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Space-time regularity of catalytic super-Brownian motion

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Abstract

We focus on the question, for which catalysts does the catalytic super-Brownian motion in $\mathbb{R}^1$ have a jointly continuous space-time Lebesgue density? As it turns out, nearly all non-atomic catalysts provide such a regular density which can be characterized as the unique solution to a stochastic partial differential equation driven by a degenerated space-time white noise.

1 Introduction

The goal of this paper is a condition on the catalyst of a catalytic super-Brownian motion in $\mathbb{R}^1$ implying the existence of a jointly continuous space-time Lebesgue density for the latter process. Before posing the problem in detail, we briefly recall the notion of super-Brownian motion (SBM) and catalytic SBM.

1.1 Ordinary super-Brownian motion

The so-called (ordinary) SBM in $\mathbb{R}^d$ is the high-density/short-lifetime measure-valued diffusion limit of a branching Brownian particle system in $\mathbb{R}^d$ where the critical branchings occur independently of space and time. The first rigorous treatment of this object is essentially due to Watanabe ([Wat68]) and Dawson ([Daw75]), whereby SBM is often called *Dawson-Watanabe process*. They characterized SBM as a Markov process whose states are measures on $\mathbb{R}^d$ and which is uniquely determined by the duality to a certain non-linear partial differential equation, the so-called cumulant equation. In Section 2.3 this duality will be posed in a more general setting. For a wide overview we refer to [Daw93].

Alternatively, one can approach SBM using stochastic calculus. Meanwhile it is a well-known fact that the SBM in $\mathbb{R}^1$ can be linked with the following stochastic partial differential equation (spde)

$$\partial_t X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}_{x,t} \qquad (1.1)$$

where Δ is the Laplacian and W a space-time white noise, i.e., more precisely, an orthogonal martingale measure - in sense of [Wal86] - with quadratic variation measure $\langle W\rangle(dxdt) = dxdt$. Indeed, it was proved by Konno and Shiga ([KS88]) and independently by Reimers ([Rei89]) that the SBM in $\mathbb{R}^1$ has a jointly continuous space-time density which solves equation (1.1). Note that the Laplacian component in (1.1) corresponds to the diffusion of the infinitesimal particles, the noise term, whose shape is motivated by the form of Feller's branching diffusion, represents the splittings of the infinitesimal particles.

In higher dimensions $d \geq 2$ the states of SBM are singular measures ([DH79]). Consequently, a relation between SBM and spde (1.1) as in dimension one fails. But let us mention that for $d \geq 2$ SBM is a solution to the generalized version of (1.1) for measure-valued processes

$$\partial_t \bar{X}_t = \frac{1}{2}\Delta \bar{X}_t + \dot{M}_{x,t}$$

where M is an orthogonal martingale measure with quadratic variation measure $\langle M\rangle(dxdt) = \bar{X}_t(dx)dt$, see [MRC88].

1.2 Catalytic super-Brownian motion

In contrast to the ordinary SBM, where the branching of an infinitesimal particle occurred independently of space and time, in the so-called catalytic case branching depends strongly on a (possibly singular) medium that may vary in space and time. Such a medium can be modelized as a (deterministic) measure-valued process $\varrho = (\varrho_t(dx) : t \geq 0)$ meaning that the branching intensity of an infinitesimal particle being at position x_0 at time t is formally given by "$\frac{\varrho_t(dx)}{dx}(x_0)$". Since the medium catalyzes the branching, it is often called *catalyst*, the reacting infinitesimal particle system *catalytic SBM* or just *reactant*.

An approximating particle system for catalytic SBM is provided by the one for the ordinary SBM modified in the following way. The particles still get random lifetimes according to a common rescaled exponential clock, but they do not age homogeneously any more. The explicit "age-function" $A(t)$ of a (Brownian) particle, say B_t, is chosen as the collision local time $L_{[B,\varrho]}(t)$ of this particle and the catalyst ϱ. Here $L_{[B,\varrho]}$ is defined via

$$L_{[B,\varrho]}(t) = \lim_{\epsilon \downarrow 0} \int_0^t \int_{\mathbb{R}^d} p_\epsilon(B_r, y) \varrho_r(dy) dr, \tag{1.2}$$

where p is the d-dimensional heat kernel. Hence, a particle only gets older if it is in contact to the catalyst. Particularly, a particle moving outside the catalyst's support does not age, so it cannot reach the end of its "life" and, consequently, it will not branch. Note that in this picture each particle lives in its own time scale and, maybe, one should better speak of *individual branching age*. Obviously, it is essential that the limit in (1.2) exists non-trivially, i.e. that $L_{[B,\varrho]} \not\equiv 0$. In one dimension ($d = 1$) one should not go into trouble for any catalyst, since even in the degenerated case $\varrho_t(dx) \equiv \delta_c(dx)$, $c \in \mathbb{R}^1$ arbitrarily fixed, one can make sense of $L_{[B,\varrho]}$ which becomes just the Brownian local time at level c. But already in dimension $d = 2$ the latter does not exist non-trivially any more. Hence, for $d \geq 2$ one has to be careful in choosing ϱ.

Dawson and Fleischmann paid attention to the catalytic SBM first. They provided a construction (avoiding a particle approximation) for $d = 1$ and studied some special cases in detail ([DF91], [DF94], etc.). In all dimensions $d \geq 1$, catalytic SBM is included in Dynkin's very general class of superprocesses, see [Dyn91],[Dyn93] or [Dyn94], also for a particle approximation (Theorem I.3.1 of [Dyn93]). Among other things, here the "age-function" A can be chosen, instead of $L_{[B,\varrho]}$, from a large class of additive functionals of the underlying spatial motion B - the class of so-called *admissible branching functionals* (cf. [Dyn94] p.49). Klenke recently extended this class and gave a particle approximation as well, see [Kle01]. Note that a relation between catalytic SBM and the mentioned approximating particle system is rigorous - so far - only if $L_{[B,\varrho]}$ exists as an admissible branching functional. But for a broad class of continuous catalysts ϱ in dimension $d = 1$, $L_{[B,\varrho]}$ was constructed in the desired form, see Theorem 4.1 of [EP94] or Proposition 6 of [DF97]. In order to be an admissible branching functional, A has to satisfy essentially some moment assumptions. In case $A(t) = L_{[B,\varrho]}(t)$, $\varrho_t(dx) \equiv \varrho(dx)$ and $d \geq 1$, Delmas ([Del96]) transformed these assumptions into a more transparent condition involving α-potentials of the measure $\varrho(dx)$. For an overview on catalytic SBM we refer to [DF00] or [Kle00b].

As in the ordinary case we now want to link the catalytic SBM with an spde. We had mentioned that the noise term in (1.1) corresponds to the branching component. But this time branchings only occur on ϱ's support, proportional to ϱ's "density". Therefore we should replace the white noise W by a degenerated white noise W^{ϱ} with intensity measure $\varrho_t(dx)dt$, i.e., more precisely, by an orthogonal martingale measure W^{ϱ} with quadratic variation measure $\langle W^{\varrho}\rangle(dxdt) = \varrho_t(dx)dt$. We heuristically end up with a formal evolution equation for catalytic SBM of the shape

$$\partial_t X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\dot{W}^{\varrho}_{x,t}. \tag{1.3}$$

1.3 The problem and sketch of main results

From now on we restrict to dimension $d = 1$. As seen in Section 1.1, for $\varrho_t(dx) \equiv dx$ (ordinary case) the SBM has a jointly continuous space-time Lebesgue density. This is not the case for the single point catalyst $\varrho_t(dx) \equiv \delta_c(dx)$, $c \in \mathbb{R}^1$, studied first in [DF94]. In fact, according to [DFLM95] or [FLG95], here the occupation density measure $\lambda^c(dt)$ of the corresponding catalytic SBM on site c is singular w.r.t. dt ($\lambda^c(dt)$ is defined via $\lambda^c((s,t]) := Y_t(c) - Y_s(c)$ where Y is the existing jointly continuous Lebesgue density of $(\bar{Y}_t(dx))_{t\geq 0}$ given by $\langle \bar{Y}_t, \psi\rangle := \int_0^t \langle \bar{X}_r, \psi\rangle dr$, $\psi \in C_b(\mathbb{R}^1, \mathbb{R}_+)$). In terms of evolution equation (1.3) this means that a concentration of the noise on a single space point destroys the regularity of the solution (density) in this point.

Now the question arises, can one return to the joint continuity by "smearing out" the atomic mass of $\delta_c(dx)$ around c? The hope is that a slight smoothing of atomic mass leads only to sharp peaks rather than to blow ups.

And indeed, there is a large class of catalysts providing regular reactants. More precisely, as we will establish in the sequel (Theorem 3.1), each catalyst ϱ satisfying

$$\exists \alpha \in (0,1]\ \forall T > 0\ \exists c > 0 : \quad \sup_{(t,x)\in[0,T]\times\mathbb{R}^1} \varrho_t(B(x,r)) \leq c\, r^{\alpha}, \qquad r \in (0,1] \tag{1.4}$$

induces a jointly continuous space-time density for the corresponding catalytic SBM. Here $B(x,r) := (x-r, x+r)$. That means that catalytic mass may not be too concentrated around a point and, particularly, that ϱ cannot have atoms. By the way, in the time-homogeneous case $\varrho_t(dx) \equiv \varrho(dx)$, the following potential-type condition

$$\sup_{x\in\mathbb{R}^1} \int_{B(x,1)} |x-y|^{-\alpha} \varrho(dy) < \infty$$

ensures that (1.4) holds, cf. Lemma A.1 in the Appendix.

Catalysts that induce a jointly continuous space-time density for the reactant will be called *moderate catalysts*. This reflects the moderation of its catalyzing impact, i.e. the non-occurrence of reactant blow ups. For moderate catalysts, more precisely for catalysts satisfying (1.4), the formal link between catalytic SBM and spde (1.3) will be made rigorous by showing that the existing regular density is the unique solution to (1.3), cf. Corollary 3.3 below.

We conclude this section with examples for moderate catalysts. The second example reveals the power of Theorem 3.1 by establishing that even catalysts without Lebesgue densities can fulfill (1.4). Moreover, for each $0 < \alpha \leq 1$ one can find a moderate catalyst with support having Hausdorff-dimension α. Note that for a positive constant c, $\sup_{x \in supp(\varrho)} \varrho(B(x,r)) \leq c\, r^{\alpha}$, respectively $\inf_{x \in supp(\varrho)} \varrho(B(x,r)) \geq c\, r^{\alpha}$, $0 < r \leq 1$, implies that the Hausdorff-dimension of the support $supp(\varrho)$ of ϱ is at least, respectively at most, α; cf. Theorem 5.7 of [Mat95].

Example 1 (moderate catalysts having Lebesgue densities) Let ϱ be a catalyst with finite measure states and having a jointly continuous density field. The latter can clearly be bounded from above by a constant, at least on compact time sets. Hence, (1.4) is trivially satisfied with $\alpha = 1$. In the ordinary case $\varrho_t(dx) \equiv dx$, Theorem 3.1 just recovers results from [KS88] and [Rei89]. If $(\varrho_t(dx))_t$ is an one-dimensional ordinary SBM (for the notion of random catalysts see Remark 2.3 below) which has a jointly continuous density, then Theorem 3.1 provides space-time regularity of catalytic SBM in a super-Brownian medium. This special case is also treated in [FKX02] using different methods.

Example 2 (moderate catalyst without Lebesgue density) Let us consider $\varrho_t(dx) \equiv \mathcal{C}_\lambda(dx)$ where $\mathcal{C}_\lambda(dx)$ is the "Cantor measure" on, say, $[0,1] \subset \mathbb{R}^1$ with index $\lambda \in (0,1)$. This measure is supported by an uncountable unification of single points (λ-Cantor set $C(\lambda)$, cf. [Mat95] 4.13) and possesses no atoms. In fact, $\mathcal{C}_\lambda(.) = \mathcal{H}^\alpha(C(\lambda) \cap .)$ where $\mathcal{H}^\alpha$ is the α-dimensional Hausdorff measure and $\alpha = |\log 2| / |\log \lambda|$ the Hausdorff-dimension of $C(\lambda) = supp(\mathcal{C}_\lambda)$. $\mathcal{C}_\lambda(dx)$ has clearly no Lebesgue density. Furthermore, see e.g. Theorem 4.14 of [Mat95], there exist $0 < c < C$ such that for all $x \in supp(\mathcal{C}_\lambda)$,

$$cr^\alpha < \mathcal{C}_\lambda(B(x,r)) < Cr^\alpha, \quad r \in (0,1]. \tag{1.5}$$

It follows immediately from (1.5) that $\varrho_t \equiv \mathcal{C}_\lambda$ satisfies (1.4).

