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Attractors of non invertible maps

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Abstract. For mappings $f : S^1 \times \mathbb{R}^n \to S^1 \times \mathbb{R}^n \quad (n \geq 2)$ of the form $f(t, x) = (\Theta t, \lambda x + v(t))$, where $\Theta \in \mathbb{Z}, \Theta \geq 2, \lambda \in (0,1), v \in C^1(S^1, \mathbb{R}^n)$ we consider the open subset $\mathcal{S}_{n,\Theta,\lambda}$ of $C^1(S^1, \mathbb{R}^n)$ which consist of all v for which the restriction of f to its attractor is injective. It is shown that for $\lambda < \min(\frac{1}{2}, \Theta^{-2/(n-1)})$ this set $\mathcal{S}_{n,\Theta,\lambda}$ is dense in $C^1(S^1, \mathbb{R}^n)$ and that for each odd n it is not dense provided $\lambda \geq 64\Theta^{-2/(n-1)}$.

1. Results

Let $S^2 = \mathbb{R}$ (modulo 1) be the unit circle. The Cartesian product $V = S^1 \times \mathbb{R}^n$ can be regarded as an open $(n+1)$–dimensional solid torus. We always assume $n \geq 2$. For an integer $\Theta > 1$, a real $\lambda \in (0,1)$ and a C^r mapping $v : S^1 \to \mathbb{R}^n \quad (r \geq 0)$ we consider the mapping

$$f : V \to V$$

which is given by

$$f(t,x) = (\Theta t, \lambda x + v(t)) \quad (t \in S^1, x \in \mathbb{R}^n).$$

If we want to point out that f is determined by Θ, λ, v we shall write $f = f_{\Theta,\lambda,v}$. Since $0 < \lambda < 1$ this mapping has a compact attractor which will be denoted by Λ_f or by Λ. To visualize this attractor we consider a compact solid torus $V_\rho = S^1 \times \mathbb{D}^n_\rho$, where $\mathbb{D}^n_\rho$ is the compact ball in $\mathbb{R}^n$ with radius p and centre o and where

$$\rho \geq \tfrac{1}{1-\lambda} \sup_{t \in S^1}(v(t)),$$

or, equivalently

$$v(t) + \lambda\rho \leq \rho \quad (t \in S^1).$$

Then $f(V_\rho) \subset V_\rho$ and

$$\Lambda = \bigcap_{i=0}^{\infty} f^i(V_\rho).$$

The image $f(V_\rho)$ is obtained by stretching V_ρ in the direction of S^1, contracting it in the direction of $\mathbb{D}^n_\rho$ and wrapping it around in V_ρ exactly Θ times without folds. (Self intersection of $f(V_\rho)$ are not excluded.) If f is injective in V_ρ, i.e. if $f(V_\rho)$ has no self interpretitions, then the images $f^i(V_\rho) \quad (i = 1, 2, \dots)$ form a nested sequence of solid tori, and $f^i(V_\rho)$ for i large is thin and runs around in V_ρ exactly Θ^i times. As mentioned above Λ is the intersection of these tori, and this intersection is the well known solenoid. So Λ has a simple structure in this case. If the restriction of f to Λ is known to be injective, then, by compactness of Λ, there is a neighbourhood of Λ on which f is injective, and it can be shown that Λ is a solenoid in this case too.

In this paper we look for conditions under which a mapping $f_{\Theta,\lambda,v}$ is injective on its attractor. The result concerns the C^1 case and can roughly speaking, be stated as follows. If for a fixed Θ the number λ is sufficiently small, then generically for all v the restriction of $f_{\Theta,\lambda,v}$ to its attractor is injective.

Theorem 1. *For n, Θ, λ as above let $\mathcal{S}_{n,\Theta,\lambda}$ be the set of all $v \in C^1(S^1, \mathbb{R}^n)$ for which $f_{\Theta,\lambda,v}$ is injective on its attractor. If $\lambda < \frac{1}{2}$ and $\lambda < \Theta^{-2/(n-1)}$ then $\mathcal{S}_{n,\Theta,\lambda}$ is open and dense in $C^1(S^1, \mathbb{R}^n)$.*

As easily senn the set

$$\{\lambda \in (0,1) | \mathcal{S}_{n,\Theta,\lambda} \text{ dense in } C^1(S^1, \mathbb{R}^n)\}$$

is an open interval $(0, \lambda_n, (\Theta))$, and the theorem is equivalent to the inequality

$$\lambda_n(\Theta) \geq \min(\frac{1}{2}, \Theta^{-2/(n-1)}).$$

Theorem 2. *For each odd dimension $n \geq 3$*

$$\lambda_n(\theta) \leq 32\theta^{-2/(n-1)}$$

holds for all $\theta \geq 2$.

Remark 1. If $f_{\theta,\lambda,v}$ is injective on its attractor Λ, then for each $t \in S^1$ the section $(\{t\} \times \mathbb{R}^n) \cap \Lambda$ is a Cantor set, and by standard methods it is easy to see that the Hausdorff dimension of this Cantor set is $\log\theta / \log\frac{1}{\lambda}$. The Hausdorff dimension of a subset of $\mathbb{R}^n$ can not exceed n, and therefore $\lambda > \theta^{-1/n}$ implies that for any $v \in C^0(S^1, \mathbb{R}^n)$ the restriction of $f_{\theta,\lambda,v}$ to its attractor can not be injective, and therefore

$$\lambda_n(\theta) \leq \theta^{-1/n}.$$

Remark 2. Let $\varepsilon > 0$ be fixed. If n and θ are sufficiently large (the lower bound for θ depends on n), then the factor 32 in Theorem 2 can be replaced by $8 + \varepsilon$. This fact can easily be derived from our proof below. Modifying this proof (the set $\mathfrak{P}_r$ e.g.) a further reduction of this factor is possible.

2. Proof of Theorem 1

Let n, Θ, λ be fixed such that $\lambda < \frac{1}{2}, \lambda < \Theta^{-2/(n-1)}$.

Since our attractors are compact it is not hard to see that for any $r \geq 0$ the set of all $v \in C^r(S^1, \mathbb{D}^n_\alpha)$ with injective restriction to its attractor is open in $C^r(S^1, \mathbb{D}^n_\alpha)$. This holds for $0 < \alpha \leq \infty$, where $\mathbb{D}^n_\infty = \mathbb{R}^n$. Therefore $\mathcal{S}_{n,\theta,\lambda}$ is open, and we have only to prove that $\mathcal{S}_{n,\Theta,\lambda}$ is dense in $C^1(S^1, \mathbb{R}^n)$. Moreover it is sufficient to prove, as we shall do below, that for an arbitrary α $(0 < \alpha < \infty)$ the intersection $\mathcal{S}_{n,\Theta,\lambda} \cap C^1(S^1, \mathbb{D}^n_\alpha)$ is dense in $C^1(S^1, \mathbb{D}^n_\alpha)$. Therefore for the rest of the proof in addition to n, θ and λ the positive real numbers α and $\rho > \frac{\alpha}{1-\lambda}$ will be fixed. Then $f_{\theta,\lambda,v}(V_\rho) \subset \operatorname{Int} V_\rho$ for any $v \in C^r(S^1, \mathbb{D}^n_\alpha)$, and

$$\Lambda_{f_{\theta,\lambda,v}} = \bigcap_{i=0}^{\infty} f^i_{\theta,\lambda,v}(V_\rho).$$

