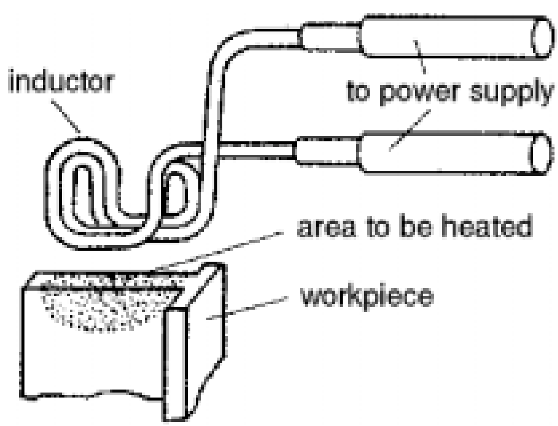
Abstract

We study a mathematical model for the inductive heating of steel.It consists of a vector potential formulation of Maxwells equations coupled with a heat equation and an evolution equation for the volume fraction of the high temperature phase in steel. An important task for practical applications of induction heating it to find the optimal coupling distance between inductor and workpiece.

To this end, we employ the speed method to investigate the sensitivity of solutions to the state equations with respect to perturbations of the inductor coil. We show the existence of strong material derivatives for the state variables and apply the structure theorem to characterize the Eulerian derivative of the cost functional.

1 Introduction

Electromagnetic induction provides a method of heating electrically conducting materials. The basic components of an induction heating system are depicted in Figure 1. An altenating current flows through the induction coil (in the sequel called inductor). It generates an alternating magnetic field which in turn induces eddy currents in the workpiece. These dissipate energy, bring about heating and lead to the growth of the high temperature phase austenite in the workpiece made of steel.



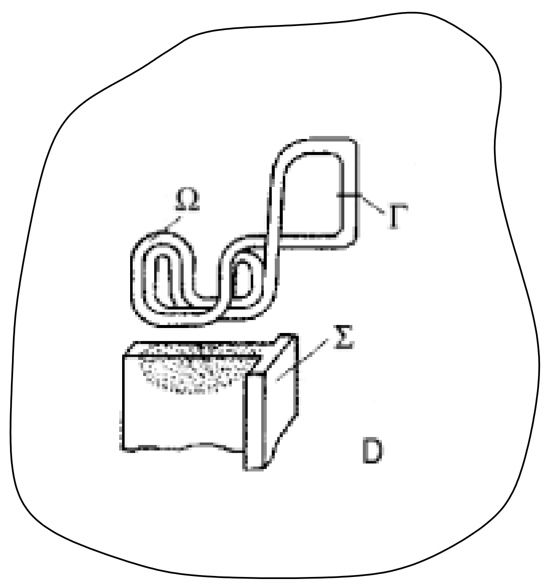


Figure 1: Induction heating: real process (left) and notation of domains in idealized setting.

Since the magnitude of the eddy currents decreases with growing distance from the workpiece surface because of the frequency dependent skin-effect, induction heating is a suitable heat source for surface heat treatments if the current frequency has been chosen big enough. On the other hand, if sufficient time for heat conduction is allowed and the current frequency is not

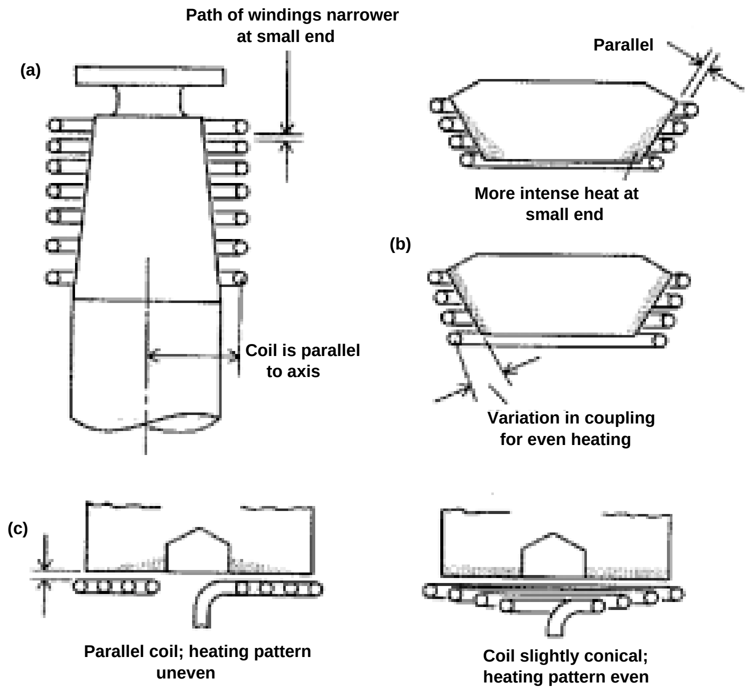


Figure 2: Adjustment of induction heating patterns by varying the turn spacing (a) or the coupling distance (b),(c) (from [5]).

too big, relatively uniform heating patterns can be obtained. Hence induction heating can also be used in heat treatments like annealing.

An important task during the planning of an induction heat treatment is to find the optimal coupling distance between inductor and workpiece in order to obtain a desired heating pattern.This is illustrated in Figure 2. In all the examples shown, the goal is to produce a uniform hardening depth. In (a) a conical workpiece shall be heated by an inductor of the shape of a cylindrical spiral. To compensate for the bigger distance between inductor and workpiece in the upper part, the turn spacing there is narrower compared to the lower part.

In (b), because of the workpiece's geometry, heat will concentrate in the lower corners of the workpiece cross-section, if the coupling distance is everywhere the same. The remedy is to increase the coupling distance in the lower part of the workpiece leading to a uniform penetration depth.

Example (c) depicts the typical situation of a hole in an otherwise plane workpiece surface. The inductor on the left-hand side with a uniform coupling distance leads to an uneven hardening pattern and possibly even to a melting of hole edges. A better result can be achieved when the coupling distance between inductor and workpiece is increased locally around the hole.

In the next Section we present the mathematical model for induction heating, which has been derived in [14]. To investigate the sensitivity with respect to perturbations of the inductor, we apply the speed-method as presented in the monographs [23] and [6]. In Section 3 we transform the state equations to the fixed domain, Section 4 is devoted to deriving stability estimates and in Section 5 we prove the existence of strong material derivatives for the state variables.

There are already a lot of papers on modeling, analysis and simulation of induction heating, e.g., [2]–[4], [7]–[9], [14], [16], [20]–[22]. In [1], an optimal control problem for a 2D induction heating setting has been considered. Mathematical models for phase transitions in steel have been considered in, e.g., [11]–[14] and [19].

2 The mathematical model

2.1 The state equations

Since we cannot model the hardening machine itself, we restrict ourselves to the following idealized geometric setting (cf. Fig. 1 (right)). Let $D \subset \mathbb{R}^3$ be the hold all domain, which contains the inductor Ω and the workpiece Σ.

We call $G = \Omega \cup \Sigma$ the set of conductors and define the space - time cylinder $Q = \Sigma \times (0, T)$.

Since we do not consider the hardening machine in our model, we assume that the inductor Ω is a closed tube. Inside we fix a section Γ and model the current density which is generated by the hardening machine by an interface condition on Γ.

In [14] an electrothermomechanical model for the induction heating of steel has been derived. Here we consider a simplified version where the equations are only sequentially coupled. It consists of a linear elliptic problem for the scalar potential ϕ, a degenerate parabolic equation for the magnetic vector potential A, a semilinear parabolic equation for the temperature θ and and ODE for the evolution of the austenite volume fraction z.

(P) Find $(A, \bar{\phi}, \theta, z) \in L^\infty(0, T; \mathbf{X}) \times H^1(0, T; H^1(\Omega)/\mathbb{R}) \times W_3^{2,1}(Q) \times W^{1,\infty}(0, T; L^\infty(\Sigma))$

such that

$$
\begin{aligned}
\kappa_0 \int_\Omega \nabla\phi \cdot \nabla u \; dx + \int_\Gamma j_s \varphi \, dx &= 0, \text{ for all } \varphi \in H^1(\Omega)/\mathbb{R}, && \text{(2.1a)}\\
A(0) &= A_0, \text{ in } D, && \text{(2.1b)}\\
\kappa_0 \int_G A_t \cdot v \; dx + \int_D \frac{1}{\mu} \operatorname{curl} A \cdot \operatorname{curl} v \; dx + \int_D \frac{1}{\mu} \operatorname{div} A \operatorname{div} v \, dx & \\
+\kappa_0 \int_\Omega \nabla\phi \cdot v \; dx &= 0, \text{ for all } v \in \mathbf{X}, \text{ a.e. in } (0, T), && \text{(2.1c)}\\
\theta(0) &= \theta_0, \quad \text{in } \Sigma, && \text{(2.1d)}\\
\frac{\partial\theta}{\partial\nu} &= 0, \quad \text{in } \partial\Sigma \times (0, T), && \text{(2.1e)}\\
\rho c_\varepsilon \theta_t - k\Delta\theta &= -\rho L z_t + \kappa_0 |A_t|^2, \quad \text{in } Q, && \text{(2.1f)}\\
z(0) &= 0, \quad \text{in } \Sigma, && \text{(2.1g)}\\
z_t &= \frac{1}{\mathcal{T}(\theta)} [z_{eq}(\theta) - z]^+, \quad \text{in } Q, && \text{(2.1h)}
\end{aligned}
$$