2 Notations and the model

2.1 Notations

Let E and E' be Polish spaces, I an interval in $[0,\infty)$ and $\mathbb{R}_+ = [0,\infty)$.

- $\mathcal{B}(E)$ is the Borel σ-field on E.
- $\mathcal{B}(E,E')$, $C(E,E')$ denote the spaces of measurable, respectively, continuous functions from E into E'.
- $D(I,E')$ is the space of cadlag functions from I into E'.

The lower index b, respectively c, always refers to the subclass of bounded functions, respectively functions having compact support. $\|.\|_\infty$ is the usual supremum norm.

- $C_c^\infty(\mathbb{R}^1,\mathbb{R}_+)$ represents the elements of $C_c(\mathbb{R}^1,\mathbb{R}_+)$ having derivatives of any order.
- $C_b^{1,2}(I \times \mathbb{R}^1,\mathbb{R}_+)$ is the class of $C(I \times \mathbb{R}^1,\mathbb{R}_+)$-functions having continuous, bounded first time- and second space-derivatives.

- $\mathcal{B}_b^T([0,\infty) \times \mathbb{R}^1, \mathbb{R}_+)$ consists of the $\mathcal{B}_b([0,\infty) \times \mathbb{R}^1, \mathbb{R}_+)$-elements with support in $[0,T] \times \mathbb{R}^1$.

Moreover,

- $\mathcal{M}(\mathbb{R}^1)$ is the space of (positive) measures on $\mathcal{B}(\mathbb{R}^1)$.
- $\mathcal{M}_f(\mathbb{R}^1)$ is the space of finite measures on $\mathcal{B}(\mathbb{R}^1)$ equipped with the weak topology.
- $B(x,r) = (x-r, x+r), \quad x \in \mathbb{R}^1, r > 0.$
- p denotes the heat kernel in $\mathbb{R}^1$, defined via

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^1,$$

  $(S_t)_{t \geq 0}$ stands for the corresponding (heat) semigroup, induced by

$$S_t\psi(x) = \int_{\mathbb{R}^1} p_t(x,y)\psi(y)dy, \quad t > 0, x \in \mathbb{R}^1, \psi \in \mathcal{B}_b(\mathbb{R}^1, \mathbb{R}_+).$$

For $\eta \in \mathcal{M}(\mathbb{R}^1)$ and proper functions ϕ and ψ on $\mathbb{R}^1$, we define

- $S_t\eta(x) = \int_{\mathbb{R}^1} p_t(x,y)\eta(dy), \quad t > 0, x \in \mathbb{R}^1,$
- $\langle \eta, \psi \rangle = \int_{\mathbb{R}^1} \psi(x)\eta(dx), \quad \langle \phi, \psi \rangle = \int_{\mathbb{R}^1} \phi(x)\psi(x)dx,$
- $supp(\eta)$ as the support of η, i.e. as the smallest closed set $F \in \mathcal{B}(\mathbb{R}^1)$ satisfying $\eta(A) = 0$ for every $A \in \mathcal{B}(\mathbb{R}^1)$ with $A \cap F = \emptyset$.

Let c always refer to a finite positive constant which may vary from place to place. Possible subscripts of c stress the dependence of c on these subscripts.

For a stochastic process $Y = (Y_t : t \geq 0)$ on a measurable space $(\Omega, \mathcal{F})$, $(\mathcal{F}_t^Y)_{t \geq 0}$ is the augmented filtration in $\mathcal{F}$ induced by Y.

Stochastic integration in infinite dimension will be done in Walsh's framework, see [Wal86], where integrators are worthy martingale measures. We denote the quadratic variation measure - cf. page 291 and Corollary 2.8 of [Wal86] - of an orthogonal martingale measure M by $\langle M \rangle(dxdt)$ and the stochastic integral, with integration domain $B \times (0,t]$, of a proper f against M by $\int_0^t \int_B f(r,y)dM_{y,r}$ or $f \bullet M(B \times (0,t])$.

2.2 The catalyst

Consider a deterministic continuous kernel ϱ from $[0,\infty)$ into $\mathcal{M}(\mathbb{R}^1)$, i.e. $\varrho_t(dx)$ is an element of $\mathcal{M}(\mathbb{R}^1)$ for every $t \in [0,\infty)$ and $t \mapsto \langle \varrho_t, \psi \rangle$ a continuous function on $[0,\infty)$ for every $\psi \in C_c(\mathbb{R}^1, \mathbb{R}_+)$. Assume that for each compact time set I there is a constant $c > 0$ such that

$$\sup_{(t,x) \in I \times \mathbb{R}^1} \varrho_t(B(x,1)) \leq c, \tag{2.6}$$

then ϱ is said to be an *admissible catalyst*.

As already mentioned, if an admissible catalyst induces a jointly continuous space-time density for the reactant, then we go to call it a *moderate catalyst.*

Moderation condition (M). *An admissible catalyst ϱ is said to satisfy condition* (**M**), *if there is an $\alpha \in (0,1]$ such that for each compact set $I \subset [0,\infty)$ there is a constant $c>0$ fulfilling*

$$\sup_{(t,x)\in I\times\mathbb{R}^1} \varrho_t(B(x,r)) \le c\ r^\alpha, \quad r\in(0,1].$$

For further considerations we also introduce the following, a bit weaker condition.

Condition (M′). *An admissible catalyst ϱ is said to satisfy condition* (**M′**), *if there is a continuous, non-decreasing function $h:[0,\infty)\to[0,\infty)$ satisfying $h(0)=0$ and*

$$\int_0 \frac{1}{r} h(r)dr < \infty, \tag{2.7}$$

such that for each compact set $I \subset [0,\infty)$ there is a constant $c>0$ fulfilling

$$\sup_{(t,x)\in I\times\mathbb{R}^1} \varrho_t(B(x,r)) \le c\ h(r), \quad r\in(0,1]. \tag{2.8}$$

Note that $h(r) = c\,r^\alpha$ trivially fulfills (2.7) whenever $\alpha \in (0,1]$ and cannot occur in (2.8) whenever $\alpha > 1$. The latter statement is justified since "the Lebesgue measure carries locally the smallest mass", see the next lemma.

Lemma 2.1 *Let $\varrho(dx) \in \mathcal{M}(\mathbb{R}^1)$ be non-trivial and assume $\sup_{x\in\mathbb{R}^1}\varrho(B(x,r)) \le c\,h(r)$, $r\in(0,1]$, for some continuous , non-decreasing function $h:[0,\infty)\to[0,\infty)$ with $h(0)=0$. Then there exists a constant $c>0$ such that $r \le c\,h(r)$, $r\in(0,1]$.*

Proof By the non-triviality there are $x\in\mathbb{R}^1$ and $\epsilon>0$ with $\varrho(B(x,1))\ge\epsilon$. The ball $B(x,1)$ can be covered by an unification of n balls $(n-1<\frac{2}{r}\le n)$ with radius r for $r\in(0,1]$. Thus, $\epsilon \le \varrho(B(x,1)) \le c\frac{2}{r}c\,h(r)$ and so $r\le\tilde{c}\,h(r)$ for all $r\in(0,1]$. □

We conclude this section with an essential remark.

Remark 2.2 *According to Lemma A.1 from the Appendix, (2.8) is equivalent to*

$$\sup_{(t,x)\in I\times\mathbb{R}^1} \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho_t(dy) \le c\ h(\sqrt{r}), \quad r\in(0,1]. \tag{2.9}$$

In terms of (2.8), (**M**) *and* (**M′**) *are comprehensible conditions that can be checked relatively easily for special catalysts. But in terms of (2.9),* (**M**) *and* (**M′**) *provide convenient tools to work with. Indeed, in all proofs of results based on* (**M**) *or* (**M′**) *we shall use the latters in terms of (2.9).*

2.3 The reactant

Let ϱ be an admissible catalyst. Then, according to Theorem 4.1 of [EP94] or Proposition 6 of [DF97], the collision local time $L_{[B,\varrho]}$ - introduced in (1.2) - exists as an admissible branching functional (cf. p.49 of [Dyn94]). Hence, by Theorem 3.4.1 of [Dyn94] and the moment formula in Proposition 6(b) of [DF97], there is an $\mathcal{M}_f(\mathbb{R}^1)$-valued (time-inhomogeneous) Markov process $\bar{X} = [\bar{X}, \Omega, \mathcal{F}, \mathbf{P}^{\varrho}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R}^1)]$ whose transition probabilities are uniquely determined by the Laplace functional

$$\mathbf{E}^{\varrho}_{s,\eta}\left[e^{-\langle \bar{X}_t, \psi\rangle}\right] = e^{-\langle \eta, U_{s,t}(.|\psi)\rangle}, \quad \psi \in C_c(\mathbb{R}^1, \mathbb{R}_+), \tag{2.10}$$

where $(U_{s,t}(z|\psi) : 0 \leq s \leq t, z \in \mathbb{R}^1)$ is the unique non-negative solution to the integral equation

$$u(s,t,z) = S_{t-s}\psi(z) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^1} p_{r-s}(y,z) u^2(r,t,y) \varrho_r(dy) dr, \tag{2.11}$$

i.e. heuristically to the formal backward *cumulant equation*

$$\left\{ \begin{array}{rcl} -\partial_s u(s,t,z) & = & \frac{1}{2}\Delta u(s,t,z) - \frac{1}{2}\text{"}\frac{\varrho_s(dy)}{dy}(z)\text{"} u^2(s,t,z) \\ u(t,t,.) & = & \psi(.) \end{array} \right\}. \tag{2.12}$$

$\bar{X}$ is called *catalytic SBM with catalyst* ϱ. Set $\mathbf{P}^{\varrho}_{\eta} := \mathbf{P}^{\varrho}_{0,\eta}$ and note that $\mathbf{P}^{\varrho}_{s,\eta}$ represents the law of the catalytic SBM starting at time s with initial state $\eta \in \mathcal{M}_f(\mathbb{R}^1)$. According to Section 3.4 of [DF97], $\bar{X}_t$ has a continuous modification w.r.t. the weak topology.

Remark 2.3 (random catalysts) *It is a natural desire to deal with random catalysts. An elegant way to do so is the so-called quenched approach: first one samples the whole catalyst process, then the reactant is run over. A rigorous treatment can be found in [DF91], Section 2.5.*

In the remainder of this section we pose some properties of $\bar{X}$. We start with a formula for the moments.

Proposition 2.4 (moments) *For all* $t \geq 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^1)$, $\psi \in \mathcal{B}_b(\mathbb{R}^1, \mathbb{R}_+)$ *and* $m \geq 1$,

$$\mathbf{E}^{\varrho}_{\eta}\left[\langle \bar{X}_t, \psi\rangle^m\right] = m! \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_1+\dots+n_k = m}} \prod_{i=1}^{k} \langle \eta, \chi_{n_i}(0,.)\rangle,$$

where the $\chi_n(.,.) = \chi_n(.,.|t,\psi)$, $n \geq 1$, *are defined recursively as follows*

$$\chi_1(s,z) = \mathbf{1}_{[0,t]}(s) S_{t-s}\psi(z)$$

$$\chi_n(s,z) = -\tfrac{1}{2}\int_s^\infty \int_{\mathbb{R}^1} p_{r-s}(y,z)\Big(\sum_{j=1}^{n-1} \chi_j(r,y)\chi_{n-j}(r,y)\Big) \varrho_r(dy) dr,$$

$n \geq 2$, *for all* $(s,z) \in [0,\infty) \times \mathbb{R}^1$.

