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Metastability in reversible diffusion processes II. Precise asymptotics for small eigenvalues

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Abstract: We continue the analysis of the problem of metastability for reversible diffusion processes, initiated in [BEGK3], with a precise analysis of the low-lying spectrum of the generator. Recall that we are considering processes with generators of the form $-\epsilon\Delta+\nabla F(\cdot)\nabla$ on $\mathbb{R}^d$ or subsets of $\mathbb{R}^d$, where F is a smooth function with finitely many local minima. Here we consider only the generic situation where the depths of all local minima are different. We show that in general the exponentially small part of the spectrum is given, up to multiplicative errors tending to one, by the eigenvalues of the classical capacity matrix of the array of capacitors made of balls of radius ϵ centered at the positions of the local minima of F. We also get very precise uniform control on the corresponding eigenfunctions. Moreover, these eigenvalues can be identified with the same precision with the inverse mean metastable exit times from each minimum. In [BEGK3] it was proven that these mean times are given, again up to multiplicative errors that tend to one, by the classical *Eyring-Kramers formula.*

1. Introduction.

In this paper we continue the investigation of reversible diffusion processes initiated in [BEGK3]. Recall that we are interested in processes $X_\epsilon(t)$ that are given as solutions of an Itô stochastic differential equation

$$dX_\epsilon(t) = \nabla F(X_\epsilon(t))dt + \sqrt{2\epsilon}dW(t) \tag{1.1}$$

on a regular domain $\Omega \subseteq \mathbb{R}^d$, where the drift ∇F is generated by a potential function that is sufficiently regular. We are interested in the case when the function $F(x)$ has several local minima. We always assume that X_ϵ is killed on Ω^c if it exists.

For a general introduction to the topic and its history we refer to the introduction of [BEGK3]. In that paper we have studied the so-called metastable exit times from attractors of local minima of F and we have given a precise asymptotic estimate for the mean value of these times. These estimates were in turn based on precise estimates of certain *Newtonian capacities* of sets containing small balls centered at the locations of the minima of F.

In the present paper we turn to the investigation of the low-lying spectrum of the generators of the process defined by (1.1), i.e.

$$L_\epsilon \equiv -\epsilon\Delta + \nabla F(x)\cdot\nabla \tag{1.2}$$

of these processes, with Dirichlet boundary conditions on Ω^c (if $\Omega \neq \mathbb{R}^d$) of these processes. It is well-known that the spectrum of such operators has precisely one exponentially small eigenvalue for each local minimum of the function F, and more or less rough estimates of their precise values are known [FW,Ma,Mi]. Wentzell [W2] and Freidlin and Wentzell [FW] obtain estimate for the exponential rate, i.e. they identify $\lim_{\epsilon\downarrow 0}\epsilon^{-1}\ln\lambda_i(\epsilon)$ using large deviation methods. Sharper estimates, with multiplicative errors of order $\epsilon^{\pm kd}$ were obtained for principal eigenvalues by Holley, Kusuoka, and Strook [HKS] using a variational principle; these methods were extended to the full set of exponentially small eigenvalues by Miclo [Mi] (see also [Ma]).

Our purpose here is to get *sharp* estimates, i.e. we seek upper and lower bounds with multiplicative errors that tend to one as ϵ tends to zero. Such estimates are known in the one-dimensional case (see e.g. [BuMa1,BuMa2] and references therein), whereas in the multi-dimensional case only heuristic results based on formal power series expansions of WKB type exist. (see e.g. [Kolo] for an analysis of the situation). While the methods introduced in the third paper on quantum mechanical tunneling by Helffer and Sjöstrand [HS3] should in principle allow to justify such expansions, their implementation seems rather tedious and has not been carried out to our knowledge.

Here we will resort to a different approach that combines ideas already suggested in [W1] with potential theoretic ideas. In fact, this approach was developed in [BEGK2] in the setting of discrete Markov chains, where indeed many technical problems we will be facing here disappear, and that may serve as a nice introduction.

We will from now on assume that F is at least three times continuously differentiable and has a finite set of local minima, which we denote by $\mathcal{M} = \{x_1, \dots, x_n\}$. We will also assume that F has exponentially tight level sets, i.e. that $\int_{y:F(y)\geq a}\exp(-F(z)/\epsilon)dz < Ce^{-a/\epsilon}$, where $C = C(a) < \infty$ is independent of ϵ. Our main interests are the distribution of stopping times

$$\tau_A \equiv \inf\{t > 0 | X(t) \in A\} \tag{1.3}$$

for the process starting in one minimum, say $x \in \mathcal{M}$, of F, when $A = B_\rho(y)$ is a small ball of radius ρ around another minimum, $y \in \mathcal{M}$. It will actually become apparent that the precise choice of the hitting set is often not important, and that the problem is virtually equivalent to considering the escape from a suitably chosen neighborhood of x, provided this neighborhood contains the relevant *saddle points* connecting x and y.

Let us now state the main results of this paper. One key notion we will need is that of the saddle between two sets $A, B \subset \mathbb{R}^d$. We say that $z^* = z^*(A, B)$ is a saddle point between A

and B, if

$$F(z^*(A,B)) = \inf_{\omega:\omega(0)\in A,\omega(1)\in B} \sup_{t\in[0,1]} F(\omega(t)) \tag{1.4}$$

where the infimum is over all continuous paths going from A to B. Note that $z^*(A,B)$ may not be uniquely defined; we call $\mathcal{Z}(A,B)$ the set of all possible solutions.

Given two disjoint closed sets A, D, we will denote by $h_{A,D}(x)$ the equilibrium potential, by $e_{A,D}(dy)$ the equilibrium measure, and by $\operatorname{cap}_A(D)$ the Newtonian capacity corresponding to the Dirichlet problem with boundary conditions one on A and zero on D. The precise definitions of these classical quantities (see e.g. [BluGet,Doo,Szni]) are recalled in Section 2 of [BEGK3].

