t))$ is absolutely continuous on S and according to the chain rule (see [3, Lemma 3.3]) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(u(t)) = \langle u'(t), P(\psi(t))\rangle_X \text{ f.a.a. } t \in S\,.$$

2. We choose some $\delta \in (0, 1)$ and define $u^\delta = u + \delta p_0$, $b^\delta = u^\delta/p_0 = b + \delta$. Then we find that $u^\delta \in H^1(S, X^*)$, $l_M(b^\delta) \in L^2(S, X)$, $\nabla l_M(b^\delta_i) = \nabla b_i/\sigma_M(b^\delta_i)\,,\ i = 1, \ldots, l$. Lemma 3.14 ensures that $l_M(b^\delta(t)) \in \partial F_{M2}(u^\delta(t))$ f.a.a. $t \in S$. Thus, the function $t \mapsto F_{M2}(u^\delta(t))$ is absolutely continuous on S and

$$\frac{\mathrm{d}}{\mathrm{d}t} F_{M2}(u^\delta(t)) = \big\langle u'(t), l_M(b^\delta(t))\big\rangle_X \text{ f.a.a. } t \in S\,.$$

3. We set $\zeta^\delta_M = l_M(b^\delta) + P(\psi)$ and obtain

$$\begin{aligned}\big[F_1(u(t)) + F_{M2}(u^\delta(t))\big]\big|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \langle u'(t), \zeta^\delta_M(t)\rangle_X \,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \langle R_N(b(t), \psi(t)) - A_M(b(t), \psi(t)), \zeta^\delta_M(t)\rangle_X \,\mathrm{d}t\,.\end{aligned}$$

The volume integral in the definition of $\langle R_N(b, \psi), \zeta^\delta_M\rangle_X$, namely

$$I = \int_\Omega \rho_N(b, \psi) \sum_{(\alpha,\,\beta)\in\mathcal{R}^\Omega} k^\Omega_{\alpha\beta}(\cdot, b, \psi)(a^\alpha - a^\beta) \sum_{i=1}^m (\beta_i - \alpha_i)\zeta^\delta_{Mi} \,\mathrm{d}x\,,\ a_i = b_i \mathrm{e}^{P_i(\psi)}\,,$$

is estimated as follows. Since for $|(b, \psi)|_\infty > N$ the integrand vanishes we may assume that $|(b, \psi)|_\infty \leq N$ and thus $b_i \leq N$, $b^\delta_i \leq N+1 \leq M$, $\zeta^\delta_{Mi} = \ln a^\delta_i$ with $a^\delta_i = b^\delta_i \mathrm{e}^{P_i(\psi)}$,

$i=1,\dots,m$, $|\psi|\le N$. Then we have

$$\big[(a^\delta)^\alpha-(a^\delta)^\beta\big]\sum\nolimits_{i=1}^m(\beta_i-\alpha_i)\ln a_i^\delta\le 0\,,$$
$$\Big|\big[a^\alpha-a^\beta-(a^\delta)^\alpha+(a^\delta)^\beta\big]\sum\nolimits_{i=1}^m(\beta_i-\alpha_i)\ln a_i^\delta\Big|\le c_N\delta(1+|\ln\delta|)$$

and $I\le c_N\delta(1+|\ln\delta|)$. The boundary integral is handled analogously and in summary we obtain

$$\langle R_N(b(t),\psi(t)),\zeta_M^\delta(t)\rangle_X\le h_1^\delta=c_N\delta(1+|\ln\delta|)\text{ f.a.a. }t\in S\,.$$

Next we consider the term $-\langle A_M(b,\psi),\zeta_M^\delta\rangle_X$, i.e. the integral

$$-\int_\Omega\sum_{i=1}^l D_{iM}(\cdot,b,\psi)p_{0i}(\nabla b_i+\sigma_M(b_i)Q_i(\psi)\nabla\psi)\cdot\nabla\zeta_{Mi}^\delta\,\mathrm{d}x\,.$$

