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Global uniqueness in determining rectangular periodic structures by scattering data with a single wave number

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Abstract. We consider an inverse scattering problem of determining a periodic structure by near-field observations of the total field. We prove the global uniqueness results in both cases of the transverse electric polarization and the transverse magnetic polarization within the class of rectangular periodic structures by a single choice of any wave number. The proof is based on the analyticity of solutions to the Helmholtz equation.

§1. Introduction.

In this paper, we consider an inverse scattering problem in a perfectly reflecting periodic rectangular structure in the following cases:

(1) the transverse electric polarization (i.e., the TE mode)
(2) the transverse magnetic polarization (i.e., the TM mode)

We will formulate the inverse problem according to Kirsch [11], and we can refer also to Bao [2], and Bao, Dobson and Cox [3]. Let us fix $a < 0$ arbitrarily and let us define a set $\mathcal{F}$ of all possible (2π)-periodic profiles by:

$$\begin{aligned} \mathcal{F} = \{f;\ & f \text{ is a piecewise linear curve in } \{(x_1, x_2); x_2 < 0\} \\ & \text{connecting } (0, a) \text{ and } (2\pi, a), \text{ and any linear part is} \\ & \text{parallel to the } x_1\text{- or } x_2\text{- axis. Moreover} \\ & f \cap \{(\kappa, x_2); x_2 \in \mathbb{R}\} \text{ is a connected segment or one point for any } \kappa \in \mathbb{R}\}. \end{aligned} \tag{1.1}$$

We call a piecewise linear curve $f \in \mathcal{F}$ a rectangular profile.

Let Ω_f be the domain over f (i.e., the component of $\mathbb{R}^2$ separated by f which is connected to $x_2 = \infty$.) We assume that Ω_f is filled by a dielectric medium. We take a plane wave given by

$$u^{in}(x_1, x_2) = \exp(i\alpha x_1 - i\beta x_2)$$

as an incident wave on f from the top.

Here and henceforth we set

$$\alpha = k \sin\theta, \quad \beta = k\cos\theta \tag{1.2}$$

where $|\theta| < \frac{\pi}{2}$: the incident angle and $k > 0$: the wave number.

Then the direct scattering problem is to determine the total field $u = u(x_1, x_2)$ satisfying (1.3) - (1.4) - (1.6) - (1.7) or (1.3) - (1.5) - (1.6) - (1.7) for given $f \in \mathcal{F}$:

$$\Delta u(x) + k^2 u(x) = 0, \qquad x \equiv (x_1, x_2) \in \Omega_f \tag{1.3}$$

$$(\text{TE mode}) \quad u = 0 \quad \text{on} \quad f \tag{1.4}$$

$$(\text{TM mode}) \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad f \tag{1.5}$$

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(α-quasiperiodicity)

$$u(x_1+2\pi,x_2)=\exp(2\pi i\alpha)u(x_1,x_2),\quad (x_1,x_2)\in\Omega_f \tag{1.6}$$

(radiation condition)

$$u(x)=\exp(i\alpha x_1-i\beta x_2)+\sum_{n\in\mathbb{Z}}A_n\exp(i(n+\alpha)x_1+i\beta_n x_2)\quad \text{if } x_2>0, \tag{1.7}$$

where $\frac{\partial}{\partial\nu}$ denotes the normal derivative, $A_n\in\mathbb{C}$ are the Rayleigh coefficients, and

$$\beta_n=\begin{cases}(k^2-(n+\alpha)^2)^{\frac{1}{2}}, & |n+\alpha|\le k\\ i(-k^2+(n+\alpha)^2)^{\frac{1}{2}}, & |n+\alpha|>k.\end{cases} \tag{1.8}$$

By the definition (1.8), we note that the series in (1.7) and any derivatives of it are uniformly convergent on any compact set of $\{(x_1,x_2);x_2>0\}$. As for the direct problem, we refer to Petit [14].

In this paper, we will consider also the case where the resonance $\beta_n=0$ may happen for some $n\in\mathbb{Z}$. In the case (1.3) - (1.4) - (1.6) - (1.7) (i.e., the TE mode), the unique existence of H^1-solution is established in Kirsch [11] for f in C^2-class, and in Elschner and Yamamoto [9] for Lipschitz continuous f.