with $W_p^{2,1}(Q) = W^{1,p}(0, T; L^p(\Sigma)) \cap L^p(0, T; W^{2,p}(\Sigma))$ and $Q = \Sigma \times (0, T)$ the space time domain. The solution space $\mathbf{X}$ for the vector potential as

$$\mathbf{X} = \{v \in H(\text{ curl }, D) \Big| \text{ div } v \in L^2(D) \text{ and } n \times v\Big|_{\partial D} = 0\},$$

which in view of (H1) below is a closed subset of $\mathbf{H}^1(D)$. Owing to [10, Lemma 3.4]

$$\|v\|_{\mathbf{X}} = \Big(\int_D |\text{ curl } v|^2\, dx + \int_D (\text{ div } v)^2\, dx\Big)^{1/2}$$

is an equivalent norm on $\mathbf{X}$. Note further that (2.1a) is Neumann problem for which solutions are sought for in the quotint space $H^1(\Omega)/\mathbb{R}$.

Remark 2.1 *In the original model derived in [14] the Coulomb gauge div $A = 0$ has been enforced by including it in the solution space $\mathbf{X}$. To simplify the application of the speed-method in the next section, we have chosen here to include a divergence part in the bilinear form in* (2.1c).

We make the following assumptions:

(H1) $\bar{\Omega} \subset D$, $\bar{\Sigma} \subset D, \bar{\Omega} \cap \bar{\Sigma} = \emptyset$, and $\partial\Omega$, $\partial\Sigma$, ∂D are of class $C^{1,1}$.

(H2) κ_0, ρ, c_ε, k, and L are positive constants,

(H3) $A_0 \in \mathbf{X} \cap \mathbf{H}^2(D)$, $\theta_0 \in W^{2,3}(\Sigma)$,

(H4) $\mu(x) = \mu_2 \chi_{\Sigma} + \mu_1(1 - \chi_{\Sigma})$, with constants $0 < \mu_1 < \mu_2$.

To obtain higher regularity we will differentiate the equations for scalar and vector potential with respect to time. Therefore we have to assume the compatibility conditions

(H5) $j_s \in H^1(0, T; H^{-1/2}(\Gamma))$, such that $\int_\Gamma j_s\, dx = 0$ and $\int_\Gamma j_{s,t}\, dx = 0$.

(H6) There exists $y_0 \in \mathbf{X}$, such that

$$\begin{aligned} \kappa_0 \int_G y_0 \cdot\ v\ dx + \int_D \frac{1}{\mu} \text{ curl } A_0 \cdot\ \text{ curl } v\, dx & \\ + \int_D \frac{1}{\mu} \text{ div } A_0 \text{ div } v\, dx + \kappa_0 \int_\Omega \nabla\phi(0) \cdot v\ dx &= 0, \end{aligned}$$

for all $v \in \mathbf{X}$.

(H7) $0 < \mathcal{T}_* \leq \mathcal{T}(x) \leq \mathcal{T}^* < \infty$, for all $x \in \mathbb{R}$, $\|\mathcal{T}\|_{C^2(\mathbb{R})} \leq M$,

(H8) $0 \leq z_{eq}(x) \leq 1$, for all $x \in \mathbb{R}$, $\|z_{eq}\|_{C^2(\mathbb{R})} \leq M$,

(H9) $\mathcal{H} \in C^{2,1}(\mathbb{R})$, a monotone approximation of the heaviside function.

2.2 A weak solution to the state equations

Problem (P) is only sequentially coupled and can be solved by solving consecutively the sub-problems (2.1a), (2.1b) + (2.1c), and (2.1d) – (2.1h).

For the first one, we have

Lemma 2.1 *Assume (H1), (H2), and (H5), then* (2.1a) *has a unique solution* $\phi \in H^1(0,T;H^1(\Omega)/\mathbb{R})$ *such that*

$$\|\nabla\phi\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \leq C, \tag{2.2}$$

with a constant $C > 0$, depending on j_s, T, and κ_0.

Proof.
The proof follows from the Lax-Milgram lemma and the fact that we may differentiate (2.1a) with respect to time because of (H5). □

For the vector potential equation (2.1b), (2.1c) we have

Lemma 2.2 *Assume* (H1)–(H6), *then* (2.1b), (2.1c) *has a unique solution* $A \in L^\infty(0,T;\mathbf{X})$, *satisfying the estimate*

$$\|A\|_{L^\infty(0,T;\mathbf{X})} + \|A_t\|_{L^\infty(0,T;\mathbf{L}^6(G))} \leq C, \tag{2.3}$$

with a constant C depending on j_s, T, A_0, κ_0, and $\mu_{1,2}$.

Proof.
To prove the xistence of a unique weak solution one can use, e.g., Rothe's method of implicit time discretization as described in the monograph [17]. The first part of the a priori estimate follows from inserting $v = A_t$ into (2.1c) and integrating in time. To obtain the second part one can formally differentiate (2.1c) with respect to t. Then, we substitute $y = A_t$ and solve the system

$$\begin{aligned} y(0) &= y_0, \text{ in } D, \\ \kappa_0 \int_G y_t \cdot v\, dx + \int_D \frac{1}{\mu} \operatorname{curl} y \cdot \operatorname{curl} v\, dx + \int_D \frac{1}{\mu} \operatorname{div} y \operatorname{div} v\, dx & \\ + \kappa_0 \int_\Omega \nabla\phi_t \cdot v\, dx &= 0, \text{ for all } v \in \mathbf{X}, \text{ a.e. in } (0,T). \end{aligned}$$

Testing with $v = y_t$ and integrating in time we obtain an estimate for y in $L^\infty(0,T;\mathbf{X})$. Owing to the compatibility condition (H6) we can recover that $y = A_t$ a.e. in G. Hence we can use the embedding $\mathbf{H}^1(G) \subset \mathbf{L}^6(G)$ and obtain the second part of (2.3). □

Lemma 2.3 *Assume (H7)–(H9), then the following are true:*

(1) Let $\theta \in L^1(Q)$, then (2.1g), (2.1h) *has a unique solution satisfying*

$$0 \leq z(x,t) < 1 \quad a.e. \ in\ Q, \tag{2.4}$$

and

$$\|z\|_{W^{1,\infty}(0,T;L^\infty(\Sigma))} \leq C, \tag{2.5}$$

with a constant $C > 0$ independent of θ.

(2) Let $\theta_k \to \theta$ strongly in $L^1(\Sigma)$. Then

$$z_k \longrightarrow z, \qquad \textit{strongly in } W^{1,p}(0,T;L^p(\Sigma)), \textit{ for } p \in [1,\infty),$$

where z_k and z are the solution to (2.1g), (2.1h) *corresponding to θ_k and θ, respectively.*

(3) Let $\theta_1, \theta_2 \in L^p(Q)$, $p \in [1,\infty)$, and z_1, z_2 the corresponding solutions to (2.1g), (2.1h), *then there exists a constant $C > 0$, such that*

$$\|z_1 - z_2\|_{W^{1,p}(0,T;L^p(\Sigma))} \leq C\|\theta_1 - \theta_2\|_{W^{1,p}(0,T;L^p(\Sigma))}.$$

Proof.
The existence of a unique local solution to (2.1g), (2.1h) is a direct consequence of the theorem of Carathéodory, see e.g. [24, p. 1044]. Using (H7)–(H9), and the theory of differential inequalities (cf. [11, Lemma 2.1], we obtain (2.4), whereas (2.5) is a direct consequence of (H7)–(H9). Assertion (2) follows from Lebesgue's lemma.