Here we write

$$\nabla b_i+\sigma_M(b_i)Q_i(\psi)\nabla\psi=\sigma_M(b_i^\delta)\nabla\zeta_{Mi}^\delta+\big[\sigma_M(b_i)-\sigma_M(b_i^\delta)\big]\,Q_i(\psi)\nabla\psi$$

and in view of $D_{iM}(\cdot,b,\psi)\le c_M$ we obtain

$$-\langle A_M(b(t),\psi(t)),\zeta_M^\delta(t)\rangle_X\le h_2^\delta(t)\text{ f.a.a. }t\in S\,,$$
$$h_2^\delta(t)=c_M\int_\Omega\sum_{i=1}^l\delta|\nabla\psi(t)|\left[|\nabla\psi(t)|+\frac{1}{\sigma_M(b_i^\delta(t))}|\nabla b_i(t)|\right]\,\mathrm{d}x\,.$$

The last estimates ensure that

$$\big[F_1(u(t))+F_{M2}(u^\delta(t))\big]\big|_{t_1}^{t_2}\le\int_{t_1}^{t_2}\big(h_1^\delta+h_2^\delta(t)\big)\,\mathrm{d}t$$

and letting $\delta\to0$ the inequality $F_M(u(t_2))-F_M(u(t_1))\le 0$ follows. The choice of M guarantees that $F_M(U)=F(U)$. The remaining assertions of the lemma are a consequence of (3.24), Lemma 3.1 and (4.2). □

3.8. Further estimates for solutions of $(\mathcal{P}_M)$. We prove

THEOREM 3.16. *There is a constant $c_{3.27}>0$ not depending on M such that*

(3.27) $$\|b\|_{L^\infty(S,L^\infty(\Omega,\mathbb{R}^m))}\le c_{3.27}\,,\quad\|b_i\|_{L^\infty(S,L^\infty(\Gamma))}\le c_{3.27}\,,\quad i=1,\dots,l\,,$$

for any solution (u,b,ψ) of $(\mathcal{P}_M)$.

Proof. Let (u,b,ψ) be a solution of $(\mathcal{P}_M)$. Let $q>2$ be chosen as in Lemma 3.1, $r=2q/(q-2)$, $r'=2q/(2+q)$. Other constants in the following estimates can depend on N (and on T).

1. Testing $(\mathcal{P}_M)$ with $(0,\dots,0,b_{l+1},\dots,b_m)$ we obtain in view of (3.5) that

(3.28) $$\|b_i(t)\|_{L^2}\le c\quad\forall t\in S\,,\quad i=l+1,\dots,m\,,$$

which ensures that $\|u_i(t)\|_{L^{r'}}\le c$ for all $t\in S$, $i=l+1,\dots,m$. Hence we get

(3.29) $$\|\psi(t)\|_{W^{1,q}}\le c\left[1+\sum_{i=1}^m\|u_i(t)\|_{L^{r'}}\right]\le c\left[1+\sum_{i=1}^l\|b_i(t)\|_{L^{r'}}\right]\quad\forall t\in S\,.$$

2. Testing $(\mathcal{P}_M)$ with $2(b_1,\dots,b_l,0,\dots,0)$, estimating $[\sigma_M(b_i)]^+$ by b_i, using (3.5), (3.29), (4.1), (4.3), Youngs's inequality and Lemma 3.15 we find that

$$\begin{aligned}
&\sum_{i=1}^{l}\left(\epsilon_0\|b_i(t)\|_{L^2}^2 - c\|U_i\|_{L^2}^2\right)\\
&\le \int_0^t \sum_{i=1}^{l}\Big\{-2\epsilon\|b_i\|_{H^1}^2 + c(\|b_i\|_{L^r}\|\psi\|_{W^{1,q}}\|b_i\|_{H^1} + \|b_i\|_{L^2}^2 + \|b_i\|_{L^2(\Gamma)}^2 + 1)\Big\}\,\mathrm{d}s\\
&\le \int_0^t \sum_{i=1}^{l}\left\{-\epsilon\|b_i\|_{H^1}^2 + \overline{c}\|b_i\|_{L^r}\|b_i\|_{H^1}\sum_{k=1}^{l}\|b_k\|_{L^{r'}} + c\right\}\mathrm{d}s\,.
\end{aligned}$$