Our main task is

Inverse Problem. Let $b>0$. Given a solution $u=u(x_1,x_2)$ to the direct problem (1.3) - (1.4) - (1.6) - (1.7) or (1.3) - (1.5) - (1.6) - (1.7), determine $f\in\mathcal{F}$ by

$$u(x_1,b)\qquad 0<x_1<2\pi. \tag{1.9}$$

The purpose of this paper is to establish the uniqueness in this inverse problem within $\mathcal{F}$ with an arbitrarily fixed value $k>0$.

Henceforth we set

$$\mathcal{U}_{f,D}=\{u;\ u\in H^1(\Omega_f)\text{ satisfies (1.3), (1.4), (1.6) and (1.7)}\} \tag{1.10}$$

and

$$\mathcal{U}_{f,N}=\{u;\ u\in H^1(\Omega_f)\text{ satisfies (1.3), (1.5), (1.6) and (1.7)}\}. \tag{1.11}$$

Remark. By Elschner and Yamamoto [9], we know that $\mathcal{U}_{f,D}$ is composed of a single element, that is, there exists a unique solution to (1.3), (1.4), (1.6) and (1.7). However, in the case of TM mode (1.5), the uniqueness of solutions is not true in general.

We are ready to state the main results.

Theorem 1. *(the TE mode) Let $f,g\in\mathcal{F}$ and $u\in\mathcal{U}_{f,D}$, $v\in\mathcal{U}_{g,D}$. Then $u(x_1,b)=v(x_1,b)$, $0<x_1<2\pi$, implies $f=g$.*

Theorem 2. *(the TM mode) Let $f, g \in \mathcal{F}$ and $u \in \mathcal{U}_{f,N}$, $v \in \mathcal{U}_{g,N}$. We assume that $u(x_1, b) = v(x_1, b)$, $0 < x_1 < 2\pi$. Then $f = g$ follows if $\alpha \neq 0$.*

Remark. In Theorem 2, in the case of $\alpha = 0$, there is a counterexample for the uniqueness. Let $k > 0$ and let us set $a = -\frac{\pi}{k}$, $u(x_1, x_2) = \exp(-ikx_2) + \exp(ikx_2)$. Set

$$f = \left\{ \left(x_1, -\frac{\pi}{k}\right) ; 0 < x_1 < 2\pi \right\}$$

and

$$g = \left\{ \left(x_1, -\frac{\pi}{k}\right) ; 0 < x_1 < p_1,\, p_2 < x_1 < 2\pi \right\} \cup \left\{ \left(x_1, -\frac{\pi}{k} - \frac{m\pi}{k}\right) ; p_1 \leq x_1 \leq x_2 \right\}$$
$$\cup \left\{ (x_1, x_2); x_1 = p_1 \text{ or } x_1 = p_2, \quad -\frac{\pi}{k} - \frac{m\pi}{k} \leq x_2 \leq -\frac{\pi}{k} \right\},$$

for arbitrarily fixed $p_1, p_2 \in (0, 2\pi)$ and $m \in \mathbb{N}$. Then $\Delta u + k^2 u = 0$ in Ω_f and in Ω_g, and $\frac{\partial u}{\partial \nu} = 0$ on f or on g. However $f \neq g$.

For the uniqueness, we have to assume that f and g pass the same point $(0, a)$. Without this condition, an example breaking the uniqueness is known (Bao [2], Hettlich and Kirsch [10]).

Example. Let $f = \{(x_1, a); 0 < x_1 < 2\pi\}$ and

$$g = \left\{ \left(x_1, a - \frac{2\pi}{\beta}\right) ; 0 < x_1 < 2\pi \right\}.$$

Then

$$u(x_1, x_2) = \exp(i\alpha x_1 - i\beta x_2) - \exp(i\alpha x_1 + i\beta(x_2 - 2a)), \quad x_2 > a$$

and

$$v(x_1, x_2) = \exp(i\alpha x_1 - i\beta x_2) - \exp(i\alpha x_1 + i\beta(x_2 - 2a)), \quad x_2 > a - \frac{2\pi}{\beta}$$

satisfy (1.3), (1.4), (1.6) and (1.7) with f and g respectively. Clearly $f \neq g$ but $u(x_1, b) = v(x_1, b)$, $0 < x_1 < 2\pi$.

