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On convergence of population processes in random environments to the stochastic heat equation with colored noise

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Abstract

We consider the stochastic heat equation with a multiplicative colored noise term on $\mathbb{R}^d$ for $d \geq 1$. First, we prove convergence of a branching particle system in a random environment, to the stochastic heat equation with a linear noise term. For this stochastic partial differential equation with more general non-Lipschitz noise coefficients we show convergence of associated lattice systems, which are infinite dimensional stochastic differential equations with correlated noise terms, provided that uniqueness of the limit is known. In the course of the proof, we establish existence and uniqueness of solutions to the lattice systems, as well as a new existence result for solutions to the stochastic heat equation. The latter are shown to be jointly continuous in time and space under some mild additional assumptions.

Contents

1 Introduction

The stochastic heat equation considered in this paper is a stochastic partial differential equation (SPDE), which can formally be written as

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) + f\left(t,x,u(t,x)\right) + \sigma\left(t,x,u(t,x)\right) \dot{W}(t,x). \tag{1}$$

Here, u is a random function on $\mathbb{R}_+ \times \mathbb{R}^d$, where $\mathbb{R}_+ \equiv [0,\infty)$, and the operator Δ denotes the Laplacian acting on $\mathbb{R}^d$. W is a noise on $\mathbb{R}_+ \times \mathbb{R}^d$ that is white in time and colored in space, for example a spatially homogeneous noise. The coefficients f and σ are real valued continuous functions on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$. They are mostly nonlinear, and we pay particular attention to coefficients which are not Lipschitz continuous in u.

We are concerned with convergence of rescaled branching particle systems in a random environment and associated lattice systems, which are infinite systems of stochastic differential equations (SDE), to solutions of (1). Intimately connected to convergence are questions of existence and uniqueness, for the lattice systems as well as for the SPDE. For the more delicate case of non-Lipschitz coefficients, a new existence result is established through the approximation procedure. In this case, uniqueness has to be shown separately to assure convergence.

The choice of SPDE and the study of convergence of associated systems to that equation has been motivated by three factors:

(i) The heat equation with a noise term that is white in space and time arises in studying the diffusion limit of a large class of spatially distributed (for the most part branching) particle systems. It is, for example, the weak limit of branching Brownian motion as well as of lattice systems of reproducing populations. Spatially colored noise reflects spatial correlations of solutions to the SPDE. Given the recent focus on interacting particle systems, it is an intriguing question how the stochastic heat equation with colored noise relates to such systems or -as an intermediate step- to infinite systems of SDEs with correlated noise terms.

(ii) Stochastic heat equations of the form (1), where W is white in space and time, have function valued solutions only in dimension one. Thus, connections of these SPDEs to population systems are restricted to a one dimensional state space. Some conditions on the coefficients and the noise are known so that the heat equation with colored noise has function valued solutions in all dimensions. This class of equations can therefore be expected to offer a description for population processes in more general settings. Biologically interesting are in particular the dimensions two and three.

(iii) The particle picture and the approximation by systems of related SDEs provide a representation of a general class of SPDEs that also arise in other areas of application, for example in filtering theory. In our case, the approximation by a system of SDEs leads to a new existence result for the stochastic heat equation with colored noise and non-Lipschitz noise coefficients. Both representations may be exploited further for numerical purposes or the study of properties of these SPDEs.

In the following we elucidate these points a bit further and point out connections to related work.

One of the classical examples for measure valued branching processes is the Dawson-Watanabe superprocess (see Watanabe [Wat68] and Dawson [Daw75]). It is a process, X, that takes values in $M_f(\mathbb{R}^d)$, the space of finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence, and can be characterised by the following martingale problem: Let $\Phi \in C_b^2(\mathbb{R}^d)$, the space of bounded continuous functions which are twice continuously differentiable, and let $X(\Phi) \equiv \int_{\mathbb{R}^d} \Phi(x)X(dx)$. Then

$$\begin{aligned} M_t(\Phi) &= X_t(\Phi) - \int_0^t X_s(\Delta\Phi)ds, & (2) \\ \langle M(\Phi)\rangle_t &= \rho \int_0^t X_s(\Phi^2)ds, & (3) \end{aligned}$$

where $M(\Phi)$ is a martingale with quadratic variation $\langle M(\Phi)\rangle$, and ρ is a constant.

The Dawson-Watanabe superprocess can be obtained as the diffusion limit of branching Brownian motion. In this population model, individuals independently perform Brownian paths during their exponentially distributed lifetime, leaving a random number of offspring after their death. As approximations one considers then the empirical measure when particle mass and lifetime are rescaled appropriately,

$$X_t^n = n^{-1} \sum_{\alpha \sim_n t} \delta_{Y_t^{\alpha,n}}, \tag{4}$$

where the sum is over all particles α alive at time t. In the n-th approximation, the branch rate is increased by a factor of n, and each particle contributes a mass $\frac{1}{n}$ at its position $Y_t^{\alpha,n}$ in the state space. In the limit, the Laplacian in (2) corresponds to the spatial motion, the quadratic variation (3) reflects the reproduction with the constant ρ depending on the variance of the offspring distribution as well as on the branching rate.

One may take another step back from these approximating population models. Branching Brownian motion itself is the diffusion limit of a branching random walk on a lattice, for example on $\mathbb{Z}^d$. As considered by Dawson [Daw90], one may change the order of limits, first taking the diffusion limit for the reproduction, and then rescaling the motion. The intermediate step can be described by a lattice system of the form

$$dx_i(t) = \sum_{j\in\mathbb{Z}^d} m_{ij}\,(x_j(t) - x_i(t))\,dt + f_i\,(t, x_i(t))\,dt + \sigma_i\,(t, x_i(t))\,dW_i(t). \tag{5}$$

Here, x_i describes the population size at lattice point i, and m_{ij} migration between site i and j. In the special case relating to the Dawson-Watanabe superprocess, the migration is given by the generator of a simple random walk for which $m_{ij} = \frac{1}{2d}$ if $|i-j| = 1$ and zero otherwise. Reflecting that branching is a local property, W^i are independent Brownian motions, and the noise coefficients $\sigma_i(t,x) = \sqrt{\rho x}$ take the same shape as in the one dimensional Feller diffusion, see [Fel51]. The latter is the diffusion limit of a Galton-Watson branching process without a spatial component.

While the measure valued process X satisfying (2) and (3) is well defined in any dimension, it has a density, which we denote by u, only in dimension one. It has been shown (see Konno and Shiga [KS88], Reimers [Rei89]) that u is a solution to the stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x)dt + \sqrt{\rho u(t,x)}\dot{W}(t,x), \tag{6}$$

where W is a one dimensional space-time white noise. Moreover, one can show (see Blount [Blo96] and Kotelenez [Kot86]) that (approximate) densities of the particles in a branching random walk converge directly to the SPDE (6).

The area of superprocesses, in general, has expanded rapidly with the main interest focused on interacting particle systems. A number of variations of (6), also for white noise and $d = 1$, have been linked to generalisations of the Dawson-Watanabe superprocess and other particle systems. We refer to [Daw91, DP99, Eth00, Per02] for an overview and further references.

Apart from its connections to population processes, the stochastic heat equation is naturally a prominent example within the area of SPDE. Function valued solutions of the heat equation with white noise have been studied in one dimension in a multitude of settings, see for example Dawson [Daw75], Walsh [Wal86], DaPrato and Zabczyk [DZ92], Shiga [Shi94], Pardoux [Par93], Gyöngy [Gyo98a], and references therein. In higher dimensions, function valued solutions for the stochastic heat equation with colored noise have been investigated. The case of a linear noise coefficient has been treated by Dawson and Salehi [DS80] and Noble [Nob97]. Manthey and Mittmann [MM99], Kotelenez [Kot92], Peszat and Zabczyk [PZ97, PZ00], Brzeźniak and Peszat [BP99] and Dalang [Dal99] investigate solutions with Lipschitz coefficients. For some results on equations with non-Lipschitz coefficient see amongst others Viot [Vio76], DaPrato and Zabczyk [DZ92], Krylov [Kry96] and Kallianpur and Sundar [KS00]. However, these earlier results are not directly applicable to the agenda considered here due to various assumptions like boundedness of the domain, compactness of the differential operator, or nuclear or spatially homogeneous noise.

The paper is organized as follows. In Section 2 we give some notation and state the main results. In Section 3 we rigorously construct a particle system in a random environment and show that it converges to the martingale problem associated to (1) with linear noise coefficient. In Section 4 we consider related lattice systems with non-Lipschitz noise coefficients and correlated noise terms, and establish their existence and uniqueness in weighted l^p spaces. We then prove existence of the corresponding stochastic heat equations with non-Lipschitz noise coefficients on weighted L^p spaces. Convergence of approximate densities of the rescaled lattice systems is shown provided that uniqueness holds for the limit. Section 5 shows that, under some additional assumptions, the solutions constructed in section 4 are jointly continuous in time and space.

2 Formulation of the main results

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space. We use C as a generic constant, which may change its precise value from line to line. Frequently, we list the quantities that the constant C depends on in parentheses. Let C_c^∞ be the infinitely differentiable functions with compact support. The space $D(\mathbb{R}_+, E)$ denotes the càdlàg functions from $\mathbb{R}_+ \to E$, endowed with the Skorohod topology, and $C(\mathbb{R}_+, E)$ the closed subspace of continuous functions endowed with the supremum norm.

The noises W considered in this work are Gaussian martingale measure on $\mathbb{R}_+ \times \mathbb{R}^d$ in the sense of Walsh [Wal86]. They can be characterized by their covariance functional

$$J_k(\phi,\psi) \equiv \mathbb{E}[W(\phi)W(\psi)] \equiv \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(s,x)k(x,y)\psi(s,y)dxdyds, \tag{7}$$

for $\phi,\psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, where $W(\phi) \equiv \int_0^\infty \int_{\mathbb{R}^d} \phi(s,x)W(dx,ds)$. The function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is called the correlation kernel. We remark that some sufficient conditions for the existence of a martingale measure W corresponding to k are that J_k is symmetric, positive definite and continuous. Thus, necessarily, $k(x,y) = k(y,x)$ for all $x,y \in \mathbb{R}^d$. Continuity on C_c^∞, is implied, for example, if k is integrable on compact sets.

We note that a general class of martingale measures, spatially homogeneous noises, can formally be described by (7). Here, $k(x,y) = \tilde{k}(x-y)$, and one can show that all spatially homogeneous noises are of this form if we allow $\tilde{k}$ to be a generalised function on $\mathbb{R}^d$. White noise is probably the most prominent example of this class, which we recover for $\tilde{k} = \delta_0$, the delta function. Also, J_k describes a nuclear martingale measure if and only if $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. See Sturm [Stu02] pp 18 for more detail.

Here, we focus on colored noises for which $k \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$. In this case, W is a random field on $\mathbb{R}_+ \times \mathbb{R}^d$. We remark that by letting $\tilde{k}$ approach a δ_0-function, this case may be considered as a "smoothing out" of white noise.

Throughout this work we consider solutions to the formal equation (1) in the mild form in the sense of the following Definition 2.1. Let p be the d-dimensional heat kernel

$$p(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp(-\frac{||x-y||^2}{2t}). \tag{8}$$

We will sometimes abuse notation and abbreviate $p(t,x-y) = p(t,x,y)$.

Definition 2.1 *A stochastic process $u : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, which is jointly measurable and $\mathcal{F}_t$-adapted, is said to be a (stochastically) weak solution to the stochastic heat equation (1) with initial condition u_0, if there exists a martingale measure W, defined on Ω, such that a.s. for almost all $x \in \mathbb{R}^d$,*

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)u_0(y)dy \quad &+ \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)f(s,y,u(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)\sigma(s,y,u(s,y))W(dy,ds). \end{aligned} \tag{9}$$

The process u is called a (stochastically) strong solution to (1) if (9) is fulfilled a.s. for almost all $x \in \mathbb{R}^d$ for a given W.

We assume that the coefficients $f,\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the following growth condition. For all $T \geq 0$, there exists a constant $c(T)$, such that for all $0 \leq t \leq T, x \in \mathbb{R}^d$, and $u \in \mathbb{R}$,

$$|f(t,x,u)| + |\sigma(t,x,u)| \leq c(T)(1+|u|). \tag{10}$$

As solution spaces we consider L^p-spaces on $\mathbb{R}^d$ for $p \geq 2$ with some weight function $\gamma : \mathbb{R}^d \to \mathbb{R}_+$. Set

$$||u||_{\gamma,p}^p = \int_{\mathbb{R}^d} |u(t,x)|^p \gamma(x)dx \tag{11}$$

and define

$$L^p_\gamma(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} \mid ||u||_{\gamma,p} < \infty\}. \tag{12}$$

We write $L^p(\mathbb{R}^d)$ if $\gamma \equiv 1$. As a choice for the weight function we consider mostly $\gamma_\lambda(x) = e^{-\lambda||x||}$ for $\lambda > 0$. However, other integrable weight functions, in particular any positive continuous function that equals γ_λ outside a bounded region could be used.

It can be shown by standard methods (see for example Walsh [Wal86] and Sturm [Stu02] Proposition 3.2.3 for detail in this specific case) that mild solutions as in Definition 2.1 satisfy the corresponding martingale problem,

$$\begin{aligned} M_t(\Phi) &= \int_{\mathbb{R}^d} u(t,x)\Phi(x)dx - \int_{\mathbb{R}^d} u_0(x)\Phi(x)dx \qquad (13)\\ &\quad - \int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\Phi(x)dxds - \int_0^t \int_{\mathbb{R}^d} f(s,x,u(s,x))\Phi(x)dxds \\ \langle M(\Phi)\rangle_t &= \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x)\Phi(y)k(x,y)\sigma(s,x,u(s,x)) \cdot \sigma(s,y,u(s,y))dxdyds. \qquad (14) \end{aligned}$$

Here, $M_t(\Phi)$ is a continuous square integrable martingale with given quadratic variation for a class of test functions Φ that depends on the regularity of the solution sought. With the appropriate class of test functions, solving the martingale problem is indeed equivalent to finding a stochastically weak solution to (9), see [Stu02] pp 103.

Interpreting u once again as the density of a measure, we note that the martingale problem (13) and (14) makes sense for measure valued solutions if $\sigma(t,x,u) = C_\sigma u$ and $f(t,x,u) = C_f u$ are linear in the solution u, where C_σ and C_f are constants. In Section 3 we define a branching particle system that converges to this solution in a measure sense. We do so in a more general setting since the arguments are identical but take $C_f \equiv 0$ for notational convenience.

In the model we consider, the particles move independently from each other on a locally compact Polish space E with their motion given by a Feller generator $(A, \mathcal{D}(A))$. At given times, each particle may branch into two particles or die, or just live on. The main difference to the Dawson-Watanabe superprocess lies in the fact that the distribution of the branching behavior is dependent on a random environment which is correlated in space but independent in time: At each branch time we consider an independent copy of a random field ξ on E. We assume that ξ is symmetric,

$$\mathbb{P}[\xi(x) > z] = \mathbb{P}[\xi(x) < -z] \text{ for all } x \in E, z \in \mathbb{R}, \tag{15}$$

and that for some $\epsilon > 0$,

$$\mathbb{E}\left[|\xi(x)|^{2+\epsilon}\right] < \infty, \tag{16}$$

uniformly in all $x \in E$. The correlation of ξ at different points in space is given by

$$k(x,y) = \mathbb{E}[\xi(x)\xi(y)] \in C_b(E \times E), \tag{17}$$

vanishing at infinity. The probabilities for a birth/death event to happen at a branch time are given by the positive/negative part of the (appropriately truncated) random field evaluated at the location of the particle. These birth/death probabilities are rescaled by $\frac{1}{\sqrt{n}}$ in the n-th diffusion approximation. For the rescaled empirical measure X^n defined as in (4) we can then establish the following result:

Theorem 2.2 *Let $k \in C_b(E \times E)$ vanishing at infinity such that $||k||_\infty \leq K$, and $X_0^n \Rightarrow m$ in $M_f(E)$. Assume that for some $\epsilon > 0$, $\sup_n \mathbb{E}\left[X_0^n(1)^{2+\epsilon}\right] < \infty$. Then $X^n \Rightarrow X$ in $D(\mathbb{R}_+, M_f(E))$, where $X \in C(\mathbb{R}_+, M_f(E))$ is the unique solution of the following martingale problem. For all $\Phi \in \mathcal{D}(A)$,*

$$\begin{aligned} M_t(\Phi) &= X_t(\Phi) - m(\Phi) - \int_0^t X_s(A\Phi)ds \qquad (18) \\ &\qquad \textit{is a continuous square integrable } \mathcal{F}_t^X - \textit{martingale with} \\ \langle M(\Phi)\rangle_t &= \int_0^t \int_{E\times E} \Phi(x)\Phi(y)k(x,y)X_s(dx)X_s(dy)ds. \qquad (19) \end{aligned}$$

The model is inspired by Mytnik [Myt96], who considers a related branching mechanism. In comparison, branching is a rather rare event in our model: As $n \to \infty$ the particles just live on for the majority of branch times. As a result, the branching does not give rise to the white noise term which is present in most superprocesses, in the archetypical Dawson-Watanabe superprocess as well as in Mytnik's limiting superprocess.