Particularly we have

$$\begin{aligned} \mathbf{E}^{\varrho}_{\eta}\left[\langle \bar X_t,\psi\rangle\right] &= \langle \eta, S_t\psi\rangle \\ \mathbf{E}^{\varrho}_{\eta}\left[\langle \bar X_t,\psi\rangle^2\right] &= \langle \eta, S_t\psi\rangle^2 + \langle \eta, \int_0^t \int_{\mathbb{R}^1} p_r(y,.)(S_{t-r}\psi)^2(y)\varrho_r(dy)dr\rangle. \end{aligned} \tag{2.13}$$

Proof of Proposition 2.4 Consider $t \geq 0$, $\psi \in \mathcal{B}_b(\mathbb{R}^1,\mathbb{R}_+)$ and the Taylor expansion of $a \mapsto U_{0,t}(z|a\psi)$ at $a=0$

$$U_{0,t}(z|\theta\psi) = 0 + \frac{1}{1!}\frac{\partial}{\partial a}U_{0,t}(z|a\psi)\Big|_{a=0}\theta + \frac{1}{2!}\frac{\partial^2}{\partial a^2}U_{0,t}(z|a\psi)\Big|_{a=0}\theta^2 + \cdots \ . \tag{2.14}$$

It is only technical work to prove (inductively) that the r.h.s. of (2.14) is nothing but $\sum_{n=1}^\infty \chi_n(0,z)\theta^n$ with

$$\chi_1(s,z) = \mathbf{1}_{[0,t]}(s)S_{t-s}\psi(z)$$

$$\chi_n(s,z) = -\tfrac{1}{2}\int_s^\infty \int_{\mathbb{R}^1} p_{r-s}(y,z)\Big(\sum_{j=1}^{n-1}\chi_j(r,y)\chi_{n-j}(r,y)\Big)\varrho_r(dy)dr,$$

$n \geq 2$, $(s,z) \in [0,\infty)\times\mathbb{R}^1$. Using the (non-trivial) estimates $b_n \leq c^{2n}4^n$ for the terms of the sequence $(b_n)_{n\geq 1}$ ($b_1 := b_2 := c > 0$, $b_n := \sum_{j=1}^{n-1} c\, b_j b_{n-j}$, $n \geq 3$) it can be shown, again by induction, that $\sum_{n=1}^\infty \chi_n(0,z)\theta^n$ is absolutely dominated by $\sum_{n=1}^\infty c_t^n\,\theta^n$ for some positive constant c_t depending on t. Hence, the serie $\sum_{n=1}^\infty \chi_n(0,z)\theta^n$ converges absolutely for sufficiently small $\theta \geq 0$. As the last step, once more by induction, one verifies

$$\frac{\partial^m}{\partial\theta^m}\mathbf{E}^{\varrho}_{\eta}\left[e^{-\langle \bar X_t,\theta\psi\rangle}\right]\Big|_{\theta=0} = \mathbf{E}^{\varrho}_{s,\eta}\left[\langle \bar X_t,\psi\rangle^m\right]$$

and

$$\begin{aligned} \frac{\partial^m}{\partial\theta^m}e^{-\langle \eta, U_{0,t}(.|\theta\psi)\rangle}\Big|_{\theta=0} \qquad &\Big(= \frac{\partial^m}{\partial\theta^m}e^{-\langle \eta,\sum_{n=1}^\infty \chi_n(0,.)\theta^n\rangle}\Big|_{\theta=0}\Big) \\ &= m!\sum_{k=1}^m \frac{(-1)^{m+k}}{k!}\sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\dots+n_k=m}}\prod_{i=1}^k \langle \eta,\chi_{n_i}(0,.)\rangle \end{aligned}$$

for $m \geq 1$, whereby the claim follows immediately from (2.10). □

Next we are going to characterize $\bar X$ as the unique solution to a martingale problem. To do so, we first have to introduce the *collision local time* of the catalyst and the reactant. Here the notion of collision local times is taken from [BEP91].

Proposition 2.5 (collision local time of catalyst and reactant) *For $\eta \in \mathcal{M}_f(\mathbb{R}^1)$ there exists a random measure $L_{[\bar X,\varrho]}(dxdt)$ on $\mathbb{R}^1\times[0,\infty)$ satisfying for all $m \geq 1$ and $f \in \ \cup_{T>0}\mathcal{B}_b^T([0,\infty)\times\mathbb{R}^1,\mathbb{R}_+)$*

(i) $t \longmapsto \int_0^t\int_{\mathbb{R}^1} f(r,y)L_{[\bar X,\varrho]}(dydr)$ is $(\mathcal{F}_t^{\bar X})_{t\geq 0}$-adapted

(ii) $\int_0^\infty \int_{\mathbb{R}^1} f(t,x) L_{[\bar{X},\varrho]}(dxdt) = \lim_{\epsilon \downarrow 0} \int_0^\infty \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} f(t,x) p_\epsilon(x,y) \bar{X}_t(dy) \varrho_t(dx) dt$, $\mathbf{P}_\eta^\varrho$*-a.s.*

(iii) $\mathbf{E}_\eta^\varrho \Big[\int_0^\infty \int_{\mathbb{R}^1} f(r,y) L_{[\bar{X},\varrho]}(dydr) \Big] = \int_{\mathbb{R}^1} \int_0^\infty \int_{\mathbb{R}^1} p_r(x,y) f(r,y) \varrho_r(dy) dr \eta(dx)$.

Part *(iii)* gives particularly

$$\mathbf{E}_\eta^\varrho \Big[\int_0^t \int_{\mathbb{R}^1} f(r,y) L_{[\bar{X},\varrho]}(dydr) \Big] < \infty \qquad \forall t > 0. \tag{2.15}$$

Proposition 2.5 was established in [DF01] (Theorem 3) for a special catalyst (ϱ=ordinary SBM). The proof for our case goes along the same lines with the obvious changes. See also Proposition 5.1 of [Del96]. Let us now turn to the mentioned martingale problem. Roelly ([RC86], Theorem 1.3) provided such a characterization for a class of super-Feller diffusions first, in fact for the case of space-time homogeneous branching.

Proposition 2.6 (martingale problem) *$\bar{X}$ is the unique solution to the martingale problem* (**MP**)*, which means that for each $\eta \in \mathcal{M}_f(\mathbb{R}^1)$ the law $\mathbf{P}_\eta^\varrho$ is the only one under that*

$$M_t(\psi_.) = \langle \bar{X}_t, \psi_t \rangle - \langle \eta, \psi_0 \rangle - \int_0^t \langle \bar{X}_r, \frac{1}{2} \Delta \psi_r(.) + \partial_r \psi_r(.) \rangle dr, \quad t \geq 0$$

is a square-integrable, continuous $(\mathcal{F}_t^{\bar{X}})_{t \geq 0}$-martingale having

$$\langle M(\psi_.) \rangle_t = \int_0^t \int_{\mathbb{R}^1} \psi^2(r,y) L_{[\bar{X},\varrho]}(dydr), \quad t \geq 0$$

as its quadratic variation process for all $\psi \in C_b^{1,2}([0,\infty) \times \mathbb{R}^1, \mathbb{R}_+)$.

An easy extension of Proposition 9.1 of [Del96] guarantees the solution property in Proposition 2.6. The uniqueness can be established using Mytnik's method of approximate dual processes ([Myt98]). Actually, one can just mimic the proof of Proposition 3 of [FKX02]. As a consequence of Proposition 2.6 we can derive the following useful representation for $\bar{X}$.

Proposition 2.7 (Green's function representation) *For each $\eta \in \mathcal{M}_f(\mathbb{R}^1)$ there is an orthogonal martingale measure M satisfying*

$$\langle \bar{X}_t, \psi \rangle = \langle \eta, S_t \psi \rangle + \int_0^t \int_{\mathbb{R}^1} S_{t-r} \psi(y) dM_{y,r} \quad \mathbf{P}_\eta^\varrho\text{-a.s.} \tag{2.16}$$

for all $t \geq 0$, $\psi \in \mathcal{B}_b(\mathbb{R}^1, \mathbb{R}_+)$ and

$$\Big\langle f \bullet M \Big\rangle \Big(\mathbb{R}^1 \times (0,t]\Big) \quad \Big(= \int_0^t \int_{\mathbb{R}^1} f^2(r,y) \langle M \rangle (dydr) \Big) = \int_0^t \int_{\mathbb{R}^1} f^2(r,y) L_{[\bar{X},\varrho]}(dydr) \quad \mathbf{P}_\eta^\varrho\text{-a.s.} \tag{2.17}$$

for all $t \geq 0$, $f \in \mathcal{B}_b([0,\infty) \times \mathbb{R}^1, \mathbb{R}_+)$.

Proof The class $\{(M_t(\psi))_{t\geq 0} : \psi \in C_b^2(\mathbb{R}^1, \mathbb{R}_+)\}$ of martingales from Proposition 2.6 extends to an orthogonal martingale measure, say M. Then,

$$M_t(f) = \int_0^t \int_{\mathbb{R}^1} f(r,y) dM_{y,r} \quad \mathbf{P}_\eta^\varrho\text{-a.s.} \tag{2.18}$$

for all $t \geq 0$ and $f \in \mathcal{B}_b([0,\infty) \times \mathbb{R}^1, \mathbb{R}_+)$ satisfying

$$\mathbf{E}_\eta^\varrho \left[\int_0^t \int_{\mathbb{R}^1} f^2(r,y) \langle M \rangle (dydr) \right] < \infty \quad \forall t \geq 0. \tag{2.19}$$

Consequently, by the form of the quadratic variation process of $(M_t(f))_{t\geq 0}$,

$$\Big\langle f \bullet M \Big\rangle \Big(\mathbb{R}^1 \times (0,t]\Big) = \int_0^t \int_{\mathbb{R}^1} f^2(r,y) L_{[\bar{X},\varrho]}(dydr) \quad \mathbf{P}_\eta^\varrho\text{-a.s.}$$

for all $t \geq 0$ and mentioned f's. Note that in (2.19) $\langle M \rangle$ can be replaced by $L_{[\bar{X},\varrho]}$. But then, according to (2.15), (2.19) is always satisfied.

In order to show (2.16), we proceed as in the proof of Theorem I-7 of [MRC88]. Let ψ be from $\mathcal{B}_b(\mathbb{R}^1, \mathbb{R}_+)$, $t \geq 0$ and set $f(r,y) = S_{t-r}\psi(y)\mathbf{1}_{[0,t]}(r)$. Then f belongs to $C_b^{1,2}([0,t] \times \mathbb{R}^1, \mathbb{R}_+)$. Hence we obtain from (**MP**) and (2.18)

$$\langle \bar{X}_t, S_{t-t}\psi \rangle = \langle \eta, S_t\psi \rangle + \int_0^t \langle \bar{X}_r, \frac{1}{2}\Delta S_{t-r}\psi + \partial_r S_{t-r}\psi \rangle dr + \int_0^t \int_{\mathbb{R}^1} S_{t-r}\psi(y) dM_{y,r},$$

$\mathbf{P}_\eta^\varrho$-almost surely. But $\frac{1}{2}\Delta$ is the generator of (S_t), so the middle term of the r.h.s. vanishes (note the reversed time r of S_{t-r}), which gives the claim. □

3 Statement and discussion of main results

3.1 Main results

Here we are going to present our main results. The first theorem justifies the name of our moderation condition (**M**) from Section 2.2.

Theorem 3.1 (space-time regular reactant) *Let ϱ be an admissible catalyst satisfying* (**M**) *and $\eta \in \mathcal{M}_f(\mathbb{R}^1)$. Then there exists a random field X on $(0,\infty) \times \mathbb{R}^1$ having a jointly continuous modification and being the density of $\bar{X}$ for almost all times, i.e. $\bar{X}_t(dx)$ is absolutely continuous w.r.t. dx and has $(X_t(x) : x \in \mathbb{R}^1)$ as its dx-density for dt-a.a. $t > 0$, $\mathbf{P}_\eta^\varrho$-almost surely. Particularly there exists a modification of $\bar{X}$ having the continuous modification of X $\mathbf{P}_\eta^\varrho$-a.s. as its space-time density. If the initial state $\eta(dx)$ has a continuous dx-density, then the statements extend to $[0,\infty) \times \mathbb{R}^1$.*

That means an admissible catalyst satisfying (**M**) is moderate. As a by-product of the proof, see Section 4.1 below, we get estimates for the moduli of continuity of X.

Corollary 3.2 (moduli of continuity of X) *Consider the continuous X from Theorem 3.1, α from* (**M**) *and $\alpha_0 \in (0, \alpha/4)$ arbitrary. Then, for each compact sets $I \subset [0,\infty)$ and $A \subset \mathbb{R}^1$ there exists a constant $c > 0$ such that for all $t, t' \in I$ and $x, x' \in A$,*

$$\left| X_t(x) - X_{t'}(x') \right| \leq c \left(|t - t'| + |x - x'| \right)^{\alpha_0}.$$

As a consequence of Theorem 3.1 and the martingale problem (**MP**) from Proposition 2.6 we get, as anticipated in section 1.2, a characterization of X as the unique solution to equation (1.3). Uniqueness here means uniqueness in law.