Our uniqueness results do not require any condition on $k \in \mathbb{R}$ or changes of values of k. Under some conditions on k, several uniqueness results for profiles given by graphs of C^2-functions, are proved in the TE mode (1.4):

(1) In the case of a lossy medium (i.e., $\mathrm{Im}k \neq 0$), the observation (1.9) for a single k guarantees the uniqueness (Bao [2]).
(2) For general $k > 0$, uniqueness results with a single k are not known. Hettlich and Kirsch [10] prove the uniqueness with observations (1.9) for finitely many (but not one, in general) wave numbers k.

As for the uniqueness, we further refer to Ammari [1], Kirsch [12]. On the other hand, within the class of C^2-profiles, local uniqueness and stability results are proved with a single k in Bao and Friedman [4]. For similar results for Lipschitz continuous profiles, see Elschner and Schmidt [8]. As for the stability without such restrictive class of profiles, we can refer to Bruckner, Cheng and Yamamoto [5], [6] where $k > 0$ is small or k is not real. In the case of TM mode (1.5), to the authors' knowledge, no uniqueness is known.

This paper is composed of three sections. In Section 2, we show key lemmata. In Section 3, we complete the proofs of Theorems 1 and 2.

§2. Key Lemmata.

We will show the following three key lemmata which are necessary for the proofs of Theorems 1 and 2. Henceforth for points $Q, R \in \mathbb{R}^2$, by $\overline{QR}$ we denote the segment connecting Q and R and not containing Q or R.

Lemma 1. *Let Q, R be any neighbouring vertices of f, that is, $\overline{QR} \subset f$ and let Q', R' be any two points on $\overline{QR}$ such that the closure of $\overline{Q'R'}$ is in $\overline{QR}$. If $u \in \mathcal{U}_{f,N}$, then there exists a neighbourhood U of $\overline{Q'R'}$ such that $u \in H^2(U \cap \Omega_f)$.*

As for the proof, we can refer, for example, to §5 of Chapter 2 of Lions and Magenes [13].

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain such that $\Omega \supset \{(x_1, x_2);\, c_1 < x_1 < c_2,\, x_2 > c_3\}$ with some c_1, c_2, c_3 such that $0 < c_1 < c_2 < 2\pi$ and $c_3 \in \mathbb{R}$. Let $v = v(x_1, x_2)$ satisfy*

$$\Delta v + k^2 v = 0 \quad \text{in } \Omega. \tag{2.1}$$

(i) If

$$v(p, x_2) = 0, \qquad q_1 < x_2 < q_2 \tag{2.2}$$

with some p, q_1, q_2 satisfying $c_1 < p < c_2$ and $c_3 < q_1 < q_2$, then $v(p, x_2) = 0$, $x_2 > c_3$.
(ii) If

$$\frac{\partial v}{\partial x_1}(p, x_2) = 0, \qquad q_1 < x_2 < q_2 \tag{2.3}$$

with some p, q_1, q_2, then $\frac{\partial v}{\partial x_1}(p, x_2) = 0$, $x_2 > c_3$.

Proof of Lemma 2. Since $v = v(x_1, x_2)$ is real analytic with respect to (x_1, x_2) in Ω (e.g., Colton and Kress [7]), it follows that $v(p, \cdot)$ and $\frac{\partial v}{\partial x_1}(p, \cdot)$ are analytic with respect to the second variable. Thus we complete the proof of the lemma.

Lemma 3. *Let $\beta_n \in \mathbb{C}$ be defined by (1.8), $|C_n| = 1$ for $n \in \mathbb{Z}$, and let $P = \{n \in \mathbb{Z};\, \beta_n \in \mathbb{R}\}$. We assume that*

$$a_0 \exp(-i\beta x_2) + \sum_{n \in \mathbb{Z}} A_n C_n (n + \alpha) \exp(i\beta_n x_2) = 0 \tag{2.4}$$

for $x_2 > c$ with some $c \in \mathbb{R}$. Then $a_0 = 0$ and $\sum_{n \in P} A_n C_n (n + \alpha) \exp(i\beta_n x_2) = 0$ for $x_2 > c$.