To prove (3), let $\theta^i \in L^p(Q)$, $i = 1, 2$, and define $\bar\theta = \theta_1 - \theta_2$, then $\bar z = z^1 - z^2$ solves

$$\bar z_t = f(\theta^1, z^1) - f(\theta^2, z^2), \tag{2.6}$$

where $f(\theta, z)$ denotes the right-hand side of (2.1h). In view of (H7)–(H9), f is Lipschitz continuous. Hence, we can test (2.6) with $\bar z^{p-1}$ and apply Young's inequality to obtain

$$\begin{aligned}\frac{1}{p}\int_\Sigma \|\bar z(t)\|^p\,dx &\leq c_1\int_0^t\int_\Sigma |\bar z|^p\,dx\,ds + c_2\int_0^t\int_\Sigma |\bar\theta||\bar z|^{p-1}\,dx\,ds \\ &\leq \left(c_1 + c_2\frac{p-1}{p}\right)\int_0^t\int_\Sigma |\bar z|^p\,dx\,ds + \frac{c_2}{p}\int_0^t\int_\Sigma |\bar\theta|^p\,dx\,ds.\end{aligned}$$

Now we can apply Gronwall's lemma and use (2.6) once again to conclude the proof. □

Remark 2.2 *Lemma 2.3 shows that the volume fraction z, which is defined as the solution to the initial value problem* (2.1g), (2.1h), *satisfies assumption (A6).*

Before considering the heat equation (2.1f), we recall the following results from the linear theory of parabolic equations:

Lemma 2.4 *[18, Theorem 9.1] Let $g \in L^p(Q)$ and $u_0 \in W^{1,p}(\Sigma)$ for some $p \in (1,\infty)$. Then there exists a constand $C > 0$ such that the unique solution to*

$$\begin{aligned} u_t - \Delta u &= g \qquad \textit{in } Q, \\ \frac{\partial u}{\partial \nu} &= 0, \qquad \textit{in } \partial\Sigma \times (0,T), \\ u(0) &= u_0, \qquad \textit{in } \Sigma \end{aligned}$$

satisfies the estimate

$$\|u\|_{W_p^{2,1}(Q)} \le C\Big(\|u_0\|_{W^{1,p}(\Sigma)} + \|g\|_{L^p(Q)}\Big).$$

For later use we also note the following embedding theorem [22, (3.9)], written down for dim $\Sigma = 3$:

Lemma 2.5 *Let* $k = 0, 1$, $p \ge q$, $2 - k - 5\Big(\frac{1}{q} - \frac{1}{p}\Big) \ge 0$, *then the embedding*

$$W_q^{2,1}(Q) \subset W_p^{k,0}(Q)$$

is continuous. The inclusion is compact if the last inequality is strict.

Lemma 2.6 *Assume(H1)–(H9), then* (2.1d)–(2.1h) *has a unique solution* (θ, z), *such that*

$$\|(\theta, z)\|_{W_3^{2,1}(Q) \times W^{1,\infty}(0,T;L^\infty(\Sigma))} \le C.$$

The constant C *depends on* A_t *and* θ_0.

Proof.
The existence can be proved, e.g., using the Schauder fixed point theorem. The a priori estimate is a direct consequence of Lemma 2.3 and Lemma 2.4.

To prove uniqueness, we take the difference of two solutions $\bar\theta = \theta^1 - \theta^2$ which satisfies

$$\rho c_\varepsilon \bar\theta_t - k\Delta\bar\theta = -\rho L(z_t^1 - z_t^2), \qquad \text{in } Q,$$

$$\frac{\partial\bar\theta}{\partial\nu} = 0, \text{ in } \partial\Sigma \times (0,T), \qquad \bar\theta(0) = 0, \text{ in } \Sigma.$$

Using Lemma 2.3(3), Lemma 2.4 and Hölder's inequality, we can infer[1]

$$\|\bar\theta\|^3_{W_3^{2,1}(Q_t)} \le c_1 \int_0^t \int_\Sigma \Big| \int_0^s \bar\theta_\xi \, d\xi \Big|^3 dx\, ds \le T^2 c_1 \int_0^t \int_0^s \int_\Sigma |\bar\theta_\xi|^3 \, dx d\xi\, ds \le c_2 \int_0^t \|\bar\theta\|^3_{W_3^{2,1}(Q_s)}\, ds,$$

where $Q_t = \Sigma \times (0, t)$. Now the assertion follows from Gronwall's lemma. □

Summarizing the results of Lemmas 2.1–2.3 and Lemma 2.6 we obtain

Theorem 2.1 *Assume* (H1) – (H9), *then Problem* **(P)** *has a unique solution.*

2.3 The shape design problem

To decide whether the coupling distance between inductor and workpiece has been chosen decently, we measure the volume fraction of austenite at the end-time T and compare it to a desired volume fraction $\bar z$, i.e., we consider the following cost functional of tracking type

$$\mathcal{J}(\Omega) = \int_\Sigma \Big(z(x,T) - \bar z\Big)^2 dx. \tag{2.7}$$

The precise formulation of our design problem then reads

[1] Note that $\bar\theta_\xi$ is short for $\frac{\partial\bar\theta(x,\xi)}{\partial\xi}$.

(CP) Minimize $\mathcal{J}(\Omega)$, given by (2.7) ,

subject to

$\Omega \in \mathcal{U}_{ad}$

and the state equations (2.1a)–(2.1h).

In this paper we completely ignore the question of defining a reasonable set of admissible domains $\mathcal{U}_{ad}$ and proving the existence of an optimal design. Using a special topology, namely tubes generated from space curves, these questions have been discussed in [15].

Instead we will investigate the shape sensitivity with respect to perturbations of the inductor without specifying the inductor topology. We only require that it satisfies (H1).

3 Speed method and transformation to the fixed domain

To investigate the sensitivity of solutions to the state system (2.1a) - (2.1h) with respect to perturbations of the shape of the inductor Ω, we use the speed method (cf. [23, Sec. 2.9]).

We introduce a speed vector field V satisfying

(H10) $V \in C(-\tau_1, \tau_1; C_0^2(D, \mathbb{R}^3))$, supp $V \subset (B_{\delta_1}(\Omega) \setminus B_{\delta_2}(\Gamma_1))$, with positive constants τ_1 and $\delta_{1,2}$.

Hence, the velocity field is chosen in such a way that the inductor can be perturbed, except for a small region around the interface Γ, where current is supplied and in reality the inductor is fixed to the hardening machine. Moreover, we tacitly assume that δ_1 has been chosen small enough to assure $\bar{\Sigma} \cap \operatorname{supp} V = \emptyset$.

Now we construct a family of mappings

$$\mathcal{T}_\tau(V) \,:\, \mathbb{R}^3 \ni X \longrightarrow x_\tau \in \mathbb{R}^3,$$

where x_τ satisfies the initial value problem

$$\begin{aligned} \frac{dx_\tau}{d\tau} &= V(\tau, x_\tau), \\ x_0 &= X. \end{aligned}$$

Then we define a family of perturbations of a given initial configuration Ω by

$$\Omega_\tau = \mathcal{T}_\tau(V)(\Omega).$$

All equations defined in Ω_τ can be transported to the fixed domain Ω, using the transformation $T_\tau^{-1} : \Omega_\tau \to \Omega$. Note that, by construction, we have $\Omega_0 = \Omega$ and $\overline{\Omega_\tau} \cap \bar{\Sigma} = \emptyset$, for all $\tau \in (-\tau_1, \tau_1)$, if τ_1 has been chosen small enough. Moreover, the interface Γ, where the source current is supplied, remains invariant under the perturbations of Ω, and we have $\mathcal{T}_\tau(V)(D) = D$.

Remark 3.1 *In the sequel we indicate functions on Ω_τ with subscript τ, and functions transported to the fixed domain Ω with superscript τ, i.e., $f^\tau = f_\tau \circ T_\tau$.*

The following lemma describes the transport of div and grad to the fixed domain. The proof can be found in [23, Sec. 2]. Note that the Jacobian of T_τ is denoted by $\mathcal{D}T_\tau$. Moreover, for any matrix B, the transposed one is denoted by $^*B$.

Lemma 3.1 *Let $B_1(\tau) = {}^*\mathcal{D}T_\tau^{-1}$, then we have*

(1)
$$(\textit{grad}\ \varphi) \circ T_\tau = \Big(B_1(\tau)\nabla\Big)\Big(\varphi \circ T_\tau\Big), \qquad \textit{for all } \varphi \in H^1(D),$$

(2)
$$(\textit{div}\ \psi) \circ T_\tau = \Big(B_1(\tau)\nabla\Big) \cdot \Big(\psi \circ T_\tau\Big), \qquad \textit{for all } \psi \in \mathbf{H}^1(D).$$

(3)
$$(\textit{curl}\ \psi) \circ T_\tau = \Big(B_1(\tau)\nabla\Big) \times \Big(\psi \circ T_\tau\Big), \qquad \textit{for all } \psi \in \mathbf{H}^1(D).$$

Using Lemma 3.1, we obtain for (2.1a), with $\varphi \in H^1(\Omega_\tau)/\mathbb{R}$,

$$\begin{aligned} -\int_\Gamma j_g \varphi\, dx &= \kappa_0 \int_{\Omega_\tau} \nabla\phi_\tau \cdot \nabla\varphi\, dx \\ &= \kappa_0 \int_\Omega \det(\mathcal{D}T_\tau)\Big(\nabla\phi_\tau \cdot \nabla\varphi\Big) \circ T_\tau\, dx \\ &= \kappa_0 \int_\Omega B_2(\tau)\nabla\phi^\tau \cdot \nabla(\varphi \circ T_\tau)\, dx \end{aligned}$$

with

$$\beta(\tau) = \det(\mathcal{D}T_\tau) \text{ and } B_2(\tau) = \beta(\tau)\, {}^*B_1(\tau)B_1(\tau).$$

