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Second kind similarity solutions of the modified porous medium equation

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Abstract

We consider the problem of a spreading ground water mound of liquid in a porous medium, situated on an impermeable horizontal solid layer. The mathematical formulation for this problem is given by the modified porous medium equation. We derive a global condition in form of an energy integral, describing the loss of liquid in the porous medium. This yields the necessary condition that determines the similarity exponents for the similarity solution of second kind, describing the long time behavior of the mound. We further apply our method to the problem when instead of an energy integral another conservation law, such as the first moment integral, obeyed by a family of antisymmetric solutions, is violated. Here, we consider as an application the problem of the impact of a flood infiltrating a porous medium. In all cases we will also solve and compare our analytical results with our numerical solution, and, if available, with examples existing in the literature.

1 Introduction

Suppose that outside the liquid, the porous medium is occupied by a gas. Under the action of gravity the liquid spreads and displaces the gas at some points, while at other points, pores that were previously filled with liquid are being occupied by the gas. In the mound at some (initial) time t_0 the liquid saturation is supposed to be equal to σ_+. Due to capillary forces some liquid remains in the pores that were originally filled with it. Let this residual saturation of the liquid be equal to σ_-.

As suggested in figure 1, let us assume, that the volume originally occupied by the liquid is radially symmetric, with its bounding interface denoted by $h(r,t)$. Furthermore, for radially symmetric flow $\partial h/\partial r < 0$, except when $r = 0$, where due to the symmetry $\partial h/\partial r = 0$. Since the flow can be assumed slow and laminar, with negligible vertical velocity component, the Navier-Stokes equations reduce to the hydrostatic law for the vertical pressure $p = \rho g\,(h - z)$, where ρ is the liquid density and g the gravitational constant. Then, according to Darcy's law the flux of liquid through a cylindrical surface of area $2\pi r\,h$ is

$$v = -\frac{\kappa}{\mu}\, 2\pi\, r\, h\, \frac{\partial p}{\partial r} = -\frac{\kappa \rho g \pi}{\mu}\, r \frac{\partial h^2}{\partial r}\,. \tag{1.1}$$

Hence, since the rate of change in the volume of the liquid mound, due to the decreasing saturation from σ_+ to σ_-, is equal to the change in the flux of the liquid, the governing equation is

$$\frac{\partial h}{\partial t} = \frac{m_-}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right)\,, \qquad \text{for} \qquad \frac{\partial h}{\partial t} < 0\,, \tag{1.2}$$

where

$$m_- = \frac{\kappa \rho\, g}{2m\mu(\sigma_+ - \sigma_-)}\,. \tag{1.3}$$

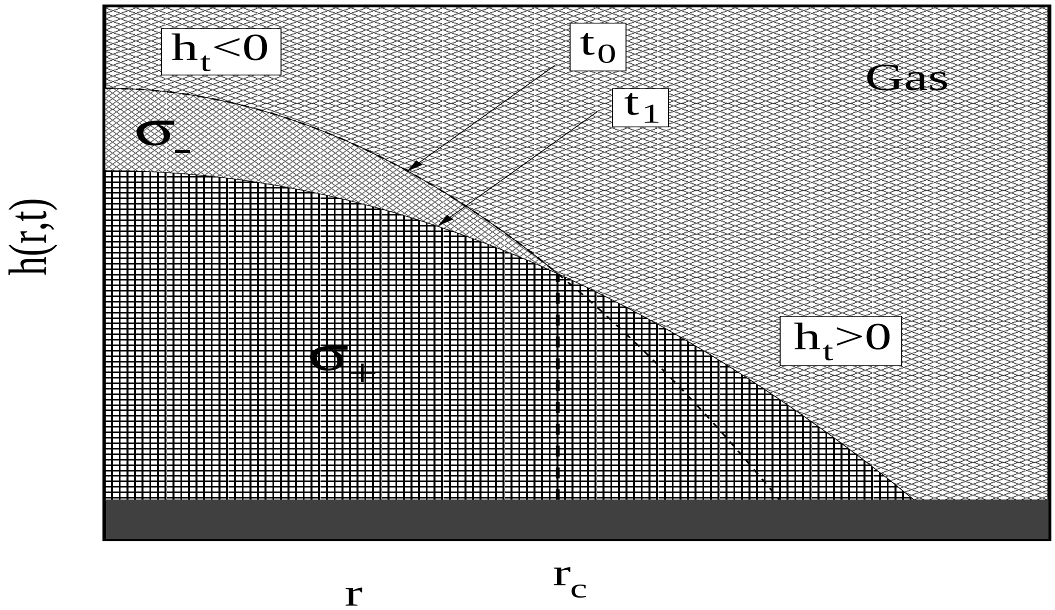

Figure 1: The spreading groundwater mound in porous medium.

We observe from figure 1, that this equation is only valid for a certain part of the radial domain, and that there is for each point in time a unique r_c, beyond which the liquid saturation is increasing from zero to σ_+. This part is governed by the equation

$$\frac{\partial h}{\partial t} = \frac{m_+}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right), \qquad \text{for} \qquad \frac{\partial h}{\partial t} > 0\,, \tag{1.4}$$

with

$$m_+ = \frac{\kappa\rho\, g}{2m\mu\sigma_+}\,. \tag{1.5}$$

After nondimensionalization via

$$r^* = \frac{r}{R}\,, \qquad t^* = \frac{m_+ Q^k}{R^{k\,d+2}}\,t\,, \qquad h = \frac{Q}{R^d}\,h^*$$

for $d = 2$ and $k = 1$, where R is the characteristic length scale in radial direction, and Q is the volume initially contained in the liquid mound, equations (1.2) and (1.4) take on the form, after dropping the $'\,*\,'$:

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right), \qquad \text{for} \qquad \frac{\partial h}{\partial t} > 0 \tag{1.6}$$

$$\frac{\partial h}{\partial t} = \frac{1+\varepsilon}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right), \qquad \text{for} \qquad \frac{\partial h}{\partial t} < 0 \tag{1.7}$$

with the boundary conditions

$$\frac{\partial h}{\partial r} = 0 \quad \text{at} \qquad r = 0\,, \quad h = 0 \qquad \text{at} \qquad r = r_0(t) \tag{1.8}$$

and interface conditions of continuity of

$$\frac{\partial h}{\partial r} \quad \text{and} \quad h \quad \text{across} \quad \frac{\partial h}{\partial t} = 0\,. \tag{1.9}$$

Here we have set

$$\frac{m_-}{m_+} = 1 + \varepsilon \quad \text{with} \quad \varepsilon = \frac{\sigma_-}{\sigma_+ - \sigma_-}\,. \tag{1.10}$$

Each of the equations (1.6) and (1.7) is an example of the porous medium equation and via a classic Lie group analysis, which we will illustrate in the following chapter, well-known similarity solutions to this equation of form

$$h(r,t) = \frac{1}{\sqrt{t}}\, f(\eta;\varepsilon)\,, \quad \text{with} \quad \eta = r\, t^{-\frac{1}{4}} \tag{1.11}$$

are obtained. However, due to the differing diffusion coefficients the respective solutions cannot be matched at the interface unless $\varepsilon = 0$. On the other hand we will solve this problem numerically and observe that the solution enters, for large times, a self-similar regime, which differs from (1.11), see also [15].

The reason for this lies in the global condition, here conservation of the fluid mass, which is obeyed if $\varepsilon = 0$. But if $\varepsilon \neq 0$ the jump discontinuity of the coefficient in (1.6) and (1.7) across the free interface $r_c(t)$ renders the integral

$$\frac{\partial}{\partial t} \int_0^{r_0} r\, h(r,t)\, dr \neq 0$$

Indeed, as we will see in more detail later, the very fact that this conservation law is violated prevents the a priori determination of the similarity index. On the other hand, by solving the whole problem numerically, we know there exists such a similarity index. Moreover, for the radially symmetric problem in d-dimensional space and power k,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left[1 + \varepsilon H\left(-\frac{\partial u}{\partial t}\right)\right] \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial u^{k+1}}{\partial r}\right), \qquad 0 < r < r_0 \\ u(r,0) &= F(r)\,, \qquad \int_0^{r_0} r^{d-1} F(r)\, dr = 1\,. \end{aligned} \tag{1.12}$$

Hulshof et al. [20] were able to show that for $k > 0$, $d \geq 1$ the Cauchy problem has a unique solution in a class of compactly supported, non-negative, maximal viscosity solutions. Furthermore, they could show that as $t \to \infty$ every maximal viscosity solution with compactly supported initial data converges to a similarity solution of the second kind.

We will show how the violation of a conservation law can be used to obtain a condition relating the unknown similarity index to a certain value of the corresponding second kind similarity profile.

Here, we will construct the second kind similarity solution to (1.12) by making use of the violation of a conservation law to obtain a global condition relating the unknown similarity index to a certain value of the corresponding second kind similarity profile. The asymptotic method presented here represents a generalization of those developed in [4] for Barenblatt's filtration equation.

2 The perturbation method

In [4] it has been shown how the second kind similarity solution of Barenblatt's filtration equation are obtained if the problem is viewed as a perturbation of a corresponding problem with known similarity solution. More precisely, one can view the Lie group of the problems with second kind similarity as a perturbation of the Lie group of a corresponding problem with known similarity solution. As a consequence of this one obtains the proper perturbation ansatz for such problems, i.e. not only the functions but also the similarity exponents are functions of the perturbation parameter. In our case we make use of the Lie-group of the porous medium equation, see [3] (p. 297). The asymptotic group properties yield the similarity form:

$$u(x,t) = t^{\alpha(\varepsilon)} f(\eta;\varepsilon) \qquad \eta = r t^{\beta(\varepsilon)} \,. \tag{2.1}$$

If we substitute this into (1.12) we obtain

$$\beta(\varepsilon) = -\frac{k\,\alpha(\varepsilon)+1}{2}\,, \tag{2.2}$$

and the similarity problem

$$\alpha(\varepsilon)\eta^{d-1} f(\eta;\varepsilon) - \frac{k\alpha(\varepsilon)+1}{2}\eta^d \frac{df}{d\eta} = (1+\varepsilon)\frac{d}{d\eta}\left(\eta^{d-1}\frac{d f^{k+1}}{d\eta}\right), \qquad 0 \le \eta \le \eta_c \tag{2.3}$$

$$\alpha(\varepsilon)\eta^{d-1} f(\eta;\varepsilon) - \frac{k\alpha(\varepsilon)+1}{2}\eta^d \frac{df}{d\eta} = \frac{d}{d\eta}\left(\eta^{d-1}\frac{d f^{k+1}}{d\eta}\right), \qquad \eta_c \le \eta \le \eta_0\,, \tag{2.4}$$

where η_0 is the point where f vanishes and η_c where the interface condition

$$\frac{d}{d\eta}\left(\eta^{d-1}\frac{d f^{k+1}}{d\eta}\right) = 0 \qquad \text{at} \qquad \eta = \eta_c \tag{2.5}$$

holds.