Theorem 1.1: *Assume that F has n local minima, $x_1,\dots,x_n$ and that for some $\theta > 0$ the minima x_i of F can be labeled in such a way that, with $\mathcal{M}_k \equiv \{x_1,\dots,x_k\}$ and $\mathcal{M}_0 \equiv \Omega^c$,*

$$F(z^*(x_k,\mathcal{M}_{k-1})) - F(x_k) \le \min_{i<k}\left(F(z^*(x_i,\mathcal{M}_k\backslash x_i)) - F(x_i)\right) - \theta \tag{1.5}$$

holds for all $k = 1,\dots,n$. We will set $B_i \equiv B_\epsilon(x_i)$ and $\mathcal{S}_k \equiv \cup_{i=1}^k B_i$, and $h_k(y) \equiv h_{B_k,\mathcal{S}_{k-1}}(y)$. Assume moreover that all saddle points $z^*(x_k,\mathcal{M}_{k-1})$ are unique, and that F has a non-degenerate Hessian at all these saddle points and at all local minima. Then there exists $\delta > 0$ such that the n exponentially small eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ of L_ϵ satisfy:

$$\begin{aligned}\lambda_k &= \frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2^2}(1+O(e^{-\delta/\epsilon}))\\ &= \frac{1}{\mathbb{E}_{x_k}\tau_{\mathcal{S}_{k-1}}}(1+O(e^{-\delta/\epsilon}))\\ &= \frac{|\lambda_1^*(z^*(x_k,\mathcal{M}_{k-1}))|}{2\pi}\sqrt{\frac{\det(\nabla^2F(x_k))}{|\det(\nabla^2F(z^*(x_k,\mathcal{M}_{k-1})))|}}e^{-[F(z^*(x_k,\mathcal{M}_{k-1}))-F(x_k)]/\epsilon}\\ &\quad\times\left(1+O\left(\epsilon^{1/2}|\ln\epsilon|\right)\right)\end{aligned} \tag{1.6}$$

where $\lambda_1^*(z^*)$ denotes the unique negative eigenvalue of the Hessian of F at the saddle point z^*. Note that if $\Omega = \mathbb{R}$, then $\operatorname{cap}_{B_1}(\mathcal{M}_0) = \operatorname{cap}_{B_1}(\emptyset) = 0$.

Remark: The theorem can be seen as containing three results: First, an asymptotically sharp identification of the exponentially small eigenvalues with the inverse mean exit times from local minima; this is a general feature of metastable systems (see e.g. [D1,D2,D3,GS,GM] for earlier results). Second, it relates these eigenvalues precisely to Newtonian capacities; this is the key difference from our results to e.g. the approach of Kolokoltsov and Makarov

[KoMa1,KoM2,Kol], since it allows thirdly to get an explicit expression for the eigenvalues in terms of the potential F.

Remark: Conditions (1.5) state that *"all valleys of F have different depth"*, which is in some sense the generic situation. In this case a number of simplifications take place, in particular we do not have to deal with degenerate eigenvalues. These conditions are completely analogous to the conditions imposed in [BEGK2]. Our general approach does, however, in principle also allow to treat degenerate situations. We postpone the treatment of such cases to future work.

In the course of the proof of Theorem 1.1 we will also obtain rather detailed control on the eigenfunctions of L_ϵ corresponding to its small eigenvalues.

Theorem 1.2: *Under the assumptions of Theorem 1.1, if ϕ_k denote the normalized eigenfunction corresponding to the eigenvalue λ_k, then there exists $\delta > 0$ s.t.*

$$\phi_k(y) = \frac{h_{B_\epsilon(x_k),\mathcal{S}_{k-1}}(y)}{\|h_{B_\epsilon(x_k),\mathcal{S}_{k-1}}\|_2}(1 + O(e^{-\delta/\epsilon})) + O(e^{-\delta/\epsilon}) \tag{1.7}$$

where $h_{B_\epsilon(x_k),\mathcal{S}_{k-1}}(y) = \mathbb{P}_y\left[\tau_{B_\epsilon(x_k)} < \tau_{\mathcal{S}_{k-1}}\right]$

Remark: We give even more precise expressions for the eigenfunctions in the course of the proofs later on. Note that there is considerable interest in the knowledge of eigenfunctions in the context of numerical schemes designed to recover metastable sets from the computation of eigenfunctions. See in particular references [S,SFHD,HMS]. Let us emphasise that, using the bounds on equilibrium potentials obtained in Corollary 4.8 of [BEGK3], Theorem 1.2 implies that the eigenfunction corresponding to a Minimum x_i is exponentially close to a constant ($\sim e^{F(x_i)/\epsilon}$) in the connected component of the level set $\{y : F(y) < F(z^*(x_i, \mathcal{M}_{i-1}))\}$ that contains x_i (i.e. in the valley below the saddle point that connects x_i to the set that lies below x_i), while it drops exponentially in the other connected components of the level set of this saddle; below the level of x_i it is exponentially small in absolute terms. Note that this implies that the zeros of ϕ are generally not in the neighborhood of the saddle points, but much closer to the minima in $\mathcal{M}_{i-1}$. This fact was also observed in [HMS]. We would like to stress that the fact that the eigenfunctions drop sharply at the saddle points makes them very good indicators of the actual valley structure of the potential F, i.e. they become excellent approximations to the indicator functions of the metastable sets corresponding to the metastable exit time $1/\lambda_i$.

Finally, it is almost a corollary from the results obtained above that metastable exit times are asymptotically exponentially distributed, when appropriate non-degeneracy conditions are met.