Hence, (2.1a) is replaced with

$$-\int_\Gamma j_g \varphi\, dx = \alpha_0(\tau, \phi^\tau, \varphi), \qquad \text{for all } \varphi \in H^1(\Omega)/\mathbb{R}, \tag{3.8}$$

and

$$\alpha_0(\tau, \phi^\tau, \varphi) := \kappa_0 \int_\Omega B_2(\tau)\nabla\phi^\tau \cdot \nabla\varphi\, dx.$$

Now we turn to the Maxwell equation (2.1c). For the first term, we obtain

$$\begin{aligned} \kappa_0 \int_{G_\tau} \frac{\partial A_\tau}{\partial t} v\, dx &= \kappa_0 \int_G \beta(\tau) A_t^\tau \cdot (v \circ T_\tau)\, dx \\ &=: \alpha_1(\tau, A_t^\tau, v \circ T_\tau), \end{aligned}$$

For the next term, we utilize Lemma 3.1 (2), (3) to obtain

$$
\begin{aligned}
&\int\limits_D \frac{1}{\mu} \text{ curl } A_\tau \cdot \text{ curl } v\, dx + \int\limits_D \frac{1}{\mu} \text{ div } A_\tau \text{ div } v\, dx \\
&\quad = \int\limits_D \frac{\beta(\tau)}{\mu}\Big(\text{ curl } A_\tau \cdot \text{ curl } v\Big) \circ T_\tau\, dx + \int\limits_D \frac{\beta(\tau)}{\mu}\Big(\text{ div } A_\tau \text{ div } v\Big) \circ T_\tau\, dx \\
&\quad = \int\limits_D \frac{\beta(\tau)}{\mu} \{(B_1(\tau)\nabla) \times A^\tau\} \cdot \{(B_1(\tau)\nabla) \times (v \circ T_\tau)\}\, dx \\
&\qquad\qquad + \int\limits_D \frac{\beta(\tau)}{\mu} \{(B_1(\tau)\nabla) \cdot A^\tau\} \{(B_1(\tau)\nabla) \cdot (v \circ T_\tau)\}\, dx \\
&\quad =: \alpha_2(\tau, A^\tau, v \circ T_\tau).
\end{aligned}
$$

For the last term in (2.1c), we have

$$
\begin{aligned}
\kappa_0 \int\limits_{\Omega_\tau} \nabla \phi_\tau \cdot v\, dx &= \kappa_0 \int\limits_\Omega \beta(\tau)\Big(\nabla \phi_\tau \cdot v\Big) \circ T_\tau\, dx \\
&= \kappa_0 \int\limits_\Omega B_3(\tau) \nabla \phi^\tau \cdot (v \circ T_\tau)\, dx \\
&=: F(\tau, \phi^\tau, v \circ T_\tau),
\end{aligned}
$$

with $B_3(\tau) = \beta(\tau) B_1(\tau)$.

Altogether, we have replaced (2.1c) with [2]

$$\alpha_1(\tau, A_t^\tau, v) + \alpha_2(\tau, A^\tau, v) + F(\tau, \phi^\tau, v) = 0, \qquad \text{for all } v \in X, \tag{3.9a}$$

$$A_0^\tau = A_0 \circ T_\tau. \tag{3.9b}$$

Remark 3.2 *Another possibility to transport the divergence operator to the fixed domain is, to use the formula (cf. [23])*

$$(\ \mathit{div}\ \psi) \circ T_\tau = \frac{1}{\beta(\tau)}\ \mathit{div}\ \Big(\beta(\tau) \mathcal{D}T_\tau^{-1}(\psi \circ T_\tau)\Big), \qquad \textit{for all } \psi \in \mathbf{H}^1(D).$$

It shows, that functions that are divergence-free on Ω_τ generally loose this property, when transported to the fixed domain. To cope with this difficulty, we could introduce an auxiliary unknown function

$$\eta^\tau = \beta(\tau) \mathcal{D}T_\tau^{-1} A^\tau,$$

which in view of the above formula would give

$$\mathit{div}\ A_\tau = 0 \quad \mathit{in}\ \Omega_\tau \quad \Longleftrightarrow \quad \mathit{div}\ \eta^\tau = 0 \quad \mathit{in}\ \Omega.$$

However, as we have mentioned already in Remark 2.2, in this case the shape sensitivity analysis becomes very difficult and is still an open problem, at least for a time-dependent vector potential.

[2]Note that $v \circ T_\tau \in \mathbf{X}$ if and only if $v \in \mathbf{X}$.

4 Stability estimates

Lemma 4.1 *B_1, B_2, B_3, β are differentiable. For $|\tau| \le \tau_1$, and τ_1 small enough, we have*

$$\begin{aligned} \beta(\tau) &= 1 + \tau\beta'(0) + o(\tau), \\ B_i(\tau) &= I + \tau B_i'(0) + o(\tau), \qquad i = 1, \dots, 3. \end{aligned}$$

The derivatives at $\tau = 0$ are given by

$$\begin{aligned} \beta'(0) &= \; div\, V(0), \\ B_1'(0) &= -{}^*\mathcal{D}V(0), \\ B_2'(0) &= \; div\, V(0) I - 2\varepsilon(V(0)), \\ B_3'(0) &= \; div\, V(0) I - {}^*\mathcal{D}V(0). \end{aligned}$$

Here, $\varepsilon(V(0))$ is the symmetrized part of $\mathcal{D}V(0)$, i.e. , $\varepsilon(V(0)) = \frac{1}{2}(\mathcal{D}V(0) + {}^*\mathcal{D}V(0))$.

For the proof, we refer again to [23, Sec. 2.13].

A particular consequence of Lemma 4.1 is

Corollary 4.1 *Let $|\tau| \le \tau_1$, and τ_1 small enough. Then there exist real-valued functions g_i satisfying $g_i(\tau) = o(\tau)$, $i = 0, \dots, 3$ and bilinear forms $\tilde{\alpha}_i(\tau, .\,, .)$, $i = 0, 1, 2$ and $\tilde{F}(\tau, .\,, .)$ such that the following are valid:*

(1) For all $\varphi_1, \varphi_2 \in H^1(\Omega)/\mathbb{R}$, we have

$$\begin{aligned} \alpha_0(\tau, \varphi_1, \varphi_2) &= \alpha_0(0, \varphi_1, \varphi_2) + \tau\alpha_{_{0,\tau}}(0, \varphi_1, \varphi_2) + \tilde{\alpha}_0(\tau, \varphi_1, \varphi_2), \\ \alpha_{_{0,\tau}}(0, \varphi_1, \varphi_2) &= \kappa_0 \int\limits_{\Omega} B_2'(0)\nabla\varphi_1\nabla\varphi_2\, dx, \\ \left|\tilde{\alpha}_0(\tau, \varphi_1, \varphi_2)\right| &\le g_0(\tau)\|\nabla\varphi_1\|_{\mathbf{L}^2(\Omega)}\|\nabla\varphi_2\|_{\mathbf{L}^2(\Omega)}; \end{aligned}$$

(2) For all $v_1, v_2 \in \mathbf{L}^2(D)$, we have

$$\begin{aligned} \alpha_1(\tau, v_1, v_2) &= \alpha_1(0, v_1, v_2) + \tau\alpha_{_{1,\tau}}(0, v_1, v_2) + \tilde{\alpha}_1(\tau, v_1, v_2), \\ \alpha_{_{1,\tau}}(0, v_1, v_2) &= \kappa_0 \int\limits_{G} \beta'(0) v_1 \cdot v_2\, dx, \\ \left|\tilde{\alpha}_1(\tau, v_1, v_2)\right| &\le g_1(\tau)\|v_1\|_{\mathbf{L}^2(G)}\|v_2\|_{\mathbf{L}^2(G)}, \end{aligned}$$

(3) For all $v_1, v_2 \in \mathbf{X}$, we have

$$\begin{aligned}
\alpha_2(\tau, v_1, v_2) &= \alpha_2(0, v_1, v_2) + \tau \alpha_{2,\tau}(0, v_1, v_2) + \tilde{\alpha}_2(\tau, v_1, v_2), \\
\alpha_{2,\tau}(0, v_1, v_2) &= \int_D \frac{\beta'(0)}{\mu} \Big(\; curl\; v_1 \cdot \; curl\; v_2 + \; div\; v_1 \; div\; v_2 \Big)\, dx \\
&\quad + \int_D \frac{1}{\mu} [(B_1'(0)\nabla) \times v_1] \cdot \; curl\; v_2 \, dx \\
&\quad + \int_D \frac{1}{\mu} \; curl\; v_1 \cdot [(B_1'(0)\nabla) \times v_2] \, dx \\
&\quad + \int_D \frac{1}{\mu} [(B_1'(0)\nabla) \cdot v_1] \; div\; v_2 \, dx \\
&\quad + \int_D \frac{1}{\mu} \; div\; v_1 \cdot [(B_1'(0)\nabla) \cdot v_2] \, dx \\
\Big| \tilde{\alpha}_2(\tau, v_1, v_2) \Big| &\leq g_2(\tau) \|v_1\|_{\mathbf{X}} \|v_2\|_{\mathbf{X}}.
\end{aligned}$$