For $E = \mathbb{R}^d$ and $A = \Delta$, any density u of X is a solution to (13) and (14) with σ linear and $f \equiv 0$, corresponding to a weak solution of the linear heat equation with no drift. For $E = \mathbb{Z}^d$ and A the discrete Laplacian we see that solutions X to the martingale problem are solutions (in l^1) to the lattice system (5) for $f_i \equiv 0$ and $\sigma_i(t,x) = x$, and correlated Brownian motions W_i.

As in the work by Mueller and Perkins [MP92], nonlinear noise coefficients can be expected to arise from the above particle picture by an additional density dependence of the branching mechanism. In Section 4 we consider such nonlinear noise coefficients which may, in particular, be non-Lipschitz. Since in this case we need to show convergence of approximate densities directly to the solution of the limiting SPDE (rather than in a measure sense), it is convenient to start with the corresponding lattice systems instead of the particle model itself (see Funaki [Fun83] and Gyöngy [Gyo98b] for this approach applied to related systems).

Thus, we first consider existence and uniqueness questions of the following system:

$$x_i(t) \quad = \quad x_i(0) + \int_0^t f_i^m(s, X(s))ds + \int_0^t \sigma_i(s, x_i(s))dW_i(s), \qquad (20)$$

for all i in a countable index set S. We write X for $(x_i)_{i\in S}$ with $x_i \in \mathbb{R}$. Here, f^m is a function $\mathbb{R}_+ \times \mathbb{R}^S \to \mathbb{R}^S$. For each i, σ_i is a function on $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and W_i are real valued martingales with $\langle W_i, W_j\rangle_t = tk_{ij}$, where k_{ij} are constants. The lattice system that interests us in particular is contained in this description by setting for each $i \in S$

$$f_i^m(t, X) = \sum_{j\in S} m_{ij}(x_j - x_i) + f_i(t, x_i). \qquad (21)$$

In analogy to the definition of solution spaces in the continuous setting, (11) and (12), we consider solutions with continuous paths in the space

$$l_\Gamma^p = \{X \in \mathbb{R}^S \mid ||X||_{\Gamma,p} < \infty\}, \text{ where } ||X||_{\Gamma,p} = (\sum_{i\in S} \gamma_i x_i^p)^{\frac{1}{p}} \qquad (22)$$

is a weighted l^p-norm on the index set S and $\Gamma = (\gamma_i)_{i\in S} \in l^1(S)$. We define the following growth and Lipschitz conditions on f^m and σ_i : For any $T \geq 0$ there exists a constant $c(T)$ so that for all $0 \leq t \leq T$,

$$\begin{aligned} ||f^m(t,X)||_{\Gamma,p} &\leq c(T)\,(1 + ||X||_{\Gamma,p})\,, &(23)\\ ||f^m(t,X) - f^m(t,Y)||_{\Gamma,p} &\leq c(T)||X-Y||_{\Gamma,p}, &(24)\\ |\sigma_i(t,x)| &\leq c(T)(1+|x|), &(25)\\ |\sigma_i(t,x) - \sigma_i(t,y)| &\leq c(T)|x-y|. &(26)\end{aligned}$$

Strong existence and uniqueness of lattice systems of the form (20) with independent noise terms W_i have, for example, been investigated by Shiga and Shimizu [SS80]. With correlated martingale terms they have not been considered in such detail. However, as we show in Section 4 existence and uniqueness results for solutions in the space l^p_Γ carry over from the uncorrelated systems, leading to the following Theorem.

Theorem 2.3 *Let $p \geq 2$. Assume also $0 \leq k_{ij} \leq K$ for all $i,j \in S$, and some constant $K > 0$, as well as $\Gamma = (\gamma_i)_{i\in S} \in l^1(S)$ with all $\gamma_i > 0$. Let f^m and σ_i for $i \in S$ be continuous in their components, f^m with respect to the product topology on $\mathbb{R}^S$. Suppose that the growth condition (23) holds for f^m and (25) holds for σ_i and all $i \in S$.*

Then there exists for each initial condition $X(0) \in l^p_\Gamma$ a solution X to the infinite dimensional system (20) with paths in $C(\mathbb{R}_+, l^p_\Gamma)$. We have the bound

$$\mathbb{E}[\sup_{0\leq t\leq T} ||X(t)||^p_{\Gamma,p}] < C(T). \tag{27}$$

If we furthermore assume that f^m satisfies (24) for $p = 1$ and σ_i satisfies (26) for all $i \in S$, then solutions to (20) are pathwise unique.

We remark that, as in the result for finite dimensional SDEs, see [YW71], pathwise uniqueness together with existence of (stochastically) weak solutions implies the existence of (stochastically) strong solutions. The following corollary demonstrates that the infinite dimensional SDEs, that are used as approximations to the stochastic heat equation, are covered by Theorem 2.3.

Corollary 2.4 *Assume that $S \subset \mathbb{R}^d$ is a lattice embedded in $\mathbb{R}^d$. Let σ and X_0 be as in Theorem 2.3, and f^m be of the form (21). Consider the weight function $\gamma_\lambda(i) = e^{-\lambda||i||}$. If $0 \leq m_{ij} \leq M$ vanishes for $|i-j| > C_m$, where C_m is a constant, and for each $i \in S$, f_i is continuous and satisfies the growth condition (25), then there exists a solution to (20) as in Theorem 2.3. If, in addition, f_i and σ_i are Lipschitz continuous, satisfying (26) for all $i \in S$, then solutions are pathwise unique.*

We continue to show that the approximate densities of the appropriately rescaled systems converge to solutions of the stochastic heat equation with colored noise (9) in the spaces $L^p_{\gamma_\lambda}$. We start with the following system defined on the rescaled lattice $\frac{1}{n}\mathbb{Z}^d$. For $z^n = (z^n_1, \ldots, z^n_d) \in \frac{1}{n}\mathbb{Z}^d$ we define

$$\begin{aligned} du^n(t,z^n) &= \Delta^n u^n(t,z^n)dt + f(t,z^n,u^n(t,z^n))dt \\ &\qquad + n^d\sigma(t,z^n,u^n(t,z^n))W(dt,I^n_{z^n}), \end{aligned} \tag{28}$$

where, denoting the unit vectors on $\mathbb{R}^d$ by e_i, the operator in the first term is given by

$$\Delta^n f(z^n) = n^2 \sum_{i=1}^{d} \left(u^n(t, z^n + \frac{e_i}{n}) + u^n(t, z^n - \frac{e_i}{n}) - 2u^n(t, z^n) \right) . \tag{29}$$

The discrete Laplacian Δ^n is the generator of a simple random walk, Y^n, on the rescaled lattice, for which time has been speeded up by a factor $2dn^2$. Hence, jumps to any neighboring site happen independently at rate n^2. The $W(t, I^n_{z^n})$ are derived from a colored noise W on $\mathbb{R}^d$ with covariance given by (7) and $k \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$. Specifically,

$$W(t, I^n_{z^n}) = \int_0^t \int_{I^n_{z^n}} W(dx, ds), \tag{30}$$

where the intervals $I^n_{z^n}$ are defined by

$$I^n_{z^n} = [z^n_1 - \frac{1}{2n}, z^n_1 + \frac{1}{2n}) \times \cdots \times [z^n_d - \frac{1}{2n}, z^n_d + \frac{1}{2n}). \tag{31}$$

The $W(t, I^n_{z^n})$ are correlated one dimensional Brownian motions and we note that for k bounded we have $\langle W(t, I^n_{z^n}) \rangle \approx n^{-2d}$, explaining the factor n^d in definition (28).

For putting (28) in its mild form, we define heat kernel approximations. Set for $z^n, \tilde{z}^n \in \frac{1}{n}\mathbb{Z}^d$,

$$p^n(t, z^n, \tilde{z}^n) \equiv n^d \mathbb{P}^{z^n} \left[Y^n_t = \tilde{z}^n \right] \equiv n^d \mathbb{P} \left[Y^n_t = \tilde{z}^n \mid Y^n_0 = z^n \right] . \tag{32}$$

We extend the lattice systems to all of $\mathbb{R}^d$ as step functions. For this define $\kappa_n(x) \equiv z^n$ for $x \in I^n_{z^n}$. The associated heat kernels are given by

$$\bar{p}^n(t, x, y) \equiv p^n(t, \kappa_n(x), \kappa_n(y)). \tag{33}$$

Note that $\bar{p}^n$ is not any more a function of $x - y$. Instead, we will use the translation invariance of p^n and the fact that $\kappa_n(x) - \kappa_n(y) = \kappa_n(x - \kappa_n(y))$ for all $x, y \in \mathbb{R}^d$, and we abbreviate occasionally $\bar{p}^n(t, x) \equiv p^n(t, 0, \kappa_n(x))$. We also write $\bar{p}^n_d, p^n_d$ and p_d to indicate the dimension if necessary.

The rescaled lattice systems, $u^n(t, x) \equiv u^n(t, \kappa_n(x))$, can now be written in the mild form for all $x \in \mathbb{R}^d$,

$$\begin{aligned} u^n(t, x) = \int_{\mathbb{R}^d} \bar{p}^n(t, x, y) u_0(y) dy \quad &+ \quad \int_0^t \int_{\mathbb{R}^d} \bar{p}^n(t - s, x, y) f(s, \kappa_n(y), u^n(s, y)) dy ds \\ &+ \quad \int_0^t \int_{\mathbb{R}^d} \bar{p}^n(t - s, x, y) \sigma(s, \kappa_n(y), u^n(s, y)) W(dy, ds). \end{aligned} \tag{34}$$

We can now state the main theorems proven in Section 4.

Theorem 2.5 *Assume that the coefficients $f(t, x, u)$ and $\sigma(t, x, u)$ are real valued functions on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ that are continuous in x and u, and satisfy the growth conditions (10). Assume further that $\mathbb{E}[||u_0||^p_{\gamma_\lambda, p}] < \infty$, for some $p \geq 2$. Let W be a colored noise of the form (7) such that $||k||_\infty \leq K < \infty$. Then there exists a (stochastically) weak solution, $u \in C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$, to the stochastic heat equation (1) with respect to W. For any $T > 0$ there exists a constant $C(T)$, so that*

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[||u(t, \cdot)||^p_{\gamma_\lambda, p} \right] \leq C(T). \tag{35}$$

Theorem 2.6 *Let f, σ and u_0 satisfy the conditions of Theorem 2.5. Assume further that there exist (stochastically) strong solutions, u^n, to the approximating lattice systems (34). If in addition pathwise uniqueness holds for the heat equation (1) then convergence in probability of u^n to u on the space $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$ holds. If weak uniqueness holds for (1) we obtain weak convergence of u^n to u on the space $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$.*

Not surprisingly, pathwise uniqueness and thus convergence of the approximations holds if the coefficients satisfy Lipschitz conditions (see [PZ00]). But pathwise uniqueness also holds for the lattice systems if the drift coefficients are Lipschitz continuous and σ satisfies the conditions of Yamada and Watanabe [YW71]. For the special case $\sigma(t,x,u) = \sqrt{u}$, and some conditions on u_0 one can also show pathwise uniqueness for the limiting equation (1) on all of $\mathbb{R}^d$, see Sturm [Stu02] Chapter 3.3. As the colored noise analogue to the Dawson-Watanabe superprocess, for which pathwise uniqueness is an open question, this is a particularly interesting case.

Finally, in Section 5 we show continuity of the solutions obtained in Theorem 2.5. Let C_{γ_λ} be the space of continuous functions on $\mathbb{R}^d$, endowed with the weighted supremum norm,

$$||u||_{\infty,\lambda} \equiv \sup_{x\in\mathbb{R}^d} |u(x)|\gamma_\lambda(x). \tag{36}$$

Theorem 2.7 *Let u be a solution of (1) with coefficients that satisfy the growth condition (10). Let $p > 2$, $d < p-2$, and assume that $\mathbb{E}[||u_0||^p_{\infty,\gamma_{\frac{\lambda}{p}}}] < \infty$, as well as (35). Then $u \in C(\mathbb{R}_+, C_{\gamma_{\frac{\lambda}{p}}})$, and for any $T > 0$ there exists a constant, $C(T)$, so that*

$$\mathbb{E}[\sup_{0\le t\le T} ||u(t,\cdot)||^p_{\infty,\frac{\lambda}{p}}] \le C(T). \tag{37}$$

3 A particle system in a random environment

In the following, we rigorously construct the branching particle system in a random environment in 3.1 and give the proof of Theorem 2.2 in 3.2.

3.1 Construction of the particle system

In keeping track of the particles and their genealogy we follow the construction of Walsh [Wal86], which has been used by Perkins [Per02], and -in a setting similar to ours- by Mytnik [Myt96]. Let all particle be labelled by

$$I = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_N) | \alpha_0 \in \mathbb{N}, \alpha_i \in \{1,2\} \text{ for } i \ge 1\}. \tag{38}$$

The quantity $|\alpha| = N$ specifies the generation of the particle. The unique ancestor of $\alpha = (\alpha_0, \dots, \alpha_N)$ k generations back is denoted by $\alpha - k \equiv (\alpha_0, \dots, \alpha_{N-k})$. We note that I is the index set for all *possible* particles since in our model there are at most two offspring. Which particles really *exist* is decided by the offspring distribution.

In the n-th approximation, branching events happen at times $\frac{i}{n}$ for $i \in \mathbb{N}$. For $t \in \mathbb{R}_+$, we set $t_n = \frac{[nt]}{n}$. Now let $\{\tilde{Y}^{\alpha,n}\}_{\alpha\in I}$ be a collection of independent Feller processes with generator A and $\tilde{Y}_0^\alpha = 0$. The path of a particular particle and its ancestors is then given by

$$Y^{\alpha,n}(t) = \begin{cases} x_{\alpha_0} + \sum_{i=0}^{|\alpha|} \tilde{Y}^{\alpha-(|\alpha|-i),n}_{(t-in^{-1})\wedge n^{-1}} & \text{for } t < |\alpha| + n^{-1}, \\ \Lambda & \text{for } t \geq |\alpha| + n^{-1}. \end{cases} \tag{39}$$

Here, x_{α_0} is the initial position of the first particle, and Λ is a "cemetery"-state.

The branching behavior is dependent on the random environment. Let ξ be as in (15) to (17). In order to define the approximating particle systems we need to truncate the random fields. For all $x \in E$ set

$$\xi^n(x) = \begin{cases} \sqrt{n} & \text{for } \xi(x) > \sqrt{n}, \\ -\sqrt{n} & \text{for } \xi(x) < -\sqrt{n}, \\ \xi(x) & \text{otherwise.} \end{cases} \tag{40}$$

Analogously to (17), we now define

$$k_n(x,y) = \mathbb{E}[\xi^n(x)\xi^n(y)]\,. \tag{41}$$

Let $(\xi_i^n)_{i\in\mathbb{N}}$ be independent copies of the truncated random field ξ^n on E. Now let $(N^{\alpha,n})_{\alpha\in I}$ be a family of random variables so that $\{N^{\alpha,n}, |\alpha| = i\}$ are conditionally independent given ξ_i^n, and the position of the particles in the i-th generation at the end of their lifespan. Denoting by ξ_i^{n+} and ξ_i^{n-} the positive and negative part of the noise respectively, the conditional offspring probabilities are given by

$$\begin{aligned} \mathbb{P}\left[N^{\alpha,n} = 2|\xi_i^n, Y^{\alpha,n}_{\frac{i+1}{n}}\right] &= \frac{1}{\sqrt{n}}\xi_i^{n+}\left(Y^{\alpha,n}_{\frac{i+1}{n}}\right), & (42)\\ \mathbb{P}\left[N^{\alpha,n} = 0|\xi_i^n, Y^{\alpha,n}_{\frac{i+1}{n}}\right] &= \frac{1}{\sqrt{n}}\xi_i^{n-}\left(Y^{\alpha,n}_{\frac{i+1}{n}}\right), & (43)\\ \mathbb{P}\left[N^{\alpha,n} = 1|\xi_i^n, Y^{\alpha,n}_{\frac{i+1}{n}}\right] &= 1 - \frac{1}{\sqrt{n}}|\xi_i^n|\left(Y^{\alpha,n}_{\frac{i+1}{n}}\right). & (44) \end{aligned}$$

According to the offspring distribution we trim the branching tree down to its existent particles. For any particle $\alpha = (\alpha_0, \ldots, \alpha_N)$ we write $\alpha \sim_n t$ whenever the particle α is alive at time t, which is the case if and only if α had an unbroken line of ancestors. Thus, $\alpha \sim_n t$ for all t with $nt_n = N$ if furthermore $N^{\alpha-i,n} \geq \alpha_{N-i+1}$ for all $i = 1, \ldots, N$.