Corollary 3.3 (spde for the reactant) *The continuous random field X from Theorem 3.1 is the unique solution to spde (1.3), i.e. there is an orthogonal martingale measure W^{ϱ} with quadratic variation measure $\langle W^{\varrho} \rangle (dxdt) = \varrho_t(dx)dt$ - possibly on an enlargement of $(\Omega, \mathcal{F})$ - such that for arbitrary $t_0 > 0$,*

$$\langle X_t, \psi \rangle = \langle X_{t_0}, \psi \rangle + \int_{t_0}^{t} \langle X_r, \frac{1}{2} \Delta \psi \rangle dr + \int_{t_0}^{t} \int_{\mathbb{R}^1} \psi(x) \sqrt{X_r(x)} dW^{\varrho}_{x,r} \tag{3.20}$$

for all $t \geq t_0$ and $\psi \in C_c^{\infty}(\mathbb{R}^1, \mathbb{R}_+)$, $\mathbf{P}^{\varrho}_{\eta}$-almost surely. If the initial state $\eta(dx)$ has a continuous dx-density, then one can choose $t_0 = 0$.

The next theorem is the basis for a second characterization - cf. Corollary 3.5 below - of the random field X from Theorem 3.1.

Theorem 3.4 *Let ϱ be an admissible catalyst satisfying* (**M'**). *Then we have*

(i) **(fundamental solutions to cumulant equation)** *For each $\nu \in \mathcal{M}_f(\mathbb{R}^1)$ there is an unique non-negative solution $(U_{s,t}(z|\nu) : 0 \leq s \leq t, z \in \mathbb{R}^1)$ to the integral equation*

$$u(s,t,z) = S_{t-s}\nu(z) - \frac{1}{2} \int_s^t \int_{\mathbb{R}^1} p_{r-s}(y,z) u^2(r,t,y) \varrho_r(dy) dr, \tag{3.21}$$

i.e. heuristically to the formal cumulant equation (2.12) where the regular final state $u(t,t) = \psi$ is replaced by $\nu(dx)$.

(ii) **(continuity of fundamental solutions)** *For each $s \geq 0$, $(U_{s,t}(z|\nu) : s < t, z \in \mathbb{R}^1, \nu \in \mathcal{M}_f(\mathbb{R}^1))$ is continuous in each of the variables t, z and ν.*

Corollary 3.5 (Laplace functional representation of X) *The random field X from Theorem 3.1 is the random element in $C((0,\infty), C(\mathbb{R}^1, \mathbb{R}_+))$, whose one-dimensional distributions are uniquely determined by the Laplace-functional*

$$\mathbf{E}^{\varrho}_{\eta} \left[e^{-\sum_{i=1}^{k} \theta_i X_t(x_i)} \right] = e^{-\left\langle \eta, U_{0,t}\left(\cdot \,|\, \sum_{i=1}^{k} \theta_i \delta_{x_i}(dx)\right) \right\rangle} \tag{3.22}$$

for $t > 0$, $\theta_i \geq 0$, $x_i \in \mathbb{R}^1$, $1 \leq i \leq k$ and $k \geq 1$.

3.2 Discussion

Conditions ($\mathbf{M}$) and ($\mathbf{M}'$) as well as the restriction to dimension one are motivated by heuristic arguments. Replacing the smooth test function $\psi(.)$ in the second moment formula (2.13) - note that (2.13) remains true for $d \geq 2$ - by the dirac-function $\delta_x(.)$, we are dealing formally with the second moment of $\bar{X}_t$'s density at position x. Heuristically we obtain

$$\mathbf{E}^{\varrho}_{\eta}[X^2_t(x)] \sim \int_0^t \frac{1}{r} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} \varrho_{t-r}(dy)dr. \tag{3.23}$$

Assuming $d = 1$ and ($\mathbf{M}$) in terms of (2.9), the r.h.s. of (3.23) is finite for every $t \in [0, \infty)$ and $x \in \mathbb{R}^1$, which heuristically indicates a regular density. On the other hand, in higher dimensions $d \geq 2$ the formal second density moment (3.23) blows up everywhere on the catalyst's support for every catalyst, whereby a search for moderate catalysts seems to be not that promising.

In dimension one, finiteness of the formal second density moment (3.23) still holds under condition ($\mathbf{M}'$). So, maybe, a catalyst satisfying ($\mathbf{M}'$) could be already moderate. This guess is somehow backed by Theorem 3.4 which establishes, under ($\mathbf{M}'$), fundamental solutions to cumulant equation (2.11) w.r.t. every $x_0 \in \mathbb{R}^1$, i.e. a solution to equation (3.21) with $\nu(dx) = \delta_{x_0}(dx)$ for every $x_0 \in \mathbb{R}^1$. Having fundamental solutions w.r.t. only dx-a.a. $x_0 \in \mathbb{R}^1$ is already sufficient to get absolutely continuous reactant states; cf. [DFR91], [Kle00a]. In fact, the absolute continuity is only ensured a.s. for fixed times. However, the strong regularity of the fundamental solutions, see part (*ii*) of Theorem 3.4, feeds the hope for an a.s.-existence of a regular space-time Lebesgue density.

4 Proofs of main results

4.1 Proof of Theorem 3.1

Let ϱ be an admissible catalyst satisfying ($\mathbf{M}$) and $\bar{X}$ the corresponding catalytic SBM starting with initial state $\eta \in \mathcal{M}_f(\mathbb{R}^1)$. We first pose the strategy of the proof. The same approach as used by Konno and Shiga to prove Theorem 1.4 of [KS88] is chosen.

For each $\epsilon > 0$ and $T > 0$ let X^ϵ be a smoothed version of $\bar{X}$, given by $X^\epsilon_t(x,\omega) = \langle \bar{X}_t(\omega), p_\epsilon(x, .)\rangle$, and $X^{\epsilon,T}$ its support restricted version, defined via $X^{\epsilon,T}_t(x,\omega) = \mathbf{1}_{[0,T]}(t) X^\epsilon_t(x,\omega)$ for all $x \in \mathbb{R}^1$, $t \geq 0$ and $\omega \in \Omega$. By the weak continuity of $\bar{X}_t$ in t, X^ϵ is clearly continuous in (t,x) and hence predictable. Furthermore, for some $\beta > 0$, set $L^2_\beta = L^2(\Omega \times \mathbb{R}^1 \times [0,\infty), \mathbf{P}^\varrho_\eta(d\omega)\, e^{-\beta|x|}dx\, t^\beta dt)$ and denote the norm $(\int_0^\infty \int_{\mathbb{R}^1} \mathbf{E}^\varrho_\eta[f^2(t,x)]\, e^{-\beta|x|}dx\, t^\beta dt)^{1/2}$ by $\|f\|_{2,\beta}$. Then the proof goes along the following lines which will be made rigorous below.

(a) $X^{\epsilon,T} \in L^2_\beta$ for all $\epsilon > 0$ and $T > 0$.

(b) $(X^{\epsilon,T})_{\epsilon \downarrow 0}$ is a Cauchy sequence in the Banach space L^2_β for all $T > 0$.

(c) There exists a predictable $X : [0,\infty) \times \mathbb{R}^1 \times \Omega \to \mathbb{R}_+$ satisfying for all $T > 0$, $X^T \in L^2_\beta$ and $\|X^{\epsilon,T} - X^T\|_{2,\beta} \to 0$ as $\epsilon \downarrow 0$, where $X^T_t(x,\omega) := \mathbf{1}_{[0,T]}(t) X_t(x,\omega)$.

(d) $\bar{X}_t(dx) = X_t(x)dx$ for dt-a.a. $t > 0$, $\mathbf{P}^{\varrho}_{\eta}$-a.s.

(e) X has a jointly continuous modification on $(0,\infty) \times \mathbb{R}^1$.

Proof of (a) Consider $t \in [0,T]$, $x \in \mathbb{R}^1$ and $\epsilon > 0$. Then, using (2.13), we have

$$
\begin{aligned}
&\mathbf{E}^{\varrho}_{\eta}\left[X^{\epsilon}_t(x)^2\right] \\
&= \langle \eta, S_t p_\epsilon(x,.)\rangle^2 + \int_{\mathbb{R}^1}\int_0^t\int_{\mathbb{R}^1} p_r(z,y)\left(S_{t-r}p_\epsilon(x,.)\right)^2(y)\varrho_r(dy)dr\,\eta(dz) \\
&\le \left(\frac{c}{\sqrt{t+\epsilon}}\int_{\mathbb{R}^1} e^{-\frac{(x-z)^2}{2(T+\epsilon)}}\eta(dz)\right)^2 + \int_0^t\int_{\mathbb{R}^1}\frac{1}{\sqrt{r}}\Big(\int_{\mathbb{R}^1} e^{-\frac{(z-y)^2}{2r}}\eta(dz)\Big)p^2_{t-r+\epsilon}(x,y)\varrho_r(dy)dr
\end{aligned}
$$

which, according to (**M**) and Remark 2.2, can be estimated by

$$
\begin{aligned}
&\frac{1}{t}c_T + c_T\int_0^t r^{-1/2}(t-r+\epsilon)^{-1}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{t-r+\epsilon}}\varrho_r(dy)dr \\
&\quad\le \tfrac{1}{t}c_T + c_T\textstyle\int_0^t r^{-1/2}(t-r+\epsilon)^{-1}(t-r+\epsilon)^{\alpha/2}dr \le \tfrac{c}{t}c_T.
\end{aligned}
$$

Thus, $\|X^\epsilon\|_{2,\beta} < \infty$ implying $X^\epsilon \in L^2_\beta$. □

Proof of (b) Again $t \in [0,T]$ and $x \in \mathbb{R}^1$. Let $\epsilon, \epsilon' > 0$ be small. Proceeding as in the proof of **(a)** and using inequality (A.43) we get

$$
\begin{aligned}
&\mathbf{E}^{\varrho}_{\eta}\left[(X^{\epsilon}_t(x) - X^{\epsilon'}_t(x))^2\right] \\
&\le \left(\int_{\mathbb{R}^1}(p_{t+\epsilon}(x,z) - p_{t+\epsilon'}(x,z))\eta(dz)\right)^2 \\
&\quad + \int_{\mathbb{R}^1}\int_0^t\int_{\mathbb{R}^1} p_r(x,.)\Big(S_{t-r}\left(p_\epsilon(x,.) - p_{\epsilon'}(x,.)\right)\Big)^2(y)\varrho_r(dy)dr\,\eta(dz) \\
&\le \frac{1}{t}c|\epsilon-\epsilon'| \;+\; c\int_0^t\int_{\mathbb{R}^1}\frac{1}{\sqrt{r}}\left(p_{t-r+\epsilon}(x,y) - p_{t-r+\epsilon'}(x,y)\right)^2\varrho_r(dy)dr.
\end{aligned}
\tag{4.24}
$$

Splitting the time-integral, the second summand on the r.h.s. of (4.24) has the bound

$$
\frac{1}{t}c\left(\int_0^{|\epsilon-\epsilon'|}\frac{1}{\sqrt{r}}dr + \int_{|\epsilon-\epsilon'|}^t\int_{\mathbb{R}^1}(p_{t-r+\epsilon}(x,y) - p_{t-r+\epsilon'}(x,y))^2\varrho_r(dy)dr\right)
$$

which, according to Lemma A.2, can be estimated by $\frac{1}{t}c(\sqrt{|\epsilon-\epsilon'|} + |\epsilon-\epsilon'|^{\alpha/2})$. Hence,

$$
\|X^\epsilon - X^{\epsilon'}\|_{2,\beta} \longrightarrow 0, \qquad \epsilon,\epsilon' \downarrow 0
$$

and thus, (X^ϵ) is Cauchy in L^2_β. □

Proof of (c) Let $(T_n)_{n\ge1}$ be an increasing sequence with $T_n \to \infty$ as $n \to \infty$. Then, by **(a)**, **(b)** and the completeness of $(L^2_\beta, \|.\|_{2,\beta})$, there exists $(X^{T_n})_{n\ge1} \subset L^2_\beta$ satisfying