(4) For all $\varphi \in H^1(\Omega)/\mathbb{R}$ and $v \in \mathbf{X}$, we have

$$\begin{aligned}
F(\tau, \varphi, v) &= F(0, \varphi, v) + \tau F_{,\tau}(0, \varphi, v) + \tilde{F}(\tau, \varphi, v), \\
F_{,\tau}(0, \varphi, v) &= \kappa_0 \int_\Omega B_3'(0) \nabla\varphi \cdot v \, dx, \\
\Big| \tilde{F}(\tau, \varphi, v) \Big| &\leq g_4(\tau) \|\nabla\varphi\|_{\mathbf{L}^2(\Omega)} \|v\|_{\mathbf{X}}.
\end{aligned}$$

Using Corollary 4.1, we can prove the following stability result:

Lemma 4.2 *Assume (H1)–(H10), then there exists a constant $C > 0$ such that*

(1) $\|\nabla\phi^\tau - \nabla\phi\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \leq C \cdot |\tau|$,

(2) $\|A^\tau - A\|_{L^2(0,T;\mathbf{X})} + \|A_t^\tau - A_t\|_{L^{10/3}(0,T;\mathbf{L}^{10/3}(G))} \leq C \cdot |\tau|$,

(3) $\|\theta^\tau - \theta\|_{W^{2,1}_{5/3}(Q)} \leq C \cdot |\tau|$,

(4) $\|z^\tau - z\|_{W^{1,5}(0,T;L^5(\Sigma))} \leq C \cdot |\tau|$.

Remark 4.1 *(z^τ, θ^τ) is the solution to* (2.1d) – (2.1h)*, where A_t in* (2.1f) *has been replaced with A_t^τ. In view of (H12), we have $A_t^\tau = A_{\tau,t}$ on Σ.*

For the proof, we need the following interpolation result:

Lemma 4.3 *Let $u \in L^\infty(0,T;L^2(\Sigma)) \cap L^2(0,T;H^1(\Sigma))$, then there holds*

$$\int_0^T \|u(t)\|_{L^{10/3}(\Sigma)}^{10/3} \, dt \leq \Big(\int_0^T \|u(t)\|_{L^6(\Sigma)}^2 \, dt \Big) \|u\|_{L^\infty(0,T;L^2(\Sigma))}^{4/3}.$$

Proof. Owing to Riesz' convexity theorem (cf. [24, (A113)]), we have

$$\|u\|_{L^r(\Sigma)} \leq \|u\|^{1-\Theta}_{L^{q_1}(\Sigma)} \|u\|^{\Theta}_{L^{q_2}(\Sigma)},$$

for all $u \in L^{q_1}(\Sigma) \cap L^{q_2}(\Sigma)$ with $1 \leq q_1, q_2 < \infty$, $0 < \Theta < 1$, and $\frac{1}{r} = \frac{1-\Theta}{q_1} + \frac{\Theta}{q_2}$. Invoking the continuous embedding $H^1(\Sigma) \subset L^6(\Sigma)$, the assertion follows by defining $q_1 = 6$, $q_2 = 2$, $\Theta = \frac{2}{5}$, and $r = \frac{10}{3}$. $\square$

Proof. [of Lemma 4.2]

According to Lemma 4.1, we can write

$$\beta(\tau) = 1 + \tau\beta'(\xi_0), \quad B_i(\tau) = I + \tau B_i'(\xi_i), \quad i = 1, 2, 3 \tag{4.10}$$

for τ small enough and $\xi_i \in [0, \tau]$, $i = 0, \ldots, 3$. Note that $\beta(\tau) \geq c_{\tau_1} > 0$ for $|\tau| \leq \tau_1$, if the latter has been chosen small enough and that the B_i's are positive definite for $|\tau| \leq \tau_1$.

Using (H2) and (H5), this gives immediately

$$\|\nabla\phi^\tau\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \leq c_1, \tag{4.11}$$

independent of τ. Moreover, we can use (3.8) and (4.10) to write

$$\begin{aligned} 0 &= \alpha_0(\tau, \phi^\tau, \varphi) - \alpha_0(0, \phi, \varphi) \\ &= \alpha_0(0, \phi^\tau - \phi, \varphi) + \tau \int_\Omega B_2'(\xi)\nabla\phi^\tau \cdot \nabla\varphi \, dx. \end{aligned}$$

Inserting $\varphi = \phi^\tau - \phi$ and using Young's inequality, we obtain

$$\|\nabla\phi^\tau - \nabla\phi\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq c_2|\tau|.$$

Since the same estimate holds true for $\phi_t^\tau - \phi_t$, assertion (1) is proved.

We insert $v = A_t^\tau$ into (3.9a), use (4.10) and integrate in time to obtain for the first term

$$\kappa_0 \int_0^t \int_\Sigma \beta(\tau) A_s^\tau \cdot A_s^\tau \, dx \, ds \geq c_{\tau_1}\kappa_0 \int_0^t \int_\Sigma |A_s^\tau|^2 \, dx \, ds.$$

The second term gives

$$
\begin{aligned}
\int_0^t \alpha_2(\tau, A^\tau, A_s^\tau)\, ds &= \int_0^t \int_D \frac{\beta(\tau)}{\mu} \{(B_1(\tau)\cdot\nabla)\times A^\tau\}\cdot\{(B_1(\tau)\cdot\nabla)\times A_s^\tau\}\; dx\, ds \\
&\quad + \int_0^t \int_D \frac{\beta(\tau)}{\mu} \{(B_1(\tau)\cdot\nabla)\cdot A^\tau\}\{(B_1(\tau)\cdot\nabla)\cdot A_s^\tau)\}\; dx\, ds \\
&= \frac{1}{2}\int_0^t \int_D \frac{\beta(\tau)}{\mu}\frac{\partial}{\partial s}\Big|\,(B_1(\tau)\cdot\nabla)\times A^\tau\Big|^2 dx\, ds \\
&\quad + \frac{1}{2}\int_0^t \int_D \frac{\beta(\tau)}{\mu}\frac{\partial}{\partial s}\Big|\,(B_1(\tau)\cdot\nabla)\cdot A^\tau\Big|^2 dx\, ds \\
&\geq \frac{1}{2\mu_2}\int_D \Big|\ \mathrm{curl}\ A^\tau(t)\Big|^2 dx + \frac{1}{2\mu_2}\int_D \Big|\ \mathrm{div}\ A^\tau(t)\Big|^2 dx + \tau\widetilde{g}(A^\tau(t)) - c_3,
\end{aligned}
$$

with a function $\widetilde{g}$ satisfying $\widetilde{g}(A^\tau(t)) \leq c_4\|A^\tau(t)\|_{\mathbf{X}}^2$. For the last term in (3.9a), we apply Young's inequality and obtain

$$
\int_0^t F(\tau, \phi^\tau, A_s^\tau)\, ds \leq \frac{c_{T_1}}{2}\kappa_0 \int_0^t \int_\Omega |A_s^\tau|^2\, dx\, ds + c_5 \int_0^t \int_\Omega |\nabla\phi^\tau|^2\, dx\, ds.
$$

Invoking (4.11) and choosing τ small enough, we finally obtain

$$
\|A^\tau\|_{L^\infty(0,T;\mathbf{X})} + \|A_t^\tau\|_{L^2(0,T;L^2(G))} \leq c_6. \tag{4.12}
$$

Now we differentiate (3.9a) formally with respect to time and insert $v = A_{tt}^\tau$. Defining

$$
A_{0,t}^\tau = y \circ T_\tau \tag{4.13}
$$

(cf. (H6) and (3.9b)), analogously to the derivation of the previous estimate, we get

$$
\|A_t^\tau\|_{L^\infty(0,T;\mathbf{X})} + \|A_{tt}^\tau\|_{L^2(0,T;L^2(G))} \leq c_7. \tag{4.14}
$$

Next, we take the difference of (3.9a) for A^τ and A and obtain

$$
\begin{aligned}
0 &= \alpha_1(\tau, A_t^\tau, v) + \alpha_2(\tau, A^\tau, v) + F(\tau, \phi^\tau, v) - \alpha_1(0, A_t, v) - \alpha_2(0, A, v) - F(0, \phi, v) \\
&= \alpha_1(0, A_t^\tau - A_t, v) + \alpha_2(0, A^\tau - A, v) + F(0, \phi^\tau - \phi, v) \\
&\qquad + G_0(\tau, \phi^\tau, v) + G_1(\tau, A_t^\tau, v) + G_2(\tau, A^\tau, v),
\end{aligned} \tag{4.15}
$$

with $G_0(\tau, \phi^\tau, v) = F(\tau, \phi^\tau, v) - F(0, \phi^\tau, v)$, $G_1(\tau, A_t^\tau, v) = \alpha_1(\tau, A_t^\tau, v) - \alpha_1(0, A_t^\tau, v)$ and $G_2(\tau, A^\tau, v) = \alpha_2(\tau, A^\tau, v) - \alpha_2(0, A^\tau, v)$ satisfying (cf. (4.10)),

$$
\begin{aligned}
|G_0(\tau, \varphi, v)| &\leq c_8|\tau|\, \|\nabla\varphi\|_{\mathbf{L}^2(\Omega)}\|v\|_{\mathbf{X}}, \\
|G_1(\tau, v_1, v_2)| &\leq c_9|\tau|\, \|v_1\|_{\mathbf{L}^2(G)}\|v_2\|_{\mathbf{L}^2(G)}, \\
|G_2(\tau, v_1, v_2)| &\leq c_{10}|\tau|\, \|v_1\|_{\mathbf{X}}\|v_2\|_{\mathbf{X}}.
\end{aligned}
$$