Theorem 1.3: *Assume that the Hessian of F is non-degenerate at all local minima and saddle points. Let x_i be a minimum of F and let D be any closed subset of $\mathbb{R}^d$ such that:*

(i) If $\mathcal{M}_i \equiv \{y_1, \dots, y_k\} \subset \mathcal{M}$ enumerates all those minima of F such that $F(y_j) \leq F(x_i)$, then $\cup_{j=1}^k B_\epsilon(y_j) \subset D$, and

(ii) $\operatorname{dist}(\mathcal{S}(x_i, \mathcal{M}_i), D) \geq \delta > 0$ for some δ independent of ϵ.

Assume further that the conditions of Theorem 1.1 are satisfied. Then, there exist $\delta > 0$ independent of ϵ and of t, such that for all $t > 0$,

$$\begin{aligned} \mathbb{P}_{x_k}\left[\tau_D > t\mathbb{E}_{x_k}\tau_D\right] = &\left(1 + O\left(e^{-\delta/\epsilon}\right)\right) e^{t(1+O(e-\delta/\epsilon))} \\ &+ \sum_{j>k} O(e^{-\delta/\epsilon}) e^{-t\lambda_j \mathbb{E}_{x_k}\tau_D} + O(1) e^{-tO(e^{d-1})\mathbb{E}_{x_k}\tau_D} \end{aligned} \tag{1.8}$$

The results of this paper together with those of [BEGK3] show that the methods to analyse metastable behaviour in discrete Markov chains introduced in [BEGK1,BEGK2] can be naturally extended to the treatment of continuous diffusion processes. In particular we see that the metastable behaviour of continuous and discrete diffusions is virtually identical, and that all results for the discrete chains treated in [BEGK1] carry over to the corresponding diffusion approximations. In fact, our results in the diffusion case are sharper, since we were able to identify the constants in the prefactors of exponentially small or large terms (we expect, however, that with some extra work this improvement can also be carried over to the discrete chains, at least under certain conditions). There are a number of generalizations of these results that can be investigated: First, one can consider diffusion processes on more general Riemannian manifolds. Second, one can consider extensions to locally infinitely divisible processes with mixed diffusion and jump components. Such extensions will require some extra work, but in principle our approach appears applicable, and qualitatively similar results should be obtainable. Another potentially interesting generalization concerns non-reversible diffusion processes. Here the main difficulty is the determination of the invariant measure, which our methods do not address at all. However, it is to be expected that at least in uniquely ergodic situations, some of our results can still be carried over. We hope to address these issues in future publications.

The remainder of this paper is organized as follows: In Section 2 we prove an a priori estimate on the spectrum of the generator when Dirichlet conditions are applied to small neighborhoods of all the local minima of F. In Section 3 we then show that the eigenvalues of the full generator are asymptotically close to those of the capacity matrix, which in turn are then evaluated in the generic situation. In the course of the proof we also identify the eigenvalues of the generator with the principle eigenvalues of appropriate Dirichlet operators. Finally, we derive from these results the exponential distribution of the mean exit times.

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2. A priori spectral estimates.

Most of the preparatory background and necessary technical a priori estimates were introduced in [BEGK3] and will be imported from there. In this section we give an additional a priori estimate on the spectrum of certain Dirichlet operators associated to L_ϵ. More precisely, we derive a priori lower bounds on principal eigenvalues and for the Dirichlet problem in (regular) open sets $D \subset \mathbb{R}^d$ with closure $\bar{D}$. We denote by ∂D the boundary of $\bar{D}$. We denote by $\bar{\lambda}(D) \equiv \bar{\lambda}_1(D)$ the principal eigenvalue of the Dirichlet problem

$$
\begin{aligned}
(L_\epsilon - \lambda) f(x) &= 0, \quad x \in D \\
f(x) &= 0, \quad x \in D^c
\end{aligned}
\tag{2.1}
$$

and sometimes use the notation $L_\epsilon^{D^c}$ to indicate the Dirichlet operator corresponding to the problem (2.1).

The following lemma is a classical result of Donsker and Varadhan [DV]:

Lemma 2.1: *The principal eigenvalue $\bar{\lambda}(D)$ satisfies*

$$
\bar{\lambda}(D) \geq \frac{1}{\sup_{x \in D} \mathbb{E}_x \tau_{D^c}}
\tag{2.2}
$$

In the case when we consider diffusions on a compact set, Lemma 2.1 will yield a sufficiently good estimate. If D is unbounded, the supremum on the right may be infinite and the estimate becomes useless. However, it is easy to modify the proof of Lemma 2.1 to yield an improvement.

Lemma 2.2: *Let ϕ_D denote the eigenfunction corresponding to the principal eigenvalue of L_ϵ^D.*

Let $A \subset D$ be any compact set, Then

$$\bar{\lambda}(D) \geq \frac{1}{\sup_{x\in A} \mathbb{E}_x \tau_{D^c}} \left(1 - \int_{D\setminus A} dy e^{-F(y)/\epsilon} |\phi_D(y)|^2\right) \tag{2.3}$$

Moreover, for any $\delta > 0$, there exists bounded $A \subset D$ such that

$$\bar{\lambda}(D) \geq \frac{1}{\sup_{x\in A} \mathbb{E}_x \tau_{D^c}} (1-\delta) \tag{2.4}$$

Proof: Let $w(x)$ denote the solution of the Dirichlet problem

$$\begin{aligned} L_\epsilon w(x) &= 1, \quad x \in D \\ w(x) &= 0, \quad x \in D^c \end{aligned} \tag{2.5}$$

Note that (see e.g. Eq (2.22) of [BEGK3]) $w(x) = \mathbb{E}_x \tau_{D^c}$. Using that for any $C > 0$, $ab \leq \frac{1}{2}(Ca^2 + b^2/C)$ with $ab = \phi(x)\phi(y)$ and $C = w(y)/w(x)$, one shows readily that

$$\begin{aligned} \int_D dx e^{-F(x)/\epsilon} \phi(x)(L_\epsilon \phi)(x) &\geq \int_D dx e^{-F(x)/\epsilon} \frac{\phi(x)}{w(x)} (L_\epsilon w)(x)\phi(x) \\ &= \int_D dx e^{-F(x)/\epsilon} \frac{\phi(x)}{w(x)} \phi(x) \\ &\geq \frac{1}{\sup_{x\in A} w(x)} \int_A dx e^{-F(x)/\epsilon} \phi^2(x) \end{aligned} \tag{2.6}$$

Choosing ϕ as the normalized eigenfunction with maximal eigenvalue yields (2.3).