If m is a proper multiple of θ, i.e. $m = \theta m'$ with $m' \in \mathbb{Z}$, $m' > 1$, we consider the points $t_i = \frac{i}{m}$ and arcs $I_i = [t_{i-1}, t_i]$ in S^1 $(i \in \mathbb{Z})$. The family of these arcs will be denoted by $\mathcal{P}_m$ and called a Markov partition of S^1. Of course $t_i = t_{i'}, I_i = I_{i'}$ if and only if $i \equiv i'$ (modulo m), so that $\mathcal{P}_m$ consists of the m arcs $I_1, \ldots, I_m$. Moreover

$$\theta t_i = t_{\theta i}, \quad \theta I = [t_{\theta(i-1)}, t_{\theta i}] = I_{\theta(i-1)+1} \cup \cdots \cup I_{\theta i}.$$

If $\mathcal{P}_m$ is fixed, then each $v \in C^0(S^1, \mathbb{R}^n)$ which is linear on each arc of $\mathcal{P}_m$ is determined by the m points $v_i = v(t_i)$ in $\mathbb{R}^n$ $(i = 1, \ldots, n)$, and we identify this piecewise linear v with the point $(v_1, \ldots, v_m)$ in $\mathbb{R}^{mn}$. So $\mathbb{R}^{mn}$ is embedded in $C^0(S^1, \mathbb{R}^n)$ and those $v \in \mathbb{R}^{mn}$ which belong to $C^0(S^1, \mathbb{D}^n_\alpha)$ are just the points in $(\mathbb{D}^n_\alpha)^m$. This set $(\mathbb{D}^n_\alpha)$, when regarded as a subset of $C^0(S^1, \mathbb{R}^n)$, will be denoted by $\mathcal{V}_m$.

For $v \in C^0(S^1, \mathbb{R}^n)$ we define

$$|v|_0 = \sup_{t \in S^1} |v(t)|$$

and for $v \in C^1(S^1, \mathbb{R}^n)$

$$|v|_1 = \max(|v|_0, |\dot{v}|_0),$$

where $\dot{v} = \frac{dv}{dt} : S^1 \to \mathbb{R}^n$. If $\mathcal{P}_m$ is a Markov partition and $v \in C^0(S^1, \mathbb{R}^n)$ is C^1 on each arc I_i of $\mathcal{P}_m$, then we define

$$|v|_1 = \max(|v|_0, \sup \{|\dot{v}(t)| \, |t \in \bigcup_{I_i \in \mathcal{P}_m} \text{Int } I_i\}).$$

Obviously for $v \in C^1(S^1, \mathbb{R}^m)$ the two definitions of $|v|_1$ coincide. If $v \in \mathcal{V}_m$ then, as easily seen,

$$|v|_1 \leq 2m|v|_0,$$

and $|\ |_0$ and $|\ |_1$ define the same topology in $\mathcal{V}_m$ which coincides with the natural topology of $(\mathbb{D}^n_\alpha)^m$.

Lemma 1. *If $v_0 \in C^1(S^1, \mathbb{D}^n_\alpha)$ and $\varepsilon > 0$, then there is a positive integer m and a mapping $v_1 \in \mathcal{V}_m$ such that $|v_0 - v_1| < \varepsilon$.*

Lemma 2. *The set $\mathcal{S}_m = \{v \in \mathcal{V}_m | f_{\theta,\lambda,v}|_{\Lambda_{f_{\theta,\lambda,v}}}$ injective$\}$ is open and dense in $\mathcal{V}_m$.*

Lemma 3. *If $v_0 \in C^1(S^1, \mathbb{D}^n_\alpha), v_2 \in \mathcal{V}_m, |v_0 - v_2|_1 < \varepsilon$ for some $\varepsilon > 0$, then for each $\delta > 0$ there is a $v \in C^1(S^1, \mathbb{D}^n_\alpha)$ such that $|v - v_0|_1 < \varepsilon, |v - v_2|_0 < \delta$.*

These three lemmas easily imply Theorem 1: Let $v_0 \in C^1(S^1, \mathbb{D}^n_\alpha)$ and ε be given. We have to find a $v \in C^1(S^1, \mathbb{D}^n_\alpha)$ such that $|v - v_0|_1 < \varepsilon$ and the restriction of $f_{\theta,\lambda,v}$ to its attractor is injective.

By Lemma 1 we find a v_1 in some $\mathcal{V}_m$ such that $|v_0 - v_1| < \varepsilon/2$. Now we apply Lemma 2 to get a $v_2 \in \mathcal{S}_m$ such that $|v_1 - v_2|_0$ is so small that $|v_0 - v_2|_1 < \varepsilon$. Since the set of all $v \in C^0(S^1, D^n_\alpha)$ with an injective restriction $f_{\theta,\lambda,v}|_{\Lambda_{f_{\theta,\lambda,v}}}$ is open in $C^0(S^1, D^n_\alpha)$ we can apply Lemma 3 to find a $v \in C^1(S^1, D^n_\alpha)$ with the property required above.

Lemma 1 is almost obvious and its proof can be omitted. To prove Lemma 3 we merely have to smoothen the corners of v_2. Therefore it remains to prove Lemma 2.

Proof of Lemma 2. We choose a fixed Markov partition $\mathcal{P}_m$ of S^1 with arcs I_i and partitioning points t_i. If $k \geq 1$ in an integer let $\mathcal{V}_m(k)$ be the set of all $v \in \mathcal{V}_m$ for which the mapping $f_{\theta,\lambda,v}$ is injective on the neighbourhood $f^{k-1}_{\theta,\lambda,v}(V_\rho)$ of the attractor $\Lambda_{f_{\theta,\lambda,v}}$. As easily seen $\mathcal{V}_m(k)$ consists of all $v \in \mathcal{V}_m$ with the following property. If s_1, s_2 are points in S^1 such that $\theta^{k-1} \cdot s_1 \neq \theta^{k-1} \cdot s_2$ but $\theta^k \cdot s_1 = \theta^k \cdot s_2$, then

$f^k_{\theta,\lambda,v}(D(s_1)) \cap f^k_{\theta,\lambda,\mu}(D(s_2)) = \emptyset$, where $D(s_i)$ are the meridional disks $\{s_i\} \times \mathbb{D}^n_\rho$ of V_ρ. Obviously $\mathcal{V}_m(k) \subset \mathcal{V}_m(k+1)$, and

$$\mathcal{S}_m = \bigcup_{k=1}^{\infty} \mathcal{V}_m(k).$$

The complement $\mathcal{V}_m \backslash \mathcal{V}_m(k)$ will be denoted by $\mathcal{W}_m(k)$. In $\mathcal{V}_m = (\mathbb{D}^n_\alpha)^m$ we consider the Lebesgue measure which will be denoted by vol. Since a subset of $\mathcal{V}_m$ whose complement has measure 0 must be dense in $\mathcal{V}_m$, Lemma 2 is an immediate consequence of $\lambda < \theta^{-2/(m-1)}$ and the following lemma.