The normalization takes on the form

$$\int_0^{\eta_0} \eta^{d-1} f(\eta;\varepsilon)\, d\eta = 1 \tag{2.6}$$

In order to determine the similarity index $\alpha(\varepsilon)$ one can make use of a conservation law for the case $\varepsilon = 0$. If $\varepsilon \neq 0$ this conservation law is violated. However, as we observe next, we obtain instead a generalized relation, we call "dissipation law". Since for our problem conservation of mass is violated we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{r_0} r^{d-1}\,u(r,t)\,dr &= \int_0^{r_c}(1+\varepsilon)\,\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial u^{k+1}}{\partial r}\right)\,dr + \int_{r_c}^{r_0}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial u^{k+1}}{\partial r}\right) \\ &= \varepsilon\, r_c^{d-1}\,\frac{\partial u^{k+1}}{\partial r}\,(r_c(t),t) \qquad (2.7)\end{aligned}$$

This condition in conjunction with the similarity form determines the similarity index $\alpha(\varepsilon)$ for $\varepsilon \neq 0$. We observe that (2.7) becomes, in view of (2.1):

$$\frac{\partial}{\partial t}\left(t^{\alpha(\varepsilon)-\beta(\varepsilon)d}\right)\int_0^{\eta_0}\eta^{d-1}\,f(\eta;\varepsilon)\,d\eta = \varepsilon\eta_c^{d-1}t^{\alpha(\varepsilon)(k+1)-\beta(\varepsilon)(d-2)}\frac{d\,f^{k+1}}{d\eta}\,(\eta_c)\ .$$

Hence, in view of (2.6) we obtain the relation

$$\alpha(\varepsilon) + \frac{d}{k\,d+2} = \varepsilon\frac{2}{k\,d+2}\eta_c^{d-1}\frac{d\,f^{k+1}}{d\eta}\,(\eta_c)\ . \qquad (2.8)$$

We assume now that $\varepsilon \ll 1$ and we make the perturbation ansatz

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots\,, \qquad (2.9)$$

$$\beta(\varepsilon) = -\frac{k\alpha_0+1}{2} - \varepsilon\frac{k\alpha_1}{2} - \varepsilon^2\frac{k\alpha_2}{2} - \ldots\,, \qquad (2.10)$$

and

$$f(\eta;\varepsilon) = f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \ldots\,. \qquad (2.11)$$

If we substitute (2.9)-(2.11) into equation (1.12) (or (2.3)-(2.4)), we obtain a sequence of ordinary differential equations, together with the normalization

$$\int_0^{\eta_0}\eta^{d-1}\,f_0(\eta)\,d\eta = 1 \qquad (2.12)$$

$$\int_0^{\eta_0}\eta^{d-1}\,f_i(\eta)\,d\eta = 0 \qquad i = 1,2,\ldots\,. \qquad (2.13)$$

Furthermore, we obtain a sequence of equations from our relation for the similarity index, namely,

$$
\begin{aligned}
\alpha_0 &= -\frac{d}{k\,d+2} && (2.14)\\
\alpha_1 &= \frac{2}{k\,d+2}\,\eta_c^{d-1}\frac{d\,f_0^{k+1}}{d\eta}\,(\eta_c) && (2.15)\\
\alpha_2 &= \frac{2}{k\,d+2}\,\eta_c^{d-1}(k+1)\frac{d}{d\eta}\left(f_0^k f_1\right)|_{\eta=\eta_c}\,. && (2.16)\\
&\vdots
\end{aligned}
$$

Hence. we finally need to determine the values $f_i(\eta_c)$. To leading order we have the similarity problem for the porous medium equation which has the well-known Barenblatt-Pattle solution. Here we obtain

$$\alpha_0\eta^{d-1}\,f_0(\eta)-\frac{k\alpha_0+1}{2}\eta^d\frac{d\,f_0}{d\eta} \;=\; \frac{d}{d\eta}\left(\eta^{d-1}\frac{d\,f_0^{k+1}}{d\eta}\right), \qquad 0\le\eta\le\eta_0\,,$$

which can be written as

$$\left(\frac{k\,d+2}{2}\alpha_0+\frac{d}{2}\right)\eta^{d-1}f_0(\eta)=\frac{d}{d\eta}\left(\eta^{d-1}\frac{d\,f_0^{k+1}}{d\eta}+\frac{k\alpha_0+1}{2}\eta^d\,f_0(\eta)\right) \tag{2.17}$$

This we integrate and recall that $\alpha_0=-d/(kd+2)$ to obtain

$$f_0(\eta)=\left[\frac{k}{2(k\,d+2)(k+1)}\left(\eta_0^2-\eta^2\right)\right]^{\frac{1}{k}}, \tag{2.18}$$

where η_0 is determined through the normalization condition

$$\int_0^{\eta_0}\eta^{d-1}\,f_0(\eta)\,d\eta=\left(\frac{k}{2(k\,d+2)(k+1)}\right)^{\frac{1}{k}}\int_0^{\eta_0}\eta^{d-1}\,\left(\eta_0^2-\eta^2\right)^{\frac{1}{k}}\,d\eta=1\,,$$

thus

$$\eta_0^{\frac{k\,d+2}{k}}=2\left(\frac{2(k\,d+2)(k+1)}{k}\right)^{\frac{1}{k}}B^{-1}\left(\frac{d}{2},\frac{k+1}{k}\right). \tag{2.19}$$

Since the interface condition (2.5) yields

$$\eta_c = \sqrt{\frac{k\,d}{k\,d+2}}\,\eta_0\,, \tag{2.20}$$

we obtain for α_1 the formula

$$\alpha_1 = -\left(\frac{k\,d}{k\,d+2}\right)^{\frac{d}{2}} \left(\frac{2}{k\,d+2}\right)^{\frac{2k+1}{k}} B^{-1}\left(\frac{d}{2}, \frac{k+1}{k}\right)\,. \tag{2.21}$$

where B denotes the Beta-function.

Next, we find the exponents α_i, $i = 2, 3, \ldots$. For this we now have only to solve the linear equations of second order for $f_{i-1}(\eta)$ in their respective interval of validity and require continuity across the interface at η_c, in order to find the value $f_{i-1}(\eta_c)$. We demonstrate this here for $i = 2$.

To $O(\varepsilon)$ we obtain from equation (2.3)-(2.4)

$$\begin{aligned}
\frac{d}{d\eta}\left[\eta^{d-1}\,(k+1)\,\frac{d}{d\eta}\left(f_0^k f_1\right) + \frac{1}{k\,d+2}\eta^d\,f_1\right] &= \alpha_1\left(\eta^{d-1}\,f_0 - \frac{k}{2}\eta^d\,\frac{d\,f_0}{d\eta}\right) - \frac{d}{d\eta}\left(\eta^{d-1}\frac{df_0^{k+1}}{d\eta}\right) \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (2.22) \\
\frac{d}{d\eta}\left[\eta^{d-1}\,(k+1)\,\frac{d}{d\eta}\left(f_0^k f_1\right) + \frac{1}{k\,d+2}\eta^d\,f_1\right] &= \alpha_1\left(\eta^{d-1}\,f_0 - \frac{k}{2}\eta^d\,\frac{d\,f_0}{d\eta}\right) \\
&\qquad \text{for} \quad \eta_c \le \eta \le \eta_0 \qquad (2.23)
\end{aligned}$$

We can now integrate (2.22) from 0 to η and (2.23) from η to η_0 to obtain

$$\begin{aligned}
\eta^{d-1}\,(k+1)\,\frac{d}{d\eta}\left(f_0^k f_1\right) + \frac{1}{k\,d+2}\eta^d\,f_1 &= \alpha_1 \int_0^\eta \eta^{d-1}\,f_0 - \frac{k}{2}\eta^d\,\frac{d\,f_0}{d\eta}\,d\eta - \eta^{d-1}\frac{df_0^{k+1}}{d\eta} \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (2.24) \\
\eta^{d-1}\,(k+1)\,\frac{d}{d\eta}\left(f_0^k f_1\right) + \frac{1}{k\,d+2}\eta^d\,f_1 &= \alpha_1 \int_\eta^{\eta_0} \eta^{d-1}\,f_0 - \frac{k}{2}\eta^d\,\frac{d\,f_0}{d\eta}\,d\eta \\
&\qquad \text{for} \quad \eta_c \le \eta \le \eta_0 \qquad (2.25)
\end{aligned}$$

Note, that now, by evaluating (2.25) at η_c we can determine α_2 from

$$\alpha_2 = -\frac{2}{k\,d+2}\left(\alpha_1\,J_1(\eta_c) + \frac{1}{k\,d+2}\eta_c^d f_1(\eta_c)\right) \tag{2.26}$$

where

$$J_1(\eta) = \left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} \int_\eta^{\eta_0} \lambda^{d-1} \left(\frac{\eta_0^2}{\eta_0^2-\lambda^2}\right)^{\frac{k-1}{k}} d\lambda\,. \tag{2.27}$$

In order to determine $f_1(\eta_c)$ we multiply (2.24)-(2.25) by the factor $\eta^{1-d}\left(\eta_0^2-\eta^2\right)^{-1/k}$ and integrate the resulting equation for the interval $0 \le \eta \le \eta_c$ from 0 to η, and the equation for the interval $\eta_c \le \eta \le \eta_0$ from η_c to η. This yields

$$\begin{aligned} f_1(\eta) &= \left(\frac{\eta_0^2}{\eta_0^2-\eta^2}\right)^{\frac{k-1}{k}} f_1(0) + \frac{2(k\,d+2)}{k\,(\eta_0^2-\eta^2)^{\frac{k-1}{k}}}\left[\alpha_1\, I_2(\eta) - I_3(\eta)\right] \\ &\qquad\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (2.28) \\ f_1(\eta) &= \left(\frac{2}{k\,d+2}\right)^{\frac{k-1}{k}} \left(\frac{\eta_0^2}{\eta_0^2-\eta^2}\right)^{\frac{k-1}{k}} f_1(\eta_c) - \alpha_1 \frac{2(k\,d+2)}{k\,(\eta_0^2-\eta^2)^{\frac{k-1}{k}}} \int_{\eta_c}^{\eta} \frac{J_1(\eta)}{\eta^{d-1}\,(\eta_0^2-\eta^2)^{\frac{1}{k}}} d\eta \\ &\qquad\qquad \text{for} \quad \eta_c \le \eta \le \eta_0 \qquad (2.29) \end{aligned}$$

where

$$\begin{aligned} I_2(\eta) &= \int_0^\eta \frac{I_1(\lambda)}{\lambda^{d-1}\,(\eta_0^2-\lambda^2)^{\frac{1}{k}}} d\lambda, \quad I_1(\eta) = \left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} \int_\eta^{\eta_0} \lambda^{d-1} \left(\frac{\eta_0^2}{\eta_0^2-\lambda^2}\right)^{\frac{k-1}{k}} d\lambda \\ &\text{and} \qquad I_3(\eta) = -\frac{\eta^2}{2(kd+2)} \left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}}\,. \end{aligned} \tag{2.30}$$

In order to eliminate the constant $f_1(0)$ we first multiply (2.28)-(2.29) by η^{d-1} and integrate the first from 0 to η_c and the second from η_c to η_0. After adding both together we can make use of the normalization (2.13) and obtain

$$0 = f_1(0)\, I_6(\eta_c) + \left(\frac{2}{k\,d+2}\right)^{\frac{k-1}{k}} f_1(\eta_c)\, J_6(\eta_c) + \frac{2(kd+2)}{k} \left(\alpha_1[I_5(\eta_c) - J_5(\eta_c)] - I_4(\eta_c)\right)\,, \tag{2.31}$$

where

$$I_4(\eta) = -\frac{1}{2(kd+2)} \left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} \int_0^\eta \frac{\lambda^{d+1}}{(\eta_0^2-\lambda^2)^{\frac{k-1}{k}}} d\lambda\,, \tag{2.32}$$

$$I_5(\eta) = \int_0^\eta \frac{\lambda^{d-1}}{(\eta_0^2-\lambda^2)^{\frac{k-1}{k}}} \left[\int_0^\lambda \frac{I_1(\sigma)}{\sigma^{d-1}\,(\eta_0^2-\sigma^2)^{\frac{1}{k}}} d\sigma\right] d\lambda\,, \tag{2.33}$$

$$J_5(\eta) = \int_\eta^{\eta_0} \frac{\lambda^{d-1}}{(\eta_0^2-\lambda^2)^{\frac{k-1}{k}}} \left[\int_{\eta_c}^\lambda \frac{J_1(\sigma)}{\sigma^{d-1}\,(\eta_0^2-\sigma^2)^{\frac{1}{k}}} d\sigma\right] d\lambda\,, \tag{2.34}$$

and

$$I_6(\eta) = \left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{-\frac{1}{k}} I_1(\eta)\,, \qquad J_6(\eta) = \left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{-\frac{1}{k}} J_1(\eta)\,. \qquad (2.35)$$

Thus, (2.31) together with (2.28) at $\eta = \eta_c$ yields the formula

$$\begin{aligned} f_1(\eta_c) &= \frac{8}{\eta_0^d B\left(\frac{d}{2}, \frac{k+1}{k}\right)} \left(\alpha_1 \left[I_6(\eta_c) I_2(\eta_c) - \eta_0^{\frac{2(k-1)}{k}} \left(I_5(\eta_c) - J_5(\eta_c)\right)\right]\right. \\ &\left. + \eta_0^{\frac{2(k-1)}{k}} I_4(\eta_c) + \frac{kd\eta_0^2}{2(kd+2)^2} \left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} I_6(\eta_c)\right) \end{aligned} \qquad (2.36)$$

In the following section we will apply our results to the problem of the spreading ground water mound in a porous medium, we expounded in the introduction. This will be followed by a comparison with our numerical solution to (2.3)-(2.4).

3 Spreading of a ground water mound

For the problem of the spreading ground water mound, where $k = 1$ and $d = 2$ our results come out particularly simple.