We now claim that for any $\gamma > 0$,

$$\int dy e^{-\gamma F(y)/\epsilon} |\phi(y)|^2 < C_\gamma < \infty \tag{2.7}$$

This clearly implies (2.4). The estimate (2.7) follows from a standard Combes-Thomas estimate for the ground-state eigenfunction, ϕ. It is convenient to introduce $v(y) \equiv e^{-F(y)/2\epsilon}\phi(y)$, which is the corresponding ground state eigenfunction of the operator

$$H_\epsilon \equiv e^{-F(\cdot)/2\epsilon} L_\epsilon e^{F(\cdot)/2\epsilon} \tag{2.8}$$

which is a symmetric operator on $L^2(\mathbb{R}^d, dy)$. By a standard computation,

$$t(\alpha)[u] \equiv \int dy u^*(y) e^{i\alpha F(y)/\epsilon} H_\epsilon e^{-i\alpha F(y)/\epsilon} u(y) \tag{2.9}$$

defines a closed sectoral form (in the sense of Kato [Ka]), which is analytic in the strip $|\Im\alpha| < 1/2$. The Combes-Thomas estimate (see e.g. [RS]) then implies that v satisfies

$$\int dy e^{(1-\gamma)F(y)/\epsilon}|v(y)|^2 < C_\gamma < \infty \tag{2.10}$$

which is equivalent to (2.11). This completes the proof of the lemma. $\diamondsuit$

We will first establish that $\bar{\lambda}(D)$ is at most polynomially small in ϵ if D does not contain local minima, more precisely, define

$$\mathcal{M}_\epsilon \equiv \{z \in \Omega |\, \mathrm{dist}(z, \mathcal{M}) \leq \epsilon\} \tag{2.12}$$

Lemma 2.3: *Assume that $D \cap \mathcal{M}_\epsilon = \emptyset$. Then there is a finite positive constant C such that*

$$\sup_{x\in D} \mathbb{E}_x \tau_{D^c} \leq C \sup_{x\in D} |\{y : F(y) \leq F(x)\}|\, \epsilon^{-d+1} \tag{2.13}$$

Proof: The starting point of the proof is the relation (which is an immediate consequence of [BEGK3], Eq. (2.27))

$$\int_D dy e^{-F(y)/\epsilon} h_{B_\rho(x),D^c}(y) \geq \inf_{z\in\partial B_\rho(x)} \mathbb{E}_z \tau_{D^c} \mathrm{cap}\,_{B_\rho(x)}(D^c) \tag{2.14}$$

between mean time, equilibrium potential and capacities. It follows from the well known relation

$$\mathbb{E}_x \tau_{D^c} = \int_D G_D(x,y)dy \tag{2.15}$$

between mean time and Green function that the Harnack inequality of [BEGK3], Lemma 4.1, carries over to $\mathbb{E}_z \tau_{D^c}$, implying that, if $\rho = c\epsilon$, then

$$\sup_{z\in\partial B_\rho(x)} \mathbb{E}_z \tau_{D^c} \leq C \inf_{z\in\partial B_\rho(x)} \mathbb{E}_z \tau_{D^c} \tag{2.16}$$

Combining this with (2.14) gives us that

$$\sup_{z\in\partial B_\rho(x)} \mathbb{E}_z \tau_{D^c} \leq C \frac{\int_D dy e^{-F(y)/\epsilon} h_{B_\rho(x),D^c}(y)}{\mathrm{cap}\,_{B_\rho(x)}(D^c)} \tag{2.17}$$

We now distinguish the regions $\{y : F(y) > F(x)\}$ and $\{y : F(y) \leq F(x)\}$ in the integral. In the former, we just use that $h_{B_\rho(x),D^c}(y) \leq 1$, while in the latter we invoke the upper bound from Proposition 4.3 in [BEGK3]. This gives

$$\begin{aligned}\sup_{z\in\partial B_\rho(x)} \mathbb{E}_z\tau_{D^c} \leq C\frac{\int_{y\in D:F(y)>F(x)} dy e^{-F(y)/\epsilon}}{\text{cap}_{\,B_\rho(x)}(D^c)} \\ + C\frac{1}{\text{cap}_{\,B_\rho(x)}(D^c)}\int_{y\in D:F(y)\leq F(x)} dy e^{-F(y)/\epsilon}\frac{\text{cap}_{\,B_\rho(y)}(B_\rho(x))}{\text{cap}_{\,B_\rho(y)}(D^c)}\end{aligned} \tag{2.18}$$

Using the upper and lower bounds on the capacities from Proposition 4.7 of [BEGK3], we get that

$$\begin{aligned}\sup_{z\in\partial B_\rho(x)} \mathbb{E}_z\tau_{D^c} \leq C'\epsilon\rho^{-d+2}e^{+F(x)/\epsilon}\int_{y\in D:F(y)>F(x)} dy e^{-F(y)/\epsilon} \\ + C'\epsilon\rho^{-d+2}\int_{y\in D:F(y)\leq F(x)} dy\end{aligned} \tag{2.19}$$

By our assumption on F, the first integral is bounded by a constant times $\exp(-F(x)/\epsilon)$ and the second is equal to the volume of the level set $\{F(y) \leq F(x)\}$. This implies the claimed bound.$\diamondsuit$