Lemma 2'. *There is a real γ such that*

$$\text{vol } \mathcal{W}_m(k) \leq \gamma \theta^{2k} \lambda^{k(n-1)} \qquad (k = 1, 2, \ldots).$$

By $\mathcal{J}_k$ we denote the set of all pairs (I, J) of arcs in S^1 such that $\theta^{k-1}I, \theta^{k-1}J$ are different arcs of our Markov partition $\mathcal{P}_m$ and $\theta^k I = \theta^k J$. The number of elements in $\mathcal{J}_k$ is bounded by

$$\#\mathcal{J}_k = \frac{m}{\theta} \cdot \theta^k \cdot \theta^{k-1}(\theta - 1) < m\theta^{2k}. \tag{1}$$

For $(I, J) \in \mathcal{J}_k$ we consider the sets

$$\mathcal{W}(I, J) = \{v \in \mathcal{V}_m | f^k_{\theta,\lambda,v}(I \times \mathbb{D}^n_\rho) \cap f^k_{\theta,\lambda,v}(J \times \mathbb{D}^n_\rho) \neq \emptyset\}.$$

This definition implies

$$\mathcal{W}_m(k) = \bigcup_{(I,J) \in \mathcal{J}_k} \mathcal{W}(I, J).$$

This last equation together with (1) reduces the proof of Lemma 2' to the proof of the following lemma.

Lemma 2". *There is a real γ which does not depend on k, I, J and for which the following inequality holds*

$$\text{vol } \mathcal{W}(I, J) \leq \gamma \lambda^{k(n-1)}.$$

Let (I, J) be a fixed pair in $\mathcal{J}_k$. We write $I = [s_1, s_2]$, $J = [s_3, s_4]$ and define for each $v \in \mathcal{V}_m$ four points $x_{v,j}$ $(1 \leq j \leq 1)$ in $\mathbb{D}^n_\rho$ by

$$f^k_{\theta,\lambda,v}(s_j, o) = (\theta^k s_j, x_{v,j}) \quad (1 \leq j \leq 4),$$

where o denotes the centre of $\mathbb{D}^n_\rho$. The four points $p_j = (\theta^k s_j, x_{v,j})$ are the end points of the two segments $f^k(I \times \{o\}), f^k(J \times \{o\})$ in the cylinder $Z = \theta^k I \times \mathbb{D}^n_\rho = \theta^k J \times \mathbb{D}^n_\rho$ (see Fig. 1). The image $f^k(I \times \mathbb{D}^n_\rho)$ is the union of all n–disks in Z which are parallel to the bottom of Z, whose centres lie on $f^k(I \times \{o\})$ and whose radius in $\rho\lambda^k$. The cylinder $f^k(J \times \mathbb{D}^n_\rho)$ is obtained similarly from $f^k(J \times \{o\})$. If, as above, the points in $\mathbb{R}^{mn}$ are identified with the elements of $C^0(S^1, \mathbb{R}^n)$ which are linear on each arc of $\mathcal{P}_m$, then the mapping $v \to (x_{v,1}, \ldots, x_{v,4})$ of $\mathcal{V}_m = (\mathbb{D}^n_\alpha)^m$ to $(\mathbb{D}^n_\rho)^4$ can be extended to a linear mapping $\varphi : \mathbb{R}^{mn} \to \mathbb{R}^{4n}$. This extension is characterized by

$$\varphi(v) = (x_1, \ldots, x_4)(v \in \mathbb{R}^{mn}, x_j \in \mathbb{R}^n),$$

$$x_j = \sum_{l=1}^{k} \lambda^{l-1} v(\theta^{k-l} s_j) \qquad (1 \le j \le 4).$$

Later in the proof of Lemma 2" we shall need the following lemma.

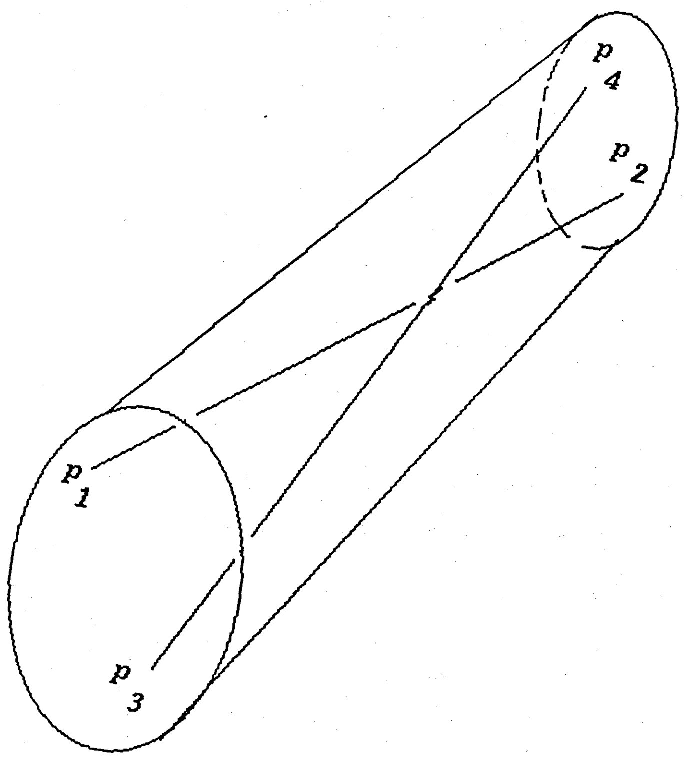

Fig. 1

Lemma 4. *There is a real γ_0 which depends on λ but not on k, I, J such that for any compact subset Q of $(\mathbb{D}^n_\rho)^4$ we have*

$$\text{vol}\,(\varphi^{-1}(Q) \cap (\mathbb{D}^n_\alpha)^m) \le \gamma_0 \text{ vol } Q,$$

where vol *denotes the Lebesgue measure in $\mathbb{R}^{mn}$ or in $\mathbb{R}^{4n}$, respectively.*

Proof of Lemma 4. Let $\delta = \lambda/(1-\lambda)$. (Here we apply $\lambda < \frac{1}{2}$ so that $\delta > 0$.) The lemma will be proved if we have found a $4n$–dimensional linear subspace L of $\mathbb{R}^{mn}$ such that $\varphi|_L : L \to \mathbb{R}^{4n}$ is regular with determinant at least δ^{4n}, where the determinant is defined with respect to the natural metrics in L and $\mathbb{R}^{4n}$.

To define L we consider the arcs $\theta^{k-1}I, \theta^{k-1}J$. These arcs belong to $\mathcal{P}_n$, and we can write

$$\theta^{k-1}I = [t_{i_1}, t_{i_2}], \quad \theta^{k-1}J = [t_{i_3}, t_{i_4}],$$

where $t_{i_1}, \ldots, t_{i_4}$ are partitioning points of $\mathcal{P}_m(1 \leq i_j \leq m)$. Since $\theta^{k-1}I \neq \theta^{k-1}J$ but $\theta^k I = \theta^k J$ we get actually four points t_{i_j}, i.e. no two of them coincide. Now let L be the space of all $v = (v_1, \ldots, v_m) \in \mathbb{R}^{mn}$ for which $v_i = o$ unless i is one of the four indices $i_1, \ldots, i_4$.