We have :

$$\alpha_0 = -\frac{1}{2}\,, \qquad \eta_0^2 = 8\,, \qquad \eta_c = 1\,, \qquad (3.1)$$

$$f_0 = \begin{cases} \frac{1}{16}\left(8 - \eta^2\right) & \eta \leq \sqrt{8} \\ 0 & \eta > \sqrt{8} \end{cases}$$

and

$$\alpha_1 = -\frac{1}{8} \qquad (3.2)$$

For the integrals we need for $f_1(\eta_c)$ and α_2 we obtain:

$$I_1(\eta) = \frac{1}{4}\eta^2\,, \qquad J_1(\eta) = \frac{1}{4}\left(8 - \eta^2\right)\,, \qquad I_2(\eta) = \frac{1}{\sqrt{8}} \ln\left(\frac{\sqrt{8} + \eta}{\sqrt{8} - \eta}\right)$$

$$I_3(\eta) = -\frac{\eta^2}{128}\,, \quad I_4(\eta) = -\frac{\eta^4}{512}\,, \quad I_6(\eta) = \frac{\eta^2}{2}\,, \quad J_6(\eta) = \frac{1}{8}\left(8 - \eta^2\right)\,.$$

and

$$
\begin{aligned}
I_5(\eta) &= -\frac{1}{16}\left[8\ln 8+\eta^2(\ln 8-1)-\left(8-\eta^2\right)\ln\left(8-\eta^2\right)\right], \\
J_5(\eta) &= \frac{1}{16}\left[8(\ln 8+\ln 4)-\eta^2\ln(\eta^2)-\left(8-\eta^2\right)\ln\left(8-\eta^2\right)-\eta^2\ln 4\right].
\end{aligned}
$$

Hence, we obtain

$$
\alpha_2 = 0.05539339761\,. \tag{3.3}
$$

For this problem there is also another result by Goldenfeld et al. available, who employ a renormalization group method, [5]. However, while the value for α_1 agrees with ours, their value for α_2 does have neither quantitative nor qualitative relations to ours, as well as their numerical solution. A method by Vazques et al.[21], employing the inverse function theorem, computed a value for α_2, which is twice as large as ours, but this difference seems only to result from a different notation for the exponents.

4 Numerical methods and comparisons

In this section, we present a numerical solution of the ground water mound problem. For this purpose, we rewrite the equations (2.3), (2.3), for $k=1$ and $d=2$, in the following form:

$$
f_{\eta\eta} = \frac{\alpha}{2(1+\varepsilon)}-\frac{f_\eta}{\eta}-\frac{f_\eta}{f}\left[f_\eta+\frac{1+\alpha}{4(1+\varepsilon)}\right] \quad \text{for} \quad 0<\eta<\eta_c, \tag{4.1}
$$

$$
f_{\eta\eta} = \frac{\alpha}{2}-\frac{f_\eta}{\eta}-\frac{f_\eta}{f}\left[f_\eta+\frac{1+\alpha}{4}\right] \quad \text{for} \quad \eta_c<\eta<1, \tag{4.2}
$$

with the boundary condition at $\eta=0$,

$$
f_\eta(0)=0\,, \tag{4.3}
$$

and the interface and continuity conditions at $\eta=\eta_c$,

$$
\alpha f(\eta_c)-\frac{1+\alpha}{2}\eta_c f_\eta(\eta_c) = 0, \tag{4.4}
$$

$$
[f]_{\eta_c^-}^{\eta_c^+} = 0, \tag{4.5}
$$

$$
[f_\eta]_{\eta_c^-}^{\eta_c^+} = 0\,. \tag{4.6}
$$

For the boundary condition at the interface to the porous medium, we normalize without loss of generality $\eta_0=1$, such that

$$
f(1)=0, \tag{4.7}
$$

Finally, we find it numerically more convenient to replace the integral condition, or the dissipation law, by an extra boundary condition at the liquid/porous-medium interface $\eta = 1$. There, we require no flux across this boundary. Hence, from (2.4) this results in the condition

$$f_\eta(1) = -\frac{1+\alpha}{4}, \tag{4.8}$$

The problem (4.1)–(4.8) is essentially a two-point boundary value problem for a second order differential equation, but with three instead of two boundary conditions at $\eta = 0$ and $\eta = 1$. The extraneous boundary condition fixes the value of $\alpha(\varepsilon)$, the calculation of which is the main goal in this section.

For this purpose, we convert the second order ODE into a system of first order ODEs using the settings $y_1(\eta) := f(\eta)$ and $y_2(\eta) := f_\eta(\eta)$, which is then solved using the explicit Adams-scheme implemented in the LSODE-package [6]. This code incorporates a local error estimator for $y_i(\eta)$ $i = 1, 2$, and automatically adapts its step-size so that the estimated error is less than $tol * (y_i(\eta) + 1)$, where tol is specified by the user.

The integration is first carried out for (4.2) up to η_c, starting from the right end point. In a second step, the integration is continued to the left with the $f(\eta_c)$ and $f_\eta(\eta_c)$ obtained from the previous run, now using (4.1). Note that η_c has to be determined as part of the first step; since preliminary runs indicated that f was monotone at η_c, this can be done very easily through bisection.

Near the left end point, f will in general not fulfill (4.3); rather, this requirement must be fulfilled in order to determine the similarity exponent $\alpha(\varepsilon)$. It turns out that, near $\eta = 0$, f_η depends monotonically on α, so a bisection method can again be used. This can be seen, for example, from numerical trials.

In both bisection schemes, we started with a rather generous choice for the bracketing interval, making sure that the value of interest was included, then calculated the value of $f(\eta_c)$, for example, and replaced one of the points of the interval, according to the sign of $f(\eta_c)$. This procedure was repeated until both the length of the interval and $f(\eta_c)$ had dropped beneath prescribed tolerances $\triangle\eta_c$ and $\triangle f$. Similarly for $\alpha(\varepsilon)$ and f_η near 0, with tolerances $\triangle\alpha$ and $\triangle f_\eta$ for the length of the bracketing interval and for f_η near zero.

Special attention is required when integrating the ODE near 0 and 1, where η and f vanish, respectively, since these quantities appear in the denominator of certain terms of (4.1) and (4.2). We avoid these regions by starting the integration at an η_r slightly smaller than 1, and using a linear approximation to obtain a good choice for $f(\eta_r)$,

$$f(\eta_r) \approx f(1) - \frac{1+\alpha}{4}(\eta_r - 1) = \frac{1+\alpha}{4}(1 - \eta_r) > 0$$

Likewise, we do not integrate up to zero, but use a small but positive value for the left end point η_l instead.

The numerical trials were carried out using the following standard for the tolerances and truncation parameters,

$$\mathrm{tol} = 10^{-10}, \qquad \eta_l = 10^{-4}, \qquad \eta_c = 1 - 10^{-5}$$

ε	$1-\eta_r=10^{-5}$	$1-\eta_r=10^{-4}$	$1-\eta_r=10^{-3}$	$\eta_l=10^{-3}$	$\eta_l=10^{-2}$
0	-0.500000005	-0.500000014	-0.500001003	-0.500000499	-0.500049962
0.001	-0.500124949	-0.500124959	-0.500125948	-0.500125444	-0.500174898
0.005	-0.500623623	-0.500623633	-0.500624626	-0.500624118	-0.500673534
0.01	-0.501244496	-0.501244506	-0.501245504	-0.501244990	-0.501294359
0.025	-0.503090869	-0.503090879	-0.503091893	-0.503091362	-0.503140590
0.05	-0.506115352	-0.506115362	-0.506116402	-0.506115842	-0.506164840
0.075	-0.509076148	-0.509076158	-0.509077225	-0.509076636	-0.509125409
0.1	-0.511975789	-0.511975800	-0.511976893	-0.511976275	-0.512024827

Table 1: $\alpha(\varepsilon)$ for various $1-\eta_r$, η_l.

$$\triangle\alpha = 10^{-11}, \qquad \triangle f_\eta = 10^{-8}, q \quad \triangle\eta_c = 10^{-11}, \qquad \triangle f = 10^{-7}\,.$$

The results for $\alpha(\varepsilon)$ of the computations for this choice are shown in table 1, plus the convergence checks for the truncation parameters η_l and η_r. Underlining indicates the digits which coincide

ε	α_1	α_2
0.001	-0.12495	0.05100
0.005	-0.1247	0.05508
0.01	-0.1244	0.05504
0.025	-0.1236	0.05461
0.05	-0.1223	0.05386
0.075	-0.1210	0.05313
0.1	-0.1198	0.05242

Table 2: $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ computed from the numerical data, for various ε.

with the values obtained for our standard choice. Regarding η_c and $\triangle\alpha$ we remark that the above choices are conservative, so that the errors here were negligible.

Both the convergence check and comparison with the analytical value $\alpha(0) = -0.5$ for the first kind similarity case indicate that the values obtained are within 10^{-8} of the real quantities.

We now compare our results with the asymptotic theory. To this end, we calculate, for each ε, the values

$$\begin{aligned} \alpha_1 &= \frac{\alpha(\varepsilon)+0.5}{\varepsilon}, \\ \alpha_2 &= \frac{\alpha(\varepsilon)+0.5+0.125\varepsilon}{\varepsilon^2}. \end{aligned}$$

As ε approaches 0, these values should converge to the theoretical predictions. The results are shown in table 2. Convergence can indeed be observed for α_1, and, to a lesser extent, for α_2. For the latter, the numerical error prevents α_2 to get closer to the theoretical value for $\varepsilon < 0.05$.

The numerical estimates can be improved by extrapolation of the tabulated values for $\alpha(\varepsilon)$. To avoid a large influence of numerical errors from the inclusion of $\alpha(\varepsilon)$ for very small ε, we only

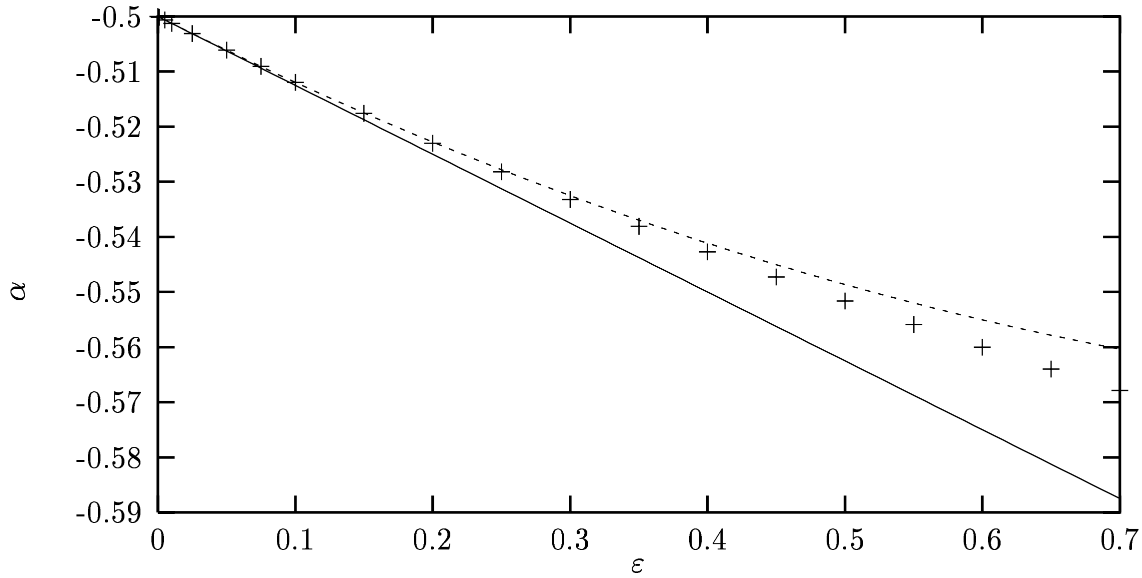

Figure 2: Numerical result (+) First order (—) and second order (- - -) asymptotic results.

used the values of table 1 for $\varepsilon \geq 0.01$ to compute the extrapolation polynomial, and read off the following values for the lower order coefficients,

$$
\begin{aligned}
\alpha_0 &= -0.5000000064, \\
\alpha_1 &= -0.1249995996, \\
\alpha_2 &= 0.05537538810.
\end{aligned}
$$

When we compare the numerical solution for the decay rate $\alpha(\varepsilon)$ with our asymptotic results we observe a significant improvement from our $O(\varepsilon^2)$ result, see figure 2.

Here, we compare our results with those from Goldenfeld et al. [5]. To that end we show, as they did in their article, the quantity $10(-\alpha(\varepsilon) - 1/2)$ as a function of ε. We observe in figure 3 that their result is also qualitatively different from our results.