Combining our results yields the

Corollary 2.4: *If $D \cap \mathcal{M}_\epsilon = \emptyset$, then there exists a finite positive constant $C < \infty$, independent of ϵ, such that*

$$\bar{\lambda}(D) \geq C\epsilon^{d-1} \tag{2.20}$$

We can generalize the bounds obtained so far to sets D containing some of the local minima of F. I.e. let $\mathcal{N} \subset \mathcal{M}$ be nonempty and let

$$\mathcal{N}_\epsilon = \left\{y \in \mathbb{R}^d \,|\, \text{dist}(y, \mathcal{N}) \leq \epsilon\right\} \tag{2.21}$$

Assume that $D \supset \mathcal{N}_\epsilon$ and set $A(x) = \{y : h_{B_\epsilon(x),D^c\setminus B_\epsilon(x)}(y) = \max_{y\in\mathcal{M}} h_{B_\epsilon(x),D^c\;baB_\epsilon(x)}(y)\}$ Then

Lemma 2.5: *Under the assumptions of lemma 2.2,*

$$\frac{1}{\bar{\lambda}(D)} \geq \sum_{i:x_i\in\mathcal{N}_\epsilon}\frac{\int_{A(x_i)} e^{-F(y)/\epsilon}dy}{\text{cap}_{\,B_\epsilon(x_i)}(D\setminus B_\epsilon(x_i))} \tag{2.22}$$

Proof: The proof is similar to that of the preceding corollary combined with the estimate on mean times given in Theorem 6.2 of [BEGK3]. We leave the details to the reader.$\diamondsuit$

Remark: The key fact we need to extract from lemma 2.5 is that

$$\bar{\lambda}(D) \geq \sim \min_{i:x_i \in \mathcal{N}} e^{-[F(z^*(x_i,\mathcal{M}_k)-F(x_i)]/\epsilon]} \tag{2.23}$$

3. Characterization of the small eigenvalues.

It is a well-known fact that if F has n local minima, then L_ϵ has n eigenvalues that are exponentially small in ϵ and that the next largest eigenvalue is of the order of a constant [FW,Kolo]. It is also known ([Kolo], Chapter 8, Proposition 2.2) that the eigenspace of these eigenfunctions is exponentially close in the $L^2(\exp(-F(y))dy)$-distance to the linear hull of the n indicator functions χ_i of the attractors of the minima x_i under the deterministic dynamical system $\dot{y}(t) = -\nabla F(y(t))$.

In this section we will derive a precise characterization of these eigenvalues that together with the estimates on capacities of [BEGK3] will ultimately yield the exact asymptotic formulae of Theorem 1.1. This is the analogue of Section 4 of [BEGK2] for the diffusion case. Our approach can in to some extent be seen as an application of the ideas of Wentzell's remarkable paper from 1973 [W2]. As we will see, the application of these ideas is not as straightforward as in the discrete case, but in principle very similar.

Before we turn to the details of this construction, it is useful to explain the general strategy.

Let us now consider a set of disjoint compact sets $B_i \equiv B_\epsilon(x_i)$, $i = 1, \ldots, k$. Let $\bar{\lambda}_k$ denote the principal eigenvalue of the Dirichlet operator L_ϵ with Dirichlet conditions on $\mathcal{S}_k \equiv \cup_{i=1}^k B_i$ (and possibly on some further set Ω). Consider, for $\lambda < \bar{\lambda}_k$, the solution of the Dirichlet problem

$$\begin{aligned} (L_\epsilon - \lambda) f^\lambda(x) &= 0, \quad x \in \Omega \backslash \partial \mathcal{S}_k \\ f^\lambda(x) &= \phi(x), \quad x \in \partial \mathcal{S}_k \end{aligned} \tag{3.1}$$

(i.e. we consider the Dirichlet problems in the exterior and the interior of the balls simultaneously; note that the principal eigenvalue of L_ϵ within a ball will always be larger than $\bar{\lambda}_k$ and so plays no rôle). The basic idea is now to construct an eigenfunction of the full operator L_ϵ as a solution of the problem (3.1) with suitably chosen ϕ. Indeed, if λ is an eigenvalue of L_ϵ and if we choose $\phi(x)$ as the eigenfunction corresponding to this eigenvalue, then $f^\lambda(x)$ is equal to ϕ everywhere. To see this, note that since $\phi(x) = f^\lambda(x)$ on $\partial \mathcal{S}_k$, we have that for $x \in \mathcal{S}_k^c$

$$(L_\epsilon - \lambda)^{S_k}(f^\lambda - \phi)(x) = (L_\epsilon - \lambda)(f^\lambda - \phi)(x) = 0 \tag{3.2}$$

But since λ is not in the spectrum of $L_\epsilon^{\mathcal{S}_k}$, this implies that $f^\lambda(x) = \phi(x)$ on $\mathcal{S}_k^c$ as well. The same argument applies in the interior of $\mathcal{S}_k$ This means that $\lambda < \bar{\lambda}_k$ is an eigenvalue of L_ϵ if and only if we can find a function ϕ on $\partial\mathcal{S}_k$, such that the solution of the Dirichlet problem (3.1) is actually an eigenfunction of L_ϵ with eigenvalue λ. In other words, any eigenfunction corresponding to eigenvalues below the principal Dirichlet eigenvalue can be represented as solution of (3.1).