We can identify L with the tensor product $\mathbb{R}^4 \otimes \mathbb{R}^n$, where for $\mu = (\mu_1, \ldots, \mu_4) \in \mathbb{R}^4, x \in \mathbb{R}^n$ the product $\mu \otimes x$ is identified with $v = (v_1, \ldots, v_m)$ given by

$$v_i = \begin{cases} \mu_j x & \text{if } i = i_j \quad (1 \leq j \leq 4), \\ o & \text{otherwise.} \end{cases}$$

Moreover, each $\mu = (\mu_1, \ldots, \mu_4) \in \mathbb{R}^4$ will be identified with the function $\mu : S^1 \to \mathbb{R}$ which is linear on each arc of $\mathcal{P}_m$ and satisfies

$$\mu(t_i) = \begin{cases} \mu_j & \text{if } i = i_j \quad (1 \leq j \leq 4), \\ 0 & \text{otherwise} \end{cases}$$

for the end points of these arcs. Then, if $\xi : \mathbb{R}^4 \to \mathbb{R}^4$ denotes the maps given by

$$\xi(\mu) = (\nu_{\mu,1}, \ldots, \nu_{\mu,4}),$$

$$\nu_{\mu,j} = \sum_{l=1}^{k} \lambda^{l-1} \mu(\theta^{k-l} s_j) \quad (1 \leq j \leq 4),$$

we get

$$\varphi|_L = \xi \otimes id : L = \mathbb{R}^4 \otimes \mathbb{R}^n \to \mathbb{R}^{4n} = \mathbb{R}^4 \otimes \mathbb{R}^n,$$

where the equation on the right hand side is realized by

$$(\nu_1, \ldots, \nu_4) \otimes (x_1, \ldots, x_n) = (\nu_1 x_1, \ldots, \nu_4 x_1, \ldots, \nu_1 x_n, \ldots, \nu_4 x_n).$$

To prove det $\varphi|_L \geq \delta^{4n}$ it is sufficient to prove det $\xi \geq \delta^4$. To this end we consider the 16 points $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_4)$ in $\mathbb{R}^4$, where $|\varepsilon_j| = 1$. Their convex hull is a cube K with volume 2^4. The images $\xi(\varepsilon) = (\nu_{\varepsilon,1}, \ldots, \nu_{\varepsilon,4})$ are given by

$$\nu_{\varepsilon,j} = \sum_{l=1}^{k} \lambda^{l-1} \varepsilon(\theta^{k-l} s_j) = \varepsilon_j + \sum_{l=2}^{k} \lambda^{l-1} \varepsilon(\theta^{k-l} s_j) = \varepsilon_j + \sum_{l=1}^{k-1} \lambda^l \varepsilon(\theta^{k-l-1} s_j).$$

(Here we apply $\varepsilon(\theta^{k-1} s_j) = \varepsilon(t_{i_j}) = \varepsilon_j$.) So we have

$$\nu_{\varepsilon,j} = \varepsilon_j + \nu'_j$$

with

$$|\nu'_j| \leq \sum_{l=1}^{\infty} \lambda^l = \frac{\lambda}{1-\lambda} = 1 - \delta.$$

If F_ε denotes the part

$$F_\varepsilon = \{(\nu_1, \ldots, \nu_4) \in \mathbb{R}^4 | \nu_j \varepsilon_j \geq \delta\}$$

of $\mathbb{R}^4$, then $\xi(\varepsilon) \in F_\varepsilon$, and it is a simple geometric fact that the convex hull $\xi(K)$ of the 16 points $\xi(\varepsilon)$ contains the cube $K' = \{(\nu_1, \ldots, \nu_4) \in \mathbb{R}^4 | \, |\nu_j| \leq \delta\}$ whose volume is $(2\delta)^4$. (See Fig. 2 where the situation is illustrated in the 2–dimensional case.) Now det $\xi \geq \delta^4$ is implied by vol $(K) = 2^4$, vol $\xi(K) \geq (2\delta)^4$. ■

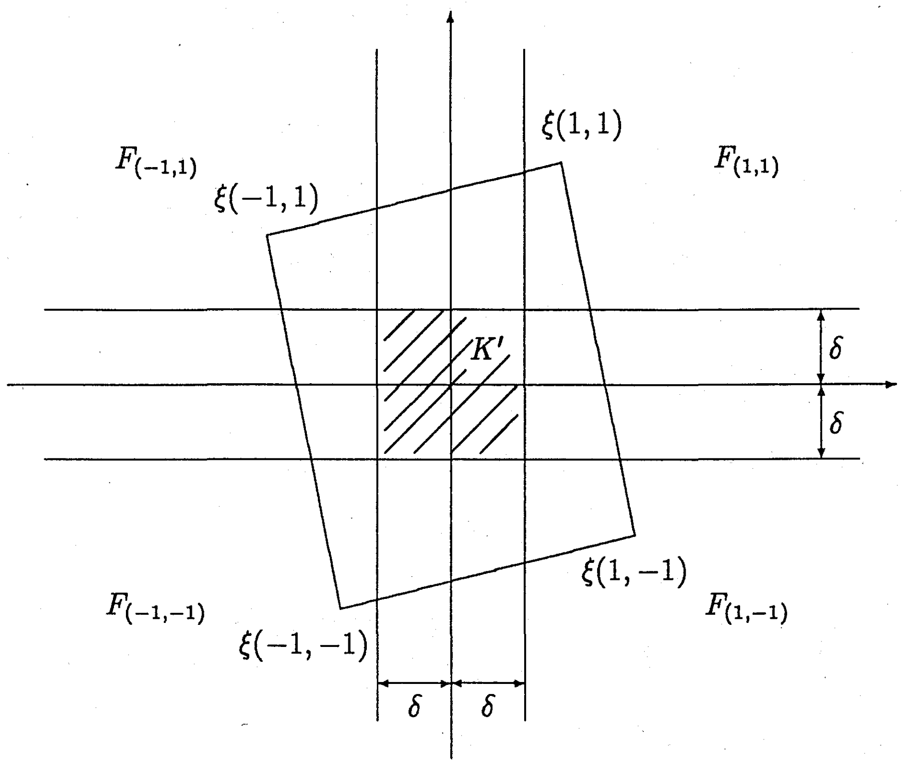

Fig. 2

We continue the proof of Lemma 2" with the definition of a function $d^* : \mathbb{R}^{4n} \to \mathbb{R}$:

$$d^*(x_1, \dots, x_4) = \inf_{0 \le \tau \le 1} |x_1 + \tau(x_2 - x_1) - (x_3 + \tau(x_4 - 3))| \qquad (x_j \in \mathbb{R}^n).$$

If $[t', t'']$ denotes the arc $\theta^k I = \theta^k J$ in S^1 and t is the point $t' + \tau(t'' - t')$ in $[t', t'']$, then $x_1 + \tau(x_2 - x_1), x_3 + \tau(x_4 - x_3)$ are the points at which the segments $f^k(I \times \{o\}), f^k(J \times \{o\})$ pierce the disk $\{t\} \times \mathbb{D}^n_\rho$. Therefore $d^*(x_1, \dots, x_4)$ may be regarded as the vertical distance between these segments. Since $f^k(I \times \mathbb{D}^n_\rho)$ is the $\rho\lambda^k$–neighbourhood of $f^k(I \times \{o\})$ with respect to this distance and the same holds for $f^k(J \times \mathbb{D}^n_\rho)$ and $f^k(J \times \{o\})$, the function d^* characterizes the set $\mathcal{W}(I, J)$ by

$$\begin{aligned} \mathcal{W}(I,J) &= \{v \in \mathcal{V}_m | d^*(x_{v,1}, \dots, x_{v,4}) \le 2\rho\lambda^k\} \\ &= \{v \in \mathcal{V}_m | d^*(\varphi(v)) \le 2\rho\lambda^k\}. \end{aligned}$$