Finally we like to address the question of convergence of solutions of (1.12) (with $k = 1$, $d = 2$) of compactly supported positive initial data $F(r)$ to a similarity solution. For the numerical integration of (1.12) we make use of the IMSL routine DMOLCH [8]. This routine uses the method of lines, where the spatial discretization is achieved by collocation using cubic Hermite polynomials. The routine assumes that the initial data satisfy the boundary conditions and have smooth derivatives. In all our calculations we let the number of Hermite knots $N = 500$ and specify the error tolerance $tol = 10^{-7}$. If we now multiply the solution $u(r,t)$ by $t^{-\alpha(\varepsilon)}$ and r by $t^{-\beta(\varepsilon)}$, then, in these scales, we observe, that the solution tends to a stationary limit as $t \to \infty$. As an example we set $\varepsilon = 0.3$ and use our asymptotic results (3.1)–(3.3) in (2.9)–(2.10). For the initial condition we use the function

$$F(r) = (1 - r^2)^3 \quad \text{for} \quad -1 < r < 1\,, \tag{4.9}$$

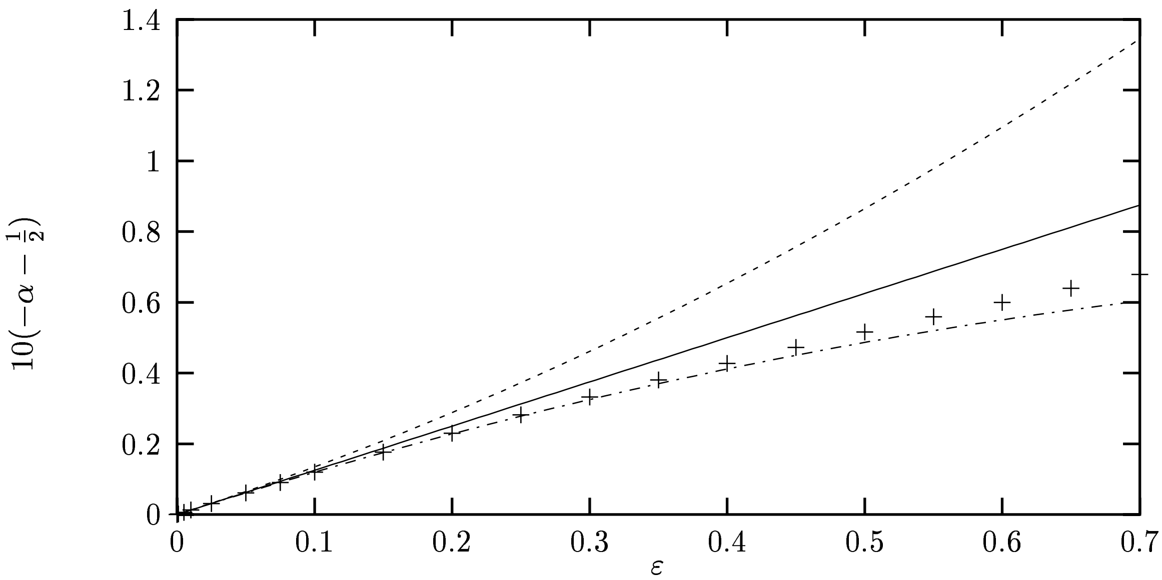

Figure 3: Numerical results (+), the first order asymptotic result (—), Goldenfeld et al. second order asymptotic result (- - -), our second order asymptotic result (- · -).

and zero otherwise. We observe in figure 4, that already for $t = 10$ the solution is, within graphical resolution, stationary, i.e. in self-similar form.

For times $t = 20$ we show in figure 5 the shapes for the ground-water mound for $\varepsilon = 0, 0.3, 1, 4, 10$. We observe, that already for small $\varepsilon = 0.3$ a significant difference in extent and shape, which decreases and flattens out, respectively, for increasing ε. In fact, our numerical calculations show that $\alpha \to -1$ and $\beta \to 0$ as $\varepsilon \to \infty$, i.e. as the residual saturation σ_- approaches σ_+.

5 The dipole problem

We like to demonstrate in this section that our method also extends to problems with different underlying conservation laws. Here, we consider instead of conservation of mass for the underlying porous medium equation ((1.12) for $\varepsilon = 0$), the conservation of the flux and for simplicity restrict ourselves to the case where $d = 1$. We therefore assume now, that the resulting problem is antisymmetric and impose Dirichlet boundary conditions at $x = 0$. We notice first that this problem does not admit an energy integral, however if we consider the first moment, we immediately see, after integrating parts, that

$$\frac{\partial}{\partial t} \int_0^\infty x\, u(x,t)\, dx = u^{k+1}(0,t)\,.$$

Thus, the flux is conserved if Dirichlet boundary conditions are obeyed at $x = 0$. This problem has similarity solutions of the first kind, the so-called dipole-solutions, describing the large time behavior of solutions with antisymmetric initial data, [1]. Existence of a unique continuous weak

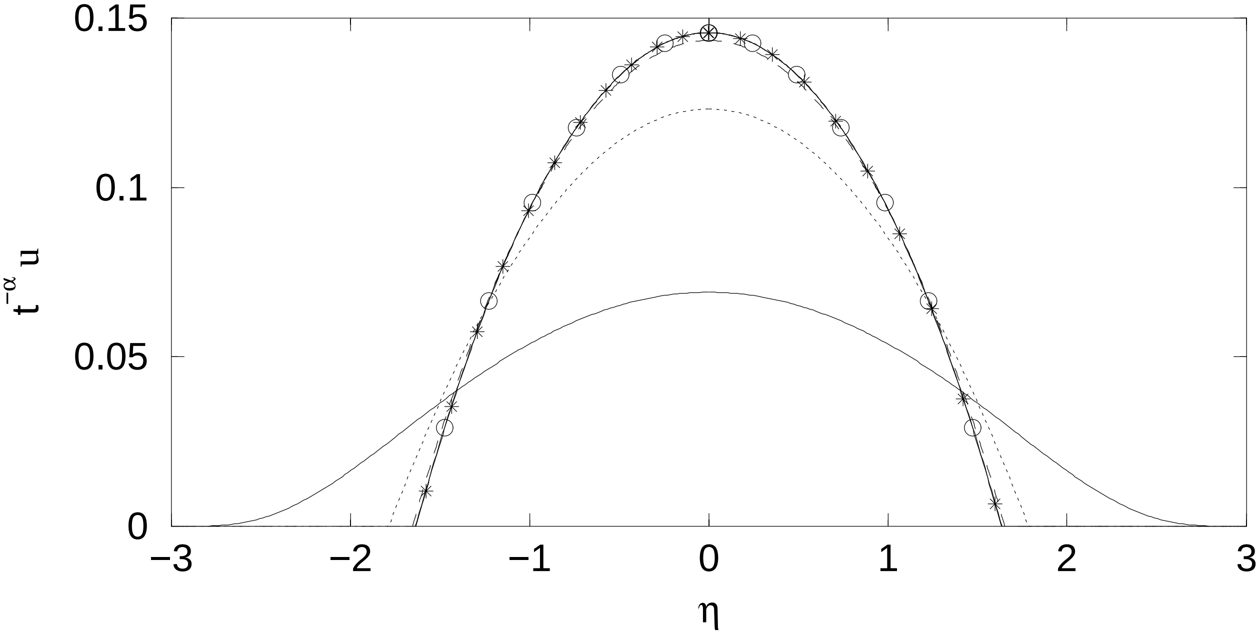

Figure 4: $t^{-\alpha}u(x,t)$ in similarity variables at times $t = 0.01$ (—), $0.1(\cdots)$, 1 (- - -), 10 ($- \circ -$), 100 ($- * -$)for $\varepsilon = 0.3$.

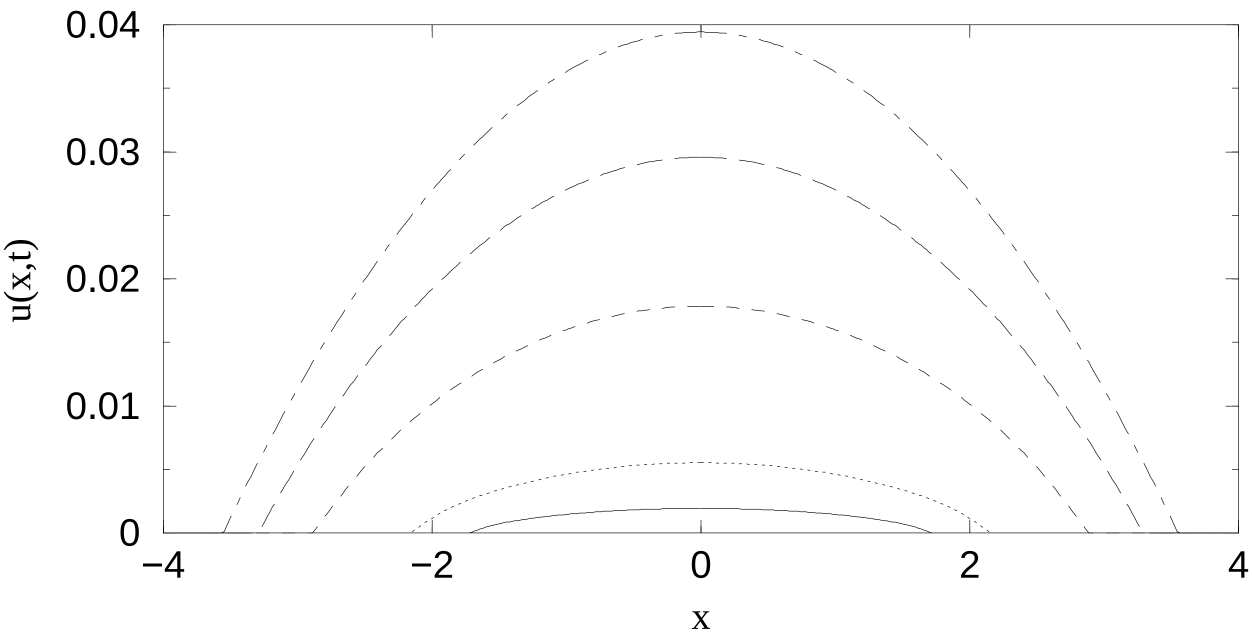

Figure 5: $u(x,t)$ at time $t = 20$ for $\varepsilon = 0$ ($- \cdot -$), 0.3 (— —), 1 (- - -), 4 ($\cdot\cdot\cdot$), 10 (—–).

solution with compact support, as well as convergence to the dipole solution in the large time limit could be shown in [13], [7].

One typical application of this problem for $k = 1$, concerns the impact of a flood on the motion of ground water. If for example at a certain time the level of a liquid begins to rise quickly at the symmetry-axis $x = 0$ of a porous layer and after a short duration is again withdrawn, the large time behavior of the liquid distribution in the porous medium shows a front moving with finite velocity further into the porous medium, while at the boundary $x = 0$ fluid is lost at a constant rate. Such application problems where for example investigated in [19]. There, also the problem of forced drainage, to prevent the spreading of the liquid was considered.

Here, we are concerned with the effect of some additional loss of liquid in the pores of the layer, which, of course to a certain extend, is the case for all materials.

When considering an analogous situation as for the problem of the ground water mound ($k = 1$), where the fluid spreads in a porous medium, we will observe that also conservation of the flux is violated and replaced by a corresponding 'dissipation law'.

Hence, when looking for the large time behavior of solutions to the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left[1 + \varepsilon H\left(-\frac{\partial u}{\partial t}\right)\right] \frac{\partial^2 u^{k+1}}{\partial x^2}, \qquad 0 < x < x_0 \\ u(x,0) &= F(x)\,, \qquad \int_0^{x_0} xF(x)\,dx = 1\,, \end{aligned} \tag{5.1}$$

we expect the similarity solutions to be of second kind. As in the previous section we construct them by using the composite expansion ansatz (2.1) in (5.1) to obtain $\beta(\varepsilon) = -[k\alpha(\varepsilon) + 1]/2$ and the similarity problem

$$\alpha(\varepsilon)\eta\, f(\eta;\varepsilon) - \frac{k\alpha(\varepsilon)+1}{2}\eta^2 \frac{df}{d\eta} = (1+\varepsilon)\frac{d}{d\eta}\left(\eta \frac{d\, f^{k+1}}{d\eta} - f^{k+1}(\eta;\varepsilon)\right), \quad \text{for} \quad 0 \le \eta \le \eta_c \tag{5.2}$$

$$\alpha(\varepsilon)\eta\, f(\eta;\varepsilon) - \frac{k\alpha(\varepsilon)+1}{2}\eta^2 \frac{df}{d\eta} = \frac{d}{d\eta}\left(\eta \frac{d\, f^{k+1}}{d\eta} - f^{k+1}(\eta;\varepsilon)\right), \quad \text{for} \quad \eta_c \le \eta \le \eta_0\,. \tag{5.3}$$