$\|X^{\epsilon,T_n} - X^{T_n}\|_{2,\beta} \to 0$ as $\epsilon \downarrow 0$ and $|X^{\epsilon_n,T_n} - X^{T_n}| \to 0$ $\mathbf{P}_\eta^\varrho(d\omega)dxt^\beta dt$-a.e. as $\epsilon_n \downarrow 0$ for some $(\epsilon_n) \subset (\epsilon)$, $\forall n \geq 1$. W.l.o.g. $(\epsilon_n) \subset (\epsilon_{n-1})$, $\forall n \geq 1$. Note that X^{T_n} agrees with X^{T_k} $\mathbf{P}_\eta^\varrho(d\omega)dxt^\beta dt$-a.e. on $\Omega \times [-T_k, T_k] \times [0, T_k]$ for all $k = 1, \ldots, n-1$, $\forall n \geq 1$. One can construct a $\mathbf{P}_\eta^\varrho(d\omega)dxt^\beta dt$-null set N and a sequence $(\tilde{\epsilon})$ such that $X^{T_n} = X^{T_k}$ on $(\Omega \times [-T_k, T_k] \times [0, T_k]) \cap N^c$ for all $k = 1, \ldots, n-1$ and $(\tilde{\epsilon}) \subset (\epsilon_n)$, $\forall n \geq 1$. Then it is not hard to show that $X_t(x,\omega) :=$

$$\begin{cases} X_t^{T_n}(x,\omega) \ \left(= \lim_{\tilde{\epsilon}\downarrow 0} X_t^{\tilde{\epsilon},T_n}(x,\omega)\right) & , \quad (t,x,\omega) \in \Big[[0,T_n] \times [-T_n,T_n] \times \Omega\Big] \cap N^c \\ \hfill 0 & , \quad (t,x,\omega) \in N \end{cases}$$

$(t,x,\omega) \in [0,\infty) \times \mathbb{R}^1 \times \Omega$, provides the wanted X. □

Proof of (d) Let X be the function from **(c)**. For each $\epsilon > 0$ and $\psi \in C_c(\mathbb{R}^1, \mathbb{R}_+)$,

$$\begin{aligned} &\mathbf{E}_\eta^\varrho\left[|\langle \bar{X}_t, \psi\rangle - \langle X_t, \psi\rangle|^2\right] \\ &\quad\leq c\mathbf{E}_\eta^\varrho\left[|\langle \bar{X}_t, \psi\rangle - \langle X_t^\epsilon, \psi\rangle|^2\right] + c\mathbf{E}_\eta^\varrho\left[|\langle X_t^\epsilon, \psi\rangle - \langle X_t, \psi\rangle|^2\right] \\ &\quad\leq c\mathbf{E}_\eta^\varrho\left[|\langle \bar{X}_t, \psi - S_\epsilon\psi\rangle|^2\right] + c\mathbf{E}_\eta^\varrho\left[\int_{\mathbb{R}^1} \psi^2(x)dx \int_{\mathbb{R}^1} \left(X_t^\epsilon(x) - X_t(x)\right)^2 dx\right]. \end{aligned} \tag{4.25}$$

The first summand on the r.h.s. of (4.25) converges to 0 as $\epsilon \downarrow 0$ by dominated convergence. The second summand can be bounded by

$$c_\psi \int_{\mathbb{R}^1} \mathbf{E}_\eta^\varrho\left[\left(X_t^\epsilon(x) - X_t(x)\right)^2\right] dx$$

and so, by **(c)**, tends to 0 as $\epsilon \downarrow 0$ for dt-a.a. $t > 0$. Hence, the l.h.s. of (4.25) equals 0 yielding

$$\mathbf{P}_\eta^\varrho\left[\langle \bar{X}_t, \psi\rangle = \langle X_t, \psi\rangle\right] = 1$$

for dt-a.a. $t > 0$. But from here, using standard arguments, one easily derives

$$\mathbf{P}_\eta^\varrho\left[\bar{X}_t(dx) = X_t(x)dx \text{ for } dt\text{-a.a. } t > 0\right] = 1,$$

which was the claim. □

Proof of (e) First of all we state a lemma whose proof is postponed to the end of this section. Recall that ϱ satisfies (**M**).

Lemma 4.1 *Consider $m \geq 1$ and $0 < t_0 < T$. Then, for $(t,x) \in [t_0,T] \times \mathbb{R}^1$ and $\epsilon \in (0,1]$,*

$$\mathbf{E}_\eta^\varrho\left[\langle \bar{X}_t, p_\epsilon(x,.)\rangle^m\right] \leq c_{m,t_0,T}.$$

For each $t_0 > 0$, the Green's function representation from Proposition 2.7 and **(d)** yield

$$\langle X_{t_0}, S_{t-t_0}\psi\rangle = \langle \eta, S_{t_0}S_{t-t_0}\psi\rangle + \int_0^{t_0}\int_{\mathbb{R}^1} S_{t_0-r}S_{t-t_0}\psi(y)dM_{y,r} \tag{4.26}$$

and

$$\langle X_t, \psi \rangle = \langle \eta, S_t \psi \rangle + \int_0^t \int_{\mathbb{R}^1} S_{t-r}\psi(y) dM_{y,r} \tag{4.27}$$

$\mathbf{P}_\eta^\varrho$-almost surely for every $t \geq t_0$ and $\psi \in C_c(\mathbb{R}^1, \mathbb{R}_+)$. Inserting (4.26) in (4.27) leads to

$$\langle X_t, \psi \rangle = \langle X_{t_0}, S_{t-t_0} \psi \rangle + \int_{t_0}^t \int_{\mathbb{R}^1} S_{t-r}\psi(y) dM_{y,r} \tag{4.28}$$

$\mathbf{P}_\eta^\varrho$-almost surely for every $t \geq t_0$ and $\psi \in C_c(\mathbb{R}^1, \mathbb{R}_+)$. Using the stochastic Fubini theorem, see e.g. [Wal86] p.296, we conclude from (4.28)

$$X_t(x) = S_{t-t_0} X_{t_0}(x) + \int_{t_0}^t \int_{\mathbb{R}^1} p_{t-r}(x,y) dM_{y,r} \tag{4.29}$$

$\mathbf{P}_\eta^\varrho$-almost surely for every $t \geq t_0$ and $x \in \mathbb{R}^1$.

Let us denote the second summand on the r.h.s of (4.29) by $Z_t(x)$. Clearly, the first summand on the r.h.s of (4.29) is continuous in (t,x) on $(t_0,\infty) \times \mathbb{R}^1$, $\mathbf{P}_\eta^\varrho$-almost surely. Assume $Z = (Z_t(x))_{t,x}$ has a continuous modification in (t,x) on $(t_0,\infty) \times \mathbb{R}^1$, then the same is true for X. Thus, in order to complete the proof, we only have to check the existence of a jointly continuous modification of Z. It is of course enough to obtain a continuous modification on $[t_0, T] \times \mathbb{R}^1$ for every $T > t_0$. For this we establish some moment estimates for space-time increments and conclude the desired modification by exploiting Kolmogorov's theorem.

Let $x, x' \in \mathbb{R}^1$ and $t, t' \in [t_0, T]$ be close together, w.l.o.g. $t < t'$ and set $p_u \equiv 0$ for $u < 0$. Then, using the Burkholder-Davis-Gundy inequality applied to the martingale

$$u \longmapsto \int_{t_0}^u \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right) dM_{y,r}, \quad t \leq u \leq t',$$

recalling (2.17) in Proposition 2.7 and using (ii) of Proposition 2.5, we estimate

$$\begin{aligned}
&\mathbf{E}_\eta^\varrho \left[\left| Z_t(x) - Z_{t'}(x') \right|^{2m} \right] \qquad (4.30)\\
&= \mathbf{E}_\eta^\varrho \left[\left| \int_{t_0}^{t'} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right) dM_{y,r} \right|^{2m} \right] \\
&\leq c_m \mathbf{E}_\eta^\varrho \left[\left(\int_{t_0}^{t'} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right)^2 L_{[\bar{X},\varrho]}(dydr) \right)^m \right] \\
&= c_m \mathbf{E}_\eta^\varrho \left[\left(\lim_{\epsilon \downarrow 0} \int_{t_0}^{t'} \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right)^2 p_\epsilon(y,z) \bar{X}_r(dz) \varrho_r(dy) dr \right)^m \right].
\end{aligned}$$

Applying Fatou's lemma and Hölder's inequality ($\frac{m-1}{m} + \frac{1}{m} = 1$), this can be bounded by

$$\begin{aligned}
&c_m \left(\int_{t_0}^{t'} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right)^2 \varrho_r(dy) dr \right)^{m-1} \times \\
&\qquad \liminf_{\epsilon \downarrow 0} \int_{t_0}^{t'} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x',y) \right)^2 \mathbf{E}_\eta^\varrho \left[\langle \bar{X}_r, p_\epsilon(y,.) \rangle^m \right] \varrho_r(dy) dr
\end{aligned}$$

and so, using Lemma 4.1 and Lemma A.2, by

$$c_{t_0,T,m}\left(\int_{t_0}^{t'}\int_{\mathbb{R}^1}\left(p_{t-r}(x,y)-p_{t'-r}(x',y)\right)^2\varrho_r(dy)dr\right)^m$$

$$\leq c_{t_0,T,m}\left(|t-t'|^{\alpha/2}+|x-x'|^{\alpha}\right)^m.$$

Now, for m sufficiently large, an application of Kolmogorov's theorem - see e.g. [Wal86], Corollary 1.2 - provides the wanted continuous modification of Z, whose moduli of continuity can be estimated as in Corollary 3.2. Assertion **(e)** is proved. □

Note that the moduli of continuity of the first summand on the r.h.s. of (4.29) are dominated as in Corollary 3.2 as well. Indeed, it is not hard to check that for each compact sets $I \subset (t_0,\infty)$ and $A \subset \mathbb{R}^1$ there exists a constant $c>0$ such that for all $t,t' \in I$ and $x,x' \in A$,

$$\left|S_{t-t_0}X_{t_0}(x)-S_{t'-t_0}X_{t_0}(x')\right| \leq c\left(|t-t'|+|x-x'|\right)^{1/2}.$$

The desired modification of $\bar{X}$ can be derived easily from the continuous modification of X, the proof of the statement involving an η with a continuous dx-density goes along the same lines with the obvious changes. We only have to verify the above lemma yet.

Proof of Lemma 4.1 Consider $(t,x)\in[t_0,T]\times\mathbb{R}^1$. By Proposition 2.4,

$$\mathbf{E}_\eta^\varrho\left[\langle \bar{X}_t,p_\epsilon(x,.)\rangle^m\right] = m!\sum_{k=1}^m\frac{(-1)^{m+k}}{k!}\sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\dots+n_k=m}}\prod_{i=1}^k\langle\eta,\chi_{n_i}(0,.)\rangle, \tag{4.31}$$

where $\chi_1(s,z)=\mathbf{1}_{[0,t]}(s)p_{t-s+\epsilon}(x,z)$ and $\chi_n(.,.)$, $n\geq 2$, is as in Proposition 2.4. We shall show that $|\chi_n(s,z)|$, for $(s,z)\in[0,t]\times\mathbb{R}^1$ and $n\geq 1$, has the bound

$$\frac{c_{n,T}}{\sqrt{t-s+\epsilon}}e^{-\frac{(x-z)^2}{K_n(t-s+\epsilon)}} \tag{4.32}$$

for some $K_n>0$. Then the claim follows immediately from (4.31).

We proceed by induction. In case $n=1$, the bound (4.32) holds trivially. For $n\geq 2$ we have

$$|\chi_n(s,z)| \leq \frac{1}{2}\int_s^\infty\int_{\mathbb{R}^1}p_{r-s}(y,z)\Big(\sum_{j=1}^{n-1}\left|\chi_j(r,y)\chi_{n-j}(r,y)\right|\Big)\varrho_r(dy)dr.$$

According to the induction assumption the r.h.s. is dominated by

$$c\int_s^t\int_{\mathbb{R}^1}\frac{1}{\sqrt{r-s}}e^{-\frac{(y-z)^2}{2(r-s)}}\left(\sum_{j=1}^{n-1}\frac{c_{j,T}}{\sqrt{t-r+\epsilon}}e^{-\frac{(x-y)^2}{K_j(t-r+\epsilon)}}\frac{c_{n-j,T}}{\sqrt{t-r+\epsilon}}e^{-\frac{(x-y)^2}{K_{n-j}(t-r+\epsilon)}}\right)\varrho_r(dy)dr$$

and so, for suitable $K_n, K_n', K_n'' > 0$, by

$$c_{n,T} e^{-\frac{(x-z)^2}{K_n(t-s+\epsilon)}} \int_s^t \frac{1}{\sqrt{r-s}} \frac{1}{t-r+\epsilon} \int_{\mathbb{R}^1} e^{-\frac{(y-z)^2}{K_n'(r-s)}} e^{-\frac{(x-y)^2}{K_n''(t-r+\epsilon)}} \varrho_r(dy)dr.$$

Splitting the time-integral in $\int_s^{s+(t-s+\epsilon)/2} + \int_{s+(t-s+\epsilon)/2}^t$ and using (**M**), the latter can be bounded by (4.32) and we are done. □

4.2 Proof of Corollary 3.3

Let X be the continuous modification from Theorem 3.1, M the martingale measure from Proposition 2.7 and $(M_t(\psi))_{t\geq 0}$ the martingales from (**MP**).