Lastly, we need to define a filtration. It will be the natural filtration generated by the process,

$$\mathcal{F}_t^n = \sigma\left(\{Y^{\alpha,n}, N^{\alpha,n} \| \alpha| < nt_n\} \cup \{Y_s^{\alpha,n} | t_n \leq s \leq t, |\alpha| = nt_n\}\right). \tag{45}$$

We remark that the environment is not a part of the filtration, and will therefore be averaged in each step. In the studies of random media this is called the "annealed" case in contrast to the "quenched" case, where one considers statements for almost all random environment. The quenched case of a similar model to the one considered here, called the parabolic Anderson model, has be studied in some detail, see for example [CM94].

For the branching times t_n, we also define a discrete filtration,

$$\tilde{\mathcal{F}}_{t_n}^n = \sigma\left(\mathcal{F}_{t_n}^n \cup \{Y^{\alpha,n} \| \alpha| = nt_n\}\right),$$

that will be used later in conditioning. Note that $\tilde{\mathcal{F}}^n_{t_n} = \mathcal{F}^n_{(t_n+n^{-1})-}$ includes the sigma-algebra generated by the motion of the particles born at time t_n, but not that generated by their offspring distribution or the random environment at time $t_n + n^{-1}$.

Before proceeding to the proof of Theorem 2.2 we put X^n into a form which gives an intuitive idea how the limit emerges. For $\Phi \in \mathcal{D}(A)$, $\alpha \sim_n t_n$ and $t \in [t_n, t_n + n^{-1})$ we define the $\mathcal{F}^n_t$-martingales

$$M^{\alpha,n}_t(\Phi) = \Phi(Y^{\alpha,n}_t) - \Phi(Y^{\alpha,n}_{t_n}) - \int_{t_n}^t A\Phi(Y^{\alpha,n}_s)ds. \tag{46}$$

Thus, we have for $t \in [t_n, t_n + n^{-1})$,

$$\begin{aligned} X^n_t(\Phi) - X^n_{t_n}(\Phi) &= n^{-1} \sum_{\alpha \sim_n t_n} \left(\Phi(Y^{\alpha,n}_t) - \Phi(Y^{\alpha,n}_{t_n})\right) \\ &= n^{-1} \sum_{\alpha \sim_n t_n} M^{\alpha,n}_t(\Phi) + \int_{t_n}^t n^{-1} \sum_{\alpha \sim_n t_n} A\Phi(Y^{\alpha,n}_s)ds. \end{aligned} \tag{47}$$

For the difference of measures between two branch times we obtain

$$\begin{aligned} X^n_{t_n+n^{-1}}(\Phi) - X^n_{t_n}(\Phi) &= n^{-1} \sum_{\alpha \sim_n t_n} \left(\Phi(Y^{\alpha,n}_{t_n+n^{-1}})N^{\alpha,n} - \Phi(Y^{\alpha,n}_{t_n})\right) \\ &= n^{-1} \sum_{\alpha \sim_n t_n} \Phi(Y^{\alpha,n}_{t_n+n^{-1}})(N^{\alpha,n} - 1) + n^{-1} \sum_{\alpha \sim_n t_n} M^{\alpha,n}_{t_n+n^{-1}}(\Phi) \\ &\quad + \int_{t_n}^{t_n+n^{-1}} n^{-1} \sum_{\alpha \sim_n t_n} A\Phi(Y^{\alpha,n}_s)ds. \end{aligned} \tag{48}$$

By adding all the differences from (47) and (48) we obtain

$$\begin{aligned} X^n_t(\Phi) &= X^n_0(\Phi) + n^{-1} \sum_{s_n < t_n} \sum_{\alpha \sim_n s_n} \Phi(Y^{\alpha,n}_{s_n+n^{-1}})(N^{\alpha,n} - 1) \\ &\quad + \left(n^{-1} \sum_{s_n < t_n} \sum_{\alpha \sim_n s_n} M^{\alpha,n}_{s_n+n^{-1}}(\Phi) + n^{-1} \sum_{\alpha \sim_n t_n} M^{\alpha,n}_t(\Phi)\right) \\ &\quad + \int_0^t X^n_s(A\Phi)ds \\ &\equiv X^n_0(\Phi) + M^{b,n}_{t_n}(\Phi) + M^{r,n}_t(\Phi) + \int_0^t X^n_s(A\Phi)ds. \end{aligned} \tag{49}$$

The term $M^{b,n}_{t_n}(\Phi)$ is a discrete martingale with respect to the filtration $\tilde{\mathcal{F}}^n_{t_n}$, as can be easily by conditioning appropriately and using the fact that for $\alpha \sim_n t_n$,

$$\begin{aligned} \mathbb{E}\left[(N^{\alpha,n} - 1)\middle|\tilde{\mathcal{F}}^n_{t_n}\right] &= \mathbb{E}\left[\frac{1}{\sqrt{n}}\xi^{n+}_{|\alpha|}\left(Y^{\alpha,n}_{t_n+n^{-1}}\right) - \frac{1}{\sqrt{n}}\xi^{n-}_{|\alpha|}\left(Y^{\alpha,n}_{t_n+n^{-1}}\right)\middle|Y^{\alpha,n}_{t_n+n^{-1}}\right] \\ &= \frac{1}{\sqrt{n}}\mathbb{E}\left[\xi^n_{|\alpha|}\left(Y^{\alpha,n}_{t_n+n^{-1}}\right)\middle|Y^{\alpha,n}_{t_n+n^{-1}}\right] = 0, \end{aligned}$$

because of the symmetry condition (15). The term $M^{r,n}_t(\Phi)$ is an $\mathcal{F}^n_t$-martingale as a sum of martingales. We subsequently show that $M^{r,n}$ becomes negligible in the limit whereas the martingale $M^{b,n}$ converges to M in (18) and its quadratic variation to (19). We get a sense of this by calculating $\langle M^{b,n}\rangle$. We note first that

$$\begin{aligned}
\mathbb{E}\left[(N^{\alpha,n}-1)^2\middle|\tilde{\mathcal{F}}^n_{t_n}\right] &= \frac{1}{\sqrt{n}}\mathbb{E}\left[\xi^{n+}_{|\alpha|}+\xi^{n-}_{|\alpha|}\middle|\tilde{\mathcal{F}}^n_{t_n}\right] \\
&= \frac{1}{\sqrt{n}}\mathbb{E}\left[|\xi^n_{|\alpha|}|(Y^{\alpha,n}_{t_n+n^{-1}})\middle|\tilde{\mathcal{F}}^n_{t_n}\right] \qquad (50) \\
\mathbb{E}\left[(N^{\alpha,n}-1)(N^{\alpha',n}-1)\middle|\tilde{\mathcal{F}}^n_{t_n}\right] &= \frac{1}{n}\mathbb{E}[\xi^{n+}_{|\alpha|}\xi^{n+}_{|\alpha'|}+\xi^{n-}_{|\alpha|}\xi^{n-}_{|\alpha'|}-\xi^{n+}_{|\alpha|}\xi^{n-}_{|\alpha'|}-\xi^{n-}_{|\alpha|}\xi^{n+}_{|\alpha'|}|\tilde{\mathcal{F}}^n_{t_n}] \\
&= \frac{1}{n}\mathbb{E}\left[\xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}})\xi^n_{|\alpha'|}(Y^{\alpha',n}_{t_n+n^{-1}})\middle|\tilde{\mathcal{F}}^n_{t_n}\right] \\
&= \frac{1}{n}k_n(Y^{\alpha,n}_{t_n+n^{-1}},Y^{\alpha',n}_{t_n+n^{-1}}), \qquad (51)
\end{aligned}$$

where we have, for notational brevity, not always explicitly stated where ξ is evaluated. By conditioning we obtain with (50) and (51),

$$\begin{aligned}
&\mathbb{E}\left[(M^{b,n}_{t_n+n^{-1}}(\Phi))^2 \mid \tilde{\mathcal{F}}^n_{t_n}\right] \\
= \;& (M^{b,n}_{t_n}(\Phi))^2+n^{-2}\sum_{\alpha\sim_n t_n}\Phi(Y^{\alpha,n}_{t_n+n^{-1}})^2\mathbb{E}\left[(N^{\alpha,n}-1)^2 \mid \tilde{\mathcal{F}}^n_{t_n}\right] \\
&+n^{-2}\sum_{\substack{\alpha\sim_n t_n\\ \alpha'\sim_n t_n\\ \alpha\neq\alpha'}}\Phi(Y^{\alpha,n}_{t_n+n^{-1}})\Phi(Y^{\alpha',n}_{t_n+n^{-1}})\mathbb{E}\left[(N^{\alpha,n}-1)(N^{\alpha',n}-1) \mid \tilde{\mathcal{F}}^n_{t_n}\right] \\
= \;& (M^{b,n}_{t_n}(\Phi))^2+n^{-2}\sum_{\alpha\sim_n t_n}\Phi(Y^{\alpha,n}_{t_n+n^{-1}})^2\Bigg(\frac{1}{\sqrt{n}}\mathbb{E}\left[|\xi^n_{|\alpha|}|(Y^{\alpha,n}_{t_n+n^{-1}}) \mid \tilde{\mathcal{F}}^n_{t_n}\right] \\
&\qquad\qquad -\frac{1}{n}k_n(Y^{\alpha,n}_{t_n+n^{-1}},Y^{\alpha,n}_{t_n+n^{-1}})\Bigg) \\
&+n^{-3}\sum_{\substack{\alpha\sim_n t_n\\ \alpha'\sim_n t_n}}\Bigg(\mathbb{E}\left[\Phi(Y^{\alpha,n}_{t_n+n^{-1}})\Phi(Y^{\alpha',n}_{t_n+n^{-1}})k_n(Y^{\alpha,n}_{t_n+n^{-1}},Y^{\alpha',n}_{t_n+n^{-1}}) \mid \tilde{\mathcal{F}}^n_{t_n}\right] \\
&\qquad\qquad -\Phi(Y^{\alpha,n}_{t_n})\Phi(Y^{\alpha',n}_{t_n})k_n(Y^{\alpha,n}_{t_n},Y^{\alpha',n}_{t_n})\Bigg) \\
&+\int_{t_n}^{t_n+n^{-1}}\int_{E\times E}\Phi(x)\Phi(y)k_n(x,y)X^n_{s_n}(dx)X^n_{s_n}(dy)ds \\
= \;& (M^{b,n}_{t_n}(\Phi))^2+\epsilon^{1,n}_{t_n}(\Phi)+\epsilon^{2,n}_{t_n}(\Phi) \\
&\qquad +\int_{t_n}^{t_n+n^{-1}}\int_{E\times E}\Phi(x)\Phi(y)k_n(x,y)X^n_{s_n}(dx)X^n_{s_n}(dy)ds.
\end{aligned}$$

The quadratic variation of $M^{b,n}$ is thus given by

$$\begin{aligned}
\langle M^{b,n}(\Phi)\rangle_{t_n} &= \sum_{s_n<t_n}\mathbb{E}\left[(M^{b,n}_{s_n+n^{-1}}(\Phi))^2-(M^{b,n}_{s_n}(\Phi))^2 \mid \tilde{\mathcal{F}}^n_{s_n}\right] \qquad (52) \\
&= \sum_{s_n<t_n}\epsilon^{1,n}_{s_n}(\Phi)+\sum_{s_n<t_n}\epsilon^{2,n}_{s_n}(\Phi) \\
&\quad +\int_0^{t_n}\int_{E\times E}\Phi(x)\Phi(y)k_n(x,y)X^n_{s_n}(dx)X^n_{s_n}(dy)ds.
\end{aligned}$$

3.2 Proof of Theorem 2.2

The proof of convergence proceeds in several well known steps. First, we show tightness of the sequence in $D(\mathbb{R}_+, M_f(\hat{E}))$, where $\hat{E}$ is the compactified space. This implies relative compactness and thus the existence of a convergent subsequence in that state space. We then show that all limit points of the sequence are in $C(\mathbb{R}_+, M_f(E))$ and that they are solutions to the martingale problem given by (18) and (19). The uniqueness of the martingale problem finally implies convergence of the particle system.

In order to show tightness of the measures X^n in $D(\mathbb{R}_+, M_f(\hat{E}))$ we start with several lemmas. We define for a process Y in $D(\mathbb{R}_+, \mathbb{R})$, $\delta Y_t \equiv Y_t - Y_{t-}$, specifying the heights of the jumps of the process Y.

Lemma 3.1 *There is an* $\epsilon > 0$ *such that for all* $T \geq 0$ *and* $\Phi \in \mathcal{D}(A)$ *bounded*

(i) $\mathbb{E}\left[\sup_{0\leq t\leq T} X^n_t(\Phi)^{2+\epsilon}\right]$ *is uniformly bounded in* n.

(ii) $\mathbb{E}\left[\sup_{0\leq t\leq T} M^{b,n}_{t_n}(\Phi)^{2+\epsilon}\right]$ *is uniformly bounded in* n.

(iii) $\mathbb{E}\left[\sup_{0\leq t\leq T} |\delta M^{b,n}_{t_n}(\Phi)|^{2+\epsilon}\right] \to 0$ *as* $n \to \infty$.

PROOF. We first obtain an $L^p(\Omega)$-estimate on $\langle M^{b,n}(\Phi)\rangle_{t_n}$ as given in (52). For $0 \leq t \leq T$ and $p \geq 1$ we have for $C_1^{(p)}(\Phi) \equiv \mathbb{E}\left[|\sum_{s_n<t_n} \epsilon^{1,n}_{s_n}(\Phi)|^p\right]$,

$$\begin{aligned} C_1^{(p)}(\Phi) &\leq \left(\frac{||\Phi||^2_\infty 2(1+K)}{\sqrt{n}}\right)^p \mathbb{E}\left[\left(n^{-2}\sum_{s_n<t_n}\sum_{\alpha\sim_n s_n} 1\right)^p\right] \\ &\leq \left(\frac{||\Phi||^2_\infty 2(1+K)}{\sqrt{n}}\right)^p T^{p-1}\int_0^{t_n} \mathbb{E}\left[X^n_{s_n}(1)^p\right] ds, \end{aligned} \tag{53}$$

where we have used the fact that, for all $x \in E$, $\mathbb{E}[|\xi^n(x)|] \leq \mathbb{E}[|\xi^n(x)|^2]^{\frac{1}{2}} = \sqrt{k_n(x,x)} \leq \sqrt{K} \leq 1+K$, as well as Jensen's Inequality.

Let $T^{(j)}$ be the semigroup on E^j of j independent motions with generator A, and define for $x, y \in E$ the function $d_n(x,y) = \Phi(x)\Phi(y)k_n(x,y)$. Then,

$$\begin{aligned} C_2^{(p)}(\Phi) &\equiv \mathbb{E}\left[|\sum_{s_n<t_n} \epsilon^{2,n}_{s_n}(\Phi)|^p\right] \\ &= \mathbb{E}\Bigg[|n^{-3}\sum_{s_n<t_n}\Big(\sum_{\substack{\alpha\sim_n s_n\\ \alpha'\sim_n s_n\\ \alpha\neq\alpha'}} (T^{(2)}_{\frac{1}{n}} - I)d_n(Y^{\alpha,n}_{s_n}, Y^{\alpha',n}_{s_n}) \\ &\qquad\qquad + \sum_{\alpha\sim_n s_n}(T^{(1)}_{\frac{1}{n}} - I)d_n(Y^{\alpha,n}_{s_n}, Y^{\alpha,n}_{s_n})\Big)|^p\Bigg] \\ &\leq \max_{i=1,2}||(T^{(i)}_{\frac{1}{n}} - I)d_n||^p_\infty \mathbb{E}\left[|\int_0^{t_n} X^n_{s_n}(1)^2 ds|^p\right] \\ &\leq (||\Phi||^2_\infty K)^p T^{p-1}\int_0^{t_n}\mathbb{E}\left[X^n_{s_n}(1)^{2p}\right] ds. \end{aligned} \tag{54}$$

Note that the last line follows from $||d_n||_\infty \le ||\Phi||^2_\infty K$ and $T^{(j)}$ being a contraction semigroup as well as Jensen's Inequality. Similarly,

$$
\begin{aligned}
C_3^{(p)}(\Phi) &\equiv \mathbb{E}\left[|\int_0^{t_n} \int_{E\times E} \Phi(x)\Phi(y)k_n(x,y)X^n_{s_n}(dx)X^n_{s_n}(dy)ds|^p \right] \\
&\le (||\Phi||^2_\infty K)^p T^{p-1} \int_0^{t_n} \mathbb{E}\left[X^n_{s_n}(1)^{2p}\right] ds.
\end{aligned}
\tag{55}
$$

We also need a bound on the jumps of $M^{b,n}(\Phi)$. For doing this we define $\delta M^{b,n}_{t_n}(\Phi) = \delta B^{1,n}_{t_n}(\Phi) + \delta B^{2,n}_{t_n}(\Phi)$, where

$$
\begin{aligned}
\delta B^{1,n}_{t_n}(\Phi) &= n^{-1} \sum_{\alpha \sim_n t_n} \Phi(Y^{\alpha,n}_{t_n+n^{-1}}) \left(N^{\alpha,n} - 1 - \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}}) \right), \\
\delta B^{2,n}_{t_n}(\Phi) &= n^{-1} \sum_{\alpha \sim_n t_n} \Phi(Y^{\alpha,n}_{t_n+n^{-1}}) \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}}).
\end{aligned}
$$