Using the inequality (4.4) for $p=2$ and Lemma 3.15 we have

$$\begin{aligned}
&\overline{c}\sum_{i=1}^{l}\|b_i\|_{L^r}\|b_i\|_{H^1}\sum_{k=1}^{l}\|b_k\|_{L^{r'}} \le \sum_{i=1}^{l}\left\{\frac{\epsilon}{2}\|b_i\|_{H^1}^2 + c\|b_i\|_{L^2}^2\sum_{k=1}^{l}\|b_k\|_{L^2}^2\right\}\\
&\le \sum_{i=1}^{l}\left\{\frac{\epsilon}{2}\|b_i\|_{H^1}^2 + \left[\frac{\sqrt{\epsilon}}{2c_{3.25}}\|b_i\ln b_i\|_{L^1}\|b_i\|_{H^1} + c\|b_i\|_{L^1}\right]^2\right\} \le \sum_{i=1}^{l}\epsilon\|b_i\|_{H^1}^2 + c\,.
\end{aligned}$$

The previous estimates and the inequalities (3.28), (3.29) ensure the existence of positive constants c, $\widetilde{\kappa}$ not depending on M such that

$$\|b_i(t)\|_{L^2} \le c\,,\ i=1,\dots,m\,,\ \|\psi(t)\|_{W^{1,q}}^{2r} + 1 \le \widetilde{\kappa}\quad \forall t\in S\,. \tag{3.30}$$

3. Following the estimates in the proofs of Lemma 3.6 and Theorem 3.7, but estimating $[\sigma_M(b_i)]^+$ by b_i and using $\widetilde{\kappa}$ from (3.30) instead of κ we find that

$$\begin{aligned}
&\|b_i(t)\|_{L^4} \le \widetilde{c}\,,\ i=1,\dots,m\,,\\
&\sum_{i=1}^{l}\|(b_i-K)^+(t)\|_{L^\infty} \le \sqrt{l}\left(2^{4r}\left(\widetilde{\kappa}+\widetilde{c}^8\right)\,cT\left(\sum_{i=1}^{l}\sup_{s\in S}\|(b_i-K)^+(s)\|_{L^4}^4 + 1\right)\right)^{c_0},\\
&\|(b_i-K)^+(t)\|_{L^\infty} \le c\,,\ i=l+1,\dots,m\,,\ \forall t\in S
\end{aligned}$$

where the constants K, c_0 have the same meaning as in the proof of Theorem 3.7. This provides the desired estimates. □

3.9. Existence result for $(\mathcal{P}_N)$.

THEOREM 3.17. *There exists a solution of* $(\mathcal{P}_N)$.

Proof. We choose $\overline{M} = \max\{M^*, c_{3.26}, c_{3.27}\}$ (cf. Lemma 3.15, Theorem 3.16). By Theorem 3.12 there is a solution (u,b,ψ) of $(\mathcal{P}_{\overline{M}})$. Since $b\ge 0$ (cf. Lemma 3.13) and

$$\begin{aligned}
&\|\psi\|_{L^\infty(S,L^\infty(\Omega))}\,,\ \|\psi\|_{L^\infty(S,L^\infty(\Gamma))} \le \overline{M}\,,\\
&\|b_i\|_{L^\infty(S,L^\infty)} \le \overline{M}\,,\ i=1,\dots,m\,,\ \|b_i\|_{L^\infty(S,L^\infty(\Gamma))} \le \overline{M}\,,\ i=1,\dots,l\,,
\end{aligned}$$

(cf. Lemma 3.15 and Theorem 3.16) this solution is a solution of $(\mathcal{P}_N)$, too. □

3.10. Existence result for $(\mathcal{P})$. *Proof* of Theorem 2.2. It suffices to prove the existence of a solution of $(\mathcal{P})$ on any finite time interval $S=[0,T]$. Such problems are denoted by $(\mathcal{P}_S)$. We choose $\overline{N} = 2\max\{c_{3.11}, c_{3.16}\}$ (cf. Lemma 3.3, Theorem 3.7). Then according to Theorem 3.17 there is a solution (u,b,ψ) of $(\mathcal{P}_{\overline{N}})$. The choice of $\overline{N}$ guarantees that the operators $R_{\overline{N}}$ and R coincide on this solution. Therefore (u,b,ψ) is a solution of $(\mathcal{P}_S)$, too. □