In this problem $f(\eta;\varepsilon)$ vanishes at $\eta = 0$ and $\eta = \eta_0$. η_c is the point where the interface condition

$$\frac{d^2 f^{k+1}}{d\eta^2} = 0 \qquad \text{at} \qquad \eta = \eta_c \tag{5.4}$$

holds and the normalization is here

$$\int_0^{\eta_0} \eta\, f(\eta;\varepsilon)\, d\eta = 1 \tag{5.5}$$

The "dissipation law" for the 'mass'-flux takes on the form

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^{x_0} x\, u(x,t)\, dx &= \int_0^{x_c} (1+\varepsilon)\frac{\partial}{\partial x}\left(x\frac{\partial u^{k+1}}{\partial x} - u^{k+1}\right) dx + \int_{x_c}^{x_0} \frac{\partial}{\partial x}\left(x\frac{\partial u^{k+1}}{\partial x} - u^{k+1}\right) \\ &= \varepsilon\left(x_c\frac{\partial u^{k+1}}{\partial x}\left(x_c(t),t\right) - u^{k+1}\left(x_c(t),t\right)\right) \end{aligned} \tag{5.6}$$

which in conjunction with the similarity form yields the formula for the similarity index $\alpha(\varepsilon)$:

$$\alpha(\varepsilon)(k+1) + 1 = \varepsilon\left(\eta_c\frac{df^{k+1}}{d\eta}(\eta_c) - f^{k+1}(\eta_c)\right) . \tag{5.7}$$

We assume now that $\varepsilon \ll 1$ and we make the perturbation ansatz (2.9)-(2.11) from which we obtain

$$\int_0^{\eta_0} \eta\, f_0(\eta)\, d\eta = 1 \tag{5.8}$$

$$\int_0^{\eta_0} \eta\, f_i(\eta)\, d\eta = 0 \qquad i = 1, 2, \dots . \tag{5.9}$$

and

$$\begin{aligned} \alpha_0 &= -\frac{1}{k+1} && (5.10) \\ \alpha_1 &= -\frac{1}{k+1}\left(f_0^{k+1}(\eta_c) - \eta_c\frac{df_0^{k+1}}{d\eta}(\eta_c)\right) && (5.11) \\ \alpha_2 &= -\left(f_0^k f_1 - \eta\frac{d}{d\eta}\left(f_0^k f_1\right)\right)\Big|_{\eta=\eta_c} . && (5.12) \\ &\ \vdots \end{aligned}$$

In order to determine $f_0(\eta_c)$ we integrate the leading order problem and obtain, in view of (5.10)

$$\eta \frac{df_0^{k+1}}{d\eta} - f_0^{k+1}(\eta) + \frac{1}{2(k+1)}\eta^2 f_0(\eta) = 0 \tag{5.13}$$

The use of the integrating factor $[\eta^{(2k+1)/(k+1)} f_0]^{-1}$ then enables us to solve for f_0 (see also [1]) :

$$f_0(\eta) = \left[\frac{k}{2(k+2)(k+1)}\left(\eta_0^{\frac{k+2}{k+1}} - \eta^{\frac{k+2}{k+1}}\right)\eta^{\frac{k}{k+1}}\right]^{\frac{1}{k}} , \tag{5.14}$$

and the normalization condition yields η_0

$$\eta_0^{\frac{2(k+1)}{k}} = \frac{k+2}{k+1}\left(\frac{2(k+2)(k+1)}{k}\right)^{\frac{1}{k}} B^{-1}\left(\frac{k+1}{k+2}+1, \frac{k+1}{k}\right) . \tag{5.15}$$

Since the interface condition (5.4) yields

$$\eta_c = \left(\frac{k(2k+3)}{2(k+1)^2}\right)^{\frac{k+1}{k+2}} \eta_0 , \tag{5.16}$$

we obtain for α_1 the formula

$$\alpha_1(k) = -\frac{1}{k+1}\left(\frac{k(2k+3)}{2(k+1)^2}\right)^{\frac{2k+3}{k+2}}\left(\frac{k+2}{2(k+1)^2}\right)^{\frac{k+1}{k}} B^{-1}\left(\frac{k+1}{k+2}+1, \frac{k+1}{k}\right) . \tag{5.17}$$

The index α_2 is again determined by solving the $O(\varepsilon)$ problem

$$\frac{d}{d\eta}\left[(k+1)\left(\eta\frac{d}{d\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)}\right] = \alpha_1\left(\eta f_0 - \frac{k}{2}\eta^2\frac{df_0}{d\eta}\right) - \frac{d}{d\eta}\left(\eta\frac{df_0^{k+1}}{d\eta} - f_0^{k+1}\right) \quad \text{for} \quad 0 \le \eta \le \eta_c \tag{5.18}$$

$$\frac{d}{d\eta}\left[(k+1)\left(\eta\frac{d}{d\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)}\right] = \alpha_1\left(\eta f_0 - \frac{k}{2}\eta^2\frac{df_0}{d\eta}\right) \quad \text{for} \quad \eta_c \le \eta \le \eta_0 \tag{5.19}$$

To solve this, we integrate (5.18) from 0 to η and (5.19) from η to η_0

$$
\begin{aligned}
(k+1)\left(\eta\frac{d}{d\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)} &= \alpha_1 \int_0^\eta \lambda f_0 - \frac{k}{2}\lambda^2 \frac{df_0}{d\lambda}\, d\lambda - \eta\frac{df_0^{k+1}}{d\eta} + f_0^{k+1} \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (5.20) \\
(k+1)\left(\eta\frac{d}{d\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)} &= -\alpha_1 \int_\eta^{\eta_0} \lambda f_0 - \frac{k}{2}\lambda^2 \frac{df_0}{d\lambda}\, d\lambda \\
&\qquad \text{for} \quad \eta_c \le \eta \le \eta_0 \qquad (5.21)
\end{aligned}
$$

Thus, by (5.21) we can determine α_2 from

$$
\alpha_2 = -\frac{1}{k+1}\left(\alpha_1\, J_1(\eta_c) + \frac{1}{2(k+1)}\eta_c^2 f_1(\eta_c)\right) \tag{5.22}
$$

where

$$
J_1(\eta) = \int_\eta^{\eta_0} \lambda f_0 - \frac{k}{2}\lambda^2 \frac{df_0}{d\lambda}\, d\lambda \tag{5.23}
$$

In order to determine $f_1(\eta_c)$, we observe first that (5.20), (5.21) can be written in the form

$$
\begin{aligned}
\frac{d}{d\eta}\left(y^{\frac{k-1}{k}}\eta^{\frac{-1}{k+1}} f_1(\eta)\right) &= \frac{2(k+2)}{k}\left(\frac{\alpha_1}{y^{\frac{1}{k}}\eta^2}\int_0^\eta \lambda f_0 - \frac{k}{2}\lambda^2\frac{df_0}{d\lambda}\, d\lambda - \frac{\eta\frac{df_0^{k+1}}{d\eta} + f_0^{k+1}}{y^{\frac{1}{k}}\eta^2}\right) \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (5.24) \\
\frac{d}{d\eta}\left(y^{\frac{k-1}{k}}\eta^{\frac{-1}{k+1}} f_1(\eta)\right) &= -\frac{2(k+2)}{k}\frac{\alpha_1}{y^{\frac{1}{k}}\eta^2}\int_0^\eta \lambda f_0 - \frac{k}{2}\lambda^2\frac{df_0}{d\lambda}\,, d\lambda \\
&\qquad \text{for} \quad \eta_c \le \eta \le \eta_0 \qquad (5.25)
\end{aligned}
$$

where we denote

$$
y(\eta) = \eta_0^{\frac{k+2}{k+1}} - \eta^{\frac{k+2}{k+1}}
$$

We can now integrate (5.20) from 0 to η and (5.21) from η_c to η to obtain

$$
\begin{aligned}
\eta f_1(\eta) &= \left(\frac{\eta_0}{y(\eta)}\right)^{\frac{k-1}{k}} \eta^{\frac{k+2}{k+1}} l_0 + \frac{2(k+2)}{k} y(\eta)^{\frac{1-k}{k}} \eta^{\frac{k+2}{k+1}} \left(\alpha_1 F_1(\eta) - F_2(\eta)\right) \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (5.26) \\
\eta f_1(\eta) &= \left(\frac{y(\eta_c)}{y(\eta)}\right)^{\frac{k-1}{k}} \eta^{\frac{k+2}{k+1}} \eta_c^{\frac{-1}{k+1}} f_1(\eta_c) - \frac{2(k+2)}{k} y(\eta)^{\frac{1-k}{k}} \eta^{\frac{k+2}{k+1}} \alpha_1 F_3(\eta)\,, \\
&\qquad \text{for} \quad 0 \le \eta \le \eta_c \qquad (5.27)
\end{aligned}
$$

where we denote

$$\begin{aligned} l_0 &= \lim_{\eta\to 0} \eta^{\frac{-1}{k+1}} f_1(\eta)\,, \qquad F_1(\eta) = \int_0^\eta \frac{1}{y(\lambda)^{\frac{1}{k}}\lambda^2}\left(\int_0^\lambda \sigma f_0 - \frac{k}{2}\sigma^2 \frac{df_0}{d\sigma}\, d\sigma\right) d\lambda \\ F_2(\eta) &= \int_0^\eta \frac{\lambda \frac{df_0^{k+1}}{d\lambda} + f_0^{k+1}}{y(\lambda)^{\frac{1}{k}}\lambda^2}\, d\lambda\,, \quad F_3(\eta) = \int_{\eta_c}^\eta \frac{1}{y(\lambda)^{\frac{1}{k}}\lambda^2}\left(\int_\sigma^{\eta_0} \sigma f_0 - \frac{k}{2}\sigma^2 \frac{df_0}{d\sigma}\, d\sigma\right) d\lambda\,. \end{aligned}$$

We finally use the integral condition (5.9) after we integrate (5.26) from 0 to η_c and (5.27) from η_c to η_0 and adding the resulting parts, yielding

$$0 = l_0\, G_1 + \eta_c^{\frac{-1}{k+1}} f_1(\eta_c)\, H_1 + \frac{2(k+2)}{k}(G_2 - H_2)\,, \tag{5.28}$$

where

$$\begin{aligned} G_1 &= \int_0^{\eta_c} \left(\frac{\eta_0}{y(\eta)}\right)^{\frac{k-1}{k}} \eta^{\frac{k+2}{k+1}}\, d\eta\,, \qquad G_2 = \int_0^{\eta_c} y(\eta)^{\frac{1-k}{k}} \eta^{\frac{k+2}{k+1}} \left(\alpha_1 F_1(\eta) - F_2(\eta)\right) d\eta \\ H_1 &= \int_{\eta_c}^{\eta_0} \left(\frac{y(\eta_c)}{y(\eta)}\right)^{\frac{k-1}{k}} \eta^{\frac{k+2}{k+1}}\, d\eta\,, \qquad H_2 = \int_{\eta_c}^{\eta_0} y(\eta)^{\frac{1-k}{k}} \eta^{\frac{k+2}{k+1}} \alpha_1 F_3(\eta)\, d\eta \end{aligned}$$

Equation (5.28) together with (5.26), evaluated at $\eta = \eta_c$, enable us to eliminate l_0 and to solve for $f_1(\eta_c)$:

$$f_1(\eta_c) = \frac{2(k+2)}{k} \eta_c^{\frac{1}{k+1}} \left(\frac{\eta_0^{\frac{1-k}{k}} \left(\alpha_1 F_1(\eta_c) - F_2(\eta_c)\right) - \frac{G_2 - H_2}{G_1}}{\left(y(\eta_c)/y(\eta_0)\right)^{\frac{k-1}{k}} + \frac{H_1}{G_1}} \right). \tag{5.29}$$

6 The flood problem: Asymptotic and numerical results

Let us look again at the case, where $k = 1$ and recall the problem of the impact of a flood on the ground water motion, briefly described at the beginning of this section.