So we get the inclusion

$$\mathcal{W}(I,J) \subset (\mathbb{D}^n_\alpha)^m \cap \varphi^{-1}(\{x \in \mathbb{R}^{4m} | d^*(x) \le 2\rho\lambda^k\}). \tag{2}$$

This inclusion suggests to look for the structure of the sets $\{x \in \mathbb{R}^{4n} | d^*(x) \le$

$\varepsilon\}$ ($\varepsilon > 0$ small), and we are led to introduce the further set

$$\{x \in \mathbb{R}^{4n} | d^*(x) = 0\}.$$

As easily seen this set is contained in

$$F = \{(x_1, \dots, x_4) \in \mathbb{R}^{4n} | x_1 - x_3, x_2 - x_4 \text{ linearly dependent in } \mathbb{R}^n\}.$$

It will turn out below that F has a simple shape (it is an $(n+1)$–dimensional cone over a smooth manifold), and we relate the sets $\{x \in \mathbb{R}^{4n} | d^*(x) \leq \varepsilon\}$ to F by showing that, for ε small, they lie close to F. Indeed, the following lemma proves

$$\{x \in \mathbb{R}^{4n} | d^*(x) \leq \varepsilon\} \subset N_{2\varepsilon}(F), \tag{3}$$

where $N_\varepsilon(F)$ denotes the closed ε–neighbourhood of F in $\mathbb{R}^{4n}$.

Lemma 5. *For* $(x_1, \dots, x_4) \in \mathbb{R}^{4n}$

$$\text{dist}\,((x_1, \dots, x_4), F) \leq 2d^*(x_1, \dots, x_4).$$

Proof of Lemma 5. Obviously we may assume $|x_1 - x_3| \leq |x_2 - x_4|, x_1 \neq x_3, x_2 \neq x_4$. First we consider the case $x_3 = x_4$. let the points $y, z \in \mathbb{R}^n$ be determined by (see Fig. 3)

(i) $(y, x_2, x_3, x_3) \in F$, i.e. y, x_2, x_3 collinear,
(ii) $y - x_1 \perp x_1 - x_2$, i.e. the scalar product $(y - x_1, x_1 - x_2)$ vanishes,
(iii) z, x_1, x_2 collinear
(iv) $z - x_3 \perp x_1 - x_2$, i.e. $(z - x_3, x_1 - x_2) = 0$.

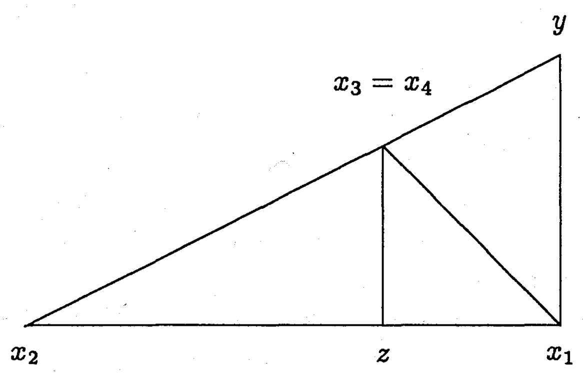

Fig. 3

Then $d^*(x_1, x_2, x_3, x_3) \geq |z - x_3|$, and by $|x_1 - x_3| \leq |x_2 - x_4|$ we have $|y - x_1| \leq 2|z - x_3|$. Therefore

$$\begin{aligned}
\text{dist}\,((x_1, x_2, x_3, x_3), F) &\leq |(x_1, x_2, x_3, x_3) - (y, x_2, x_3, x_3)| \\
&= |y - x_1| \\
&\leq 2|z - x_3| \\
&\leq 2d^*(x_1, x_2, x_3, x_3).
\end{aligned}$$

To prove the lemma in the general case we introduce the point $x^* = x_2 - (x_4 - x_3)$. Then

$$\begin{aligned} d^*(x_1, x^*, x_3, x_3) &= d^*(x_1, x_4, x_3, x_4), \\ \text{dist}\,((x_1, x^*, x_3, x_3), F) &= \text{dist}\,((x_1, x_2, x_3, x_4), F), \end{aligned}$$

and the general case is reduced to the special case considered above. ∎

Using (2) and (3) we get

$$\mathcal{W}(I, J) \subset (\mathbb{D}^n_\alpha)^m \cap \varphi^{-1}(N_{4\lambda^k}(F)). \tag{4}$$

The set $\varphi(\mathbb{D}^n_\alpha)^m$ is contained in the subset $(\mathbb{D}^n_\rho)^4$ of $\mathbb{R}^{4n}$, and we get the further inclusion

$$\mathcal{W}(I, J) \subset (\mathbb{D}^n_\alpha)^m \cap \varphi^{-1}((\mathbb{D}^n_\rho)^4 \cap N_{4\lambda^k}(F)).$$

We shall show that there is a real γ_1 such that for each $\varepsilon > 0$

$$\text{vol}\,(N_\varepsilon(F) \cap (\mathbb{D}^n_\rho)^4) \le \gamma_1 \varepsilon^{n-1}, \tag{5}$$

where vol denotes the Lebesgue measure in $\mathbb{R}^{4n}$. Since neither F nor $(\mathbb{D}^n_\rho)^4$ depends on k, I, J, the number γ_1 is also independent of k, I, J. This inequality (5) (with $\varepsilon = 4\lambda^k$) together with (4) and Lemma 4 (with $Q = N_{4\lambda^k}(F) \cap (\mathbb{D}^n_\rho)^4$) immediately implies Lemma 2" (with $\gamma = \gamma_0\gamma_1$) and therefore Lemma 2 and Theorem 1.

So we must prove (5). To this aim we describe the set F. If $\sigma : \mathbb{R}^{4n} \to \mathbb{R}^{2n}$ is the projection which is defined by

$$\sigma(x_1, \dots, x_4) = (x_1 - x_3, x_2 - x_4),$$

then $F = \sigma^{-1}(F_0)$, where

$$F_0 = \{(x_1, x_2) \in \mathbb{R}^{2n}|_{x_1, x_2} \text{ linearly dependent in } \mathbb{R}^n\}.$$