4. Appendix. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitzian domain. We apply Sobolev's imbedding theorems (see [17]) and some other imbedding results, especially the trace inequalities

$$\|w\|^q_{L^q(\Gamma)} \le c_\Omega\, q\, \|w\|^{q-1}_{L^{2(q-1)}(\Omega)} \|w\|_{H^1(\Omega)} \quad \forall w \in H^1(\Omega)\,,\ q \ge 2\,, \tag{4.1}$$

$$\|w\|_{L^\infty(\Gamma)} \le \|w\|_{L^\infty(\Omega)} \quad \forall w \in H^1(\Omega) \cap L^\infty(\Omega)\,, \tag{4.2}$$

and the following version of the Gagliardo–Nirenberg inequality (see [5, 21])

$$\|w\|_{L^p} \le c_{p,k}\, \|w\|^{k/p}_{L^k} \|w\|^{1-k/p}_{H^1} \quad \forall w \in H^1(\Omega)\,,\ 1 \le k < p < \infty\,. \tag{4.3}$$

As an extended form of the this inequality one obtains that for any $\delta > 0$ and any $p \in (1, \infty)$ there exists a constant $c_{\delta,p} > 0$ such that

$$\|w\|^p_{L^p} \le \delta\, \|w \ln |w|\|_{L^1}\, \|w\|^{p-1}_{H^1} + c_{\delta,p}\, \|w\|_{L^1} \quad \forall w \in H^1(\Omega)\,. \tag{4.4}$$

This inequality is proven in [2] for bounded domains with smooth boundary and $p = 3$. But (4.4) is valid also for bounded Lipschitzian domains and $p \in (1, \infty)$, since (4.3) is true in this case, too.

Let $p_0 \in L^\infty(\Omega)$, $\operatorname{ess\,inf}_{x\in\Omega} p_0(x) > 0$. We define $B\colon L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$(Bw, \overline{w})_{L^2} = \int_\Omega p_0\, w\, \overline{w} \mathrm{d}x \quad \overline{w} \in L^2(\Omega)\,.$$

Let $S = [0, T]$ be a compact interval. The extended operator $B\colon L^2(S, L^2(\Omega)) \longrightarrow L^2(S, L^2(\Omega))$ is defined by $(Bw)(t) = B(w(t))$ f.a.a. $t \in S$. Because of properties of p_0 the operator B is linear, continuous, self-adjoint, positive definite, and it exists the inverse operator $B^{-1}\colon L^2(S, L^2(\Omega)) \longrightarrow L^2(S, L^2(\Omega))$ with the same properties. Let

$$W_B = \Big\{ w \in L^2(S, H^1)\colon (Bw)' \in L^2(S, H^{1*}) \Big\}\,.$$

The following assertions can be verified as in [6, 18, 22]

Lemma 4.1.

i) *Equipped with the scalar product*

$$(w, \overline{w})_{W_B} = (w, \overline{w})_{L^2(S,H^1)} + ((Bw)', (B\overline{w})')_{L^2(S,H^{1*})}$$

the linear space W_B is a Hilbert space.

ii) *W_B is continuously embedded in $C(S, L^2(\Omega))$.*

iii) *The operator $B\colon W_B \to C(S, L^2(\Omega))$ is continuous.*

iv) *For $w \in W_B$ and t_1, $t_2 \in S$ the formula*

$$\int_{t_1}^{t_2} \big\langle (Bw)'(s), w(s) \big\rangle_{H^1} \mathrm{d}s = \frac{1}{2}\, ((Bw)(t_2), w(t_2))_{L^2} - \frac{1}{2}\, ((Bw)(t_1), w(t_1))_{L^2}$$

holds.

v) *The imbeddings of W_B in $L^2(S, L^2(\Omega))$ and in $L^2(S, L^2(\Gamma))$, respectively, are compact.*

Finally, the following existence result can be obtained as in [6, Chapt. VI].

LEMMA 4.2. *Let* $A: L^2(S, H^1(\Omega)) \to L^2(S, H^1(\Omega)^*)$ *be the operator*

$$\langle Aw, \overline{w}\rangle_{L^2(S,H^1)} = \int_0^T \int_\Omega (a\,\nabla w \cdot \nabla \overline{w} + d\,w\,\overline{w})\,\mathrm{d}x\mathrm{d}s\,,\ \ w,\, \overline{w} \in L^2(S, H^1(\Omega))\,,$$

where $a,\, d \in L^\infty(S \times \Omega)$ *with* $a(t,x),\, d(t,x) \geq c > 0$ *a.e. on* $S \times \Omega$*. Then for every* $f \in L^2(S, H^1(\Omega)^*)$ *and every* $U \in L^2(\Omega)$ *there is a unique solution of the problem*

$$(Bw)' + Aw = f\,, \quad (Bw)(0) = U\,, \quad w \in W_B\,.$$

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