Inserting $v = A^\tau - A$ into (4.15) and integrating in time leads to

$$\frac{\kappa_0}{2}\int_G |A^\tau(t)-A(t)|^2\,dx + \int_0^t\int_D \frac{1}{\mu}|\ \mathrm{curl}\ (A^\tau - A)|^2\,dx\,dt$$
$$\le \kappa_0 \int_0^t\int_D |\nabla(\phi^\tau-\phi)\cdot(A^\tau-A)|\,dx\,dt + \frac{\kappa_0}{2}\int_G |A_0^\tau - A_0|^2\,dx$$
$$+|\tau|c_8\int_0^t \|\nabla\phi^\tau\|_{\mathbf{L}^2(\Omega)}\|A^\tau - A\|_{\mathbf{X}} + |\tau|c_9\int_0^t \|A_t^\tau\|_{\mathbf{L}^2(G)}\cdot\|A^\tau - A\|_{\mathbf{L}^2(G)}$$
$$+|\tau|c_{10}\int_0^t \|A^\tau\|_{\mathbf{X}}\cdot\|A^\tau - A\|_{\mathbf{X}}.$$

Applying the inequalities of Young and Gronwall and using (3.9b), we obtain

$$\|A^\tau - A\|_{L^\infty(0,T;L^2(G))} + \|A^\tau - A\|_{L^2(0,T;\mathbf{X})} \le c_{11}|\tau|. \tag{4.16}$$

Moreover, using (4.15) once again as well as (4.16), we obtain

$$\int_0^t \alpha_1(0, A_s^\tau - A_s, v)\,ds \le c_{12}|\tau|\|v\|_{L^2(0,t;\mathbf{X})}. \tag{4.17}$$

As before, we now differentiate (4.15) formally with respect to time[3] insert $v = A_t^\tau - A_t$ and make the same computations as before, but use (4.13) instead of (3.9b). Thus we obtain

$$\|A_t^\tau - A_t\|_{L^\infty(0,T;L^2(G))} + \|A_t^\tau - A_t\|_{L^2(0,T;\mathbf{X})} \le c_{13}|\tau|,$$

and, similar to (4.17)

$$\int_0^t \alpha_1(0, A_{ss}^\tau - A_{ss}, v)\,ds \le c_{14}|\tau|\|v\|_{L^2(0,t;\mathbf{X})}. \tag{4.18}$$

To conclude the proof of assertion (2), we apply Lemma 4.3 with $u = A_t^\tau - A_t$, i.e.,

$$\|A_t^\tau - A_t\|_{L^{10/3}(0,T;\mathbf{L}^{10/3}(\Sigma))} \ \le\ c_{15}\cdot|\tau|. \tag{4.19}$$

To prove assertion (3), we define $\bar\theta = \theta^\tau - \theta$ and $\bar z = z^\tau - z$ (cf. Remark 4.1). Then $\bar\theta$ solves

$$\rho c_p \bar\theta_t - k\Delta\bar\theta = -\rho L \bar z_t + \kappa_0(A_t^\tau - A_t)\cdot(A_t^\tau + A_t) \text{ in } Q$$
$$\frac{\partial\bar\theta}{\partial\nu} = 0, \text{ in } \Sigma\times(0,T), \quad \bar\theta(0) = 0 \text{ in } \Sigma.$$

[3]Note that $B_i(\tau), i = 1,2,3$ and $\beta(\tau)$ are independent of time, hence the bilinear forms are not affected.

In view of Lemma 2.4, we can apply Hölder's inequality, Lemma 2.3(3), and (4.19) to infer

$$\begin{aligned}
\|\bar{\theta}\|^{5/3}_{W^{2,1}_{5/3}(Q_t)} &\leq c_{16}\int_0^t\int_\Sigma |\bar{z}_s|^{5/3}\,dx\,ds \\
&\quad +c_{17}\Big(\int_0^t\int_\Sigma |A^\tau_s - A_s|^{10/3}\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_\Sigma |A^\tau_s + A_s|^{10/3}\,dx\,ds\Big)^{1/2} \\
&\leq c_{18}\int_0^t \|\bar{\theta}\|^{5/3}_{W^{2,1}_{5/3}(Q_s)} + c_{19}|\tau|^{5/3}.
\end{aligned}$$

Then assertion (3) follows from Gronwall's lemma whereas assertion (4) is a direct consequence of (3), Lemma 2.3(3), and the continuous embedding $W^{2,1}_{5/3}(Q) \subset L^5(Q)$ (cf. Lemma 2.5). □

5 Strong material derivatives

Remark 5.1 *All the unknowns depend on the shape of Ω_τ, either explicitly as A^τ and ϕ^τ or implicitly as θ^τ and z^τ. For all these functions, we call*

$$\dot{f} = \lim_{\tau \to 0} \frac{f^\tau - f}{\tau}$$

the strong material derivative of f, whenever the limit exists in the strong sense in the respective Banach space.

Our main result in this section is

Theorem 5.1 *Assume (H1)–(H10), then the following are valid:*

(1) The strong material derivative
$\nabla\dot{\phi}$ exists in $H^1(0,T;\mathbf{L}^2(\Omega))$,
$\dot{A}$ exists in $L^\infty(0,T;X)$ and $W^{1,10/3}(0,T;L^{10/3}(G))$,
$\dot{z}$ exists in $W^{1,5/2}(0,T;L^{5/2}(\Sigma))$,
$\dot{\theta}$ exists in $W^{2,1}_{5/3}(Q)$.

(2) $(\dot{\phi}, \dot{A}, \dot{z}, \dot{\theta})$ satisfy the linearized state equations

$$\begin{aligned}
\alpha_0(0,\dot{\phi},\varphi) + \alpha_{0,\tau}(0,\phi,\varphi) &= 0, \quad \text{for all } \varphi \in H^1(\Omega)/\mathbb{R}, && (5.20a)\\
\alpha_1(0,\dot{A}_t,v) + \alpha_2(0,\dot{A},v) + F(0,\dot{\phi},v) + F_{,\tau}(0,\phi,v) & \\
+\alpha_{1,\tau}(0,A_t,v) + \alpha_{2,\tau}(0,A,v) &= 0, \quad \text{for all } v \in \mathbf{X}, && (5.20b)\\
\dot{A}_0 - \mathcal{D}A_0V(0) &= 0, \quad \text{in } D, && (5.20c)\\
\dot{z}_t - \frac{\partial f}{\partial \theta}\dot{\theta} - \frac{\partial f}{\partial z}\dot{z} &= 0, \quad \text{in } Q, && (5.20d)\\
\dot{z}(0) &= 0, \quad \text{in } \Sigma, && (5.20e)\\
\rho c_p \dot{\theta}_t - k\Delta\dot{\theta} + \rho L \dot{z}_t - 2\kappa_0 A_t \cdot \dot{A}_t &= 0, \quad \text{in } Q, && (5.20f)\\
\frac{\partial \dot{\theta}}{\partial \nu} &= 0, \quad \text{in } \partial\Sigma \times (0,T), && (5.20g)\\
\dot{\theta}(0) &= 0, \quad \text{in } \Sigma, && (5.20h)
\end{aligned}$$

where f is the right-hand side of (2.1h).

(3) There exists a constant $C > 0$ such that

$$\begin{aligned}
&\|\nabla\dot{\phi}\|_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\dot{A}\|_{L^\infty(0,T;X)} + \|\dot{A}_t\|_{L^{10/3}(0,T;L^{10/3}(G))} \\
&\quad + \|\dot{\theta}\|_{W^{2,1}_{5/3}(Q)} + \|\dot{z}\|_{W^{1,5}(0,T;L^5(\Sigma))} \le C\|V(0)\|_{C^1(D)}. \qquad (5.21)
\end{aligned}$$

Proof.

Similar to the proof of Theorem 2.1, one can show that (5.20a) – (5.20h) has a unique solution $(\dot\phi, \dot A, \dot z, \dot\theta)$, hence we omit this part of the proof. To prove assertion (3), we first test (5.20a) with $\dot\phi$. According to Corollary 4.1 and Lemma 4.1, we obtain

$$\kappa_0 \int_0^t \int_\Omega |\nabla \dot\phi|^2 \, dx \, ds \le \|B_2'(0)\|_{C^1(D)} \Big(\int_0^t \int_\Omega |\nabla \dot\phi|^2 \, dx \, ds \Big)^{1/2} \Big(\int_0^t \int_\Omega |\nabla \phi|^2 \, dx \, ds \Big)^{1/2}.$$

Using Young's inequality and (2.2), we obtain the estimate for $\nabla\dot\phi$. Then we again differentiate formally with respect to t and obtain the estimate for $\nabla \dot\phi_t$.