Indexed lexicographically by $\alpha \sim_n t_n$ and conditioned on $\sigma(\tilde{\mathcal{F}}^n_{t_n} \cup \xi^n_{|\alpha|})$, each $\delta B^{1,n}_{t_n}(\Phi)$ is a discrete martingale with respect to its natural filtration since $\mathbb{E}[(N^{\alpha,n} - 1)|\xi^n_{|\alpha|}(x)] = \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(x)$. The term $C_4^{(p)}(\Phi) \equiv \mathbb{E}[\sup_{0 \le t \le T} |\delta B^{1,n}_{t_n}(\Phi)|^{2p}]$ can be bounded by using the martingale properties,

$$
\begin{aligned}
C_4^{(p)}(\Phi) &\le \mathbb{E}\left[\sum_{0 \le t_n \le T} |\delta B^{1,n}_{t_n}(\Phi)|^{2p} \right] \\
&\le \sum_{0 \le t_n \le T} \mathbb{E}\left[\mathbb{E}\left[|\delta B^{1,n}_{t_n}(\Phi)|^{2p} \Big| \sigma(\tilde{\mathcal{F}}^n_{t_n} \cup \xi^n_{|\alpha|}) \right] \right] \\
&\le C(\frac{||\Phi||_\infty}{n})^{2p} \sum_{0 \le t_n \le T} \Bigg(\mathbb{E}\left[\sup_{\alpha \sim_n t_n} |N^{\alpha,n} - 1 - \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}})|^{2p} \right] \\
&\quad + \mathbb{E}\left[\left(\sum_{\alpha \sim_n t_n} \mathbb{E}\left[(N^{\alpha,n} - 1 - \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}}))^2 \Big| \sigma(\tilde{\mathcal{F}}^n_{t_n} \cup \xi^n_{|\alpha|}) \right] \right)^p \right] \Bigg) \\
&\le C(\frac{2||\Phi||_\infty}{n})^{2p} n \int_0^T \left(1 + n^p \mathbb{E}\left[X^n_{t_n}(1)^p\right]\right) dt,
\end{aligned}
\tag{56}
$$

where in the third inequality we have used Burkholder's Inequality for discrete martingales (see [Bur73]) resulting in some constants C. For the fourth inequality note that $|N^{\alpha,n} - 1 - \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}})| \le 2$. We are left to estimate $\delta B^{2,n}(\Phi)$:

$$
\begin{aligned}
C_5^{(p)}(\Phi) &\equiv \mathbb{E}\left[\sup_{0 \le t \le T} |\delta B^{2,n}_{t_n}(\Phi)|^{2p} \right] \\
&\le \sum_{0 \le t_n \le T} \mathbb{E}\left[|n^{-1} \sum_{\alpha \sim_n t_n} \Phi(Y^{\alpha,n}_{t_n+n^{-1}}) \frac{1}{\sqrt{n}} \xi^n_{|\alpha|}(Y^{\alpha,n}_{t_n+n^{-1}})|^{2p} \right] \\
&\le n^{1-p} ||\Phi||^{2p}_\infty \sup_{x \in E} \mathbb{E}\left[|\xi(x)|^{2p}\right] \int_0^T \mathbb{E}\left[\sup_{0 \le s \le t} X^n_{s_n}(1)^{2p} \right] dt,
\end{aligned}
\tag{57}
$$

where we have first applied Jensen's Inequality to the particle sum. Now, because A is a conservative operator we have for $\Phi \equiv 1$ that $A\Phi \equiv 0$. Thus we obtain from (46) and (49), $X_t^n(1) = X_0^n(1) + M_{t_n}^{b,n}(1)$. By the same version of Burkholder's Inequality as above and for ϵ small enough as in (16), setting $p = 1 + \frac{\epsilon}{2}$, it now follows that

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le t\le T} X_t^n(1)^{2+\epsilon}\right] &\le C\left(1 + \mathbb{E}\left[\sup_{0\le t\le T} M_{t_n}^{b,n}(1)^{2+\epsilon}\right]\right) \qquad (58)\\ \le \quad C &\left(1 + \mathbb{E}\left[\langle M^{b,n}(1)\rangle_T^{1+\frac{\epsilon}{2}}\right] + \mathbb{E}\left[\sup_{t_n\le T} |\delta M_{t_n}^{b,n}(1)|^{2+\epsilon}\right]\right)\\ \le \quad C(T,K) &\left(1 + \int_0^T \mathbb{E}\left[\sup_{0\le s\le t} X_s^n(1)^{2+\epsilon}\right] ds\right). \end{aligned}$$

The last line and the choice of the constant $C(T,K)$, which is independent of n, followed from (53) to (55) and (56) as well as (57). But the function $T \mapsto \mathbb{E}\left[\sup_{0\le t\le T} X_t^n(1)^{2+\epsilon}\right] < \mathbb{E}[(X_0^n(1)2^{nT})^{2+\epsilon}]$ is a bounded measurable function, and thus we can apply Gronwall's Lemma to obtain the bound $\mathbb{E}\left[\sup_{0\le t\le T} X_t^n(1)^{2+\epsilon}\right] \le C(T,K)$. This completes the proof of (i) since $\mathbb{E}\left[\sup_{0\le t\le T} X_t^n(\Phi)^{2+\epsilon}\right] \le ||\Phi||_\infty^{2+\epsilon}\mathbb{E}\left[\sup_{0\le t\le T} X_t^n(1)^{2+\epsilon}\right]$.

Property (ii) follows now from the calculations in (58) with an additional constant depending on $||\Phi||_\infty$ and the boundedness of the mass shown in (i).

Property (iii) follows from (56) and (57) combined with the boundedness of the total mass shown in (i). □

Using the above Lemma 3.1(i) we can now show that both, $M^{r,n}(\Phi)$ as well as the ϵ terms in (52), become indeed negligible.

Lemma 3.2 *For all* $T > 0, \Phi \in \mathcal{D}(A)$, $\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} M_t^{r,n}(\Phi)^2\right] = 0$.

Since the motion of the particle system is no different from that of the Dawson-Watanabe superprocess considered by Perkins [Per02] in the same set-up we may simply refer to his Lemma II.4.3 for proof. To show convergence of the remaining terms we need the following lemma which is a consequence of condition (16), see [Stu02] Lemma 6.2.1 for proof.

Lemma 3.3 $\sup_{(x,y)\in E\times E} |k_n(x,y) - k(x,y)| \to 0$, *as* $n \to 0$.

Lemma 3.4 *We have for* $i = 1, 2$, $\Phi \in \mathcal{D}(A)$ *and all* $T \ge 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T} \sum_{s_n < t_n} \epsilon_{s_n}^{i,n}(\Phi)\right] \to 0.$$

PROOF. For $i = 1$ the statement follows immediately from (53) because of the boundedness of the total mass (Lemma 3.1(i)). For $i = 2$ we refer to (54) combined with Lemma 3.1(i) and note the fact that for $i = 1, 2$ and $n \to \infty$, $||(T_{\frac{1}{n}}^{(i)} - I)d_n||_\infty \to 0$. The latter follows since $||d_n - d||_\infty$, where $d(x,y) \equiv \Phi(x)\Phi(y)k(x,y)$, converges to zero by Lemma 3.3. Also, $||(T_{\frac{1}{n}}^{(i)} - I)d||_\infty$ converges to zero because of the strong continuity of the semigroups. □

We now show that the martingale as well as the integral part of X^n are tight.

Lemma 3.5 *For all $\Phi \in \mathcal{D}(A)$ we have that $M^{b,n}(\Phi)$ and $\langle M^{b,n}(\Phi)\rangle$ as well as $C_t^n(\Phi) \equiv \int_0^t X_s^n(A\Phi)ds$ are C-tight sequences in $D(\mathbb{R}_+, \mathbb{R})$.*

PROOF. Lemma 3.1(ii), together with Markov's Inequality and Burkholder's Inequality, implies that $M_{t_n}^{b,n}(\Phi)$ and $\langle M^{b,n}(\Phi)\rangle_{t_n}$ are tight in $\mathbb{R}$ for any fixed $t \geq 0$. To complete the tightness proof we estimate for $0 \leq t \leq T$ and $0 \leq u \leq \delta$, using (52) and the calculations in (53) to (55),

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T} |\langle M^{b,n}(\Phi)\rangle_{(t+u)_n} - \langle M^{b,n}(\Phi)\rangle_{t_n}|\right] \qquad (59)\\ \leq\ & (u + \frac{1}{n}) \quad C(||\Phi||_\infty, K) \left(1 + \mathbb{E}\left[\sup_{0\leq t\leq T+\delta} X_t^n(1)^2\right]\right). \end{aligned}$$

Due to Lemma 3.1(i) this converges to zero uniformly in n as $\delta \to 0$. Theorem 8.6 of Chapter 3 in [EK86] now implies the tightness of $\langle M^{b,n}(\Phi)\rangle$. The tightness of $M^{b,n}(\Phi)$ follows exactly with the same calculation by observing that

$$\mathbb{E}\left[|M_{(t+u)_n}^{b,n}(\Phi) - M_{t_n}^{b,n}(\Phi)|^2\right] \leq \mathbb{E}\left[|\langle M^{b,n}(\Phi)\rangle_{(t+u)_n} - \langle M^{b,n}(\Phi)\rangle_{t_n}|\right].$$

It remains to show C-tightness of X^n. For the quadratic variation we just need to observe according to Proposition VI.3.26 of [JS87] that

$$\mathbb{P}\left[\sup_{0\leq t\leq N} |\delta\langle M^{b,n}(\Phi)\rangle_{t_n}| > \epsilon\right] \leq \frac{1}{\epsilon}\mathbb{E}\left[\sup_{0\leq t\leq N} |\langle M^{b,n}(\Phi)\rangle_{t_n+\frac{1}{n}} - \langle M^{b,n}(\Phi)\rangle_{t_n}|\right],$$

which converges to zero by (59). For $M^{b,n}(\Phi)$ itself, the same condition has already been shown in Lemma 3.1(iii). The arguments for $C_t^n(\Phi)$ follow the same pattern using the boundedness of $||A\Phi||_\infty$ and Lemma 3.1(i). □

Denote by $\hat{E} = E \cup \Theta$ the one point compactification of E. The generator A and its semigroup T_t are extended to $\hat{E}$ by setting for $f \in C(\hat{E})$, $T_t f = f(\Theta) + T_t(f - f(\Theta))$.

Proposition 3.6 *The X^n are a tight sequence in $D(\mathbb{R}_+, M_f(\hat{E}))$ and all limit points are continuous.*

PROOF. By (49), $X_t^n(\Phi) = X_0^n(\Phi) + M_{t_n}^{b,n}(\Phi) + M_t^{r,n}(\Phi) + C_t^n(\Phi)$. Here, the first term converges weakly by assumption, the branching part $M^{b,n}(\Phi)$ and $C^n(\Phi)$ are C-tight in $D(\mathbb{R}_+, \mathbb{R})$ by Lemma 3.5. The martingale $M^{r,n}(\Phi)$ converges to zero in $C(\mathbb{R}_+, \mathbb{R})$ in $L^2(\Omega)$ by Lemma 3.2 so certainly also in law. Thus $X^n(\Phi)$ is C-tight in $D(\mathbb{R}_+, \mathbb{R})$ for Φ in a dense subset of $C_b(\hat{E})$. As $M_f(\hat{E})$ is compact, the compact containment condition is fulfilled naturally, and thus [RC86] now implies tightness in $D(\mathbb{R}_+, M_f(\hat{E}))$. All limit points X must have continuous sample paths since $X(\Phi)$ is continuous for Φ in a dense subset of $C_b(E)$. □

Now, let X^{n_k} be a subsequence which converges weakly in the space $D(\mathbb{R}_+, M_f(\hat{E}))$. By Skorohod's Representation Theorem we can find a probability space and on it a sequence $\tilde{X}^{n_k}$ such that $\mathcal{L}(X^{n_k}) = \mathcal{L}(\tilde{X}^{n_k})$ with $\tilde{X}^{n_k}$ converging to $\tilde{X}$ almost surely in $D(\mathbb{R}_+, M_f(\hat{E}))$.

Lemma 3.7 *For any a.s. convergent subsequence $\tilde{X}^{n_k}$, $\tilde{M}^{b,n_k}_t(\Phi)$ converges in probability for each $t \geq 0$ and $\Phi \in \mathcal{D}(\hat{A})$. The limit is a square integrable continuous martingale, $M(\Phi)$, with quadratic variation given by (19).*

PROOF. By the continuity of the limit, for all $t \geq 0$, $\sup_{0\leq s\leq t}|\tilde{X}^{n_k}_s(\Phi) - \tilde{X}_s(\Phi)| \to 0$ a.s., and so $\int_0^t \tilde{X}^{n_k}_s(\hat{A}\Phi)ds \to \int_0^t \tilde{X}_s(\hat{A}\Phi)ds$ a.s.. By Lemma 3.2, $\sup_{0\leq s\leq t} M^{r,n}_s(\Phi) \to 0$ in $L^2(\Omega)$. Thus,

$$\tilde{M}^{b,n_k}_t(\Phi) = \tilde{X}^{n_k}_t(\Phi) - \tilde{X}^{n_k}_0(\Phi) - \tilde{M}^{r,n_k}_t(\Phi) - \int_0^t \tilde{X}^{n_k}_s(\hat{A}\Phi)ds \tag{60}$$

converges in probability on $D(\mathbb{R}_+, \hat{E})$. The limit is a square integrable martingale because of Lemma 3.1(ii). It is continuous since all the terms in (60) have continuous limits. It only remains to show that the quadratic variation converges to the appropriate expression. Thus, consider a.s.

$$\begin{aligned}
&|\int_0^{t_n}\int_{\hat{E}\times\hat{E}} \Phi(x)\Phi(y)k_{n_k}(x,y)\tilde{X}^{n_k}_{s_{n_k}}(dx)\tilde{X}^{n_k}_{s_{n_k}}(dx)ds \\
&\qquad\qquad - \int_0^t\int_{\hat{E}\times\hat{E}} \Phi(x)\Phi(y)k(x,y)\tilde{X}_s(dx)\tilde{X}_s(dx)ds| \\
\leq\ & \int_0^{t_n}\int_{\hat{E}\times\hat{E}} \Phi(x)\Phi(y)|k_{n_k}(x,y) - k(x,y)|\tilde{X}^{n_k}_{s_{n_k}}(dx)\tilde{X}^{n_k}_{s_{n_k}}(dx)ds \\
&+|\int_0^t (1_{\{s\leq t_n\}}\int_{\hat{E}\times\hat{E}} \Phi(x)\Phi(y)k(x,y)\tilde{X}^{n_k}_{s_{n_k}}(dx)\tilde{X}^{n_k}_{s_{n_k}}(dx) \\
&\qquad\qquad - \int_{\hat{E}\times\hat{E}} \Phi(x)\Phi(y)k(x,y)\tilde{X}_s(dx)\tilde{X}_s(dx))ds|.
\end{aligned}$$

Here, the first term converges to zero by Lemma 3.3 and Lemma 3.1(i). The second term converges since $\tilde{X}^{n_k}_{t_{n_k}} \times \tilde{X}^{n_k}_{t_{n_k}} \to \tilde{X}_t \times \tilde{X}_t$ a.s. on $M_f(\hat{E}\times\hat{E})$ according to Lemma 2.1.2 of [Daw91]. The remainder terms of the quadratic variation, see (52), converge to zero in $L^1(\Omega)$ due to Lemma 3.4. Thus, the expression (19) is the a.s. limit of $\langle\tilde{M}^{b,n_k}(\Phi)\rangle_t$. □

Lemma 3.8 *The limit takes values in space $C(\mathbb{R}_+, M_f(E))$.*

The proof follows standard arguments, see [Stu02] p102 for detail. The proof of Theorem 2.2 is now complete upon noting that following result which is contained in the main theorem of Mytnik [Myt96].

Theorem 3.9 *Solutions to the martingale problem (18) and (19) are unique in distribution.*

The proof is based on the observation that X would be dual to itself if it was sufficiently regular: If u and v are the densities of two independent processes that satisfy the martingale problem (18) to (19), then it would follow that $\mathbb{E}\left[\exp(-\int_E u_t(x)v_0(x)dx)\right] = \mathbb{E}\left[\exp(-\int_E u_0(x)v_t(x)dx)\right]$. For proving uniqueness, it suffices then to construct a suitably regular approximation to X, and apply an approximate duality argument.

4 Convergence to the heat equation

Here, we outline first the proof of Theorem 2.3. follows the arguments of Shiga and Shimizu [SS80] closely, and we will therefore only be explicit about the necessary modifications. We then prove Corollary 2.4, which shows that the stepping stone models approximating the heat equation fulfill the conditions of Theorem 2.3.