To show the solution property we proceed as in the proof of Lemma 2.4 of [KS88]. Pick an orthogonal martingale measure $\tilde{W}^\varrho$ being independent of M and having quadratic variation measure $\langle \tilde{W}^\varrho \rangle (dxdt) = \varrho_t(dx)dt$, if necessary on an enlargement of $\bar{X}$'s domain $(\Omega, \mathcal{F})$. Set for all $\psi \in C_c^\infty(\mathbb{R}^1, \mathbb{R}_+)$ and $t \geq t_0$

$$W_t^\varrho(\psi) = \int_{t_0}^t \int_{\mathbb{R}^1} \psi(y) \mathbf{1}_{X_r(y)\neq 0} \frac{1}{\sqrt{X_r(y)}} dM_{y,r} + \int_{t_0}^t \int_{\mathbb{R}^1} \psi(y) \mathbf{1}_{X_r(y)=0} d\tilde{W}_{y,r}^\varrho.$$

Then, using $L_{[\bar{X},\varrho]}(dxdt) = X_t(x)\varrho_t(dx)dt$, which holds by the continuity of X, it is easy to verify that W^ϱ provides an orthogonal martingale measure with $\langle W^\varrho \rangle (dxdt) = \varrho_t(dx)dt$ and satisfying

$$M_t(\psi) = \int_{t_0}^t \int_{\mathbb{R}^1} \psi(y) \sqrt{X_r(y)} dW_{y,r}^\varrho$$

$\mathbf{P}_\eta^\varrho$-almost surely for every $t \geq t_0$ and $\psi \in C_c^\infty(\mathbb{R}^1, \mathbb{R}_+)$. Now it follows immediately from the martingale problem (**MP**) in Proposition 2.6 that (3.20) holds for a.a. t and ψ, $\mathbf{P}_\eta^\varrho$-almost surely. Since X is continuous, (3.20) even holds for all t and ψ, $\mathbf{P}_\eta^\varrho$-almost surely. The predictability of X and (implicitly) Lemma 4.1 guarantee that the involved stochastic integrals are well-defined, i.e. the integrands belong to the usual class of admissible integrands w.r.t. their integrators. The weak uniqueness carries over from the one for the martingale problem. Indeed, let $(\tilde{X}, \tilde{W}^\varrho)$ be a weak solution to equation (1.3). Then,

$$\tilde{M}_t(\psi) := \int_{t_0}^t \int_{\mathbb{R}^1} \psi(y) \sqrt{\tilde{X}_r(y)} d\tilde{W}_{y,r}^\varrho = \langle \tilde{X}_t, \psi \rangle - \langle \tilde{X}_{t_0}, \psi \rangle - \int_{t_0}^t \langle \tilde{X}_r, \tfrac{1}{2}\Delta\psi \rangle dr$$

provides a square-integrable martingale with quadratic variation process

$$\begin{aligned}\langle \tilde{M}(\psi) \rangle_t &= \int_{t_0}^t \int_{\mathbb{R}^1} \psi^2(y) \tilde{X}_r(y) \langle \tilde{W}^\varrho \rangle (dydr) \\ &= \int_{t_0}^t \int_{\mathbb{R}^1} \psi^2(y) \tilde{X}_r(y) \varrho_r(dy)dr = \int_{t_0}^t \int_{\mathbb{R}^1} \psi^2(y) L_{[\tilde{\bar{X}},\varrho]}(dydr)\end{aligned}$$

where $\tilde{\bar{X}}_t(dx) := \tilde{X}_t(x)dx$. Thus, the law of $\tilde{\bar{X}}$ ($\simeq$ law of $\tilde{X}$) solves (**MP**) and is hence unique.

4.3 Proof of Theorem 3.4

Proof of (i) According to Proposition 1.2 of [Kle00a] and the moment formula in Proposition 6(b) of [DF97], one only has to check for each $t > s$ that $z \mapsto S_{t-r}\nu(z)$ is bounded for every $r \in [s,t)$ and that for dx-a.a. $x \in \mathbb{R}^1$,

$$\int_s^t \int_{\mathbb{R}^1} p_{r-s}(x,y)(S_{t-r}\nu)^2(y)\varrho_r(dy)dr < \infty. \tag{4.33}$$

The first task is easy, for the second one note that - even for all $x \in \mathbb{R}^1$ - the l.h.s. of (4.33) can be estimated, with help of $(\mathbf{M}')$ and Hölder's inequality, by

$$\begin{aligned} & c\int_s^t \frac{1}{\sqrt{r-s}}\Big(\int_{\mathbb{R}^1} p_{t-r}(z,y)\nu(dz)\Big)^2 \varrho_r(dy)dr \\ \leq\ & c\int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{t-r}c_\nu \int_{\mathbb{R}^1}\int_{\mathbb{R}^1} e^{-\frac{(z-y)^2}{2(t-r)}}\varrho_r(dy)\nu(dz)dr \\ \leq\ & c_\nu \int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{t-r}h(\sqrt{t-r})dr < \infty. \end{aligned}$$

Proof of (ii) First of all note that, by Lemma 2.1, there is a constant $c > 0$ such that $r \leq c\, h(r)$ for all $r \in (0,1]$. Let us show continuity in z. Fix $\nu \in \mathcal{M}_f(\mathbb{R}^1)$ and $0 \leq s < t$. Then, by (3.21),

$$\begin{aligned} & \Big|U_{s,t}(z|\nu) - U_{s,t}(z^{'}|\nu)\Big| \\ \leq\ & \Big|S_{t-s}\nu(z) - S_{t-s}\nu(z^{'})\Big| + \Big|\int_s^t \int_{\mathbb{R}^1} \Big(p_{r-s}(y,z^{'}) - p_{r-s}(y,z)\Big)\, U_{r,t}^2(y|\nu)\varrho_r(dy)dr\Big|. \end{aligned} \tag{4.34}$$

The first term on the r.h.s. of (4.34) tends to 0 as $|z - z^{'}| \to 0$ since $t > s$. The second term on the r.h.s. of (4.34) - note $U_{r,t}(y|\nu) \leq S_{t-r}\nu(y)$ - is dominated by

$$\begin{aligned} & c\int_s^t \int_{\mathbb{R}^1} \frac{1}{\sqrt{r-s}}\Big|e^{-\frac{(y-z)^2}{2(r-s)}} - e^{-\frac{(y-z^{'})^2}{2(r-s)}}\Big|\frac{1}{t-r}\int_{\mathbb{R}^1} e^{-\frac{(y-a)^2}{t-r}}\nu(da)\varrho_r(dy)dr \\ \leq\ & c_{\nu,t-s}\int_s^{s+(t-s)/2} \frac{1}{\sqrt{r-s}}\int_{\mathbb{R}^1}\Big|e^{-\frac{(y-z)^2}{2(r-s)}} - e^{-\frac{(y-z^{'})^2}{2(r-s)}}\Big|\varrho_r(dy)dr \\ +\ & c_{t-s}\int_{s+(t-s)/2}^t \frac{1}{t-r}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}\Big|e^{-\frac{(y-z)^2}{2(r-s)}} - e^{-\frac{(y-z^{'})^2}{2(r-s)}}\Big|e^{-\frac{(y-a)^2}{t-r}}\varrho_r(dy)\nu(da)dr \\ \leq\ & c_{\nu,t-s}\int_s^{s+(t-s)/2} \frac{1}{\sqrt{r-s}}\int_{\mathbb{R}^1}|z-z^{'}|\frac{|\bar z - y|}{r-s}e^{-\frac{(y-\bar z)^2}{2(r-s)}}\varrho_r(dy)dr \\ +\ & c_{t-s}\int_{s+(t-s)/2}^t \frac{1}{t-r}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}|z-z^{'}||\bar z - y|e^{-\frac{(y-\bar z)^2}{2(r-s)}}e^{-\frac{(y-a)^2}{t-r}}\varrho_r(dy)\nu(da)dr, \end{aligned}$$

where for the second inequality the mean value theorem for differentials was used and $\bar z$ belongs to $[z, z^{'}]$, when w.l.o.g. $z \leq z^{'}$. The first of the latter two summands converges to 0 as $|z-z^{'}|\to 0$ by $(\mathbf{M}')$ and (iii) of Lemma A.1. Using the elementary inequality $x/e^{x^2}\le c$, $x\ge 0$, the second summand can be bounded by

$$c_{t-s}|z-z^{'}|\int_{s+(t-s)/2}^{t}\frac{1}{t-r}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}c\sqrt{2(r-s)}e^{-\frac{(y-a)^2}{t-r}}\varrho_r(dy)\nu(da)dr$$

which, by $(\mathbf{M}')$, converges to 0 as well whenever $|z-z^{'}|\to 0$. Thus, $U_{s,t}(z|\nu)$ is continuous in z.

We now prove continuity in ν on $\mathcal{M}_f(\mathbb{R}^1)$ and in t on $[t_0,T]$ for every $s<t_0\le T$, i.e. on (s,∞). Consider $\nu,\nu^{'}\in\mathcal{M}_f(\mathbb{R}^1)$ and $t,t^{'}\in[t_0,T]$, w.l.o.g. $t\le t^{'}$. Furthermore, choose $t_1\in(s,t_0]$ arbitrarily and set $\tilde{h}(r)=\int_0^r\frac{1}{u}h(u)du$. Note that $\tilde{h}(r)\to 0$ as $r\downarrow 0$. We first verify the following lemma.

Lemma 4.2 *The following inequality holds true,*

$$\begin{aligned}
&\left\|U_{s,t}(.|\nu)-U_{s,t^{'}}(.|\nu^{'})\right\|_\infty^2 &(4.35)\\
\le\ & c_{t_1,T,\nu,\nu^{'}}\Big\{\left\|S_{t-s}\nu(.)-S_{t^{'}-s}\nu^{'}(.)\right\|_\infty^2+\int_{t_1}^{t}\|S_{u-s}\nu(.)-S_{u-s+|t-t^{'}|}\nu^{'}(.)\|_\infty^2du\\
&+\tilde{h}(\sqrt{|t-t^{'}|})^2+\int_s^{t_1}h(\sqrt{u-s})\|U_{t+s-u,t}(.|\nu)-U_{t+s-u,t^{'}}(.|\nu^{'})\|_\infty^2du\Big\}.
\end{aligned}$$

Proof We intend an application of Gronwall's lemma. By (3.21), for $\tau\in[t_1,t]$, $z\in\mathbb{R}^1$,

$$\begin{aligned}
&\left|U_{t+s-\tau,t}(z|\nu)-U_{t+s-\tau,t^{'}}(z|\nu^{'})\right|^2\le c\Big\{\left|S_{\tau-s}\nu(z)-S_{\tau-s+|t-t^{'}|}\nu^{'}(z)\right|^2\\
&+\left(\int_{t+s-\tau}^{t}\int_{\mathbb{R}^1}p_{r-(t+s-\tau)}(y,z)\left(U_{r,t^{'}}^2(y|\nu^{'})-U_{r,t}^2(y|\nu)\right)\varrho_r(dy)dr\right)^2 \qquad (4.36)\\
&+\left(\int_t^{t^{'}}\int_{\mathbb{R}^1}p_{r-(t+s-\tau)}(y,z)U_{r,t^{'}}^2(y|\nu^{'})\varrho_r(dy)dr\right)^2\Big\}.
\end{aligned}$$

According to $U_{r,t}(y|\nu)\le S_{t-r}\nu(y)$, the last summand on the r.h.s. of (4.36) can be estimated by

$$\left(\int_t^{t^{'}}\int_{\mathbb{R}^1}p_{r-(t+s-\tau)}(y,z)(S_{t^{'}-r}\nu^{'})^2(y)\varrho_r(dy)dr\right)^2.$$