This set F_0 is a cone with vertex o, i.e. $x \in F_0$ and $\tau \in \mathbb{R}$ imply $\tau x \in F_0$. To find a basis of this cone we consider the neighbourhood $B = \mathbb{D}^n_1 \times \mathbb{D}^n_1$ of o in $\mathbb{R}^{2n}$. The boundary of B is

$$\partial B = (\partial\mathbb{D}^n_1 \times \mathbb{D}^n_1) \cup (\mathbb{D}^n_1 \times \partial\mathbb{D}^n_1) = (S^{n-1} \times \mathbb{D}^n_1) \cup (\mathbb{D}^1_n \times S^{n-1}),$$

where $S^{n-1} = \partial\mathbb{D}^n_1$ is the $(n-1)$–dimensional unit sphere. Each of the sets

$$\begin{aligned} F_0 \cap (S^{n-1} \times \mathbb{D}^n_1) &= \{(x, \tau x) | x \in S^{n-1}, -1 \le \tau \le 1\} \\ F_0 \cap (\mathbb{D}^n_1 \times S^{n-1}) &= \{(\tau x, x) | x \in S^{n-1}, -1 \le \tau \le 1\} \end{aligned}$$

is a smooth compact n–dimensional manifold, and $(x, \tau) \mapsto (x, \tau x), (\tau, x) \mapsto (\tau x, x)$ $(x \in S^{n-1}, \tau \in [-1, 1])$ define homeomorphisms $S^{n-1} \times [-1, 1] \to F_0 \cap (S^{n-1} \times \mathbb{D}^n_1), [-1, 1] \times S^{n-1} \to F_0 \cap (\mathbb{D}^{n-1}_1 \times S^{n-1})$, respectively. Both manifolds have the same boundary

$$F_0 \cap (S^{n-1} \times S^{n-1}) = \{(x, \tau x) | x \in S^{n-1}, \tau = \pm 1\}$$

and no further common points. Therefore their union $F_0 \cap \partial B$ is a topological manifold without boundary, and this manifold is a basis of the cone F_0. So F_0 is the cone over an n–dimensional topological manifold which is the union of two smooth compact n–dimensional manifolds with common boundary, and it is not hard to see that $F_0\backslash\{0\}$ is a smooth $(n+1)$–dimensional manifold. The codimension of F_0 in $\mathbb{R}^{2n}$ is $n-1$, and we can find a real γ' such that for any $\varepsilon > 0$ we have

$$\mathrm{vol}\,(N_\varepsilon(F_0)\cap(\mathbb{D}^n_{2\rho})^2)\le\gamma'\varepsilon^{n-1},$$

where vol denotes the Lebesgue measure in $\mathbb{R}^{2n}$. Since $\sigma((\mathbb{D}^n_\rho)^4)\subset(\mathbb{D}^n_{2\rho})^2$ this inequality shows that there is a γ_1 which satisfies (5).

3. Proof of Theorem 2

We fix an odd integer $n=2n'+1\ge 3$. The main part in the proof of the theorem will be the proof of the following lemma.

Lemma 6. *Let $r\ge 1$ be an integer, and let*

$$\theta_r=2^{2n'+2}\binom{n}{n'}r^{n'},\qquad \lambda>\tfrac{1}{2}r^{-1}.$$

Then there is a non empty open subset $\mathcal{U}_r$ of $C^1(S^1,\mathbb{R}^n)$ such that for any $v\in\mathcal{U}_r$ the restriction of $f_{\theta_r,\lambda,v}$ to its attractor is not injective.

Before proving this lemma we show how it implies Theorem 2. Since $\frac{1}{2}r^{-1}=2^{1+2/n'}\binom{n}{n'}^{1/n'}\theta_r^{-1/n'}$, as an immediate consequence of the lemma we get

$$\lambda_n(\theta_r)\le 2^{1+2/n'}\binom{n}{n'}^{1/n'}\theta_r^{-1/n'}.$$

For $n'\to\infty$ the value $\binom{n}{n'}^{1/n'}$ tends from below to 4, and it is easy to see that

$$\lambda_n(\theta_r)\le 24\theta_r^{-1/n'}.$$

If $\theta_r\le\theta<\theta_{r+1},\quad r\ge 3$, then

$$\begin{aligned}\lambda_n(\theta)\le\lambda_n(\theta_r)\le\ & 24\theta_r^{-1/n'}=24\left(\tfrac{\theta}{\theta_r}\right)^{1/n'}\theta^{-1/n'}<24\left(\tfrac{\theta_{r+1}}{\theta_r}\right)^{1/n'}\theta^{-1/n'}\\ \le\ & 24\tfrac{r+1}{r}\theta^{-1/n'}\le 32\theta^{-1/n'}\end{aligned}$$

If $2\le\theta<\theta_3$ then we use $\lambda_n(\theta)\le\theta^{-1/n}$ (see Remark 1). So we get

$$\begin{aligned}\lambda_n(\theta)\le\theta^{-1/(2n'+1)} &= \theta^{(n'+1)/[n'(2n'+1)]}\theta^{-1/n'}\\ &< \theta_3^{(n'+1)/[n'(2n'+1)]}\theta^{-1/n'}\\ &= 2^{2(n'+1)^2/[n'(2n'+1)]}\binom{n}{n'}^{(n'+1)/[n'(2n'+1)]}3^{(n'+1)/(2n'+1)}\theta^{-1/n'}\\ &\le 2^{8/3}\cdot 3^{2/3}\cdot 3^{2/3}\cdot\theta^{-1/n'}\\ &< 32\theta^{-1/n'}.\end{aligned}$$

Therefore for each $\theta\ge 2$ we have $\lambda_n(\theta)\le 32\theta^{-1/n'}$, and since $1/n'=2/(n-1)$ the theorem is proved.

In the proof of Lemma 6 we shall apply the following Lemma 8. Let $\Omega(\rho)$ be the set of all n–dimensional cubes Q in $\mathbb{R}^n$ whose edges are parallel to the axes of $\mathbb{R}^n$ and have length ρ. The k–dimensional skeleton of a cube Q, i.e. the union of its k–dimensional faces, will be denoted by $S_k(Q)$.

Lemma 7. *Let Q', Q'' be q–dimensional cubes in $\mathbb{R}^q$, where $q = 2q'$ is even. We assume that Q', Q'' intersect, that Q', Q'' have the same edge length and that the edges of both cubes are parallel to the axes of $\mathbb{R}^q$. Then $S_{q'}(Q') \cap S_{q'}(Q'') \neq \emptyset$.*

Lemma 8. *If $Q'(\tau), Q''(\tau)$ $(\tau \in [0,1])$ are two continuous families of cubes in $\Omega(\rho)$, then $Q'(0) \cap Q''(0) \neq \emptyset, Q'(1) \cap Q''(1) = \emptyset$ implies that there is a $\tau_0 \in [0,1]$ such that $S_{n'}(Q'(\tau_0)) \cap S_{n'}(Q''(\tau_0)) \neq \emptyset$.*

The proof of Lemma 7 is easy and can be omitted. The topological background of Lemma 8 is the fact that for two cubes Q', Q'' of $\Omega(\rho)$ with $Q' \cap Q'' \neq \emptyset, S_{n'}(Q') \cap S_{n'}(Q'') = \emptyset$ these two n'–dimensional skeletons must be linked as indicated for $n = 3, n' = 1$ in Figure 4.