We note by (1.8) that P is a finite set and we set $\ell = |P|$ (the number of the elements of P).

Proof of Lemma 3. By the definition (1.8) of β_n, we note that the left hand side of (2.4) and any derivatives of it are convergent uniformly on any compact set of $\{x_2 > c\}$. Without loss of generality, we may assume that $c = 0$. By (2.4), we see that

$$\begin{aligned} 0 = &\left(a_0 \exp(-i\beta x_2) + \sum_{n \in P} A_n \exp(i\beta_n x_2) \right) \\ &+ \sum_{n \in \mathbb{Z} \setminus P} A_n \exp(i\beta_n x_2) \equiv S_1(x_2) + S_2(x_2), \quad x_2 > 0. \end{aligned} \tag{2.5}$$

Here by the same letters A_n, we denote $A_n C_n(n+\alpha)$ for simplicity, and we note that there exists a constant $c_0 > 0$ such that $-i\beta_n \ge c_0 > 0$ for all $n \in \mathbb{Z} \setminus P$ and $\beta_n \sim |n|i$ as $|n| \longrightarrow \infty$. First we have

$$-\beta \notin \{\beta_n\}_{n\in P}, \tag{2.6}$$

because $\beta > 0$ and $\beta_n \ge 0$ by the definition (1.8). Then it is sufficient to prove

$$S_1(x_2) = 0, \qquad x_2 > 0. \tag{2.7}$$

Because if (2.7) will be proved, then we have $a_0 = 0$ by (2.6).

Proof of (2.7). Henceforth let $\frac{\beta}{2\pi}$, $\frac{\beta_n}{2\pi}$ be all irrational numbers. Otherwise we need not choose integers m_k^0, m_{kn}, N_k as follows and our proof is more direct. Then, for $n \in P$, there exist sequences $\{N_k\}_{k\in\mathbb{N}}$, $\{m_{kn}\}_{k\in\mathbb{N}}$ and $\{m_k^0\}_{k\in\mathbb{N}}$ of integers such that $\lim_{k\to\infty} N_k = \infty$ and

$$\left| -\frac{\beta}{2\pi} - \frac{m_k^0}{N_k} \right|, \quad \left| \frac{\beta_n}{2\pi} - \frac{m_{kn}}{N_k} \right| \le \frac{1}{N_k^{1+\frac{1}{\ell+1}}}$$

(e.g. Corollary 1B (p.27) in Schmidt [15]).

Then the function

$$S_1^k(x_2) \equiv a_0 \exp(m_k^0 2\pi i x_2/N_k) + \sum_{n\in P} A_n \exp(m_{kn} 2\pi i x_2/N_k), \quad k \in \mathbb{N}$$

is N_k-periodic and we have

$$\begin{aligned}
&|S_1(x_2) - S_1^k(x_2)| \\
\le& |a_0| |\exp(-i\beta x_2) - \exp(m_k^0 2\pi i x_2/N_k)| + \sum_{n\in P} |A_n| |\exp(i\beta_n x_2) - \exp(m_{kn} 2\pi i x_2/N_k)| \\
\le& |a_0| \left| (-\beta) - \frac{m_k^0}{N_k} 2\pi \right| |x_2| + \sum_{n\in P} |A_n| \left| \beta_n - \frac{m_{kn}}{N_k} 2\pi \right| |x_2| \\
\le& \frac{C_1}{N_k^{1+\frac{1}{\ell+1}}} |x_2|.
\end{aligned}$$

Here the constant $C_1 > 0$ is independent of k. Let $\varepsilon > 0$ be given arbitrarily. Since $-i\beta_n \ge c_0 > 0$ for all $n \in \mathbb{Z} \setminus P$, there exists $k \in \mathbb{N}$ sufficiently large such that

$$|S_2(x_2)| < \varepsilon, \qquad x_2 > N_k. \tag{2.8}$$

Therefore, for $\varepsilon > 0$, we can further choose $k \in \mathbb{N}$ sufficiently large, so that

$$|S_1(x_2) - S_1^k(x_2)| \le \varepsilon, \qquad 0 \le x_2 \le 2N_k. \tag{2.9}$$