From our asymptotic results, we find in this case

$$\begin{aligned} \alpha_0 &= -\frac{1}{2}\,, \\ \alpha_1 &= -0.1427743036\,. \end{aligned}$$

The integrals, needed to find $f_1(\eta_c)$, turn out to be

$$F_1(\eta) = -\frac{\eta_0^{3/2}}{60}\ln(1-z^3)\,, \qquad F_2(\eta) = -\frac{\eta^{3/2}}{72}\,, \qquad \text{where} \qquad z = \left(\frac{\eta}{\eta_0}\right)^{1/2}.$$

$$\begin{aligned} F_3(\eta) &= \frac{\eta_0^{3/2}}{60}\left[\sqrt{3}\arctan\left(\frac{2\zeta+1}{\sqrt{3}}\right) - \frac{1}{2}\ln\left(\frac{(1-\zeta)^2}{1+\zeta+\zeta^2}\right) - \frac{3}{2\zeta^2} + \ln(1-\zeta^3)\right]_v^z \\ &=: \frac{\eta_0^{3/2}}{60}\left(\phi(z) - \phi(v)\right)\,, \qquad \text{where} \qquad v = \left(\frac{\eta_c}{\eta_0}\right)^{1/2}. \end{aligned}$$

$$G_1 = \frac{2}{5}\eta_c^{5/2}\,, \qquad H_1 = \frac{2}{5}\left(\eta_0^{5/2} - \eta_c^{5/2}\right).$$

$$\begin{aligned} H_2 &= \frac{\alpha_1\eta_0^4}{150}\left(\sqrt{3}(1-v^5)\arctan\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{3}{2}(1+v^5)\ln(1+v+v^2)\right. \\ &\left. + \frac{3}{2}v^2\left(1+v+\frac{2}{5}v^3\right) + 3\left(\ln(3) - \frac{6}{5}\right) - \phi(v)(1-v^5)\right). \end{aligned}$$

$$\begin{aligned} G_2 &= -\frac{\alpha_1\eta_0^4}{150}\left(\frac{v^5}{2}\ln\left((1-v^3)^2\right) + \frac{1}{2}\ln(1+v+v^2) - \frac{1}{2}\ln\left((1-v^2)\right)\right. \\ &\left. -\sqrt{3}\arctan\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{3}{2}v^2\left(1+\frac{2}{5}v^3\right) + \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)\right) \\ &+ \frac{\eta_c^4}{288}. \end{aligned}$$

These yield the value

$$\alpha_2 = 0.06773941887\,.$$

For a comparison of our results with a numerical solution of this problem we recast the equations in the following form,

$$f_{\eta\eta} = -\frac{f_\eta^2}{f} + \frac{1}{2(1+\varepsilon)}\left[\alpha - \frac{1+\alpha}{2}\eta\frac{f_\eta}{f}\right] \qquad 0 < \eta < \eta_c, \tag{6.1}$$

$$f_{\eta\eta} = -\frac{f_\eta^2}{f} + \frac{1}{2}\left[\alpha - \frac{1+\alpha}{2}\eta\frac{f_\eta}{f}\right] \qquad \eta_c < \eta < 1, \tag{6.2}$$

with the boundary and interface conditions at 0, η_c and 1,

$$f(0) = 0, \tag{6.3}$$

$$\alpha f(\eta_c) - \frac{1+\alpha}{2}\eta_c f_\eta(\eta_c) = 0, \tag{6.4}$$

$$[f]_{\eta_c^-}^{\eta_c^+} = 0, \tag{6.5}$$

$$[f_\eta]_{\eta_c^-}^{\eta_c^+} = 0, \tag{6.6}$$

$$f(1) = 0, \tag{6.7}$$

$$f_\eta(1) = -\frac{1+\alpha}{4}, \tag{6.8}$$

where we have normalized η_0 to 1.

Again, we have to solve a two point boundary value problem with one boundary condition more than the order of the differential equation, which is necessary to fix the unknown similarity exponent $\alpha(\varepsilon)$. The numerical algorithm follows a pattern very similar to the method use for the ground water mound problem. We first solve (6.2) starting from a point η_l just left of 1. The initial data we obtain from the linear approximations

$$\begin{aligned} f(\eta_l) &\approx \frac{\alpha+1}{4}(1-\eta_l), \\ f_\eta(\eta_l) &\approx -\frac{\alpha+1}{4}. \end{aligned}$$

We integrate up to η_c which we determine as before, then continue with (6.1), with initial data provided by (6.5) and (6.6), until we arrive a an η_l where f is zero. Up to here, α is an arbitrary parameter; it is fixed by the requirement that (6.3) be fulfilled, i.e., that η_l must be at the origin. Preliminary trials showed that, here, η_l depends monotonously on α, so that the value for which this is achieved can be determined through bisection. At every step of the iteration, the sign of η_l is used to select one of the two parts of the interval. The iteration terminates when $|\eta_l|$ is smaller than a specified tolerance tol_l.

A minor complication arises from the fact that the denominator of the first term on the RHS of (6.2) vanishes at a zero of f, i.e., at η_l. Indeed, numerical tests indicate that the tangent to the graph of f tends to be vertical at this point. Therefore, the numerical ODE-integrator (LSODE) cannot compute f up to, or past, η_l. Attempts to do so lead to premature termination and an error message of the routine. To get as close as possible to the true value of η_l, we adopt the following procedure: We start by requiring LSODE to integrate over a small but finite distance, backwards on the η-axis. As an error is flagged as soon as we choose an endpoint to the left of η_l, we return to the end point of the last successful call to LSODE and resume integration, but only over half the distance. We repeat this until the estimate $|f(\eta)|/|f_\eta(\eta)|$ for the distance of the current point to η_l, meets a certain threshold tol_0.

For the calculation, we chose the following set of values for the truncation parameters and tolerances,

$$\begin{aligned} \mathrm{tol} &= 10^{-14}, \\ \eta_r &= 1-10^{-5}, \\ \triangle\eta_c &= 10^{-11}, & \triangle f &= 10^{-8}, \\ \triangle\alpha &= 10^{-11}, & \mathrm{tol}_{\eta_l} &= 10^{-9}, \\ \mathrm{tol}_0 &= 10^{-9}. \end{aligned}$$

Convergence with respect to η_l is checked by comparing the results from these calculations with those where $1-\eta_r$ is set to 10^{-4} and 10^{-3}, for five different values of ε. The results are shown in table 3, where we have underlined the digits which agree with the result for the smallest $1-\eta_l$. Convergence appears to be guaranteed, and the case $\varepsilon = 0$, where we know α to be exactly -0.5, provides us with an estimate for the typical error less than 10^{-9}.

We now compare our results with the asymptotics. Table 4 shows the values of α for different ε, together with the approximations to the coefficients of the asymptotic expansion obtained from them through

$$\alpha_1 = \frac{\alpha + 0.5}{\varepsilon}, \qquad \alpha_2 = \frac{\alpha + 0.5 + 0.1427743036\varepsilon}{\varepsilon^2}.$$

The results indeed seem to converge to the corresponding values from the perturbation analysis as ε goes to zero. We obtain better approximations to the coefficients of the asymptotic expansion from the lower coefficients of the polynomial in ε which interpolates $\alpha(\varepsilon)$ for the choice of several ε in $0.001, \ldots, 0.1$.

$$\alpha_0 = -0.5000000001, \quad \alpha_1 = -0.1427742997, \quad \alpha_2 = 0.06773914094.$$

The following figure (6) shows again a very good agreement of numerical and asymptotic results for values of ε of up to 0.3.

ε	$1-\eta_r = 10^{-5}$	$1-\eta_r = 10^{-4}$	$1-\eta_r = 10^{-3}$
0	-0.500000000	$-0.\underline{5000000}19$	$-0.\underline{50000}1874$
0.001	-0.500142707	$-0.\underline{5001427}25$	$-0.\underline{50014}4582$
0.1	-0.513637545	$-0.\underline{5136375}65$	$-0.\underline{51363}9585$
1	-0.599348416	$-0.\underline{5993484}51$	$-0.\underline{5993}51895$
2	-0.655767439	$-0.\underline{6557674}88$	$-0.\underline{6557}72403$
4	-0.722109746	$-0.\underline{722109}822$	$-0.\underline{7221}17434$

Table 3: $\alpha(\varepsilon)$ for various $1-\eta_r$.

ε	α_1	α_2
0.001	-0.142707	0.0676145
0.005	-0.142437	0.0675370
0.01	-0.142101	0.0673409
0.025	-0.141105	0.0667556
0.05	-0.139484	0.0658037
0.075	-0.137908	0.0648818
0.1	-0.136375	0.0639886

Table 4: $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ computed from the numerical data, for various ε.

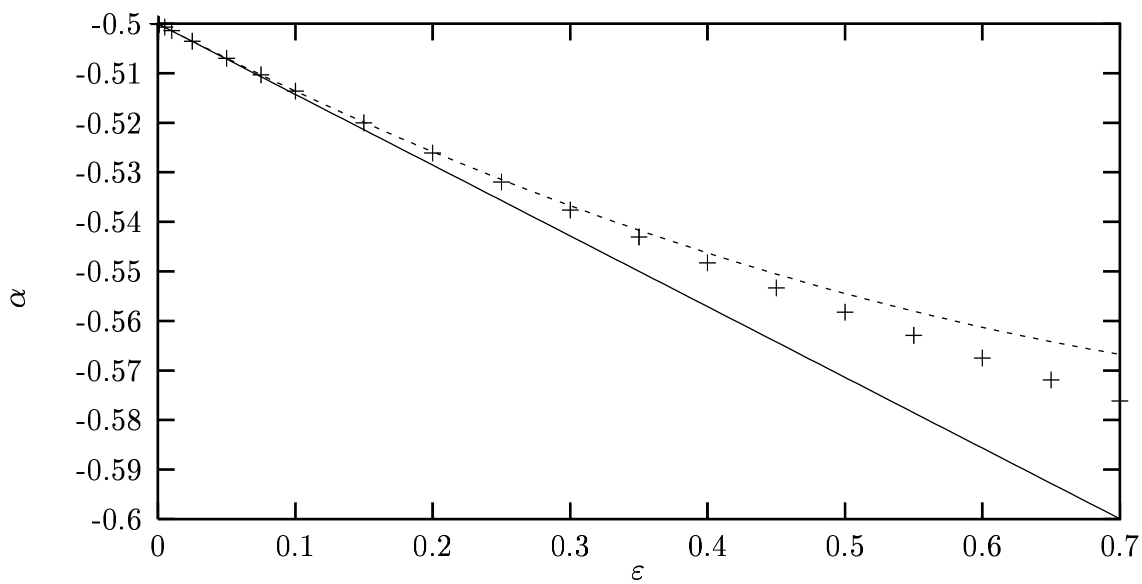

Figure 6: Numerical result (+), first order (—) and second order (- - -) asymptotic results.

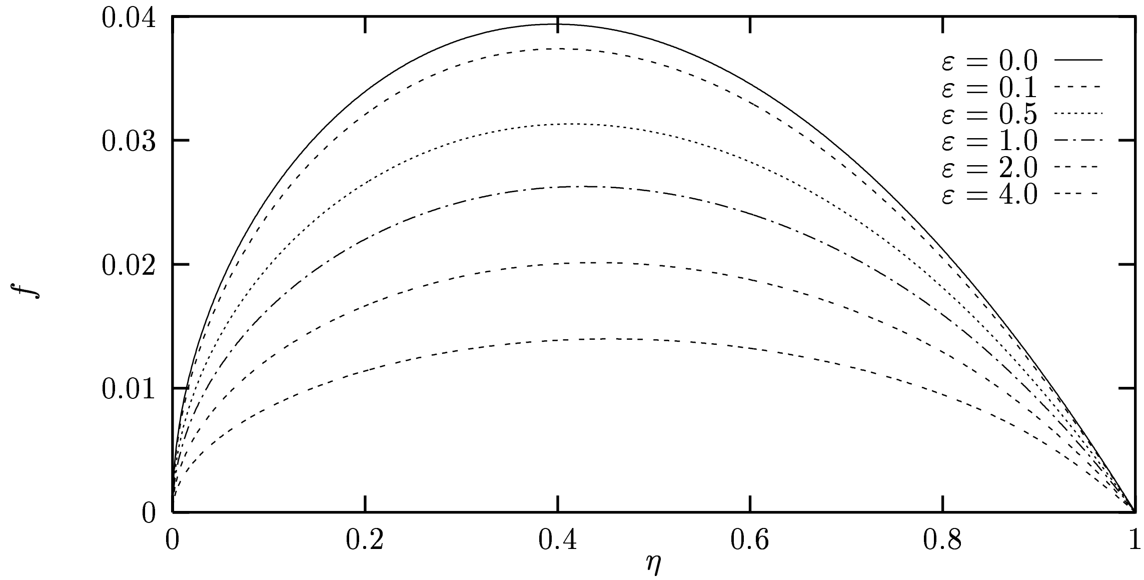

Figure 7: The Dipole similarity solution for various ε.