Using $(\mathbf{M}')$ and $t_1>s$ one easily bounds the latter term by $c_{t_1,T,\nu^{'}}\tilde{h}(\sqrt{|t-t^{'}|})^2$. By $U_{r,t}(y|\nu)\le S_{t-r}\nu(y)$, $(\mathbf{M}')$ and $t_1>s$, the second term on the r.h.s. of (4.36) is dominated by

$$\left\{\int_{t+s-\tau}^{t}\int_{\mathbb{R}^1}p_{r-(t+s-\tau)}(y,z)\left(S_{t^{'}-r}\nu^{'}(y)+S_{t-r}\nu(y)\right)\varrho_r(dy)\|U_{r,t^{'}}(.|\nu^{'})-U_{r,t}(.|\nu)\|_\infty dr\right\}^2$$

$$\le c_{t_1,\nu,\nu'}\Big\{ \int_{t+s-\tau}^{t+(s-\tau)/2} \frac{h(\sqrt{r-(t+s-\tau)})}{\sqrt{r-(t+s-\tau)}} \|U_{r,t'}(.|\nu') - U_{r,t}(.|\nu)\|_\infty dr \tag{4.37}$$

$$+ \int_{t+(s-\tau)/2}^{t} \Big(\frac{h(\sqrt{t'-r})}{\sqrt{t'-r}} + \frac{h(\sqrt{t-r})}{\sqrt{t-r}}\Big) \|U_{r,t'}(.|\nu') - U_{r,t}(.|\nu)\|_\infty dr \Big\}^2 .$$

Note that $h(\sqrt{u})/\sqrt{u}$ increases as $u \downarrow 0$ and assume w.l.o.g. $h(0+\epsilon) > 0$ for all $\epsilon > 0$. Then Hölder's inequality and a substitution $u = t + s - r$ bound the r.h.s. of (4.37) by

$$c_{t_1,\nu,\nu'}\Big\{ \Big[\int_{t+s-\tau}^{t+\frac{s-\tau}{2}} \frac{h(\sqrt{r-(t+s-\tau)})^2}{r-(t+s-\tau)} dr\Big]^{1/2} \Big[\int_{t+s-\tau}^{t+\frac{s-\tau}{2}} \|U_{r,t}(.|\nu) - U_{r,t'}(.|\nu')\|_\infty^2 dr\Big]^{1/2}$$

$$+ \Big[\int_{t+\frac{s-\tau}{2}}^{t} \frac{h(\sqrt{t-r})}{t-r} dr\Big]^{1/2} \Big[\int_{t+\frac{s-\tau}{2}}^{t} h(\sqrt{t-r}) \|U_{r,t}(.|\nu) - U_{r,t'}(.|\nu')\|_\infty^2 dr\Big]^{1/2} \Big\}^2$$

$$\begin{aligned} &\le c_{t_1,T,\nu,\nu'} \int_{t+s-\tau}^{t} h(\sqrt{t-r}) \|U_{r,t}(.|\nu) - U_{r,t'}(.|\nu')\|_\infty^2 dr \\ &= c_{t_1,T,\nu,\nu'} \int_{s}^{\tau} h(\sqrt{u-s}) \|U_{t+s-u,t}(.|\nu) - U_{t+s-u,t'}(.|\nu')\|_\infty^2 du. \end{aligned} \tag{4.38}$$

Altogether, using the bound for the l.h.s. of (4.36) that was just established,

$$\begin{aligned} &\Big\|U_{t+s-\tau,t}(.|\nu) - U_{t+s-\tau,t'}(.|\nu')\Big\|_\infty^2 \\ &\le c_{t_1,T,\nu,\nu'}\Big\{ \Big\|S_{\tau-s}\nu(.) - S_{\tau-s+|t-t'|}\nu'(.)\Big\|_\infty^2 + \tilde{h}(\sqrt{|t-t'|})^2 \\ &\quad + \int_{s}^{t_1} h(\sqrt{u-s}) \|U_{t+s-u,t}(.|\nu) - U_{t+s-u,t'}(.|\nu')\|_\infty^2 du \Big\} + \\ &\quad c_{t_1,T,\nu,\nu'} \int_{t_1}^{\tau} \|U_{t+s-u,t}(.|\nu) - U_{t+s-u,t'}(.|\nu')\|_\infty^2 du. \end{aligned}$$

Now Gronwall's lemma, applied to

$$\tau \longmapsto \|U_{t+s-\tau,t}(.|\nu) - U_{t+s-\tau,t'}(.|\nu')\|_\infty^2, \quad \tau \in [t_1, t],$$

gives

$$\begin{aligned} &\Big\|U_{t+s-\tau,t}(.|\nu) - U_{t+s-\tau,t'}(.|\nu')\Big\|_\infty^2 \\ &\le c\Big\{ \Big\|S_{\tau-s}\nu(.) - S_{\tau-s+|t-t'|}\nu'(.)\Big\|_\infty^2 + \int_{t_1}^{\tau} \|S_{u-s}\nu(.) - S_{u-s+|t-t'|}\nu'(.)\|_\infty^2 \, e^{c(\tau-u)} du \\ &\quad + \tilde{h}(\sqrt{|t-t'|})^2 + \int_{s}^{t_1} h(\sqrt{u-s}) \|U_{t+s-u,t}(.|\nu) - U_{t+s-u,t'}(.|\nu')\|_\infty^2 du \Big\} \end{aligned}$$

for all $\tau \in [t_1, t]$ where $c = c_{t_1,T,\nu,\nu'}$. Setting $\tau := t$ we reach (4.35). □

In order to complete the proof of (ii), we only have to show that the r.h.s. of (4.35) converges to 0 as $t \to t'$ and $\nu \to \nu'$. The first and the second summand on the r.h.s. of (4.35) converges to 0 since $t_1 > s$, and the third summand anyway. Hence, it remains to prove that the last summand on the r.h.s. of (4.35), which is henceforth denoted by $a(t, t', \nu, \nu' | t_1)$, tends to 0 as $t \to t'$ and $\nu \to \nu'$. As in the proof of Lemma 4.2 one can bound $a(t, t', \nu, \nu' | t_1)$ by

$$\begin{aligned} & c \int_s^{t_1} h(\sqrt{u-s}) \| S_{u-s}\nu(.) - S_{u-s+|t-t'|}\nu'(.)\|_\infty^2 du \quad + \quad c_T \tilde{h}(\sqrt{|t-t'|})^2 \\ & + \quad c \int_s^{t_1} h(\sqrt{u-s}) \Big\| \int_{t+s-u}^{t} \int_{\mathbb{R}^1} p_{r-(t+s-u)}(y, \bullet)(S_{t-r}\nu(y) + S_{t'-r}\nu'(y))\varrho_r(dy) \\ & \quad \times \| U_{r,t}(.|\nu) - U_{r,t'}(.|\nu')\|_\infty dr \Big\|_\infty^2 du. \end{aligned} \tag{4.39}$$

The second summand in (4.39) obviously tends to 0 as $t \to t'$, $\nu \to \nu'$ and the first summand, too. Indeed, the latter can be estimated by

$$\begin{aligned} & c \int_s^{t_1} h(\sqrt{u-s}) \| S_{u-s}\nu(.) - S_{u-s}\nu'(.)\|_\infty^2 du \\ & \qquad + \quad c \int_s^{t_1} h(\sqrt{u-s}) \| S_{u-s}\nu'(.) - S_{u-s+|t-t'|}\nu'(.)\|_\infty^2 du. \end{aligned} \tag{4.40}$$

The first summand in (4.40) is bounded by

$$c \int_s^{t_1} \frac{1}{u-s} h(\sqrt{u-s}) \Big| \langle \nu, \mathbf{1} \rangle - \langle \nu', \mathbf{1} \rangle \Big|^2 du$$

which, recalling $(\mathbf{M'})$, tends to 0 whenever ν and ν' approach each other w.r.t. the weak topology. Using inequality (A.43), the second summand in (4.40) can be estimated by

$$\begin{aligned} & c_{\nu'} \int_0^{t_1 - s} h(\sqrt{u}) \frac{|t-t'|}{u(u+|t-t'|)} du \\ & \quad \le c_{\nu'} \int_0^{|t-t'|} \frac{1}{u} h(\sqrt{u}) \frac{|t-t'|}{u+|t-t'|} du + c_{\nu'} |t-t'| \frac{h(\sqrt{|t-t'|})}{\sqrt{|t-t'|}} \int_{|t-t'|}^{t_1-s} \frac{1}{u^{3/2}} du. \end{aligned}$$

But the two summands of the latter estimate converge to 0 as $|t-t'| \to 0$; the first one since it has the bound $c_{\nu',T}\tilde{h}(\sqrt{|t-t'|})$, the second one because it can be bounded by $c_{\nu',T} h(\sqrt{|t-t'|})$.

Let $b(t, t', \nu, \nu' | t_1)$ denote the sum of the first two summands in (4.39). Then, as just established,

$$b(t, t', \nu, \nu' | t_1) \longrightarrow 0 \quad \text{as } t \to t', \nu \to \nu'. \tag{4.41}$$

Assume there was a $t_1 \in (s, t_0]$ such that the last summand in (4.39) is dominated by $\frac{1}{2} a(t, t', \nu, \nu' | t_1)$. Then we had

$$0 \le a(t, t', \nu, \nu' | t_1) \le b(t, t', \nu, \nu' | t_1) + \frac{1}{2} a(t, t', \nu, \nu' | t_1)$$

and so, by (4.41), $a(t,t^{'},\nu,\nu^{'}|t_1) \to 0$ as $t \to t^{'}$ and $\nu \to \nu^{'}$, yielding the claim.

We conclude the proof with establishing, for t_1 sufficiently close to s, the bound $\frac{1}{2}a(t,t^{'},\nu,\nu^{'}|t_1)$ for the last summand in (4.39). Similar to obtaining the bound (4.38) for the l.h.s. of (4.37), the last summand on the r.h.s. of (4.39) can be estimated by

$$c_{T,\nu,\nu'} \int_s^{t_1} h(\sqrt{u-s}) \frac{1}{u-s} \int_{t+s-u}^{t} h(\sqrt{t-r}) \|U_{r,t}(.|\nu) - U_{r,t'}(.|\nu^{'})\|_\infty^2 dr du. \tag{4.42}$$

Here the factor $\frac{1}{u-s}$ occurs additionally since the lower bound of the domain of the inner integral can reach t. (4.42) is trivially dominated by

$$c_{T,\nu,\nu'} \int_{t+s-t_1}^{t} h(\sqrt{t-r}) \|U_{r,t}(.|\nu) - U_{r,t'}(.|\nu^{'})\|_\infty^2 dr \int_s^{t_1} \frac{1}{u-s} h(\sqrt{u-s}) du$$

and so, substituting $v = t+s-r$ and recalling $(\mathbf{M}')$, by

$$c_{T,\nu,\nu'} \int_s^{t_1} h(\sqrt{v-s}) \|U_{t+s-v,t}(.|\nu) - U_{t+s-v,t'}(.|\nu^{'})\|_\infty^2 dv \quad \tilde{h}(\sqrt{t_1 - s})$$

$$= \quad c_{T,\nu,\nu'} \quad a(t,t^{'},\nu,\nu^{'}|t_1) \quad \tilde{h}(\sqrt{t_1 - s}).$$

Choosing t_1 sufficiently close to s, the latter can be bounded by $\frac{1}{2}a(t,t^{'},\nu,\nu^{'}|t_1)$. We are done.

4.4 Proof of Corollary 3.5

Of course, the only thing to check is relation (3.22). By the continuity of X,

$$\mathbf{E}_{s,\eta}^{\varrho}\left[e^{-\sum_{i=1}^k \theta_i X_t(x_i)}\right] = \lim_{\epsilon \downarrow 0} \mathbf{E}_{s,\eta}^{\varrho}\left[e^{-\sum_{i=1}^k \theta_i \langle X_t, p_\epsilon(x_i,.)\rangle}\right].$$

According to Theorem 3.1, (2.10) and Theorem 3.4 (*ii*), the equation continues

$$= \lim_{\epsilon \downarrow 0} e^{-\left\langle \eta, U_{s,t}\left(.|\sum_{i=1}^k \theta_i p_\epsilon(x_i,.)\right)\right\rangle} = e^{-\left\langle \eta, U_{s,t}\left(.|\sum_{i=1}^k \theta_i \delta_{x_i}(dx)\right)\right\rangle}.$$

A Appendix

Here we are going to establish two useful lemmas.