Thus the eigenvalue problem reduces to finding out for which values of λ for suitable ϕ on the boundaries of B_i, $(L_\epsilon - \lambda)f^\lambda = 0$ everywhere. In fact, $(L_\epsilon - \lambda)f^\lambda$ is in general a measure concentrated on the surface $\partial\mathcal{S}_k$; demanding that this surface measure be zero yields in general an integral equation for $\phi(x)$ on $\partial\mathcal{S}_k$, which is not particularly easy to handle. In the case of discrete Markov processes, we have considered a very similar problem in [BEGK2]. There, the balls B_i were, however, simply the points x_i. The measure $(L_\epsilon - \lambda)f^\lambda$ was then a simple measure on the finite set $\mathcal{M}_k$, and the boundary condition reduces to the k numbers $\phi(x_i)$, and the integral equation was reduced to a simple linear equation for the unknown vector $\phi(x_i), i = 1, \dots, k$. The condition for λ to be an eigenvalue was thus simply that a certain determinant vanishes. It would be more than nice if we could reduce ourselves to a similarly simple condition in the present case. Indeed this would be so, if we knew beforehand that $\phi(x)$ is constant on each surface ∂B_i. While this cannot be truly the case, if ϵ is small we may expect that ϕ varies little. In that case, we could, as we shall see, use perturbative arguments to arrive at the desired conclusion. Unfortunately, to obtain such control on eigenfunctions looks rather difficult. While the Harnack- and Hölder inequalities will give us the desired control if we know that the eigenfunction does not change sign in a suitable neighborhood of the minimum, one cannot exclude that some minima are close to such zeros. To deal with these cases creates a number of complications.

Regularity properties of positive harmonic functions. We first state a simple application of the Harnack- and Hölder inequalities (see [GT] , Corollaries 9.25 and 9.24) that we have stated as Lemmata 4.1 and 4.2 in [BEGK3].

Lemma 3.1: *Assume that x is a local minimum of F. Let ϕ be a positive strong solution of $(L_\epsilon - \lambda)\phi = 0$, $|\lambda| \leq 1$, on a ball $B_{4\sqrt{\epsilon}}(x)$. Then there exists a constant $C < \infty$ and $\alpha > 0$, both independent of ϵ such that*

$$\mathrm{osc}_{y\in B_\epsilon(x)}\phi(y) \leq C\epsilon^{\alpha/2} \min_{y\in B_\epsilon(x)} \phi(x) \tag{3.3}$$

Proof: We can use Lemmata 4.1 and 4.2 stated in [BEGK3] with $\Lambda = \lambda = \epsilon$, $\gamma = 1$, $c = \lambda$, and

$$\nu = \epsilon^{-2} \sup_{y \in B_{4\sqrt{\epsilon}}(x)} \|\nabla F(y)\|_\infty^2 \leq const.\epsilon^{-1} \tag{3.4}$$

Then, with $R = 2\sqrt{\epsilon}$, we obtain first from Lemma 4.2 that

$$\sup_{y \in B_{2\sqrt{\epsilon}}(x)} \phi(y) \leq C \inf_{y \in B_{2\sqrt{\epsilon}}(x)} \phi(y) \tag{3.5}$$

and then from Lemma 4.1 that

$$\operatorname{osc}_{y \in B_\epsilon(x)} \phi(x) \leq C\epsilon^{\alpha/2} \sup_{y \in B_{2\sqrt{\epsilon}}(x)} \phi(y) \left(1 + \sqrt{\epsilon}^{d+1} |\lambda|\right) \tag{3.6}$$

This implies the lemma if λ is not too large.$\diamondsuit$

Principal eigenvalues revisited. We will now improve on the estimates on principal eigenfunctions $\bar\lambda(D)$ obtained in Section 2 by showing that in the case when D contains a local minimum of F, these estimates are essentially exact.

Proposition 3.2: *Assume that D contains $l \geq 1$ local minima of the function F and that there is a single minimum $x \in D$ that realizes*

$$F(z^*(x, D^c)) - F(x) = \max_{i=1}^{l} [F(z^*(x_i, D^c)) - F(x_i)] \tag{3.7}$$

We write $B \equiv B_\epsilon(x)$. Then there exists $\alpha > 0, C < \infty, \delta > 0$, independent of ϵ, such that principal eigenvalue $\bar\lambda(D)$ of the Dirichlet problem on D satisfies

$$\frac{\operatorname{cap}_B(D^c)}{\|h_{B,D^c}\|_2^2}(1 - C\epsilon^{\alpha/2})(1 - e^{-\delta/\epsilon}) \leq \bar\lambda(D) \leq \frac{\operatorname{cap}_B(D^c)}{\|h_{B,D^c}\|_2^2}(1 + C\epsilon^{\alpha/2})(1 + e^{-\delta/\epsilon}) \tag{3.8}$$

where here and henceforth $\|\cdot\|_2$ denotes the L^2 norm with respect to the measure $e^{-F(y)/\epsilon}dy$.

Proof: Set $D^0 = D \backslash B$. Then we know by Lemma 2.5 that there exists $\delta > 0$ such that

$$\bar\lambda(D^0) \geq e^{-[F(z^*(x,D^c)) - F(x)]/\epsilon} e^{\delta/\epsilon} \tag{3.9}$$

while we know that $\bar\lambda(D) < \bar\lambda(D^0)$ (and expect $\bar\lambda \sim e^{-[F(z^*(x,D^c)) - F(x)]/\epsilon}$, i.e. much smaller). By the general philosophy outlined above, we know that the principal eigenfunction can be represented as the solution of the Dirichlet problem (both inside B and outside B)

$$\begin{aligned} (L_\epsilon - \lambda) f^\lambda(y) &= 0, \quad y \in D \backslash \partial B \\ f^\lambda(y) &= \phi_D(y), \quad y \in \partial B \\ f^\lambda(y) &= 0, \quad y \in D^c \end{aligned} \tag{3.10}$$

where the boundary conditions ϕ_D are given by the actual principal eigenfunction. We will assume that $\text{dist}(x, D^c) \geq \delta > 0$, independent of ϵ. Then $B_{4\sqrt{\epsilon}}(x) \subset D$, and since ϕ_D is the principal eigenfunction, it may be chosen positive on D. Therefore Lemma 3.1 applies and shows that

$$\inf_{y\in\partial B} \phi_D(y) \equiv c \leq \sup_{y\in\partial B} \phi_D(y) \leq (1 + C\epsilon^{\alpha/2})c \tag{3.11}$$