Next, we test (5.20b) with $\dot A$ and obtain

$$\begin{aligned}
&\frac{\kappa_0}{2} \int_G |\dot A(t)|^2 \, dx + \int_0^t \int_D \frac{1}{\mu} |\operatorname{curl} \dot A|^2 \, dx \, ds + \int_0^t \int_D \frac{1}{\mu} |\operatorname{div} \dot A|^2 \, dx \, ds \\
\le \;& \kappa_1 \int_0^t \|\nabla \dot\phi\|_{\mathbf{L}^2(\Omega)} \|\dot A\|_{\mathbf{L}^2(\Omega)} \, ds + \kappa_1 \|B_3'(0)\|_{C^1(D)} \int_0^t \|\nabla \phi\|_{\mathbf{L}^2(\Omega)} \|\dot A\|_{\mathbf{L}^2(\Omega)} \, ds \\
& + \|\beta'(0)\|_{C^1(D)} \int_\Sigma \|A_s\|_{\mathbf{L}^2(G)} \|\dot A\|_{\mathbf{L}^2(G)} \, ds \\
& + c_1 \|B_1'(0)\|_{C^1(D)} \int_0^t \|A\|_{\mathbf{X}} \|\dot A\|_{\mathbf{X}} \, ds + \frac{\kappa_0}{2} \int_G |\dot A(0)|^2 \, dx.
\end{aligned}$$

Now we apply Young's inequality, Gronwall's lemma and (5.20c) to infer

$$\|\dot A\|_{L^\infty(0,T;\mathbf{L}^2(G))} + \|\dot A\|_{L^2(0,T;\mathbf{X})} \le c_1 \|V(0)\|_{C^1(D)}.$$

Using the corresponding initial condition for $\dot A_t$ (cf. (4.13)), we differentiate (5.20b) formally with respect to time and insert $v = \dot A$ to obtain

$$\|\dot A_t\|_{L^\infty(0,T;\mathbf{L}^2(G))} + \|\dot A_t\|_{L^2(0,T;\mathbf{X})} \le c_2 \|V(0)\|_{C^1(D)}.$$

A further application of Lemma 4.3 then yields

$$\|\dot A_t\|_{L^{10/3}(0,T;L^{10/3}(G))} \le c_3 \|V(0)\|_{C^1(D)}. \tag{5.22}$$

Next, we remark that similarly to the derivation of Lemma 2.3(3), we can infer that

$$\|\dot z\|^p_{W^{1,p}(0,T;L^p(\Sigma))} \le c_4 \|\dot\theta\|_{L^p(Q)}.$$

Then in the light of (5.22), the last part of inequality (5.21) follows as in the proof of Lemma 4.2(3) and (4).

It remains to show that the solutions to (5.20a)–(5.20h) are the strong material derivatives. To this end, let

$$\psi^\tau = \frac{1}{\tau}(\phi^\tau - \phi) - \dot{\phi}, \tag{5.23}$$

then, according to Corollary 4.1, (3.8), and (5.20a), ψ^τ satisfies

$$\begin{aligned}\alpha_0(0,\psi^\tau,\varphi) &= -\frac{1}{\tau}\Big(\alpha_0(\tau,\phi^\tau,\varphi) - \alpha_0(0,\phi^\tau,\varphi)\Big) - \alpha_0(0,\dot{\phi},\varphi)\\ &= \alpha_{0,\tau}(0,\phi-\phi^\tau,\varphi) - \frac{1}{\tau}\tilde{\alpha}_0(\tau,\phi^\tau,\varphi).\end{aligned}$$

Integrating in time, inserting $\varphi = \psi^\tau$, and using Corollary 4.1 once again, we obtain

$$\|\nabla\psi^\tau\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \xrightarrow[\tau\to 0]{} 0. \tag{5.24}$$

Since the same computations hold for $\nabla\phi_t$, the first part of assertion (1) is proved.

Next, defining

$$p^\tau = \frac{1}{\tau}(A^\tau - A) - \dot{A},$$

and using (5.20b), and Corollary 4.1, we see that p^τ satisfies

$$\begin{aligned}\alpha_1(0,p_t^\tau,v) + \alpha_2(0,p^\tau,v) = &-\frac{1}{\tau}\Big(F(\tau,\phi^\tau,v) - F(0,\phi,w)\Big)\\ &-\frac{1}{\tau}\Big(\alpha_1(\tau,A_t^\tau,v) - \alpha_1(0,A_t^\tau,v)\Big) - \frac{1}{\tau}\Big(\alpha_2(\tau,A^\tau,v) - \alpha_2(0,A^\tau,v)\Big)\\ &+F(0,\dot{\phi},v) + F_{,\tau}(0,\phi,v) + \alpha_{1,\tau}(0,A_t,v) + \alpha_{2,\tau}(0,A,v)\\ = &-F(0,\psi^\tau,v) - F_{,\tau}(0,\phi^\tau-\phi,v) + \frac{1}{\tau}\tilde{F}(\tau,\phi^\tau,v)\\ &-\alpha_{1,\tau}(0,A_t^\tau - A_t,v) - \alpha_{2,\tau}(0,A^\tau - A),v)\\ &-\frac{1}{\tau}\tilde{\alpha}_1(\tau,A_t^\tau,v) - \frac{1}{\tau}\tilde{\alpha}_2(\tau,A^\tau,v).\end{aligned} \tag{5.25}$$

We take $v = p^\tau$, integrate in time, and use Hölder's inequality to obtain

$$\begin{aligned}&\frac{\kappa_0}{2}\int_G |p^\tau|^2\,dx - \frac{\kappa_0}{2}\int_G |p_0^\tau|^2\,dx + \int_0^t\int_D \frac{1}{\mu}|\operatorname{curl} p^\tau|^2\,dxds + \int_0^t\int_D \frac{1}{\mu}(\operatorname{div} p^\tau)^2\,dxds\\ \leq\;& \kappa_0\int_0^t \|\nabla\psi^\tau\|_{\mathbf{L}^2(\Omega)}\|p^\tau\|_{\mathbf{L}^2(\Omega)}\,ds + c_5\int_0^t \|\nabla\phi^\tau - \nabla\phi\|_{\mathbf{L}^2(\Omega)}\|p^\tau\|_{\mathbf{L}^2(\Omega)}\,ds\\ &+\frac{1}{\tau}g_3(\tau)\int_0^t \|\nabla\phi^\tau\|_{\mathbf{L}^2(\Omega)}\|p^\tau\|_{\mathbf{L}^2(\Omega)}\,ds + \int_0^t\int_G \beta'(0)(A_s^\tau - A_s)\cdot p^\tau\,dx\,ds\\ &+c_6\int_0^t \|A^\tau - A\|_{\mathbf{X}}\|p^\tau\|_{\mathbf{X}}\,ds + \frac{1}{\tau}g_1(\tau)\int_0^t \|A_s^\tau\|_{\mathbf{L}^2(G)}\|p^\tau\|_{\mathbf{L}^2(G)}\\ &+\frac{1}{\tau}g_2(\tau)\int_0^t \|A^\tau\|_{\mathbf{X}}\|p^\tau\|_{\mathbf{X}}\,ds.\end{aligned}$$

Using (3.9b), the second term in (5.25) gives

$$\int_G |p_0^\tau|^2 \, dx = \int_G |\frac{1}{\tau}(A_0 \circ T_\tau - A_0) - \dot{A}_0|^2 dx.$$

According to [23, Sec. 2.14], $\tau \mapsto A_0 \circ T_\tau$ is differentiable with

$$\frac{d}{d\tau}(A_0 \circ T_\tau)\Big|_{\tau=0} = DA_0 V(0),$$

hence

$$A_0 \circ T_\tau = A_0 + \tau \mathcal{D} A_0 V(0) + o(\tau).$$

Hence, we obtain

$$\int_G |p_0^\tau|^2 \, dx \xrightarrow[\tau \to 0]{} 0.$$

Regarding (H12) and Lemma 4.1, $\beta'(0)p^\tau \in \mathbf{X}$ a.e. in $(0,T)$. Thus, we apply (4.17) to infer

$$\begin{aligned}
\int_0^t \int_G \beta'(0)(A_s^\tau - A_s) \cdot p^\tau \, dx \, ds &= \frac{1}{\kappa_1} \int_0^t \alpha_1(0, A_s^\tau - A_s, \beta'(0)p^\tau) \, ds \\
&\le c_7 |\tau| \, \|p^\tau\|_{L^2(0,t;\mathbf{X})}.
\end{aligned}$$