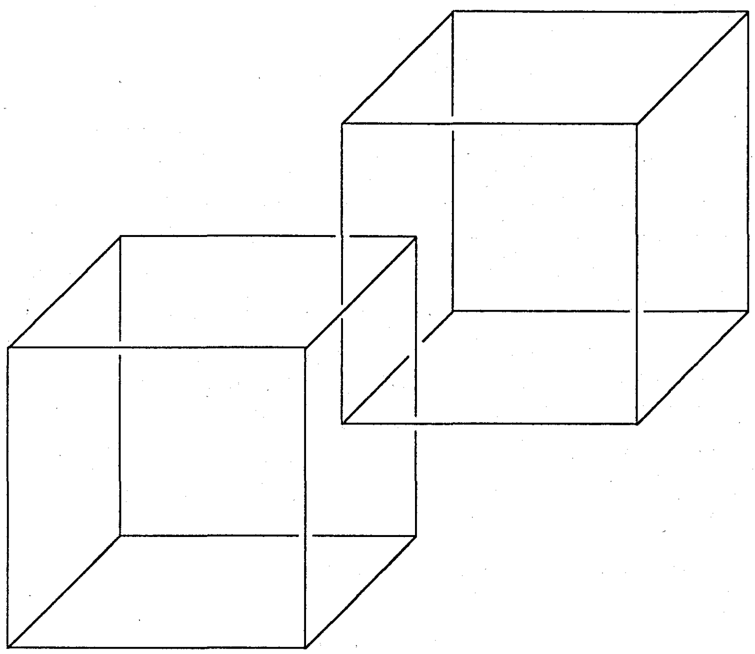

Fig. 4

Proof of Lemma 8. We define

$$\tau_0 = \sup\{\tau \in [0,1] | Q'(\tau) \cap Q''(\tau) \neq \emptyset\}.$$

Then $Q'(\tau_0) \cap Q''(\tau_0) \neq \emptyset$, but Int $Q'(\tau_0) \cap$ Int $Q''(\tau_0) = \emptyset$. If $H_1, \ldots, H_{2n}$ are the $(n-1)$–dimensional Hyperplanes in $\mathbb{R}^n$ each of which contains an $(n-1)$–dimensional face of $Q'(\tau_0)$, then there is at least one H_i such that $Q'(\tau_0), Q''(\tau_0)$ lie on different sides of H_i. (Otherwise the interiors of the two cubes would intersect.) Let F', F'' be the $(n-1)$–dimensional faces of $Q'(\tau_0), Q''(\tau_0)$, respectively, which lie in H_i. Then $F' \cap F'' \neq \emptyset$, and since $n - 1 = 2n'$ is even we can apply Lemma 1. So we get $S_{n'}(F') \cap S_{n'}(F'') \neq \emptyset$ and therefore $S_{n'}(Q'(\tau_0)) \cap S_{n'}(Q''(\tau_0)) \neq \emptyset$. ∎

Proof of Lemma 6. Let $\mathfrak{Q}_r (r \geq 1$ an integer) be the lattice of all cubes from $\mathfrak{Q}(r^{-1})$ whose vertices belong to the point lattice $(r^{-1}\mathbb{Z})^n$ in $\mathbb{R}^n$. By $\mathfrak{P}_r$ we denote the set of all $Q \in \mathfrak{Q}_r$ with $Q \subset I^n, Q \cap S_{n'}(I^n) \neq \emptyset$, where I^n is the cube $[-1,1]^n$ in $\mathbb{R}^n$. Since an n-dimensional cube has $2^{n-n'}\binom{n}{n'}$ n'-dimensional faces and since the edge length of I^n is 2, the number $\#\mathfrak{P}_r$ of cubes in $\mathfrak{P}_r$ can be estimated by

$$\#\mathfrak{P}_r \leq 2^{n'+1}\binom{n'}{n}(2r)^{n'} = 2^{2n'+1}\binom{n}{n'}r^{n'}.$$

We fix numbers θ_r, λ as in the lemma and consider the set $S^1 \times I^n$ which can be regarded as a solid torus with corners in $S^1 \times \mathbb{R}^n$. Then we define $\mathcal{U}_r$ to be the set of all $v \in C^1(S^1, \mathbb{R}^n)$ with the following two properties.

(i) If $t \in S^1, Q \in \mathfrak{P}_r$, then there is a $t' \in S^1$ such that the cube $f_{\theta_r,\lambda,v}(\{t'\} \times I^n)$ contains the cube $\{t\} \times Q$ in its interior.

(ii) There is a $t^* \in S^1$ such that

$$f_{\theta_r,\lambda,v}(\{t^*\} \times I^n) \cap f_{\theta_r,\lambda,v}((S^1 \backslash \{t^*\}) \times I^n) = \emptyset.$$

Using an compactness argument it is not hard to see that $\mathcal{U}_r$ is open in $C^1(S^1, \mathbb{R}^n)$.

To prove that $\mathcal{U}_r$ is not empty, i.e. to find a mapping v in $C^1(S^1, \mathbb{R}^n)$ which belongs to $\mathcal{U}_r$, we remark first that $\theta_r > 2\#\mathfrak{P}_r$ and that λ times the edge length of I^n is greater than the edge length of the cubes in $\mathfrak{P}_r$. Let $\mathfrak{P}_r = \{Q_1, \ldots, Q_p\}$, and let z_i be the centre of Q_i. We decompose $[0,1]$ in the θ subintervals $I_i = [\frac{i-1}{\theta}, \frac{i}{\theta}]$ $(1 \leq i \leq \theta)$. Since $\theta \geq 2p$ it is easy to find a $v \in C^1(S^1, \mathbb{R}^n)$ such that $v(t) = z_i$ for all $t \in I_{2i}$ $(1 \leq i \leq p)$. Obviously for such a v the mapping $f_{\theta_r,\lambda,v}$ has the property (i). Then to get property (ii) we define $t^* = \frac{1}{2\theta}$ and modify v in the interval I_1 so that the equation of (ii) is satisfied.

To prove Lemma 6 we must show that for $v \in \mathcal{U}_r$ the restriction of $f_{\theta_r,\lambda,v}$ to its attractor Λ is not injective. Therefore we shall construct points $x' \neq x''$ in Λ with $f_{\theta_r,\lambda,v}(x') = f_{\theta_r,\lambda,v}(x'')$.

In the first step of the construction we apply (i) to find points t_1', t_1'' in S^1 such that

$$t_1' \neq t_1'', \theta t_1' = \theta t_1''$$
$$f_{\theta_r,\lambda,v}(\{t_1'\} \times I^n) \cap f_{\theta_r,\lambda,v}(\{t_1''\} \times I^n) \neq 0.$$

Obviously t_1' can not coincide with the point t^* of (ii).

In the second step we denote the point t^* of (ii) by t_1^* and coinsider the point t_1^{**} for which the arcs $[t_1', t_1^*], [t_1'', t_1^{**}]$ have the same length so that $\theta t_1^* = \theta t_1^{**}$. By (ii) the cubes

$$f_{\theta_r,\lambda,v}(\{t_1^*\} \times I^n), \quad f_{\theta_r,\lambda,v}(\{t_1^{**}\} \times I^n)$$

are disjoint, and we consider the following two families of cubes

$$\begin{aligned} Q'(\tau) &= f_{\theta_r,\lambda,v}(\{t_1' + \tau(t_1^* - t_1')\} \times I^n) \\ & \qquad\qquad\qquad\qquad \tau \in [0,1] \\ Q''(\tau) &= f_{\theta_r,\lambda,v}(\{t_1'' + \tau(t_1^{**} - t_1'')\} \times I^n). \end{aligned}$$