Subsequently, after citing some auxiliary lemmas, we give the proof of Theorem 2.5 and Theorem 2.6. We first show tightness of the rescaled systems. Thus, we are able to prove existence of weak solutions to the heat equation with colored noise for continuous coefficients that obey a linear growth bound, see Theorem 2.5. When the strong existence of the approximating systems and uniqueness of the SPDE, well-known for Lipschitz coefficients, and for non-Lipschitz coefficients investigated in [Stu02], is known, Theorem 2.6 establishes convergence of the approximations.

4.1 Proof of Theorem 2.3

We approximate the solution X to (20) by finite dimensional diffusions. So choose $S_n \subset S$ finite such that $S_n \uparrow S$, and let X^n be the solution of the diffusion

$$x_i^n(t) = x_i(0) + \int_0^t f_i^m(s, X^n(s))ds + \int_0^t \sigma_i(s, x_i^n(s))dW_i^n(s)$$

for $i \in S_n$. Set $x_i^n(t) = x_i(0)$ for $i \notin S_n$. Note that the W_i^n can be represented by a linear combination of at most n independent Brownian motions. Thus, existence of weak solutions with continuous sample paths is a classic result, see for example, Theorem 3.10 of Chapter 5 in [EK86].

The key of the proof is to obtain a uniform bound on the approximating finite dimensional solutions X^n in the norm (27), which can then be used to bound temporal differences in the same norm.

In order to be able to apply Gronwall's Lemma we use a stopping time argument and define $T^{(N,n)} = \inf\{t \geq 0 \mid \quad ||X^n(t)||_{\Gamma,p} \geq N\}$. Now, we consider

$$\begin{aligned} g^{n,N}(T) &\equiv \mathbb{E}\left[\sup_{0\leq t\leq T\wedge T^{(N,n)}} ||X^n(t)||_{\Gamma,p}^p\right] \qquad (61)\\ &\leq C\,(\sum_{i\in S}\gamma_i|x_i(0)|^p + \mathbb{E}\left[\sup_{0\leq t\leq T\wedge T^{(N,n)}}\sum_{i\in S_n}\gamma_i\left|\int_0^t f_i^m(s,X^n(s))ds\right|^p\right] \\ &\qquad\qquad + \mathbb{E}\left[\sup_{0\leq t\leq T\wedge T^{(N,n)}}\sum_{i\in S_n}\gamma_i\left|\int_0^t \sigma_i(s,x_i^n(s))dW_i^n(s)\right|^p\right]\,). \end{aligned}$$

While the first term is bounded by assumption, the next two terms can be bounded by $C(c,T,K)(1+\int_0^T g^{n,N}(s)ds)$:

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge T^{(N,n)}}\sum_{i\in S_n}\gamma_i\left|\int_0^t \sigma_i(s,x_i^n(s))dW_i^n(s)\right|^p\right]$$

$$\begin{aligned}
&\le \sum_{i\in S_n}\gamma_i\mathbb{E}\left[\sup_{0\le t\le T\wedge T^{(N,n)}}\left|\int_0^t \sigma_i(s',x_i^n(s'))dW_i^n(s)\right|^p\right]\\
&\le C(p)T^{\frac{p}{2}-1}\sum_{i\in S_n}\gamma_i k_{ii}\mathbb{E}\left[\int_0^{T\wedge T^{(N,n)}} |\sigma_i(s',x_i^n(s'))|^p ds\right]\\
&\le C(p,c,T,K)\left(1+\int_0^T g^{n,N}(s)ds\right),
\end{aligned}$$

where we have used Burkholder's Inequality and Jensen's Inequality as well as the growth condition (25) on the σ_i. The term involving f^m is estimated similarly using (23). Thus, by Gronwall's Lemma, $g^{n,N}(T)$ is bounded by a constant that is independent of n and N. The sample paths are a.s. continuous for each n and therefore bounded on $[0,T]$, albeit not uniformly. This implies $\mathbb{P}[T^{(N,n)} \le T] \to 0$ as $N \to \infty$, and as a consequence $\lim_{N\to\infty}\sup_{0\le t\le T\wedge T^{(N,n)}} ||X^n(t)||^p_{\Gamma,p} = \sup_{0\le t\le T} ||X^n(t)||^p_{\Gamma,p}$ a.s.. Thus, by Fatou's Lemma,

$$\sup_n \mathbb{E}\left[\sup_{0\le t\le T} ||X^n(t)||^p_{\Gamma,p}\right] \le \sup_n \liminf_{N\to\infty} g^{n,N}(T) \le C(T). \tag{62}$$

Using this bound and an almost identical calculation leads to

$$\sup_{\substack{0\le s\le t\le T\\ |t-s|\le\delta<1}} \mathbb{E}\left[||X^n(t)-X^n(s)||^p_{\Gamma,p}\right] \le C(p,c,K,T)\delta. \tag{63}$$

The estimates (62) and (63) combined with Theorem 8.6 of Chapter 3 of [EK86] show that each coordinate is tight in $C(\mathbb{R}_+,\mathbb{R})$. By a diagonalisation argument one can then find a weakly convergent subsequence in $C(\mathbb{R}_+,\mathbb{R}^S)$, where $\mathbb{R}^S$ is equipped with the product topology. Using the continuity of the coefficients one can show that all limit points solve (20), which completes the proof of existence. We remark that this argument does not imply any convergence on $C(\mathbb{R}_+, l^p_\Gamma)$, and thus (27) needs to be verified separately for the solution to the infinite dimensional lattice system. By (62) and Fatou's Lemma we obtain first $\sup_{0\le t\le T}\mathbb{E}\left[||X(t)||^p_{\Gamma,p}\right] < \infty$. From this (27) follows by a calculation analogous to that of (61). The a.s. continuity of the sample paths in the space l^p_Γ follows from this bound with a similar calculation as in (63). Pathwise uniqueness follows now with the same calculations as for the uniform bound if the Lipschitz conditions on the coefficients are assumed. Here we estimate the difference of two strong solutions with respect to the same noise using the Lipschitz conditions where we have previously used the growth conditions.

4.2 Proof of Corollary 2.4

For existence of solutions we merely have to verify (23).

$$\begin{aligned}
&\sum_{i\in S}\gamma_\lambda(i)\quad\Big|\sum_{|i-j|\le C_m} m_{ij}(x_i-x_j)+f_i(t,x_i)\Big|^p\\
\le\;& (C(M,C_m)+1)^{p-1}\sum_{i\in S}\gamma_\lambda(i)\Big(\sum_{|i-j|\le C_m} m_{ij}|x_i-x_j|^p + c^p(1+x_i)^p\Big)\\
\le\;& C(M,C_m,p,c,\lambda)\sum_{i\in S}\gamma_\lambda(i)\Big(\sum_{|i-j|\le C_m}(m_{ij}+m_{ji}\frac{\gamma_\lambda(j)}{\gamma_\lambda(i)}) + c^p\Big)|x_i|^p.
\end{aligned}$$

Here, we have first used Jensen's Inequality and the growth condition (25). Then we use that $\gamma_\lambda(\cdot)$ is summable over S. Finally, we note that the term in parentheses is bounded by a constant since for $|i-j| \le C_m$, $\gamma_\lambda(j)/\gamma_\lambda(i) \le e^{\lambda C(C_m,d)}$.

For uniqueness we have to verify (24), which works in an analogous way, using the Lipschitz condition (26) instead of the growth condition (25).

4.3 Auxiliary lemmas

We start with stating a number of technical lemmas, proofs of which can be found in the appendix. Lemma 4.1 estimates spatial and temporal differences of the heat kernels $\bar{p}^n$, as well as of the differences of $\bar{p}^n$ and p. Lemma 4.2 provides an estimate for the heat kernels $\bar{p}^n$ and p integrated against the weight function γ_λ. In order to show tightness of the approximations we need a compactness criterion on $L^p_{\gamma_\lambda}(\mathbb{R}^d)$, which is stated in Lemma 4.3. This is an adaptation of the Frechet-Kolmogorov Theorem to our setting.

Lemma 4.1 *We have the following properties of $\bar{p}^n$ and the heat kernel p :*

(i) $\int_{\mathbb{R}^d} \bar{p}^n(t,x,y)dy = \int_{\mathbb{R}^d} p(t,x,y)dy = 1$ *for all* $x \in \mathbb{R}^d, t \ge 0$.

(ii) $\mathcal{F}\bar{p}^n(t,\cdot)(\xi) \equiv \int_{\mathbb{R}^d} \bar{p}^n(t,y)e^{i\xi\cdot y}dy = \exp(-n^2 t\sum_{i=1}^d (1-\cos\frac{\xi_i}{n}))$ *for all* $\xi \in \mathbb{R}^d$.

(iii) $\sup_{x,y\in\mathbb{R}^d} |\bar{p}^n(t,x,y)-p(t,x,y)| \to 0$ *as* $n\to\infty$ *for each* $t>0$.
$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\bar{p}^n(t,x,y)-p(t,x,y)|\,dy \to 0$ *as* $n\to\infty$ *for each* $t>0$.
$\int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n(s,x,y)-p(s,x,y)|\,dy\right)^\alpha ds \to 0$ *as* $n\to\infty$ *for any* $\alpha>0$.

(iv) $\int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n(s+h,x,y)-\bar{p}^n(s,x,y)|\,dy\right)^\alpha ds \to 0$ *uniformly in n for any* $\alpha > 0$ *as* $h\to 0$. *The analogous result holds for p.*

(v) $\sup_{||x'||\le\delta} \int_{\mathbb{R}^d} |\bar{p}^n(t,x+x',y)-\bar{p}^n(t,x,y)|\,dy \to 0$ *as* $\delta\to 0$, *for almost all x and all* $t>0$. *Similarly, as* $\delta \to 0$,
$\sup_{x\in\mathbb{R}^d}\sup_{||x'||\le\delta} \int_{\mathbb{R}^d} |p(s,x+x',y)-p(s,x,y)|\,dy \to 0$.

Lemma 4.2 *Let* $\gamma_\lambda(x) = e^{-\lambda||x||}$, *and* $\lambda\in\mathbb{R}$. *Then there exists a constant* $C(\delta,\lambda)\to 1$ *as* $\delta\to 0$ *such that*

$$\sup_{x\in\mathbb{R}^d}\sup_{||y||\le\delta} \frac{\gamma_\lambda(x-y)}{\gamma_\lambda(x)} < C(\delta,\lambda). \tag{64}$$

Also, for all $T \ge 0$, *there exists a constant* $C(T,\lambda)$ *independent of n such that for all* $x\in\mathbb{R}^d$ *and* $0\le t\le T$,

$$\int_{\mathbb{R}^d} \bar{p}^n(t,x,y)\gamma_\lambda(y)dy \le C(T,\lambda)\gamma_\lambda(x), \tag{65}$$

and likewise

$$\int_{\mathbb{R}^d} p(t,x,y)\gamma_\lambda(y)dy \le C(T,\lambda)\gamma_\lambda(x). \tag{66}$$

Lemma 4.3 *A set C_K in $L^p(\mathbb{R}^d)$ is relatively compact if and only if the following conditions hold,*

- *(i)* $\sup_{f\in C_K}\int_{\mathbb{R}^d}|f(x)|^p dx<\infty$,
- *(ii)* $\lim_{y\to 0}\int_{\mathbb{R}^d}|f(x+y)-f(x)|^p dx=0$ *uniformly for all* $f\in C_K$,
- *(iii)* $\lim_{\alpha\to\infty}\int_{\mathbb{R}^d\setminus B_\alpha}|f(x)|^p dx=0$ *uniformly for all* $f\in C_K$, *where B_α is the ball with radius α.*

A set C_K in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ is relatively compact if the above conditions hold for Lebesgue measure replaced by $\gamma_\lambda(x)dx$.

4.4 Proof of Theorems 2.5 and 2.6

By the assumptions on the coefficients in Theorem 2.6, Corollary 2.4 assures existence of (stochastically) strong solutions to the system (34) with initial conditions u_0. In this case, we set $\bar{f}^n(t,x,u)=f(t,\kappa_n(x),u)$ and likewise for $\bar{\sigma}^n$. By the continuity of f and σ we obtain pointwise convergence: For all $(t,x,u)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}$ as $n\to\infty$,

$$\bar{\sigma}^n(t,x,u)\to\sigma(t,x,u)\quad\text{and}\quad\bar{f}^n(t,x,u)\to f(t,x,u).\tag{67}$$

In order to obtain approximations driven by a given noise W to the SPDE of Theorem 2.5 we exploit the continuity of f and σ and define $\tilde{f}^n$ and $\tilde{\sigma}^n$ which converge pointwise as in (67), and satisfy in addition to the growth condition (10) the Lipschitz condition

$$|\tilde{f}^n(t,x,u)-\tilde{f}^n(t,x,v)|+|\tilde{\sigma}^n(t,x,u)-\tilde{\sigma}^n(t,x,u)|\le C(n)|u-v|$$

for all $t\in\mathbb{R}_+,x\in\mathbb{R}^d$, and $u,v\in\mathbb{R}$. Corollary 2.4 now implies pathwise uniqueness and thus existence of strong solutions to (34) with initial condition u_0 and coefficients $\tilde{f}$ and $\tilde{\sigma}$, and so we define in this case $\bar{f}^n(t,x,u)=\tilde{f}(t,\kappa_n(x),u)$ and similarly $\bar{\sigma}^n$. Note that these functions also satisfy (67).

The proof of the theorems now proceeds as follows. Proposition 4.4 gives a uniform bound on u^n in the $||\cdot||_{\gamma_\lambda,p}$-norm. Proposition 4.5 estimates temporal and spatial differences of u^n in this norm. Using these results and Lemma 4.3, we prove compact containment in Proposition 4.6. Finally, we show tightness and identify the limit points -establishing the existence statement of Theorem 2.5- and prove the convergence result of Theorem 2.6. We proceed to show a uniform bound on the approximating solutions:

Proposition 4.4 *Assume the linear growth condition (10) on $\bar{f}^n$ and $\bar{\sigma}^n$ as well as $||k||_\infty\le K<\infty$. Then, for all γ_λ with $\lambda>0$, $p\ge 2$ and u_0 with $\mathbb{E}[||u_0||^p_{\gamma_\lambda,p}]<\infty$, there exists a constant $C_p(T)$ so that*

$$\sup_n\sup_{0\le t\le T}\mathbb{E}\left[||u^n(t,\cdot)||^p_{\gamma_\lambda,p}\right]\le C_p(T).\tag{68}$$

PROOF. Let $0\le t\le T$. We treat the three terms in (34) separately. Thus, by Jensen's Inequality, $g^n(t)\equiv\mathbb{E}\left[\int_{\mathbb{R}^d}|u^n(t,x)|^p\gamma_\lambda(x)dx\right]\le 3^p\left(A_1+A_2+A_3\right)$, where, by Lemma 4.1(i), Jensen's Inequality and Lemma 4.2,

$$\begin{aligned} A_1 &\equiv \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \bar{p}^n(t,x,y)u_0(y)dy|^p \gamma_\lambda(x)dx\right] \\ &\le \mathbb{E}\left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\bar{p}^n(t,x,y)\gamma_\lambda(x)dx\right)|u_0(y)|^p dy\right] \\ &\le C(T,\lambda)\int_{\mathbb{R}^d}\mathbb{E}[|u_0(y)|^p]\,\gamma_\lambda(y)dy < C(T,u_0,p,\lambda). \end{aligned}$$

For A_2 we use the growth condition (10) as well as $\gamma_\lambda \in L^1$ in addition to the same line of arguments:

$$\begin{aligned} A_2 &\equiv \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} \bar{p}^n(t-s,x,y)\bar{f}^n(s,y,u^n(s,y))dyds|^p \quad \gamma_\lambda(x)dx\right] \\ &\le C(T,c,p)\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\bar{p}^n(t-s,x,y)\gamma_\lambda(x)dx\right)(1+|u^n(s,y)|)^p dyds\right] \\ &\le C(T,c,p,\lambda)\left(1+\int_0^t g^n(s)ds\right). \end{aligned}$$

For A_3 we first apply Burkholder's Inequality and the growth condition, and then Jensen's Inequality as well as $||k||_\infty \le K$. The last inequality follows as in the calculations for A_2.

$$\begin{aligned} A_3 &\equiv \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} \bar{p}^n(t-s,x,y)\bar{\sigma}^n(s,y,u^n(s,y))W(dy,ds)|^p \gamma_\lambda(x)dx\right] \\ &\le C(c,p)\int_{\mathbb{R}^d}\mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\bar{p}^n(t-s,x,y)\bar{p}^n(t-s,x,z)k(y,z) \\ &\qquad\qquad (1+|u^n(s,y))|)(1+|u^n(s,z)|)dydzds\Big)^{\frac{p}{2}}\Big] \quad \gamma_\lambda(x)dx \\ &\le C(c,K,p)\int_{\mathbb{R}^d}\mathbb{E}\left[\left(\int_0^t\left(\int_{\mathbb{R}^d}\bar{p}^n(t-s,x,y)\cdot(1+|u^n(s,y)|)dy\right)^2 ds\right)^{\frac{p}{2}}\right]\gamma_\lambda(x)dx \\ &\le C(c,K,p)T^{\frac{p}{2}-1}\int_{\mathbb{R}^d}\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^d}\bar{p}^n(t-s,x,y)\cdot(1+|u^n(s,y)|)^p dyds\right]\gamma_\lambda(x)dx \\ &\le C(T,c,K,p,\lambda)(1+\int_0^t g^n(s)ds). \end{aligned}$$

Taken together, we obtain that there is a constant $C = C(T,c,K,p,\lambda,u_0)$ independent of n such that for all $t \le T$, $g^n(t) \le C(1+\int_0^t g^n(s)ds)$. But each g^n is bounded according to (27). Thus, $\sup_{0\le t\le T} g^n(t) \le Ce^{CT} =: C(T)$ by Gronwall's Lemma. □

Using this bound we can prove the following approximation of differences.