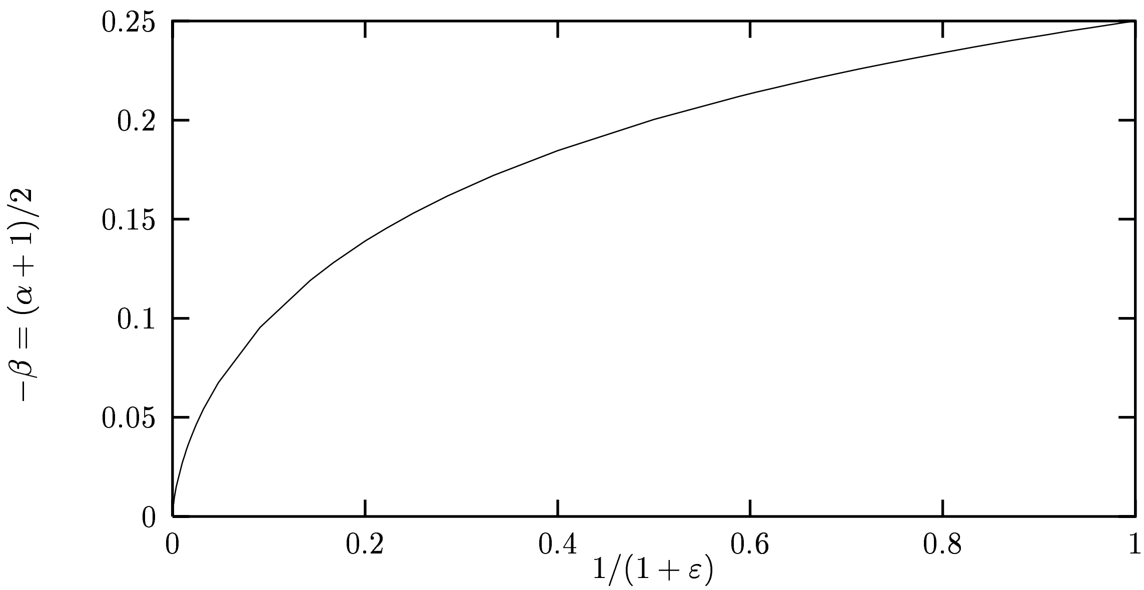

Figure 8: The exponent $-\beta$, determining the rate of extension of the solution for various ε.

In figure 7 we show some typical self-similar shapes for the dipole solution for various ε. We observe, that when the residual saturation σ_- approaches σ_+, i.e. $1/(1+\varepsilon) \to 0$, the similarity shapes become symmetrical about their maximum. In this limit, the spreading rate $\beta(\varepsilon) = -[k\alpha(\varepsilon)+1]/2$ vanishes. This can be seen in figure 8, which shows the complete range of spreading rates.

7 Calculation of waiting-times

In this section, we like to make a few remarks on how solutions of (5.1) (with $k = 1$) for compactly supported positive initial data $F(x)$ converge to the similarity solutions for various ε. For the numerical integration of (5.1) we use again the IMSL routine DMOLCH [8]. Again, we let in all our calculations the number of Hermite knots $N = 500$ and specify the error tolerance $tol = 10^{-7}$. For the initial condition we use the function

$$F(x) = x(2-x)^3 \quad \text{for} \quad 0 \leq x < 2\,, \tag{7.1}$$

and zero otherwise.

At first we consider the case $\varepsilon = 0$. We observe in figure 9 , that the similarity solution emerges rather quickly. Furthermore, we note, that until about $t^* \simeq 0.11$ the support of the initial data does not move, i.e. we have a positive waiting-time t^*. We illustrate these properties in the following three figures 10-12, where we compare the numerical solution to (5.1) with the solution to (6.1)-(6.8). Here, we scale the solution $f(\eta)$ such that its maximum agrees with the maximum of $u(x,t)$ and multiply $\eta_0 = 1$ by the right boundary of the support of $u(x,t)$, at a given time.

From the previous sections we know, that for $\varepsilon > 0$, the decay rate $\alpha(\varepsilon)$ increases, while the spreading rate $\beta(\varepsilon)$ decreases. Starting with the same initial data as above (7.1), we observe this behavior in figure 13 for $\varepsilon = 10$.

Furthermore, we find that also the modified dipole problem has waiting-time behavior. In particular, the waiting-time for the case $\varepsilon > 0$ is larger than for $\varepsilon = 0$. This has the effect that in this case, the solution is already in self-similar form when the boundary of its support begins to move, as can be observed in figures 14-16, and hence the location of the boundary is completely described by the similarity solution.

Note however, that waiting-times result to a certain extent from local effects [14]. For initial shapes other than (7.1), for example when the right boundary of the support is approached linearly, we observe zero waiting times. A classification for initial conditions with positive waiting-time for the modified dipole problem, as well as the modified porous medium equation still needs to be derived.

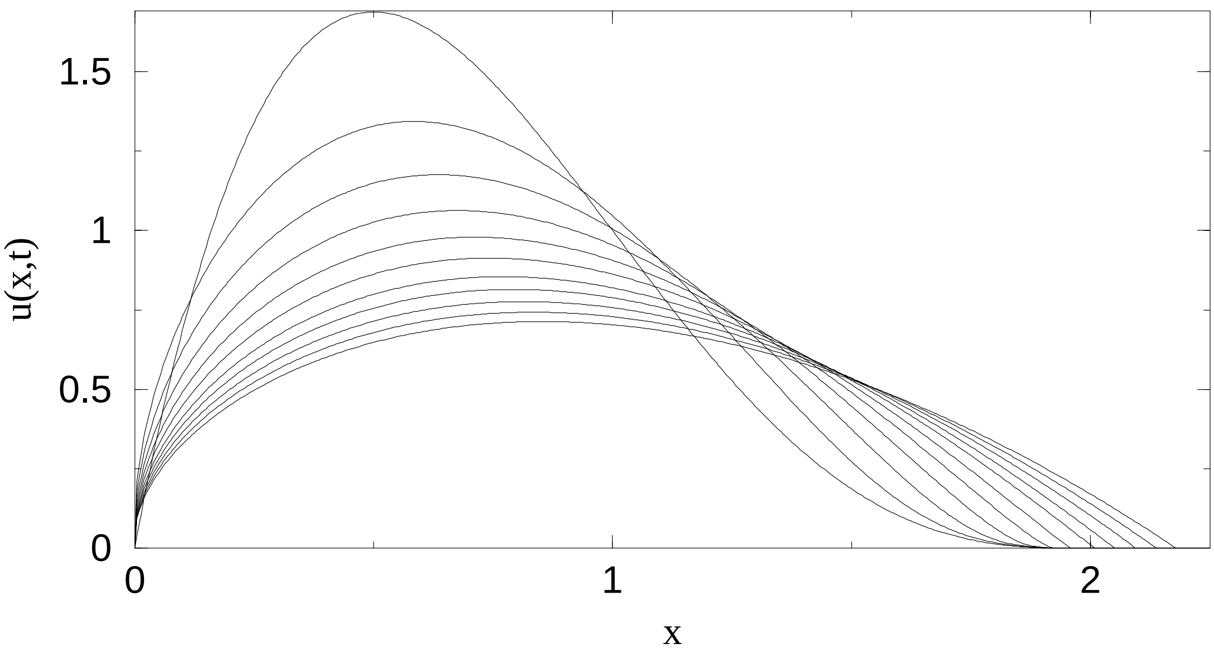

Figure 9: Time evolution of $u(x,t)$ for $\varepsilon = 0$ and $t = 0, \ldots, 0.2$.

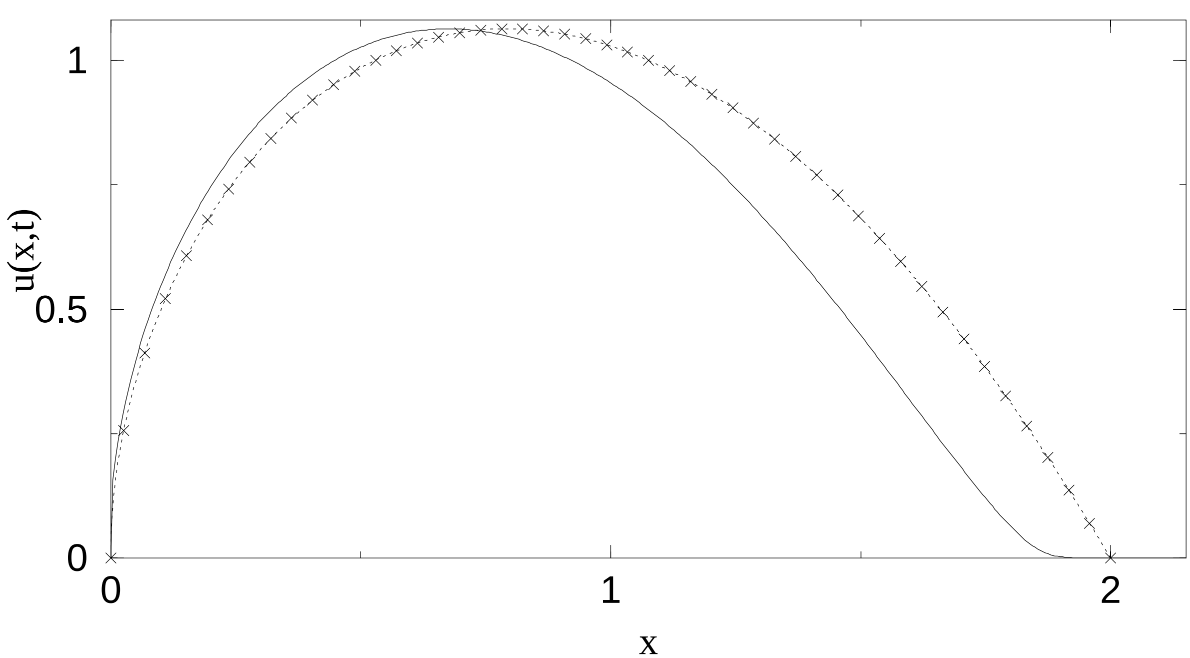

Figure 10: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.06$ for $\varepsilon = 0$.

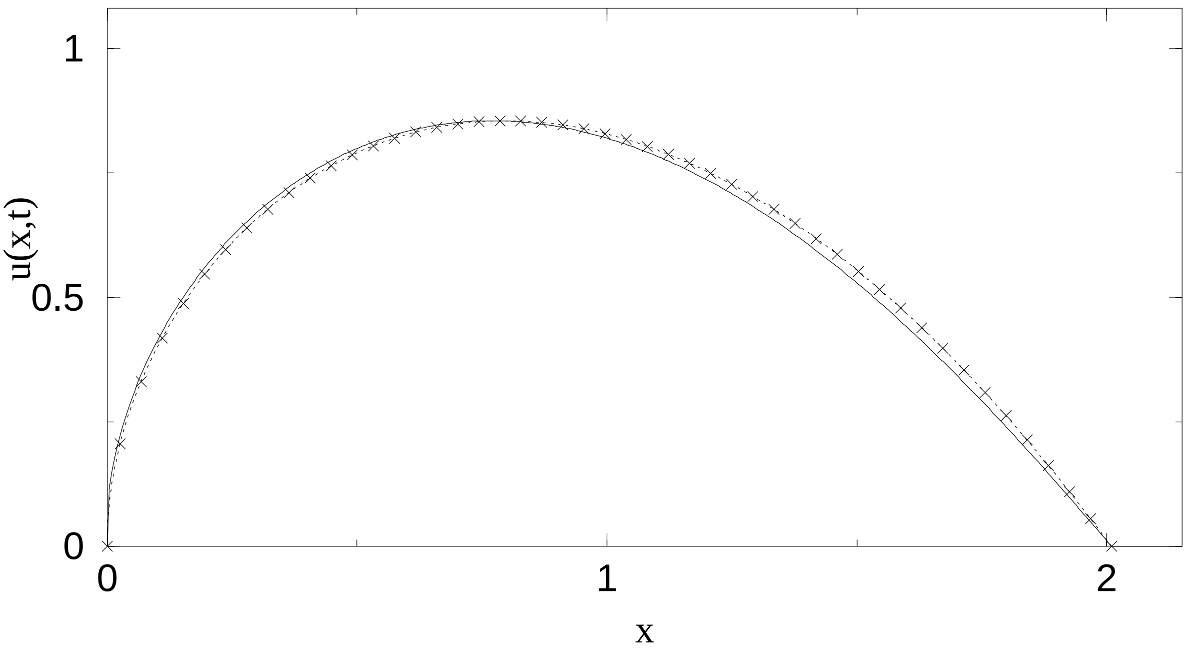

Figure 11: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.12$ for $\varepsilon = 0$.