Therefore (2.5), (2.8) and (2.9) yield

$$|S_1^k(x_2)| \le 2\varepsilon, \qquad N_k \le x_2 \le 2N_k.$$

By the N_k-periodicity,

$$|S_1^k(x_2)| \le 2\varepsilon, \qquad 0 \le x_2 \le 2N_k.$$

Hence (2.9) implies

$$\begin{aligned}|S_1(x_2)| \le& |S_1(x_2) - S_1^k(x_2)| + |S_1^k(x_2)|\\ \le& 3\varepsilon, \qquad 0 \le x_2 \le N_k.\end{aligned}$$

This means that $S_1(x_2) = 0$ for $0 \le x_2 \le 2\pi$. Since S_1 is real analytic in x_2, we can complete the proof of (2.7).

§3. Proof of Theorems 1 and 2.

Since the proof of Theorem 1 is carried out by the same way, we will prove only Theorem 2. First we note by the interior regularity of an elliptic equation (e.g., Colton and Kress [7]) that $u \in \mathcal{U}_{f,N}$ is sufficiently smooth in any open set $\mathcal{O}$ such that $\overline{\mathcal{O}} \subset \Omega_f$.

Assume contrarily that $f \ne g$. Since $f, g \in \mathcal{F}$, both curves start at $A(0,a)$, so that there exist points $B(p_1,a)$, $C(p_2,a)$ and $D(p_1,q)$ such that $0 < p_1 < p_2$, $q > a$, $\overline{AB} \cup \overline{BC} \subset f$, $\overline{AB} \cup \overline{BD} \subset g$ and $\overline{BD} \subset \Omega_f$.

Let $u(x_1,b) = v(x_1,b)$, $0 < x_1 < 2\pi$. Then the uniqueness of the direct problem with the profile $x_2 = b$ yields $u(x_1,x_2) = v(x_1,x_2)$, $x_2 > b$. Therefore

$$\frac{\partial u}{\partial x_2}(x_1,x_2) = \frac{\partial v}{\partial x_2}(x_1,x_2), \quad x_2 > b.$$

Consequently by the unique continuation of solutions to the Helmholtz equation: $\Delta w + k^2 w = 0$, we see that

$$\text{(3.1)} \qquad \begin{aligned}&u = v \text{ in any open set of } \Omega_f \cap \Omega_g\\ &\text{which is connected with the straight line } x_2 = b.\end{aligned}$$

We arbitrarily take two points $B'(p_1,a')$ and $D'(p_1,q')$ such that $a < a' < q' < q$. Then the domain $V = \Omega_f \cap \Omega_g$ is connected with $x_2 = b$ and satisfies

$$\text{(3.2)} \qquad \partial V \cap g \supset \overline{B'D'}.$$

Hence, by (3.1), we have $u - v = 0$ in V. Lemma 1 and the trace theorem yield $\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0$ in $H^{\frac{1}{2}}(\overline{B'D'})$. By $v \in \mathcal{U}_{g,N}$ and (3.2), we obtain $\frac{\partial u}{\partial x_1} = 0$ on $\overline{B'D'}$. By $f, g \in \mathcal{F}$, the half line $\{(p_1,x_2); x_2 > a'\}$ is in Ω_f, so that we can apply Lemma 2 to obtain $\frac{\partial u}{\partial x_1}(p_1,x_2) = 0$, $x_2 > a'$. Therefore, by the radiation condition (1.7), we have

$$\alpha e^{i\alpha} e^{i\alpha p_1} \exp(-i\beta x_2) + \sum_{n\in\mathbb{Z}} (n+\alpha) A_n e^{i(n+\alpha)p_1} \exp(i\beta_n x_2) = 0,$$

for $x_2 > a'$. Lemma 3 implies that $\alpha e^{i\alpha} e^{i\alpha p_1} = 0$, that is, $\alpha = 0$. By the assumption in the theorem, we have $\alpha \ne 0$ and we have a contradiction. Thus the proof of Theorem 2 is complete.

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