We will normalize the eigenfunction s.t. $c = 1$, Thus we can write $f^\lambda(x) = h^\lambda_{B,D^c}(x) + \psi^\lambda(x)$, where $h^\lambda_{B,D^c} \equiv h^\lambda$ is the λ-equilibrium potential (see [BEGK3], Chapter 2) that solves

$$\begin{aligned} (L_\epsilon - \lambda)h^\lambda(y) &= 0, \quad y \in D\backslash\partial B \\ h^\lambda(y) &= 1, \quad y \in \partial B \\ h^\lambda(y) &= 0, \quad y \in D^c \end{aligned} \tag{3.12}$$

while ψ^λ solves

$$\begin{aligned} (L_\epsilon - \lambda)\psi^\lambda(y) &= 0, \quad y \in D\backslash\partial B \\ \psi^\lambda(y) &= \phi_D(y) - 1, \quad y \in \partial B \\ \psi^\lambda(y) &= 0, \quad y \in D^c \end{aligned} \tag{3.13}$$

We want that $(L_\epsilon - \lambda)f^\lambda(x) = 0$ on all of D. Here we have to interpret $(L_\epsilon - \lambda)f^\lambda$ as a surface measure on ∂B. I.e., if g is a smooth test function that vanishes on D^c,

$$\begin{aligned} &\int_D dy e^{-F(y)/\epsilon} g(y)(L_\epsilon - \lambda)f^\lambda(y) \equiv \int_D dy e^{-F(y)/\epsilon} f^\lambda(y)(L_\epsilon - \lambda)f^\lambda(y)g(y) \\ &= \int_{D\backslash B} dy e^{-F(y)/\epsilon} f^\lambda(y)(L_\epsilon - \lambda)g(y) + \int_{\text{int } B} dy e^{-F(y)/\epsilon} f^\lambda(y)(L_\epsilon - \lambda)g(y) \\ &= \epsilon \int_{\partial B} e^{-F(y)/\epsilon} \left(g(y)\partial_{n(y)}f^\lambda(y) - f^\lambda(y)\partial_{n(y)}g(y)\right) d\sigma_B(y) \\ &+ \epsilon \int_{\partial B} e^{-F(y)/\epsilon} \left(g(y)\partial_{-n(y)}f^\lambda(y) - f^\lambda(y)\partial_{-n(y)}g(y)\right) d\sigma_B(y) \\ &= \epsilon \int_{\partial B} e^{-F(y)/\epsilon} \left(g(y)\partial_{n(y)}f^\lambda(y) + g(y)\partial_{-n(y)}f^\lambda(y)\right) d\sigma_B(y) \end{aligned} \tag{3.14}$$

where $d\sigma_B(y)$ denotes the Euclidean surface measure on ∂B, and $\partial_{\pm n(y)}$ denote the normal derivative at $y \in \partial B$ from the exterior and interior of B, respectively. Thus we can identify

$$dy e^{-F(y)/\epsilon}(L_\epsilon - \lambda)f^\lambda(y) = \epsilon e^{-F(y)/\epsilon}(\partial_{n(y)}f^\lambda(y) + \partial_{-n(y)}f^\lambda(y))d\sigma_B(y) \tag{3.15}$$

To get control on $\bar{\lambda}$, we can ask at least that the total mass of this measure on ∂B vanishes, i.e. that

$$0 = \int_{\partial B} e^{-F(y)/\epsilon}(\partial_{n(y)}f^\lambda(y) + \partial_{-n(y)}f^\lambda(y))d\sigma_B(y) \tag{3.16}$$

To evaluate this expression it will be convenient to observe that on ∂B, $h_{B,D^c}(y) = 1$ for $y \in \partial B$ (where $h_{B,D^c} \equiv h^{\lambda=0}_{B,D^c}$ is the Newtonian potential (see [BEGK3], Chapter 2)). Moreover, on B, $h_{B,D^c}(y) \equiv 1$, so that $\partial_{-n(y)} h_{B,D^c}(y)$ vanishes on ∂B. Using these facts together with Green's second identity (see Eq. (2.8) in [BEGK3]), we get from (3.15) the condition

$$\begin{aligned} 0 &= \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) f^\lambda(y) - \frac{\lambda}{\epsilon} \int_D dy e^{-F(y)/\epsilon} h_{B,D^c}(y) f^\lambda(y) \\ &= \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) - \frac{\lambda}{\epsilon} \int_D dy e^{-F(y)/\epsilon} h_{B,D^c}(y) h^\lambda_{B,D^c}(y) \\ &+ \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) \psi^\lambda(y) - \frac{\lambda}{\epsilon} \int_D dy e^{-F(y)/\epsilon} h_{B,D^c}(y) \psi^\lambda(y) \end{aligned} \tag{3.17}$$

(Note that the derivative $\partial_{n(y)}$ is in the direction of the interior of B). The two terms involving ψ^λ will be naturally treated as error terms. In fact, since $\partial_{n(y)} h_{B,D^c}(y) > 0$, using Lemma 3.1, we get that

$$0 \leq \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) \psi^\lambda(y) \leq C \epsilon^{\alpha/2} \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) \tag{3.18}$$

If we define $\delta\psi^\lambda = \psi^\lambda - \psi^0$, we see that $\delta\psi^\lambda$ solves the Dirichlet problem

$$\begin{aligned} (L_\epsilon - \lambda)\delta\psi^\lambda(y) &= \lambda \psi^0(y), \quad y \in D \backslash \partial B \\ \delta\psi^\lambda(y) &= 0, \quad y \in \partial B \\ \delta\psi^\lambda(y) &= 0, \quad y \in D^c \end{aligned} \tag{3.19}$$

and thus

$$\delta\psi^\lambda(y) = \lambda (L_\epsilon^{D^c \cup B} - \lambda)^{-1} \psi^0(y) \tag{3.20}$$

and so

$$\|\delta\psi^\lambda\|_2 \leq \frac{\lambda}{\bar\lambda(D^0) - \lambda} \|\psi^0\|_2 \tag{3.21}$$