Proposition 4.5 *Assume the conditions of Proposition 4.4. Then the approximating solutions satisfy*

$$\lim_{\delta\to 0}\sup_n \sup_{0\le t\le T}\sup_{0\le h\le\delta}\mathbb{E}\left[||u^n(t+h,\cdot)-u^n(t,\cdot)||^p_{\gamma_\lambda,p}\right] = 0. \tag{69}$$

For the difference of spatial translations we obtain for all $0 \le t \le T$,

$$\lim_{\delta\to 0}\sup_n \mathbb{E}\left[\sup_{0\le||x'||\le\delta}||u^n(t,\cdot+x')-u^n(t,\cdot)||^p_{\gamma_\lambda,p}\right] = 0. \tag{70}$$

PROOF. In order to show (69) we use the decomposition (34) and split the integral into five parts. Abbreviate the difference $\bar{p}^n_h(t,x,y) \equiv \bar{p}^n(t+h,x,y) - \bar{p}^n(t,x,y)$, and observe that by Jensen's Inequality

$$\begin{aligned}
& \mathbb{E}\left[\int_{\mathbb{R}^d} |u^n(t+h,x) - u^n(t,x)|^p \gamma_\lambda(x)dx\right] \\
\leq\; & 5^p \Big(\mathbb{E}\left[\int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \bar{p}^n_h(t,x,y)u_0(y)dy|^p \gamma_\lambda(x)dx\right] \\
& + \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} \bar{p}^n_h(t-s,x,y)\bar{f}^n(s,y,u^n(s,y))dyds|^p \gamma_\lambda(x)dx\right] \\
& + \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(t+h-s,x,y)\bar{f}^n(s,y,u^n(s,y))dyds|^p \gamma_\lambda(x)dx\right] \\
& + \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} \bar{p}^n_h(t-s,x,y)\bar{\sigma}^n(s,y,u^n(s,y))W(dy,ds)|^p \gamma_\lambda(x)dx\right] \\
& + \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(t+h-s,x,y)\bar{\sigma}^n(s,y,u^n(s,y))W(dy,ds)|^p \gamma_\lambda(x)dx\right]\Big) \\
\equiv\; & 5^p \sum_{i=1}^5 B_i.
\end{aligned}$$

For bounding B_1 let us first assume that $\mathbb{E}[||\Delta u_0||^p_{\gamma_\lambda,p}] < \infty$. Then,

$$\begin{aligned}
B_1 &= \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(s,x,y)\Delta^n u_0(y)dyds|^p \gamma_\lambda(x)dx\right] \\
&\leq \sup_{x\in\mathbb{R}^d} \left(\int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(s,x,y)dyds\right)^{p-1} \\
&\quad \cdot \mathbb{E}\left[\int_{\mathbb{R}^d} \int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(s,x,y) \cdot |\Delta^n u_0(y)|^p dyds \gamma_\lambda(x)dx\right] \\
&= h^{p-1}\mathbb{E}\left[\int_t^{t+h} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \bar{p}^n(s,x,y)\gamma_\lambda(x)dx\right) |\Delta^n u_0(y)|^p dyds\right] \\
&\leq C(T+h)h^p \mathbb{E}\left[||\Delta^n u_0||^p_{\gamma_\lambda,p}\right] \leq C(T+h)h^p \mathbb{E}\left[||\Delta u_0||^p_{\gamma_\lambda,p}\right].
\end{aligned}$$

We have applied Jensen's Inequality twice before using Lemma 4.2. For general u_0 consider $u_0^\epsilon(x) \equiv \int_{\mathbb{R}^d} p(\epsilon, x-y)u_0(y)dy$. Observe that $\Delta p(\epsilon,x,y) = (\frac{1}{\epsilon^2}||x-y||^2 - \frac{d}{\epsilon})p(\epsilon,x,y)$. Thus, with arguments almost identical to those of Lemma 4.2 we obtain $\int_{\mathbb{R}^d} \Delta p(\epsilon,x,y)\gamma_\lambda(y)dy \leq C(\epsilon)\gamma_\lambda(x)$, as well as $\mathbb{E}[||\Delta u_0^\epsilon||^p_{\gamma_\lambda,p}] \leq C(\epsilon)\mathbb{E}[||u_0||^p_{\gamma_\lambda,p}] < \infty$. Now,

$$\begin{aligned}
B_1 \leq\; & \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} \left(\bar{p}^n(t+h,x,y) - \bar{p}^n(t,x,y)\right)(u_0 - u_0^\epsilon)dy|^p \gamma_\lambda(x)dx\right] \\
& + \mathbb{E}\left[\int_{\mathbb{R}^d} |\int_t^{t+h} \int_{\mathbb{R}^d} \bar{p}^n(s,x,y)\Delta^n u_0^\epsilon(y)dyds|^p \gamma_\lambda(x)dx\right].
\end{aligned}$$

By applying Lemma 4.2 we obtain that for all $\epsilon \geq 0$ bounded, a.s. $||u_0^\epsilon||^p_{\gamma_\lambda,p} \leq C||u_0||^p_{\gamma_\lambda,p} < \infty$. Thus, by first applying Lebesgue's Differentiation Theorem and then using the above bound and Lebesgue's Dominated Convergence Theorem we obtain $\mathbb{E}[||u_0^\epsilon - u_0||^p_{\gamma_\lambda,p}] \to 0$ as $\epsilon \to 0$. It follows that B_1 converges to 0 uniformly in n and $0 \leq t \leq T$ by first letting

$h \to 0$ and then $\epsilon \to 0$. Similarly to the previous calculations we obtain for B_2,

$$\begin{aligned}
B_2 &\leq \sup_{x\in\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| dyds \right)^{p-1} \\
&\qquad \cdot \mathbb{E}\left[\int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| |\bar{f}^n(s,y,u^n(s,y))|^p dyds \gamma_\lambda(x) dx \right] \\
&\leq \sup_{x\in\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| dyds \right)^{p-1} \\
&\qquad \cdot c^p 2C(T+h)T \sup_{0\leq t\leq T} \mathbb{E}\left[|| \quad 1+|u^n(t,y)| \quad ||^p_{\gamma_\lambda,p} \right] \to 0,
\end{aligned}$$

uniformly in n as $h \to 0$ according to Lemma 4.1 (iv) and Proposition 4.4. Likewise, we can bound

$$\begin{aligned}
B_3 &\leq h^{p-1} C(T+h) c^p \mathbb{E}\left[\int_t^{t+h} \int_{\mathbb{R}^d} (1+|u^n(s,y))|)^p \gamma_\lambda(y) dyds \right] \\
&\leq h^p C(T+h) c^p \sup_{0\leq t\leq T} \mathbb{E}\left[|| \quad 1+|u^n(t,y)| \quad ||^p_{\gamma_\lambda,p} \right] \to 0,
\end{aligned}$$

as in the term B_2. For B_4 we use in addition Burkholder's Inequality as well as the boundedness of k.

$$\begin{aligned}
B_4 &\leq C(p,c)\mathbb{E}\Big[\int_{\mathbb{R}^d} (\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y) \bar{p}_h^n(t-s,x,z)| k(y,z) \\
&\qquad \cdot (1+|u^n(s,y)|)(1+|u^n(s,z)|) dydzds)^{\frac{p}{2}} \gamma_\lambda(x) dx \Big] \\
&\leq C(p,c,K) T^{\frac{p}{2}-1} \mathbb{E}\Big[\int_{\mathbb{R}^d} \int_0^t (\int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| \\
&\qquad \cdot (1+|u^n(s,y)|) dy)^p ds \gamma_\lambda(x) dx \Big] \\
&\leq C(p,c,K,T) \mathbb{E}\Big[\int_{\mathbb{R}^d} \int_0^t \sup_{x\in\mathbb{R}^d} (\int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| dy)^{p-1} \\
&\qquad \cdot (\int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| (1+|u^n(s,y)|)^p dy) ds \gamma_\lambda(x) dx \Big] \\
&\leq C(p,c,K,T) \left(\int_0^t \sup_{x\in\mathbb{R}^d} (\int_{\mathbb{R}^d} |\bar{p}_h^n(t-s,x,y)| dy)^{p-1} ds \right) \sup_{0\leq t\leq T} \mathbb{E}\left[||1+|u^n(s,\cdot)|||^p_{\gamma_\lambda,p} \right].
\end{aligned}$$

We have used Jensen's Inequality first for the time, and then for the spatial integral. In the last equality we applied Lemma 4.2. The quantity now converges to zero by Lemma 4.1 (iv) and Proposition 4.4. The calculation for B_5 follows closely that for B_4, and we obtain

$$\begin{aligned}
B_5 &\leq C(p,c) h^{\frac{p}{2}-1} \mathbb{E}\left[\int_{\mathbb{R}^d} \int_t^{t+h} \int_{\mathbb{R}^d} |\bar{p}^n(t-s,x,y)| \cdot (1+|u^n(s,y)|)^p dy \right] ds \gamma_\lambda(x) dx \\
&\leq C(T,p,c) h^{\frac{p}{2}} \sup_{0\leq t\leq T} \mathbb{E}\left[|| \quad 1+|u^n(s,\cdot)| \quad ||^p_{\gamma_\lambda,p} \right],
\end{aligned}$$

which converges to zero as $h \to 0$ uniformly in n, due to Proposition 4.4.

For proving (70), define pathwise $x'_{n,\delta}(t) \in \mathbb{R}^d$ such that $||x'_{n,\delta}(t)|| \leq \delta \leq 1$ as well as $\sup_{||x'|| \leq \delta} ||u^n(t, \cdot + x') - u^n(t, \cdot)||^p_{\gamma_\lambda, p} = ||u^n(t, \cdot + x'_{n,\delta}(t)) - u^n(t, \cdot)||^p_{\gamma_\lambda, p}$. Since the shift operator is continuous on $L^p_{\gamma_\lambda}$, such a $x'_{n,\delta}(t)$ does exist. Set $\bar{p}^n_{x'}(t, x, y) = \bar{p}^n(t, x + x', y) - \bar{p}^n(t, x, y)$. According to (34) we have to bound the following terms,

$$\begin{aligned}
& \mathbb{E}\left[||u^n(t, \cdot + x'_{n,\delta}(t)) - u^n(t, \cdot)||^p_{\gamma_\lambda, p}\right] \\
\leq \; & 3^p \,(\, \mathbb{E}\left[|| \int_{\mathbb{R}^d} \bar{p}^n_{x'_{n,\delta}(t)}(t, x, y) u_0(y) dy ||^p_{\gamma_\lambda, p} \right] \\
& + \mathbb{E}\left[|| \int_0^t \int_{\mathbb{R}^d} \bar{p}^n_{x'_{n,\delta}(t-s)}(t-s, x, y) \bar{f}^n(s, y, u^n(s, y)) dy ds ||^p_{\gamma_\lambda, p} \right] \\
& + \mathbb{E}\left[|| \int_0^t \int_{\mathbb{R}^d} \bar{p}^n_{x'_{n,\delta}(t-s)}(t-s, x, y) \bar{\sigma}^n(s, y, u^n(s, y)) W(dy, ds) ||^p_{\gamma_\lambda, p} \right] \,) \\
\equiv \; & 3^p \sum_{i=1}^{3} C_i .
\end{aligned}$$

We now estimate the first term similarly to B_1,

$$\begin{aligned}
C_1 \; \leq \; & \mathbb{E}\Big[\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n_{x'_{n,\delta}(t)}(t, x, y')| dy' \right)^{p-1} \\
& \qquad \left(\bar{p}^n(t, x + x'_{n,\delta}(t), y) + \bar{p}^n(t, x, y) \right) \gamma_\lambda(x) dx) \quad |u_0(y)|^p dy \Big] \\
\leq \; & \sup_{||x'|| \leq \delta} \sup_{y \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n_{x'}(t, x, y')| dy' \right)^{p-1} \bar{p}^n(t, x, y) \frac{\gamma_\lambda(x)}{\gamma_\lambda(y)} dx \right) \\
& \qquad \cdot \left(\sup_{||x'|| \leq \delta} \sup_{x \in \mathbb{R}^d} \frac{\gamma_\lambda(x + x')}{\gamma_\lambda(x)} + 1 \right) \mathbb{E}\left[||u_0(y)||^p_{\gamma_\lambda, p} \right] \\
\leq \; & C(u_0) \int_{\mathbb{R}^d} \sup_{||x'|| \leq \delta} \left(\int_{\mathbb{R}^d} |\bar{p}^n_{x'}(t, x'', y'')| dy'' \right)^{p-1} \bar{p}^n(t, x'') \gamma_{-\lambda}(x'') dx''
\end{aligned} \tag{71}$$

In the second inequality we have used that $||x'_{n,\delta}(t)|| \leq \delta$, together with a shift of variable and (64) of Lemma 4.2 for the first term in the sum. In the third inequality we have estimated $\frac{\gamma_\lambda(x)}{\gamma_\lambda(y)} \leq C \gamma_{-\lambda}(x - \kappa_n(y))$ with (64). We have then performed the variable shifts $x'' = x - \kappa_n(y)$ as well as $y'' = y' - \kappa_n(y)$, and exploited the shift invariance of p^n (see (33)). For fixed n the supremum converges to zero as $\delta \to 0$ for almost all x'' due to Lemma 4.1(v). Since it is bounded by 2 and $\bar{p}^n(t, x'') \gamma_{-\lambda}(x'')$ is integrable by Lemma 4.2 the result follows for fixed n and any $t > 0$ by Lebesgue's Dominated Convergence Theorem.

Similarly, using the growth conditions on $\bar{f}^n$ and $\bar{\sigma}^n$ as well as Burkholder's inequality for the stochastic integral, we obtain that C_2 and C_3 are bounded by

$$\begin{aligned}
& C(T) \sup_n \sup_{0 \leq t \leq T} \mathbb{E}\left[||1 + |u^n(t, \cdot)|||^p_{\gamma_\lambda, p} \right] \\
& \cdot \int_0^t \sup_{||x'|| \leq \delta} \sup_{y \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} |\bar{p}^n_{x'}(t-s, x, y')| dy')^{p-1} \bar{p}^n(t-s, x, y) \frac{\gamma_\lambda(x)}{\gamma_\lambda(y)} dx \right) ds,
\end{aligned} \tag{72}$$

where the expectation is bounded according to Proposition 4.4. Thus, with the same arguments as for (71) plus an additional application of Lebesgue's Dominated Convergence Theorem for the time integral convergence follows for each n fixed as $\delta \to 0$.

To obtain convergence uniformly in n we note that the arguments in (71) and (72) are true, uniformly in n, if p replaces $\bar{p}^n$. Furthermore, when $|\bar{p}^n - p|$ replaces the spatial differences $|\bar{p}^n_{x'}|$ in the C_i we obtain convergence to zero as $n \to \infty$. For example, the stochastic integral is bounded by

$$C \int_0^t \sup_{x \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n(t-s,x,y) - p(t-s,x,y)| dy \right)^{p-1} ds \cdot \sup_n \sup_{t \leq T} \mathbb{E}\left[||1 + |u^n(t,\cdot)|||^p_{\gamma_\lambda, p} \right],$$

which converges to zero according to Lemma 4.1(iii) and Proposition 4.4. Inserting $p(\cdot, x, y)$ and $p(\cdot, x+x', y)$ and using (64) as well as a 3ϵ argument now implies convergence uniformly in n. □

We obtain the compact containment condition of the approximating sequence of solutions.