Then we apply Young's inequality, Corollary 4.1, (5.24) and Gronwall's lemma to conclude

$$\|p^\tau\|^2_{L^\infty(0,T;\mathbf{L}^2(G))} + \|p^\tau\|^2_{L^2(0,T;\mathbf{X})} \xrightarrow[\tau \to 0]{} 0.$$

Now we differentiate (5.25) formally with respect to time, repeat the same considerations as before (but use (4.13) as initial value instead of (3.9b)) and obtain

$$\|p_t^\tau\|^2_{L^\infty(0,T;\mathbf{L}^2(G))} + \|p_t^\tau\|^2_{L^2(0,T;\mathbf{X})} \xrightarrow[\tau \to 0]{} 0.$$

A further application of Lemma 4.3 finally yields

$$\|p_t^\tau\|_{L^{10/3}(0,T;\mathbf{L}^{10/3}(G))} \xrightarrow[\tau \to 0]{} 0. \tag{5.26}$$

To prove the differentiability of θ^τ and z^τ, we define

$$\begin{aligned}
q^\tau &= \frac{1}{\tau}(\theta^\tau - \theta) - \dot{\theta}, \\
r^\tau &= \frac{1}{\tau}(z^\tau - z) - \dot{z},
\end{aligned}$$

then, (q^τ, r^τ) solve

$$\rho c_p q_t^\tau - k\Delta q^\tau = -\rho L r_t^\tau + \kappa_0 \tau |\dot{A}_t|^2 + \kappa_0 p_t^\tau \cdot \Big(2A_t + 2\tau \dot{A}_t + \tau p_t^\tau\Big) \tag{5.27a}$$

$$\begin{aligned}
r_t^\tau &= \frac{1}{\tau}\Big(f(\theta^\tau, z^\tau) + f(\theta, z)\Big) - \frac{\partial f}{\partial \theta}(\theta, z)\dot{\theta} - \frac{\partial f}{\partial z}(\theta, z)\dot{z} \\
&=: G(\tau)
\end{aligned} \tag{5.27b}$$

$$\frac{\partial q^\tau}{\partial \nu} = 0, \qquad q^\tau = 0, \qquad r^\tau(0) = 0. \tag{5.27c}$$

Owing to (H7)–(H9), we can apply Taylor's formula to develop $G(\tau)$ and obtain (with a constant $\xi \in [0,1]$)

$$\begin{aligned}
|G(\tau)| &= \Big|\frac{1}{\tau}\Big(f(\theta+\tau(q^\tau+\dot\theta), z+\tau(r^\tau+\dot z)) - f(\theta,z)\Big) - \frac{\partial f}{\partial\theta}(\theta,z)\dot\theta - \frac{\partial f}{\partial z}(\theta,z)\dot z\Big| \\
&= \Big|(q^\tau+\dot\theta)\frac{\partial f}{\partial\theta}(\theta+\xi\tau(q^\tau+\dot\theta), z+\xi\tau(r^\tau+\dot z)) \\
&\qquad +(r^\tau+\dot z)\frac{\partial f}{\partial z}(\theta+\xi\tau(q^\tau+\dot\theta), z+\xi\tau(r^\tau+\dot z)) - \frac{\partial f}{\partial\theta}(\theta,z)\dot\theta - \frac{\partial f}{\partial z}(\theta,z)\dot z\Big| \\
&\le c_8|q^\tau| + c_9|r^\tau| + |\dot\theta|\Big|\frac{\partial f}{\partial\theta}(\theta+\xi(\theta^\tau-\theta), z+\xi(z^\tau-z)) - \frac{\partial f}{\partial\theta}(\theta,z)\Big| \\
&\qquad +|\dot z|\Big|\frac{\partial f}{\partial z}(\theta+\xi(\theta^\tau-\theta), z+\xi(z^\tau-z)) - \frac{\partial f}{\partial z}(\theta,z)\Big| \\
&\le c_8|q^\tau| + c_9|r^\tau| + c_{10}|\dot\theta||\theta^\tau-\theta| + c_{11}|\dot\theta||z^\tau-z| + c_{12}|\dot z||\theta^\tau-\theta| + c_{13}|\dot z||z^\tau-z|.
\end{aligned}$$

Owing to (5.26) and (5.21), the last term of right-hand side of (5.27a) will be in $L^{5/3}(0,T;L^{5/3}(\Sigma))$. Thus we try to get an estimate for $G(\tau)$ in the same space. To this end, we apply the inequalities of Hölder and Young and use (5.21) to obtain

$$\begin{aligned}
\int_0^t\int_\Sigma |G(\tau)|^{5/3}\,dx\,ds &\le c_{14}\int_0^t\int_\Sigma |q^\tau|^{5/3}\,dx\,ds + c_{15}\int_0^t\int_\Sigma |r^\tau|^{5/3}\,dx\,ds \\
&\quad + c_{16}\Big(\int_0^t\int_\Sigma |\theta^\tau-\theta|^{10/3}\,dx\,ds\Big)^{1/2} + c_{17}\Big(\int_0^t\int_\Sigma |z^\tau-z|^{10/3}\,dx\,ds\Big)^{1/2}. \qquad (5.28)
\end{aligned}$$

Next, we test (5.27b) with $(r^\tau)^{2/3}$, use the estimate above and apply the inequalities of Young and Gronwall, as well as the stability estimates of Lemma 4.2, to obtain

$$\frac{3}{5}\int_\Sigma |r^\tau|^{5/3}\,dx \le c_{18}|\tau|^{5/3} + c_{19}\int_0^t\int_\Sigma |q^\tau|^{5/3}\,dx\,ds.$$

Using the last estimate and (5.28) we go back to (5.27b) and conclude

$$\int_0^t\int_\Sigma |r^\tau_s|^{5/3}\,dx\,ds \le c_{20}|\tau|^{5/3} + c_{21}\int_0^t\int_\Sigma |q^\tau|^{5/3}\,dx\,ds. \qquad (5.29)$$

Now we can proceed again as in the proof of Lemma 4.2(3), i.e., we apply Lemma 2.4 to (5.27a), and use (5.29) and (5.21) to obtain

$$\|q^\tau\|_{W^{2,1}_{5/3}(Q)} = o(1).^4 \qquad (5.30)$$

Using the embedding $W^{2,1}_{5/3}(Q) \subset L^5(Q)$ (cf. Lemma 2.5), we can go back, estimate $G(\tau)$ again (this time in $L^{5/2}(Q)$), and obtain finally

$$\|r^\tau\|_{W^{1,5/2}(0,T;L^{5/2}(\Sigma))} \xrightarrow[\tau\to 0]{} 0.$$

□

[4] Recall that $g(\tau) = o(1)$ if and only if $g(\tau) \to 0$ for $\tau \to 0$.

6 The structure theorem

In the last section we have shown that the state variables possess strong material derivatives. In particular, we can conclude (cf. Theorem 5.1(1))

$$z^{\tau}(T) = z(T) + \tau \dot{z}(T) + o(\tau), \qquad \text{in } L^{5/2}(\Sigma),$$

where z^{τ} is the volume fraction of austenite in Σ, corresponding to the perturbation $\Omega_{\tau} = T_{\tau}(V)(\Omega)$.

Hence, our cost functional (2.7) is *shape differentiable* at any admissible domain Ω, i.e. the limit

$$d\mathcal{J}(\Omega; V) = \lim_{\tau \to 0} \frac{1}{\tau}(\mathcal{J}(\Omega_{\tau}) - \mathcal{J}(\Omega)) \tag{6.31}$$

exists and satisfies

$$d\mathcal{J}(\Omega; V) = 2 \int_{\Sigma} \Big(z(x, T - \bar{z}(x)\Big) \dot{z}(T)\, dx. \tag{6.32}$$

Let us recall that the cost functional depends only implicitly on Ω, namely through the equation for the scalar potential φ and the vector potential A. The dependence of the material derivative $\dot{z}$ on V is revealed in Theorem 5.1(3). We can conclude

Corollary 6.1 *Assume (H1) – (H12), then the mapping*

$$d\mathcal{J}(\Omega; .) \; : \; C_0^1(D) \; \to \; \mathrm{I\!R}, \; V \mapsto d\mathcal{J}(\Omega; V)$$

is linear and continuous.

Corollary 6.1 allows us to apply the structure theorem (cf. [23]).

From this, we infer

Corollary 6.2 *Assume (H1) – (H12), and let in addition $\partial\Omega$ be of class C^2, then there exists a distribution $\mathcal{G}_{\partial\Omega}$ with support in $\partial\Omega$, the shape gradient, such that $\mathcal{G}_{\partial\Omega} \in C^1(\partial\Omega)^*$, and for all $V \in C_0^1(D; \mathrm{I\!R}^3)$ there holds*

$$d\mathcal{J}(\Omega; V) = < \mathcal{G}_{\partial\Omega}, V \cdot \nu >_{C^1(\partial\Omega)^* \times C^1(\partial\Omega)},$$

where ν is the outer unit normal vector on the boundary of the tube Ω.

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