Since $Q'(0) \cap Q''(0) = \emptyset, Q'(1) \cap Q''(1) \neq \emptyset$ we can apply Lemma 8 and find a value τ_0 such that for

$$s_1' = t_1' + \tau_0(t_1^* - t_1'), \quad s_1'' = t_1'' + \tau_0(t_1^{**} - t_1'')$$

we have

$$s_1' \neq s_1'', \quad \theta s_1' = \theta s_1'',$$
$$S_n(f_{\theta_r,\lambda,v}(\{s_1'\} \times I^n)) \cap S_n(f_{\theta_r,\lambda,v}(\{s_1''\} \times I^n)) \neq \emptyset.$$

This implies that there are cubes Q_1', Q_1'' in $\mathfrak{P}_r$ for which

$$f_{\theta_r,\lambda,v}(\{s_1'\} \times Q_1') \cap f_{\theta_r,\lambda,v}(\{s_1''\} \times Q_1'') \neq \emptyset,$$

and applying (i) we find points $t_1', t_2'' \in S^1$ such that $\theta t_2' = s_1', \theta t_2'' = s_2''$

$$f_{\theta_r,\lambda,v}(\{t_2'\} \times I^n) \supset \{s_i'\} \times Q_1', \quad f_{\theta_r,\lambda,v}(\{t_2''\} \times I^n) \supset \{s_1''\} \times Q_1''.$$

These points t_2', t_2'' have the following properties

$$\theta t_2' \neq \theta t_2'', \quad \theta^2 t_2' = \theta^2 t_2'',$$
$$f^2_{\theta_r,\lambda,v}(\{t_2'\} \times I^n) \cap f^2_{\theta_r,\lambda,v}(\{t_2''\} \times I^n) \neq \emptyset,$$

and we conclude the second step of the construction with the remark that $\theta t_2'$ can not be the point t^*.

In the third step we consider the point $t_2^* \in S^1$ for which $\theta t_2^* = t_1^* = t^*$ and the arc $[t_2', t_2^*]$ does not contain any further point t with $\theta t = t^*$. Then there is a unique point t_2^{**} for which the arcs $[t_2', t_2^*], [t_2'', t_2^{**}]$ have the same length so that $\theta^2 t_2^* = \theta^2 t_2^{**}$. By (ii)

$$f^2_{\theta_r,\lambda,v}(\{t_2^*\} \times I^n) \cap f^2_{\theta_r,\lambda,v}(\{t_2^{**}\} \times I^n) = \emptyset,$$

and as in the second step we find points $s_2' \in [t_2', t_2^*], s_2'' \in [t_2'', t_2^{**}]$ such that

$$\theta s_2' \neq \theta s_2'', \quad \theta^2 s_2' = \theta^2 s_2'',$$
$$S_n(f^2_{\theta_r,\lambda,v}(\{s_2'\} \times I^n)) \cap S_n(f^2_{\theta_r,\lambda,v}(\{s_2''\} \times I^n)) \neq \emptyset.$$

Then there are cubes Q_2', Q_2'' in $\mathfrak{P}_r$ such that

$$f^2_{\theta_r,\lambda,v}(\{s_1'\} \times Q_2') \cap f_{\theta_r,\lambda,v}(\{s_2''\} \times Q_2'')) \neq \emptyset,$$

and by (i) we find $t_3', t_3'' \in S^1$ such that $\theta t_3' = s_2', \theta t_3'' = s_2''$,

$$f_{\theta_r,\lambda,v}(\{t_3'\} \times I^n) \supset \{s_2'\} \times Q_2', \quad f_{\theta_r,\lambda,v}(\{t_3''\} \times I^n) \supset \{s_2''\} \times Q_2''.$$

So we have

$$\theta^2 t_3' \neq \theta^2 t_3'', \quad \theta^3 t_3' = \theta^3 t_3''$$
$$f^3_{\theta_r,\lambda,v}(\{t_3'\} \times I^n) \cap f^3_{\theta_r,\lambda,v}(\{t_3''\} \times I^n) \neq \emptyset.$$

Continuing in this way we find points $t_1', t_2', \ldots, t_1'', t_2'', \ldots$ such that

$$\theta^{k-1} t_k' \neq \theta^{k-1} t_k'', \quad \theta^k t_k' = \theta^k t_k'' \tag{6}$$
$$f^k_{\theta_r,\lambda,v}(\{t_k'\} \times I^n) \cap f^k_{\theta_r,\lambda,v}(\{t_k''\} \times I^n) \neq \emptyset \quad (k = 1, 2, \ldots). \tag{7}$$

To get the points x', x'' we consider $\bar{t}_k' = \theta^{k-1} t_k', \bar{t}_k'' = \theta^{k-1} t_k''$ and the centres x_k', x_k'' of the cubes $f^{k-1}_{\theta_r,\lambda,v}(\{t_k'\} \times I^n), f^{k-1}_{\theta_r,\lambda,v}(\{t_k''\} \times I^n)$ in $\{\bar{t}_k'\} \times \mathbb{R}^n$ or in $\{\bar{t}_k''\} \times \mathbb{R}^n$, respectively. All these points x_k', x_k'' belong to a compact subset of $S^1 \times \mathbb{R}^n$, and we can find a sequence $k_1 < k_2 < \ldots$ of indices for which the sequences $(x_{k_j}'), (x_{k_j}''), (j = 1, 2, \ldots)$ converge to points x', x'', respectively. To see that x', x'' lie in Λ we consider a compact subset K of $S^1 \times \mathbb{R}^n$ such that $S^1 \times I^n \subset K, f_{\theta_r,\lambda,v}(K) \subset K$. Then $x', x'' \in \bigcap_{k=1}^{\infty} f^k_{\theta_r,\lambda,v}(K) \subset \Lambda$. Now we show $x' \neq x''$. If $\bar{t}', \bar{t}''$ are the projections of

x', x'', respectively, to S^1, then $\bar{t}' = \lim_{j\to\infty} \bar{t}'_{k_j}, \bar{t}'' = \lim_{j\to\infty} \bar{t}''_{k_j}$. By (6) we have $\bar{t}'_{k_j} \neq \bar{t}''_{k_j}$ but $\theta\bar{t}'_{k_j} = \theta\bar{t}_{k_j}$ and therefore $\mathrm{dist}(\bar{t}'_{k_j}, \bar{t}''_{k_j}) \geq \theta^{-1}$ which implies $\mathrm{dist}(\bar{t}', \bar{t}'') \geq \theta^{-1}$ and hence $x' \neq x''$. Finally we prove $f_{\theta_r,\lambda,v}(x') = f_{\theta_r,\lambda,v}(x'')$. By (7) the cubes $f^k_{\theta_r,\lambda,v}(\{t'_{k_j}\} \times I^n), f^k_{\theta_r,\lambda,v}(\{t''_{k_j}\} \times I^n)$ with centres $f_{\theta_r,\lambda,v}(x'_{k_j}), f_{\theta_r,\lambda,v}(x''_{k_j})$, respectively, and edge length λ^{k_j} intersect so that the distance between these two points is at most $\sqrt{n}\lambda^{k_j}$. Since

$$f_{\theta_r,\lambda,v}(x') = \lim_{j\to\infty} f_{\theta_r,\lambda,v}(x'_{k_j}), \quad f_{\theta_r,\lambda,v}(x'') = \lim_{j\to\infty} f_{\theta_r,\lambda,v}(x''_{k_j}),$$

this implies $f_{\theta_r,\lambda,v}(x') = f_{\theta_r,\lambda,v}(x'')$.

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