Proposition 4.6 *For each $t \geq 0$, and $\epsilon > 0$ there exists a compact set C_K in the space $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ such that for all n,*

$$\mathbb{P}\left[u^n(t,\cdot) \in C_K\right] \geq 1 - \epsilon. \tag{73}$$

PROOF. We start by showing that for each $\epsilon > 0$,

$$\lim_{\delta \to 0} \sup_n \mathbb{P}\left[\sup_{||x'|| < \delta} \int_{\mathbb{R}^d} |u^n(t, x+x') - u^n(t,x)|^p \gamma_\lambda(x) dx > \epsilon \right] = 0. \tag{74}$$

Using Markov's Inequality, the convergence is implied by (70) of Proposition 4.5. We will also need to show that for all $\epsilon > 0$

$$\lim_{\alpha \to \infty} \sup_n \mathbb{P}\left[\int_{\mathbb{R}^d \setminus B_\alpha} |u^n(t,x)|^p \gamma_\lambda(x) dx > \epsilon \right] = 0. \tag{75}$$

We define an auxiliary function

$$\gamma_\lambda^{(\alpha)}(x) \equiv \begin{cases} \gamma_\lambda(x) & \text{for } ||x|| > \alpha, \\ e^{-\lambda\alpha} & \text{for } ||x|| \leq \alpha. \end{cases} \tag{76}$$

which, as an immediate consequence of Lemma 4.2, also satisfies (65). Thus, we obtain as in the proof of Proposition 4.4,

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{R}^d} |u^n(t,x)|^p \gamma_\lambda^{(\alpha)}(x) dx \right] \\ \leq \;& C(T) \left(\int_{\mathbb{R}^d} (1 + \mathbb{E}[|u_0(x)|^p]) \gamma_\lambda^{(\alpha)}(x) dx + \int_0^t \mathbb{E}\left[\int_{\mathbb{R}^d} |u^n(s,x)|^p \gamma_\lambda^{(\alpha)}(x) dx \right] ds \right). \end{aligned} \tag{77}$$

Since the first term is independent of n and converges to zero as $\alpha \to \infty$ by Lebesgue's Dominated Convergence Theorem, we obtain uniform convergence of (77) to zero by Gronwall's Inequality. But $1_{\mathbb{R}^d \setminus B_\alpha} \gamma_\lambda \leq \gamma_\lambda^{(\alpha)}$, and so (75) follows by Markov's Inequality.

Now, by (74) and (75) we can for any $\epsilon > 0$ and $k \in \mathbb{N}$ choose δ_k and α_k such that

$$\sup_n \mathbb{P}\left[\sup_{||x'|| < \delta_k} \int_{\mathbb{R}^d} |u^n(t, x+x') - u^n(t,x)|^p \gamma_\lambda(x) dx > \frac{1}{k} \right] \leq \frac{\epsilon}{3} 2^{-k},$$

$$\sup_n \mathbb{P}\left[\int_{\mathbb{R}^d \setminus C_{\alpha_k}} |u^n(t,x)|^p \gamma_\lambda(x) dx > \frac{1}{k} \right] \leq \frac{\epsilon}{3} 2^{-k}.$$

Also choose N such that $\mathbb{P}\left[||u^n(t,\cdot)||^p_{\gamma_\lambda,p} > N\right] \leq \frac{\epsilon}{3}$, and define the sets

$$
\begin{aligned}
C_K^1 &\equiv \{u^n| \quad ||u^n(t,\cdot)||^p_{\gamma_\lambda,p} \leq N\}, \\
C_K^2 &\equiv \bigcap_{k=1}^{\infty} \{u^n| \sup_{||x'||<\delta_k} \int_{\mathbb{R}^d} |u^n(t,x+x') - u^n(t,x)|^p \gamma_\lambda(x)dx \leq \frac{1}{k}\}, \\
C_K^3 &\equiv \bigcap_{k=1}^{\infty} \{u^n| \int_{\mathbb{R}^d \setminus C_{\alpha_k}} |u^n(t,x)|^p \gamma_\lambda(x)dx \leq \frac{1}{k}\}, \\
C_K &\equiv C_K^1 \cap C_K^2 \cap C_K^3.
\end{aligned}
$$

By Lemma 4.3 C_K is a compact set in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$, and by the above definitions, we finally conclude that $\inf_n \mathbb{P}\left[u^n(t,\cdot) \in C_K\right] \geq 1 - \frac{\epsilon}{3}(1 + 2\sum_{k=1}^{\infty} 2^{-k}) = 1 - \epsilon$. □

For the proof of Theorem 2.6 we require another Lemma (see Lemma 4.4 of [Gyo98b]).

Lemma 4.7 *Let E be a Polish space equipped with its Borel σ-algebra. A sequence of E-valued random elements u^n converges in probability if and only if for every pair of subsequences u^l and u^m there exists a subsequence $v^k \equiv (u^{l(k)}, u^{m(k)})$ converging weakly to a random element v supported on the diagonal $\{(u,u') \in E \times E \big| u = u'\}$.*

PROOF OF THEOREM 2.5 and THEOREM 2.6.
Taking together the tightness condition for each $t \geq 0$, that has been shown in Proposition 4.6, and the estimation of the differences in time given by (69) of Proposition 4.5, we obtain tightness of u^n in $D(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$ according to Theorem 8.6 of Chapter 3 in [EK86]. Since all u^n are continuous in time (Theorem 2.3), they are relatively compact in $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$. This implies that we can find a subsequence which converges weakly on $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$ to a process u.

By Skorohod's Representation Theorem we can find another probability space $\tilde{\Omega}$, and on it a further subsequence, $\tilde{u}^n$, as well as a noise $\tilde{W}$ equivalent in law to u^n and W, so that $\tilde{u}^n$ converges almost surely to $\tilde{u}$ in $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$. We now show that, by taking a further subsequence if necessary, the right hand side of (34) converges a.s. for all $t \geq 0$ in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ to the appropriate expressions for the limit process $\tilde{u}$. This implies that $\tilde{u}$ satisfies (9) and is thus a solution to the heat equation with colored noise as in Definition 2.1.

Following the calculations for B_1 in the proof of Proposition 4.5, we obtain for any $t \leq T$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(\bar{p}^n(t-s,x,y) - p(t-s,x,y)\right) u_0(y)dy \right)^p \gamma_\lambda(x)dx \\
\leq\ & C(T) \left(\int_{\mathbb{R}^d} \left(\bar{p}^n(t-s,x,y) - p(t-s,x,y)\right) dy \right)^{p-1} \int_{\mathbb{R}^d} |u_0(y)|^p \gamma_\lambda(y)dy.
\end{aligned}
$$

Here, the first term converges to zero as $n \to \infty$ by Lemma 4.1(iii), and the second integral is bounded a.s. by assumption. We consider next

$$
\begin{aligned}
\int_{\mathbb{R}^d} \Big(\int_0^t \int_{\mathbb{R}^d} &\big(\bar{p}^n(t-s,x,y)\bar{\sigma}^n(s,y,\tilde{u}^n(s,y)) \qquad (78) \\
&- p(t-s,x,y)\sigma(s,y,\tilde{u}(s,y))\big)\tilde{W}(dy,ds)\Big)^p \gamma_\lambda(x)dx \leq D_1 + D_2.
\end{aligned}
$$

Here, we split the integrand into a term, D_1, involving the differences of the convolution kernels, and one, D_2, involving the differences of the solutions. With a calculation analogous to that of B_4 in the proof of Proposition 4.5 we obtain that $\mathbb{E}[D_1]$ is bounded by

$$\begin{aligned} & C(p,K,T)||\mathbb{E}\Big[\int_0^t \big(\int_{\mathbb{R}^d} |\bar{p}^n(t-s,x,y) - p(t-s,x,y)|\bar{\sigma}^n(s,y,\tilde{u}^n(s,y))dy\big)^p ds\Big]||^1_{\gamma_\lambda,1} \\ \leq\ & C(p,K,T,c)\int_0^t \sup_{x\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\bar{p}^n(t-s,x,y') - p(t-s,x,y')|dy'\right)^{p-1} ds \\ & \qquad\qquad\qquad\qquad\qquad\qquad \cdot \sup_{0\leq t\leq T}\mathbb{E}\left[||1+\tilde{u}^n(s,y))||^p_{\gamma_\lambda,p}\right], \end{aligned}$$

which converges to zero by Proposition 4.4 and Lemma 4.1(iii). By choosing a further subsequence if necessary, a.s. convergence follows. To estimate the second difference, D_2, we define $V_T \equiv \sup_n \sup_{s\leq T}||\tilde{u}^n(s,y)||^p_{\gamma_\lambda,p}$, which is bounded a.s. because of the convergence of the $\tilde{u}^n$ in $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$. As a consequence, we have $\lim_{N\to\infty}\mathbb{P}[V_T > N] = 0$. Since, by Markov's Inequality, $\mathbb{P}[D_2 > \epsilon] \leq \mathbb{P}[V_T > N] + \frac{1}{\epsilon}\mathbb{E}[D_2 \mid V_T \leq N]$, it suffices to show for any fixed N, $\lim_{n\to\infty}\mathbb{E}[D_2 \mid V_T \leq N] = 0$. With a similar calculation as for D_1, we bound this expectation by

$$C(p,K,T)\int_0^t \mathbb{E}\left[||\bar{\sigma}^n(s,y,\tilde{u}^n(s,y)) - \sigma(s,y,\tilde{u}(s,y))||^p_{\gamma_\lambda,p}\Big| V_T \leq N\right] ds. \tag{79}$$

By taking a further subsequence if necessary, $\tilde{u}^n(s,y) \to \tilde{u}(s,y)$ a.s. for a.a. y and all s. Thus, the continuity of $\bar{\sigma}^n$ and σ and (67) imply that $\bar{\sigma}^n(s,y,\tilde{u}^n(s,y)) \to \sigma(s,y,\tilde{u}(s,y))$ a.s. for a.a. y and all s. But by (10),

$$|\bar{\sigma}^n(s,y,\tilde{u}^n(s,y)) - \sigma(s,y,\tilde{u}(s,y))| \leq c\,(2 + |\tilde{u}^n(s,y)| + |\tilde{u}(s,y)|)\,. \tag{80}$$

Since $\tilde{u}^n(s,\cdot) \to u(s,\cdot)$ in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ a.s. for each s, the right hand side and so also the left hand side of (80) is a uniformly integrable in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ a.s. for each s. Therefore, the norm converges a.s. for each s. The conditioning on the event $\{V_T \leq N\}$ and Lebesgue's Dominated Convergence Theorem, now imply that (79) converges to zero. Thus, $D_2 \to 0$ in probability as $n \to \infty$, and a further subsequence converges a.s..

Taking the two estimates together, we have proven that, for a further subsequence if necessary, (78) converges to zero a.s. for $t \in [0,T]$ and so, since T is arbitrary, for all $t \geq 0$. We can perform essentially the same, albeit slightly simpler, calculation to show that for the chosen subsequence $\tilde{u}^n$,

$$\int_{\mathbb{R}^d}\Big(\int_0^t\int_{\mathbb{R}^d}\bar{p}^n(t-s,x,y)\bar{f}^n(s,y,\tilde{u}^n(s,y)) - p(t-s,x,y)f(s,y,\tilde{u}(s,y))dyds\Big)^p\gamma_\lambda(x)dx \to 0,$$

as $n \to \infty$ a.s. for all $t \geq 0$. Thus, $\tilde{u}$ is a solution to (1), which by Proposition (4.4) and Fatou's Lemma also satisfies (35). Since $(\tilde{u}, \tilde{W})$ have the same distribution as (u, W) we have shown the existence result of Theorem 2.5.

It remains to complete the proof for Theorem 2.6. The weak convergence result follows immediately from weak uniqueness of the limit. For convergence in probability when pathwise uniqueness of the limit is known we consider a pair of subsequences u^l and u^m. By the tightness on $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$ we can find further subsequences $u^{l(k)}$ and $u^{m(k)}$ that converge weakly on $C(\mathbb{R}_+, L^p_{\gamma_\lambda}(\mathbb{R}^d))$. The above calculation shows that both limit points satisfy the heat equation with respect to W. Thus, the pathwise uniqueness implies that they are equal a.s., and so on the diagonal of $E \times E$. Theorem 2.6 follows now by Lemma 4.7. $\square$

5 Continuity of solutions

5.1 Proof of Theorem 2.7

We first show that under the assumptions of Theorem 2.7, (35) implies (37). Set $\lambda_p \equiv \frac{\lambda}{p}$, and bound $\mathbb{E}[\sup_{t\le T}||u(t,\cdot)||^p_{\infty,\lambda_p}]$ by

$$\begin{aligned}
\sum_{i=1}^{3} S_i \quad &\equiv \quad \mathbb{E}\left[\sup_{t\le T}||\int_{\mathbb{R}^d} p(t,\cdot,y)u_0(y)dy||^p_{\infty,\lambda_p}\right] \\
&\quad +\mathbb{E}\left[\sup_{t\le T}||\int_0^t\int_{\mathbb{R}^d} p(t-s,\cdot,y)f(s,y,u(s,y))dyds||^p_{\infty,\lambda_p}\right] \\
&\quad +\mathbb{E}\left[\sup_{t\le T}||\int_0^t\int_{\mathbb{R}^d} p(t-s,\cdot,y)\sigma(s,y,u(s,y))W(dy,ds)||^p_{\infty,\lambda_p}\right].
\end{aligned}$$

Note that Lemma 4.2 and $\mathbb{E}[||u_0||^p_{\infty,\gamma_{\lambda_p}}] < \infty$ bound S_1. To bound S_2 and S_3 we use a factorisation method first introduced by DaPrato, Kwapień and Zabczyk [DKZ87], which is based on the fact that for $0 < \alpha < 1$,

$$\int_s^t (t-u)^{\alpha-1}(u-s)^{-\alpha}du = \frac{\pi}{\sin(\pi\alpha)}.$$

To demonstrate the argument we focus on the stochastic integral S_3 and define

$$\begin{aligned}
J^{\alpha-1}u(t,x) \quad &= \quad \frac{\sin(\pi\alpha)}{\pi}\int_0^t\int_{\mathbb{R}^d}(t-s)^{\alpha-1}p(t-s,x,y)u(s,y)dyds, \\
J_\alpha u(t,x) \quad &= \quad \int_0^t\int_{\mathbb{R}^d}(t-s)^{-\alpha}p(t-s,x,y)\sigma(s,y,u(s,y))W(dy,ds),
\end{aligned}$$

so that with the stochastic Fubini Theorem (see Theorem 2.6 of [Wal86]),

$$J^{\alpha-1}J_\alpha u(t,x) = \int_0^t\int_{\mathbb{R}^d} p(t-s,x,y)\sigma(s,y,u(s,y))W(dy,ds).$$

Thus, S_3 is equal to

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}||J^{\alpha-1}J_\alpha u(t,\cdot)||^p_{\infty,\lambda_p}\right] \\
\le \quad & C\mathbb{E}\left[\sup_{t\le T}||\int_0^t(t-s)^{\alpha-1}\left(\int_{\mathbb{R}^d} p(t-s,\cdot,y)\gamma_{-\frac{\lambda}{2}}(y)\cdot|J_\alpha u(s,y)|^{\frac{p}{2}}\gamma_{\frac{\lambda}{2}}(y)dy\right)^{\frac{2}{p}} ds||^p_{\infty,\lambda_p}\right] \\
\le \quad & C\mathbb{E}\left[\sup_{t\le T}||\int_0^t(t-s)^{\alpha-1}\left(\int_{\mathbb{R}^d} p(t-s,\cdot,y)^2\gamma_{-\lambda}(y)dy\right)^{\frac{1}{p}}\cdot||J_\alpha u(s,\cdot)||_{\gamma_\lambda,p}ds||^p_{\infty,\lambda_p}\right]
\end{aligned}$$

$$
\begin{aligned}
&\le C(T)\mathbb{E}\left[\sup_{t\le T}\left(\int_0^t (t-s)^{\alpha-1-\frac{d}{2p}}||\int_{\mathbb{R}^d} p(t-s,\cdot,y)\gamma_{-\lambda}(y)dy||_{\infty,\lambda}^{\frac{1}{p}}\cdot ||J_\alpha u(s,\cdot)||_{\gamma_\lambda,p}ds\right)^p\right]\\
&\le C(T)\left(\int_0^T s^{(\alpha-1-\frac{d}{2p})\frac{p}{p-1}}ds\right)^{p-1}\mathbb{E}\left[\int_0^T ||J_\alpha u(s,\cdot)||^p_{\gamma_\lambda,p}ds\right]. \qquad (81)
\end{aligned}
$$

We have first used Jensen's and the Cauchy-Schwartz Inequality. We have then used (66) of Lemma 4.2 and (84) of the proof of Lemma 4.1 to see that $\int_{\mathbb{R}^d} p(t-s,x,y)^2\gamma_{-\lambda}(y)dy \le C(T)(t-s)^{-\frac{d}{2}}\gamma_{-\lambda}(x)$. Lemma 4.2 and a subsequent application of Hölder's Inequality completes the calculation. Now, for all $t \le T$, by Burkholder's Inequality and $||k||_\infty \le K$,

$$
\begin{aligned}
&\mathbb{E}\left[||J_\alpha u(t,\cdot)||^p_{\gamma_\lambda,p}\right] \qquad (82)\\
\le\;& C(K)\mathbb{E}\left[||\int_0^t (t-s)^{-2\alpha}\left(\int_{\mathbb{R}^d} p(t-s,\cdot,y)\sigma(s,y,u(s,y))dy\right)^2 ds||^{\frac{p}{2}}_{\gamma_\lambda,\frac{p}{2}}\right]\\
\le\;& C(K)\left(\int_0^T s^{-2\alpha}ds\right)^{\frac{p}{2}-1}\cdot\mathbb{E}\left[\int_0^t (t-s)^{-2\alpha}||\sigma(s,\cdot,u(s,\cdot))||^p_{\gamma_\lambda,p}ds\right]\\
\le\;& C(K,c)\left(\int_0^T s^{-2\alpha}ds\right)^{\frac{p}{2}}\cdot\left(1+\sup_{t\le T}\mathbb{E}\left[||u(t,\cdot))||^p_{\gamma_\lambda,p}\right]\right).
\end{aligned}
$$

Thus, by (35), the term is bounded provided that $-2\alpha > -1$ and $(\alpha-1-\frac{d}{4p})\frac{p}{p-1} > -1$ (from (81)), which can be fulfilled if and only if $d < p-2$. The term S_2 works similarly, implying the same conditions on α.