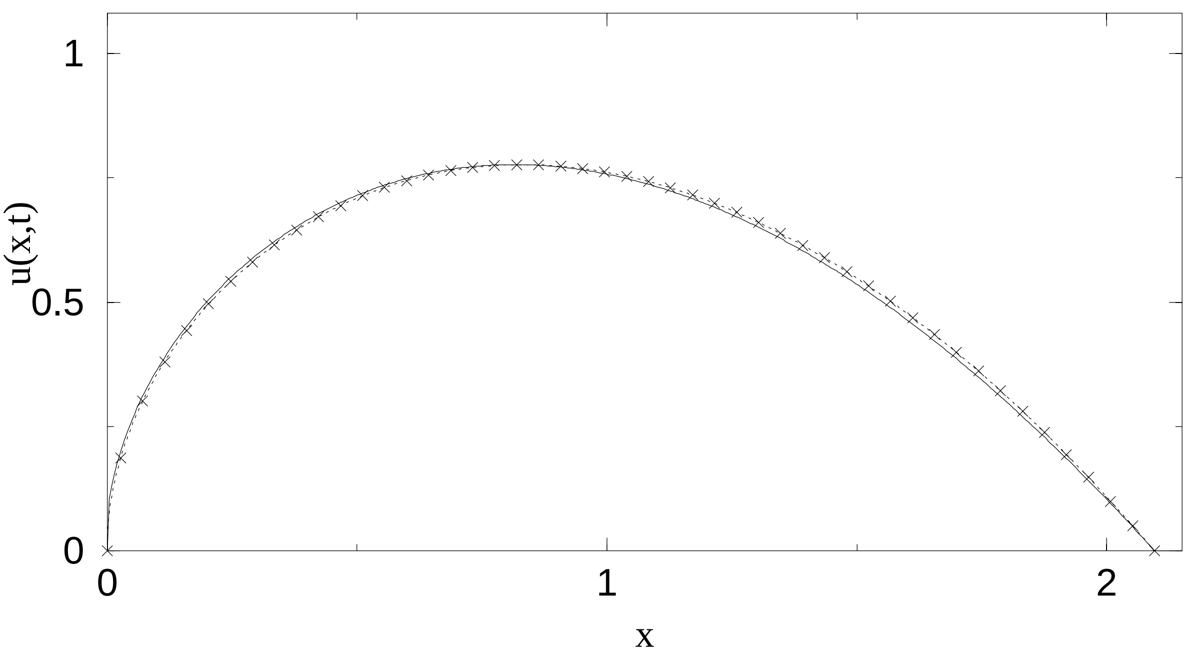

Figure 12: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.16$ for $\varepsilon = 0$.

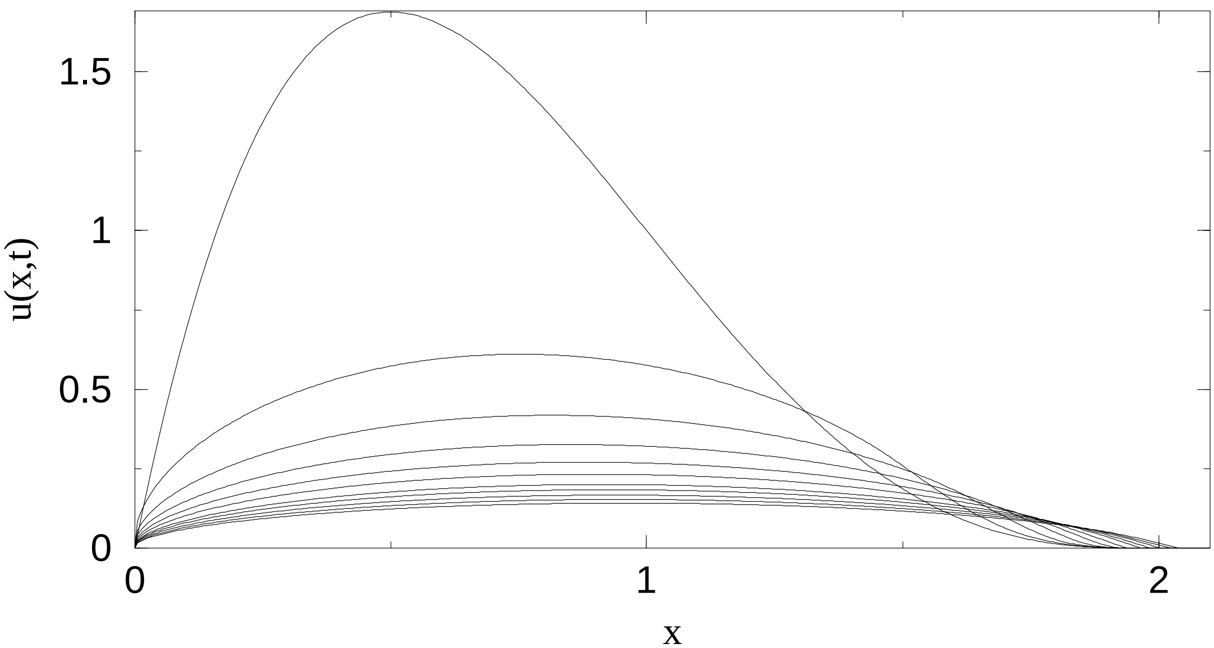

Figure 13: Time evolution of $u(x,t)$ for $\varepsilon = 10$ and $t = 0, \ldots, 0.2$.

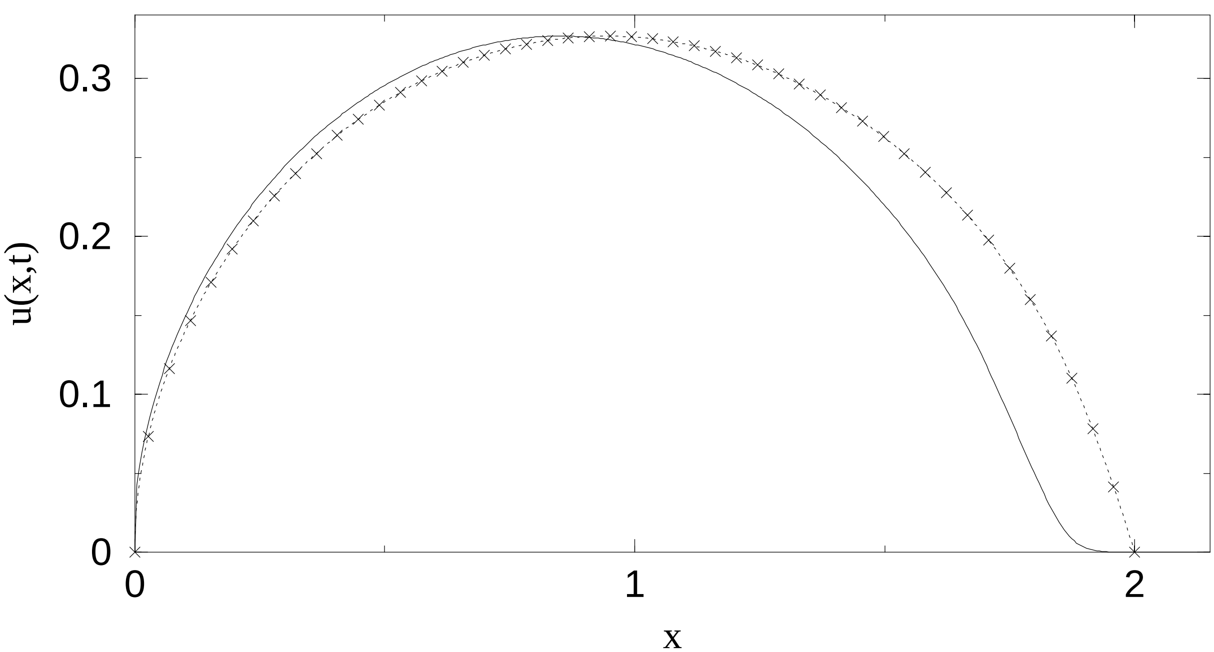

Figure 14: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.06$ for $\varepsilon = 10$.

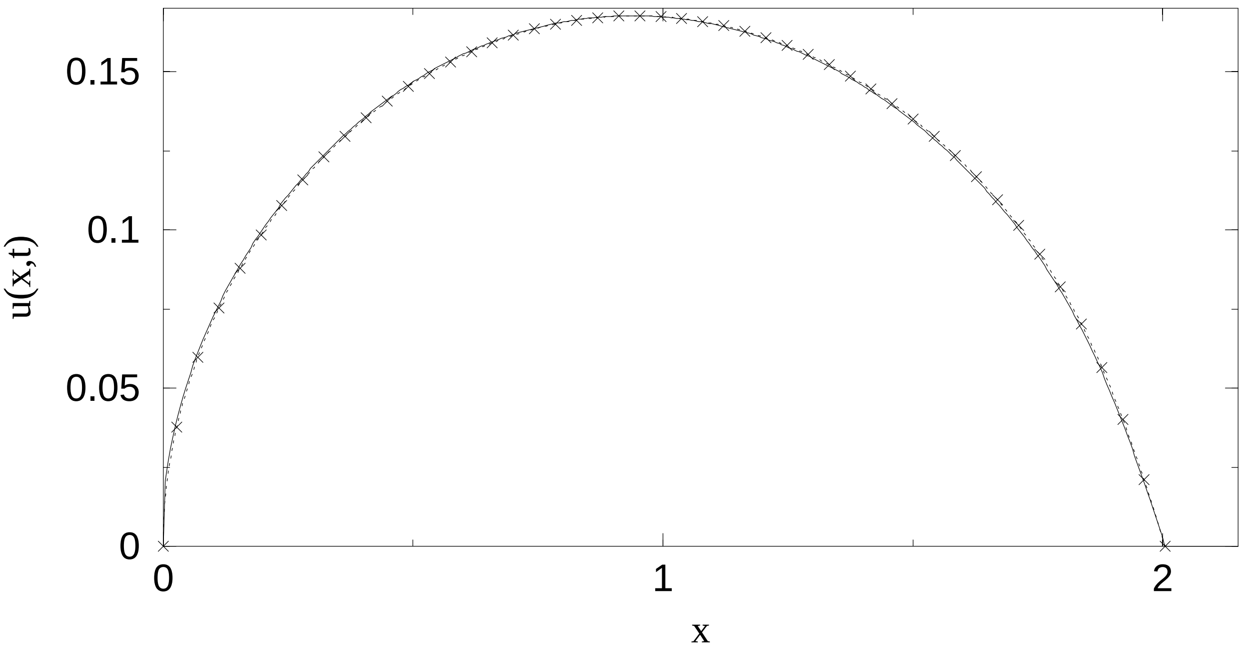

Figure 15: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.16$ for $\varepsilon = 10$.

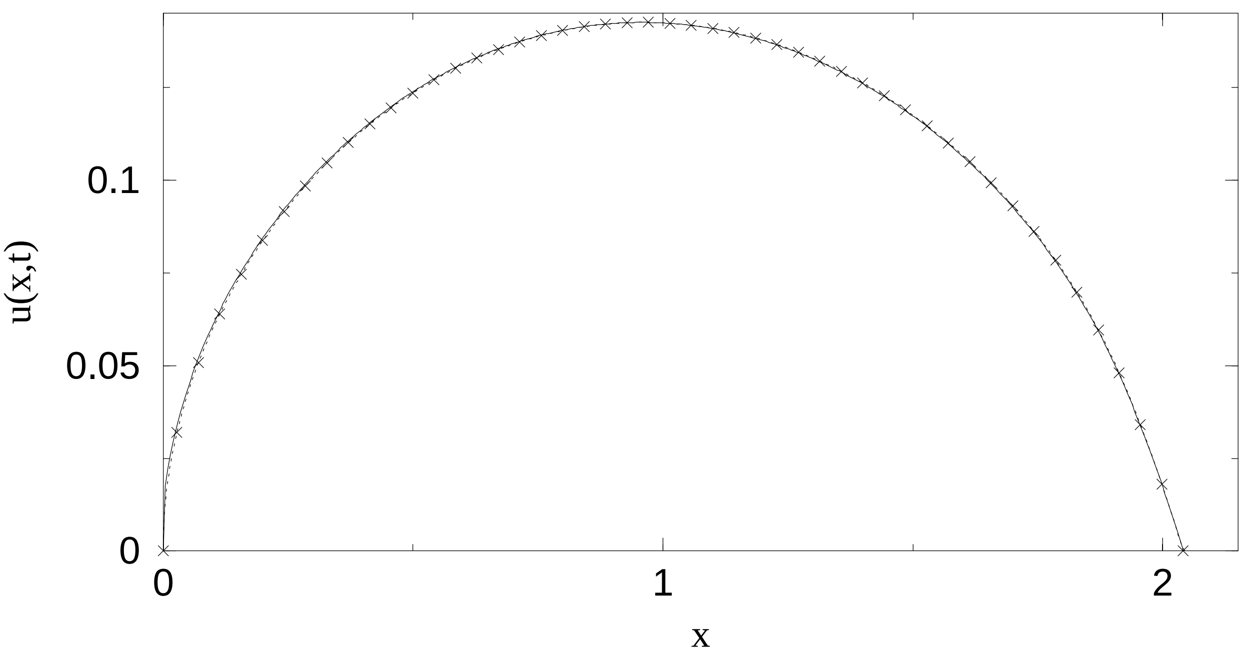

Figure 16: Comparison of $u(x,t)$ and the corresponding similarity solution $(\cdots\times\cdots)$ at $t = 0.2$ for $\varepsilon = 10$.

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