Lemma A.1 *Let $\varrho(dy)$ be from $\mathcal{M}(\mathbb{R}^1)$ with $\sup_{x\in\mathbb{R}^1} \varrho(B(x,1)) < \infty$, $x \in \mathbb{R}^1$ and $h : [0,\infty) \to [0,\infty)$ a continuous, non-decreasing function satisfying h(0)=0 and $\int_0 \frac{1}{r} h(r) dr < \infty$. Consider,*

(i) $\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho(dy) \le c\ h(\sqrt{r}), \quad r \in (0,1]$

(ii) $\varrho(B(x,r)) \le c\ h(r), \quad r \in (0,1]$

(iii) $\int_{\mathbb{R}^1} |x-y| e^{-\frac{(x-y)^2}{r}} \varrho(dy) \le c\ \sqrt{r}\ h(\sqrt{r}), \quad r \in (0,1]$

(iv) $\int_{\mathbb{R}^1} |x-y|^2 e^{-\frac{(x-y)^2}{r}} \varrho(dy) \le c\ r\ h(\sqrt{r}), \quad r \in (0,1].$

(v) $\int_{B(x,1)} |x-y|^{-\alpha} \varrho(dy) < \infty.$

Then, $(i) \Leftrightarrow (ii) \Rightarrow (iii), (iv)$. *In case* $h(r) = c\, r^{\alpha}$, $\alpha \in (0,1]$, $(v) \Rightarrow (ii)$.

Note that by Lemma 2.1 there is a constant $c > 0$ such that $r \le c\ h(r)$ for all $r \in (0,1]$.

Proof (i)⇒(ii) Assuming (i) we trivially get for $r \in (0,1]$

$$e^{-1} \varrho(B(x,\sqrt{r})) = \int_{\mathbb{R}^1} e^{-1} \mathbf{1}_{B(x,\sqrt{r})}(y) \varrho(dy) \le \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho(dy) \le c\ h(\sqrt{r}).$$

(ii)⇒(i) One easily calculates

$$\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho(dy) = \int_0^\infty \varrho(y : e^{-\frac{(x-y)^2}{r}} \ge u) du = \int_0^1 \varrho(B(x, (r \log \frac{1}{u})^{1/2})) du.$$

Substituting $s = \log \frac{1}{u}$ and applying (ii), we get for $r \in (0,1]$

$$\begin{aligned}
\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho(dy) &= \int_0^\infty e^{-s} \varrho(B(x, \sqrt{sr})) ds \\
&\le \int_0^1 c\ h(\sqrt{sr}) ds + \int_1^\infty e^{-s} c \sqrt{sr} ds \ \le\ c\ h(\sqrt{r}) + c\sqrt{r} \ \le\ c\ h(\sqrt{r}).
\end{aligned}$$

(ii)⇒(iv) The integral in (iv) equals

$$\int_0^\infty \varrho\Big(y : |x-y|^2 e^{-\frac{(x-y)^2}{r}} \ge u\Big) du = \int_0^{r/e} \varrho\Big(y : \frac{|x-y|^2}{r} e^{-\frac{(x-y)^2}{r}} \ge \frac{u}{r}\Big) du.$$

According to the elementary inequality $z^2 e^{-z^2} \le 2e^{-z}$, $z \ge 0$, the latter term is bounded by

$$\begin{aligned}
&\int_0^{r/e} \varrho\Big(y : 2e^{-\frac{|x-y|}{\sqrt{r}}} \ge \frac{u}{r}\Big) du \\
&\le \int_0^{r/e} \varrho\Big(y : \sqrt{r} \log \frac{2r}{u} \ge |x-y|\Big) du \ = \ \int_0^{r/e} \varrho\Big(B(x, \sqrt{r} \log \frac{2r}{u})\Big) du.
\end{aligned}$$

Substituting $s = \sqrt{r} \log \frac{2r}{u}$ and recalling $r \le c\, h(r)$, the inequality continues for $r \in (0,1]$

$$\begin{aligned}
&\le \int_{2\sqrt{r}}^\infty \varrho(B(x,s)) 2\sqrt{r} e^{-s/\sqrt{r}} ds \ \le\ 2\sqrt{r} \int_{2\sqrt{r}}^1 c\ h(s) e^{-s/\sqrt{r}} ds + 2\sqrt{r} \int_1^\infty c\ s e^{-s/\sqrt{r}} ds \\
&\le c\sqrt{r} h(2\sqrt{r}) \int_{2\sqrt{r}}^1 \frac{h(s)}{h(2\sqrt{r})} e^{-s/\sqrt{r}} ds + c\, r^{3/2} \int_1^\infty e^{-a} a\ da \\
&\le c\sqrt{r} h(\sqrt{r}) \int_{2\sqrt{r}}^1 c \frac{s}{2\sqrt{r}} e^{-s/\sqrt{r}} ds + c\, r^{3/2} \ \le\ c\, h(\sqrt{r}) r \int_0^\infty a e^{-a} da + c\, r^{3/2} \\
&\le c\, h(\sqrt{r}) r + c\, r^{3/2} \ \le\ \tilde{c}\ r\ h(\sqrt{r}).
\end{aligned}$$

(ii)⇒(iii) Can be proved analogously to *(ii)* ⇒ *(iv)*.
(v)⇒(ii) One easily estimates for $r \in (0,1]$

$$c \geq \int_{B(x,1)} |x-y|^{-\alpha} \varrho(dy) \geq \int_{B(x,r)} r^{-\alpha} \varrho(dy) = r^{-\alpha} \varrho(B(x,r))$$

proving the claim. □

Before turning to the next lemma we recall an useful inequality, cf. e.g. (1) of [Del96]. For each $\epsilon > 0$, $0 < t < t^{'}$ and $x, y \in \mathbb{R}^1$,

$$|p_t(x,y) - p_{t'}(x,y)| \leq c_\epsilon \int_t^{t'} \frac{1}{u} p_{(1+\epsilon)u}(x,y) du. \tag{A.43}$$

Lemma A.2 *Let h be as in Lemma A.1, $\varrho = (\varrho_t(dy) : t \geq 0)$ a measurable kernel from $[0,\infty)$ into $\mathcal{M}(\mathbb{R}^1)$ fulfilling (2.6) and I a compact time set. Assume $(\varrho_t(dy) : t \in I)$ satisfies (i) of Lemma A.1 uniformly for all $x \in \mathbb{R}^1$. Then for all $(t,x), (t^{'}, x^{'}) \in I \times \mathbb{R}^1$ with $|t - t^{'}| \leq 1$, $|x - x^{'}| \leq 1$ and w.l.o.g. $t \leq t^{'}$,*

$$\int_0^{t'} \int_{\mathbb{R}^1} \left(p_{t-r}(x,y) - p_{t'-r}(x^{'},y) \right)^2 \varrho_r(dy) dr \leq c_I \left(\tilde{h}(\sqrt{|t-t^{'}|}) + \tilde{h}(|x - x^{'}|) \right) \tag{A.44}$$

where $\tilde{h}(u) := \int_0^u \frac{1}{s} h(s) ds$, $u > 0$, and $p_u :\equiv 0$, $u < 0$.

Note that $\tilde{h}(u) = cu^\alpha$ whenever $h(u) = u^\alpha$, $\alpha \in (0,1]$, and that $\int_0^u \frac{1}{s} h(\sqrt{s}) ds = 2\tilde{h}(\sqrt{u})$.
Proof The l.h.s. of (A.44) is bounded by

$$\begin{aligned} &\int_0^{t'} \int_{\mathbb{R}^1} \left(p_{t'-r}(x^{'},y) - p_{t'-r}(x,y) \right)^2 \varrho_r(dy) dr \\ &+ \int_0^t \int_{\mathbb{R}^1} \left(p_{t'-r}(x,y) - p_{t-r}(x,y) \right)^2 \varrho_r(dy) dr + \int_t^{t'} \int_{\mathbb{R}^1} p^2_{t'-r}(x,y) \varrho_r(dy) dr. \end{aligned} \tag{A.45}$$

The last term in (A.45) can be rewritten as

$$c \int_0^{|t-t^{'}|} \frac{1}{r} \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \varrho_{t-r}(dy) dr$$

which, by the assumptions, is dominated by $c_I \tilde{h}(\sqrt{|t-t^{'}|})$.
The middle term in (A.45) equals

$$\begin{aligned} &\int_0^{t-|t-t^{'}|} \int_{\mathbb{R}^1} \left(p_{t'-r}(x,y) - p_{t-r}(x,y) \right)^2 \varrho_r(dy) dr \\ &+ \int_{t-|t-t^{'}|}^t \int_{\mathbb{R}^1} \left(p_{t'-r}(x,y) - p_{t-r}(x,y) \right)^2 \varrho_r(dy) dr. \end{aligned} \tag{A.46}$$

The second summand in (A.46) can be estimated by

$$c\int_{t-|t-t'|}^{t}\Big(\frac{1}{t-r}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{t-r}}\varrho_r(dy) + \frac{1}{t'-r}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{t'-r}}\varrho_r(dy) + \frac{1}{t-r}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{2(t-r)}}\varrho_r(dy)\Big)dr$$

and so, using the assumptions, by $c_I\tilde{h}(\sqrt{|t-t'|})$. Let $\epsilon>0$, then inequality (A.43) bounds the first summand in (A.46) by

$$c_\epsilon\int_0^{t-|t-t'|}\int_{\mathbb{R}^1}\Big(\int_{t-r}^{t'-r}\frac{1}{u}p_{(1+\epsilon)u}(x,y)du\Big)^2\varrho_r(dy)dr.$$

Again exploiting the assumption, this can be estimated by

$$\begin{aligned}
& c\int_0^{t-|t-t'|}\int_{\mathbb{R}^1}\Big(\int_{t-r}^{t'-r}\frac{1}{u^{3/2}}e^{-\frac{(x-y)^2}{2(1+\epsilon)(t'-r)}}du\Big)^2\varrho_r(dy)dr \\
\leq\ & c\int_0^{t-|t-t'|}\Big(\int_{t-r}^{t'-r}\frac{1}{u^{3/2}}du\Big)^2\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{(1+\epsilon)(t'-r)}}\varrho_r(dy)dr \\
\leq\ & c\int_0^{t-|t-t'|}\Big((t-r)^{-1/2}-(t'-r)^{-1/2}\Big)^2 c_I h(\sqrt{t'-r})dr \\
\leq\ & c_I\int_0^{t-|t-t'|}\frac{|t-t'|}{(t-r)(t'-r)}h(\sqrt{t'-r})dr \\
\leq\ & c_I|t-t'|\frac{h(\sqrt{t'-t})}{\sqrt{t'-t}}\int_0^{t-|t-t'|}\frac{1}{(t-r)^{3/2}}dr \\
\leq\ & c_I h(\sqrt{|t-t'|}) \quad \leq \quad c_I\tilde{h}(\sqrt{|t-t'|}).
\end{aligned}$$

Therefore, (A.46) is dominated by $c_I\tilde{h}(\sqrt{|t-t'|})$.

The first term in (A.45) is smaller than

$$\begin{aligned}
& c\int_{t'-|x-x'|^2}^{t'}\frac{1}{t'-r}\int_{\mathbb{R}^1}\Big(e^{-\frac{(x'-y)^2}{t'-r}}+e^{-\frac{(x-y)^2}{t'-r}}\Big)\varrho_r(dy)dr \\
+\ & c\int_0^{t'-|x-x'|^2}\frac{1}{t'-r}\int_{\mathbb{R}^1}\Big(e^{-\frac{(x'-y)^2}{t'-r}}-e^{-\frac{(x-y)^2}{t'-r}}\Big)^2\varrho_r(dy)dr.
\end{aligned} \tag{A.47}$$

Using the assumptions, the first summand in (A.47) can be bounded immediately by $c_I\tilde{h}(|x-x'|)$. According to the mean value theorem for differentials, the second summand in (A.47) has the bound

$$c\int_{|x-x'|^2}^{t'}\frac{1}{r}\int_{\mathbb{R}^1}\Big(|x-x'|\frac{2|\bar{x}-y|}{r}e^{-\frac{(\bar{x}-y)^2}{r}}\Big)^2\varrho_{t'-r}(dy)dr \tag{A.48}$$

for some $\bar{x}$ between x and $x^{'}$. Then, by Lemma A.1 $(i) \Rightarrow (iv)$, (A.48) is dominated by

$$c_I|x-x^{'}|^2\int_{|x-x^{'}|^2}^{t^{'}}\frac{1}{r^3}\,r\,h(\sqrt{r})dr \;\leq\; c_I|x-x^{'}|^2\frac{h(\sqrt{|x-x^{'}|^2})}{\sqrt{|x-x^{'}|^2}}\int_{|x-x^{'}|^2}^{t^{'}}\frac{1}{r^{3/2}}dr$$

$$\leq c_I h(|x-x^{'}|) \quad\leq\quad c_I\tilde{h}(|x-x^{'}|).$$

Altogether we have the desired bound for (A.45), i.e. for the l.h.s. of (A.44). □

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