In order to see that $u(t,\cdot) \in C_{\gamma_{\lambda_p}}$ for any $0 \le t \le T$, consider a.s. $|u(t,x)-u(t,x+x')|$ for $||x'|| < 1$. The difference can again be bounded by three terms according to (9). The term involving the initial condition converges as $||x'|| \to 0$ due to Lemma 4.1(v) and Lemma 4.2. We focus again on the stochastic integral, which may be approximated analogously to (81), and is thus bounded by

$$
\begin{aligned}
C(T)\left(\int_0^T s^{(\alpha-1-\frac{d}{2p})\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}&\left(\int_0^T ||J_\alpha u(s,\cdot)||^p_{\gamma_\lambda,p}\right.\\
&\left.\cdot\left(\int_{\mathbb{R}^d}|p(t-s,x+x',y)-p(t-s,x,y)|\gamma_{-\lambda}(y)dy\right)\quad ds\right)^{\frac{1}{p}}.
\end{aligned}
$$

By Lemma 4.2 the integral of the heat kernel differences is bounded by $C(T,x)$. Since $J_\alpha u \in L^p([0,T],L^p_{\gamma_\lambda})$, a.s. it is sufficient by Lebesgue's Dominated Convergence Theorem to note that the integral of the heat kernel differences converges to zero for each $s \le t$. This is again a consequence of Lebesgue's Theorem combined with Taylor's Theorem and Lemma 4.2.

We end the proof by showing that $u \in C([0,T],C_{\gamma_{\lambda_p}})$ for any $T > 0$, and thus in $C(\mathbb{R}_+,C_{\gamma_{\lambda_p}})$. Once again, we use the definition in (9) and show continuity of the stochastic integral. We note that the drift term can be treated similarly and that the first term converges according to Lemma 4.1(iv) and Lemma 4.2. Hence, we bound a.s.

$$||J^{\alpha-1}J_\alpha u(t+h,\cdot) - J^{\alpha-1}J_\alpha u(t,\cdot)||_{\infty,\lambda_p}$$

$$\begin{aligned} &\leq \quad \Big|\Big|\int_t^{t+h}\int_{\mathbb{R}^d}(t+h-s)^{\alpha-1}p(t+h-s,\cdot,y)J_\alpha u(s,y)dyds\Big|\Big|_{\infty,\lambda_p} \\ &\quad +\Big|\Big|\int_0^t\int_{\mathbb{R}^d}(t-s)^{\alpha-1}\,|p(t+h-s,\cdot,y)-p(t-s,\cdot,y)|\cdot|J_\alpha u(s,y)|dyds\Big|\Big|_{\infty,\lambda_p} \\ &\leq \quad C(T)\Big(\Big(\int_0^h s^{(\alpha-1-\frac{d}{2p})\frac{p}{p-1}}ds\Big)^{\frac{p-1}{p}}\int_0^T ||J_\alpha u(s,y)||^p_{\gamma_\lambda,p}ds \\ &\quad +\Big(\int_0^t \Big|\Big|\int_{\mathbb{R}^d}|p(t+h-s,\cdot,y)-p(t-s,\cdot,y)|\gamma_{-\lambda}(y)dy\Big|\Big|_{\infty,\lambda}\cdot||J_\alpha u(s,\cdot)||^p_{\gamma_\lambda,p}ds\Big)^{\frac{1}{p}}\Big). \end{aligned}$$

We have used that $(t+h-s)^{\alpha-1} \leq (t-s)^{\alpha-1}$ for $s \in [0,t]$. Arguments analogous to those in (81) explain the second inequality. We observe for the first term that $\int_0^T ||J_\alpha u(s,y)||^p_{\gamma_\lambda,p}ds$ is bounded a.s., for the second that the inner integral is bounded by Lemma 4.2 and converges pointwise for each $s > 0$. Thus, both terms converge to zero by Lebesgue's Dominated Convergence Theorem as $h \to 0$.

6 Appendix

6.1 Proof of Lemma 4.1

We use the random walk Y^n as in the definition (32) of $\bar{p}^n$. Property (i) merely states that the transition probabilities $\bar{p}^n$ and p sum (respectively integrate) to one.

The Fourier transform in (ii) is given by

$$\mathcal{F}p^n(t,0,\kappa_n(\cdot))(\xi) = \mathbb{E}[e^{i\xi\cdot Y_t^n}] = \prod_{i=1}^d \mathbb{E}[e^{i\xi_i Y_{i,t}^n}] = \prod_{i=1}^d \Phi_y(n^2t, \frac{\xi_i}{n}),$$

where $\Phi_y(t,r) = \exp(-t(1-\cos r))$ is the characteristic function for a one dimensional simple random walk y at time t (see for example [Fel51]), and (ii) follows.
In order to show (iii) we use a result in [DEF$^+$02]. For $d = 1$ and $\eta > 0$ there exist constants $K_0(\eta)$ and $C(K_0)$ such that

$$\sup_{z,\tilde{z}\in\frac{1}{n}\mathbb{Z}} |p_1^n(t,z,\tilde{z}) - p_1(t,z,\tilde{z})| \leq \eta t^{-\frac{1}{2}} + C(K_0)\frac{1}{n^2}t^{-\frac{3}{2}} \tag{83}$$

for all $n > \frac{K_0}{\pi}t^{-\frac{1}{2}}$. Also stated in [DEF$^+$02] is that there exists a universal constant c_1, independent of n and t, such that

$$\sup_{x,y\in\mathbb{R}^d} \max(\bar{p}_1^n(t,x,y), p_1(t,x,y))t^{\frac{1}{2}} \leq c_1. \tag{84}$$

We observe via Taylor's Theorem that there exists another universal constant c_2 such that for all $x, y \in \mathbb{R}^d$

$$|p_1(t,x,y) - p_1(t,\tilde{x},\tilde{y})| \leq c_2(|x-\tilde{x}| + |y-\tilde{y}|)t^{-1}. \tag{85}$$

Therefore, combining (83) and (85) implies for all $n > \frac{K_0}{\pi} t^{-\frac{1}{2}}$,

$$\begin{aligned}\sup_{x,y\in\mathbb{R}^d} |\bar{p}_1^n(t,x,y) - p_1(t,x,y)| &\leq \sup_{x,y\in\mathbb{R}^d} \left(|\bar{p}_1^n(t,x,y) - p_1(t,\kappa_n(x),\kappa_n(y))|\right. \\ &\qquad \left. + |p_1(t,\kappa_n(x),\kappa_n(y)) - p_1(t,x,y)|\right) \\ &\leq \eta t^{-\frac{1}{2}} + C(K_0)\frac{1}{n^2} t^{-\frac{3}{2}} + \frac{2c_2}{n} t^{-1}. \qquad (86)\end{aligned}$$

In d dimensions we have $\bar{p}_d^n(t,x,y) = \prod_{i=1}^d \bar{p}_1^n(t,x_i,y_i)$ and the analogous form for p_d. Thus, with (84) and (86) we finally obtain for $n > \frac{K_0}{\pi} t^{-\frac{1}{2}}$ that

$$\begin{aligned}&\sup_{x,y\in\mathbb{R}^d} |\bar{p}_d^n(t,x,y) - p_d(t,x,y)| \\ \leq\ & \sum_{i=1}^d |\bar{p}_1^n(t,x_i,y_i) - p_1(t,x_i,y_i)| \cdot \prod_{j<i} \bar{p}_1^n(t,x_j,y_j) \prod_{k>i} p_1(t,x_k,y_k) \\ \leq\ & d\left(c_1 t^{-\frac{1}{2}}\right)^{d-1} \left(\eta t^{-\frac{1}{2}} + C(K_0)\frac{1}{n^2} t^{-\frac{3}{2}} + \frac{2c_2}{n} t^{-1}\right). \qquad (87)\end{aligned}$$

Since $\eta > 0$ may be chosen as small as we like the first part of (iii) now follows. For the two remaining statements we first note that since $\kappa_n(x) - \kappa_n(y) = \kappa_n(\kappa_n(x) - y)$, we can deduce that $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\bar{p}^n(t,x,y) - p(t,x,y)|\, dy$ is bounded by

$$\int_{\mathbb{R}^d} |\bar{p}^n(t,y) - p(t,y)|\, dy + \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |p(t,\kappa_n(x),y) - p(t,x,y)|\, dy.$$

Convergence of the second term is deferred to (v). For the first term we use that, for all $\epsilon > 0$ and $T \geq 0$, there exists a compact set $C_{\epsilon,T}$ independent of n so that

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}^d\setminus C_{\epsilon,T}} (\bar{p}^n(t,y) + p(t,y))\, dy < \epsilon. \qquad (88)$$

This is a consequence of the tightness of the associated measures in $D(\mathbb{R}_+,\mathbb{R}^d)$, following from the classical functional Central Limit Theorem (see for example [EK86]). Thus, on for any $t \leq T$,

$$\int_{\mathbb{R}^d} |\bar{p}^n(t,y) - p(t,y)|\, dy \leq \int_{C_{\epsilon,T}} |\bar{p}^n(t,y) - p(t,y)|\, dy + 2\epsilon.$$

Hence, the first part of (iii), (84) and Lebesgue's Dominated Convergence Theorem imply that the integral on the right hand side converges to zero. The second part of (iii) now follows by letting $\epsilon \to 0$. The last part of (iii) is obtained by another application of Lebesgue's Dominated Convergence Theorem upon noting that, by (i), the spatial integrals are bounded by 2.

For property (iv) consider first

$$\begin{aligned}&\int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p(s+h,x,y) - p(s,x,y)|\, dy\right)^\alpha ds \\ \leq\ & \int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\int_s^{s+h} \Delta p(\bar{s},x,y) d\bar{s}\right| dy\right)^\alpha ds \\ \leq\ & \int_0^t \left(h \sup_{\bar{s}\in[s,s+h]} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\Delta p(\bar{s},x,y)| dy\right)^\alpha ds,\end{aligned}$$

where the term in brackets is bounded by 2^α by (i) and converges to zero as $h \to 0$ for all $s > 0$, since $\sup_{\bar{s}\in[s,s+h]} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta p(\bar{s}, x, y)dy < C(\frac{1}{s^2} + \frac{1}{s})$. Lebesgue's Dominated Convergence Theorem now implies statement (iv) for p. For $\bar{p}^n$ we use a decomposition as in (87) as well as property (i) to obtain that

$$\begin{aligned}
&\int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\bar{p}^n(s+h, x, y) - \bar{p}^n(s, x, y)|\, dy \right)^\alpha ds \\
\leq \quad & C(\alpha) \sum_{i=1}^d \int_0^t \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}} |\bar{p}_1^n(s+h, x_i, y_i) - \bar{p}_1^n(s, x_i, y_i)| dy_i \right)^\alpha ds.
\end{aligned}$$

But by the definition of $\bar{p}^n$ the term in absolute values equals

$$\int_s^{s+h} \frac{n^2}{2} (p_1^n(\,\bar{s}, \kappa_n(x_i), \kappa_n(y_i) + \frac{1}{n}) + p_1^n(\bar{s}, \kappa_n(x_i), \kappa_n(y_i) - \frac{1}{n}) - 2p_1^n(\bar{s}, \kappa_n(x_i), \kappa_n(y_i))\,)\, d\bar{s}.$$

Thus, by property (i) the integral is bounded by $C(\alpha)td(2hn^2)^\alpha$, which proves (iv) for any given $\bar{p}^n$. That the convergence is uniform in n follows now by a 3ϵ argument from the statement for p and the appropriate convergence shown in (iii).

For the first statement of (v) we merely note that, for all x in the interior of the intervals I^n (see the definition of κ_n), the spatial differences of $\bar{p}^n$ are identically zero for δ small enough. But the boundary of these intervals form a null set. To show (v) for p we use arguments analogous to those in(88). For all $\epsilon, \delta > 0$, find a compact set C such that, for all $||x'|| \leq \delta$ and $t \leq T$, $\int_{\mathbb{R}^d \setminus C} p(t, x', y)dy < \epsilon$. Thus, $\sup_{||x'||\leq\delta} \int_C |p(t, x', y) - p(t, 0, y)|\, dy \to 0$,as $\delta \to 0$. Because of shift invariance in x this establishes the convergence result for p.

6.2 Proof of Lemma 4.2

For property (64) note that

$$\frac{\gamma_\lambda(x-y)}{\gamma_\lambda(x)} = e^{-\lambda(||x-y||-||x||)} \leq e^{|\lambda|\cdot||y||}. \tag{89}$$

Let Y^n be a simple random walk as in the definition (32) of $\bar{p}^n$. Using the norm equivalence on $\mathbb{R}^d$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \bar{p}^n(t, x, y) e^{-\lambda(||y||-||x||)} dy \\
\leq \quad & \sum_{y^n\in\frac{1}{n}\mathbb{Z}^d} \mathbb{P}^{\kappa_n(x)}[Y_t^n = y^n] n^d \int_{I_{y^n}^n} e^{C|\lambda| \sum_{i=1}^d (|y_i|-|x_i|)} dy \\
\leq \quad & \prod_{i=1}^d \left(\sum_{y_i^n\in\frac{1}{n}\mathbb{Z}} \mathbb{P}^{\kappa_n(x_i)}[Y_{i,t}^i = y_i^n] n \int_{I_{y_i^n}^n} C e^{C|\lambda|(y_i^n - \kappa_n(x_i))} dy_i \right) \\
= \quad & \left(C \sum_{\tilde{y}^n\in\frac{1}{n}\mathbb{Z}} \mathbb{P}^0[Y_{1,t}^n = \tilde{y}^n] e^{C|\lambda|\tilde{y}^n} \right)^d = \left(Ce^{\left(\frac{1}{2}\left(e^{\frac{C\lambda}{n}} + e^{-\frac{C\lambda}{n}}\right) - 1\right) n^2 t} \right)^d \\
= \quad & \left(Ce^{\sum_{k=1}^\infty \frac{C\lambda^{2k} n^{(-2k+2)}}{(2k)!} t} \right)^d \leq \left(Ce^{\sum_{k=1}^\infty \frac{C\lambda^{2k}}{(2k)!} t} \right)^d \leq \left(Ce^{e^{C\lambda} T} \right)^d.
\end{aligned}$$

In the first inequality we have used the symmetry in x as well as (64) and subsequently Lemma 4.1(ii). By similar arguments (66) follows, see [Stu02] p. 75 for detail.

6.3 Proof of Lemma 4.3

The first part of the theorem is just the Frechet-Kolmogorov Theorem (see IV.8.21 of [DS58]). Observe now that $f_n \to f$ in $L^p_{\gamma_\lambda}(\mathbb{R}^d)$ if and only if $f_n\gamma_\lambda^{\frac{1}{p}} \to f\gamma_\lambda^{\frac{1}{p}}$ in $L^p(\mathbb{R}^d)$. Thus, conditions (i) and (iii) transfer immediately to their analogues on $L^p_{\gamma_\lambda}(\mathbb{R}^d)$. For condition (ii) consider

$$\begin{aligned} & \int_{\mathbb{R}^d} |f(x+y)\gamma_\lambda^{\frac{1}{p}}(x+y) - f(x)\gamma_\lambda^{\frac{1}{p}}(x)|^p dx \\ \leq\ & 2^p \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p \gamma_\lambda(x) dx + \int_{\mathbb{R}^d} |f(x+y)|^p \gamma_\lambda^{\frac{1}{p}}(x+y) - \gamma_\lambda^{\frac{1}{p}}(x)|^p dx \right) \\ \leq\ & 2^p \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p \gamma_\lambda(x) dx + \sup_{x \in \mathbb{R}^d} |1 - \left(\frac{\gamma_\lambda(x-y)}{\gamma_\lambda(x)} \right)^{\frac{1}{p}} |^p \int_{\mathbb{R}^d} |f(x)|^p \gamma_\lambda(x) dx \right). \end{aligned}$$

Provided condition (i) is fulfilled, the second integral converges to zero uniformly for $f \in C_K$ due to (64) of Lemma 4.2. Uniform convergence of the first integral, which corresponds to condition (ii) with the measure $\gamma_\lambda(x)dx$, is thus sufficient for compactness.

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