By the same argument we also have that

$$\|h^\lambda_{B,D^c} - h_{B,D^c}\|_2 \leq \frac{\lambda}{\bar\lambda(D^0) - \lambda} \|h_{B,D^c}\|_2 \tag{3.22}$$

On the other hand, using the Poisson kernel representation of ψ^0,

$$\psi^0(z) = -\epsilon \int_{\partial B} (\phi_D(y) - 1) \partial_{n(y)} G_{D\backslash B}(x,y) d\sigma_B(y) \tag{3.23}$$

where $G_{D\backslash B}(x,y)$ denotes the Green's function for the Dirichlet problem with in $D\backslash B$ (see [BEGK3], Chapter 2). Since the normal derivative of the Green's function is negative on ∂B, we get that

$$0 \leq \psi^0(z) \leq C \epsilon^{\alpha/2} h_{B,D^c}(z) \tag{3.24}$$

With

$$\epsilon \int_{\partial B} e^{-F(y)/\epsilon} \partial_{n(y)} h_{B,D^c}(y) = \operatorname{cap}_B(D^c) \tag{3.25}$$

(3.17) implies that

$$\begin{aligned} &0 \geq \operatorname{cap}_B(D^c) - \lambda \|h_{B,D^c}\|_2^2 (1 - C\epsilon^{\alpha/2})(1 - \lambda/(\bar{\lambda}(D^0) - \lambda)) \\ &0 \leq \operatorname{cap}_B(D^c)(1 + C\epsilon^{\alpha/2}) - \lambda \|h_{B,D}\|_2^2 \end{aligned} \tag{3.26}$$

This implies the claimed bound on $\bar{\lambda}(D)$. Note that, while we have only used a necessary condition for $\bar{\lambda}(D)$, the fact that there must be such an eigenvalue implies that it actually lies in the bounds given by (3.26). $\diamondsuit$

Remark: In the case when several of the minima within D satisfy (3.7) (i.e. if D contains several minima that are "equally deep"), one has to remove balls $B_\epsilon(x_i)$ for each of these minima. Then one may proceed as before. The only difference is that now there appears one value c_i for each of the minima that is yet to be determined. One sees that in such a case $\bar{\lambda}(D)$ is determined by a variational formula

$$\bar{\lambda}(D) = \min_{c_1,\dots,c_l \geq 0} \frac{\int_D e^{-F(y)/\epsilon} \|\nabla h(c_1,\dots,c_l)\|_2^2}{\|h(c_1,\dots,c_l)\|_2^2} (1 + O(\epsilon^{\alpha/2}, e^{-\delta/\alpha})) \tag{3.27}$$

where

$$\begin{aligned} L_\epsilon h(x_1,\dots,c_l)(y) = 0, &\quad y \in D \setminus \cup_{i=1}^l \partial B_\epsilon(x_i) \\ h(c_1,\dots,c_l)(y) = c_i, &\quad y \in \partial B_\epsilon(x_i) \end{aligned} \tag{3.28}$$

It is easy to see that the result differs only by a constant factor from that in the non-degenerate case stated in the proposition.

Uniform estimates on principal eigenfunctions. The proof of Proposition 3.2 has already provided us with an approximation for the principal eigenfunction, namely h_{B,D^c}. We have seen that in L^2 this approximation is good on the order $\epsilon^{\alpha/2}$. We will now show that this approximation is also uniformly good.

Proposition 3.3: *Under the hypothesis of Proposition 3.2, the principal eigenfunction, ϕ_D, of $L_\epsilon^{D^c}$, normalized such that $\inf_{y\in\partial B} \phi_D = 1$, satisfies*

$$h_{B,D^c}(y) \leq \phi_D(y) \leq h_{B,D^c}(y)(1 + C\epsilon^{\alpha/2})(1 + e^{-\delta/\epsilon}) \tag{3.29}$$

Proof: Let us first assume that D is bounded. Observe that set $\delta f^\lambda = f^\lambda - f^0$. Then δf^λ satisfies the Dirichlet problem

$$\begin{aligned} L_\epsilon \delta f^\lambda(y) = \lambda \psi^\lambda(y), &\quad y \in D \setminus \partial B \\ \delta f^\lambda(y) = 0, &\quad y \in \partial B \\ \delta f^\lambda(y) = 0, &\quad y \in D^c \end{aligned} \tag{3.30}$$

Thus we can write

$$\frac{\delta f^\lambda(y)}{h_{B,D^c}(y)} = \int_{D\backslash B} \frac{1}{h_{B,D^c}(y)} G_{D\backslash B}(y,z) h_{B,D^c}(z) \frac{\delta f^\lambda(z)}{h_{B,D^c}(z)} \tag{3.31}$$

Assume that $M \equiv \sup_{y\in D\backslash B} \frac{\phi_D(y)}{h_{B,D^c}(y)} < \infty$. Then (3.31) together with (3.24) implies that

$$\begin{aligned} M &\leq 1 + C\epsilon^{\alpha/2} + \lambda M \sup_{y\in D\backslash B} \int_{D\backslash B} \frac{1}{h_{B,D^c}(y)} G_{D\backslash B}(y,z) h_{B,D^c}(z) \\ &= 1 + C\epsilon^{\alpha/2} + \lambda M \sup_{y\in D\backslash B} \mathbb{E}_y \left[\tau_B | \tau_B \leq \tau_{D^c}\right] \end{aligned} \tag{3.32}$$

Using the representation of the conditional mean time from Proposition 6.1 of [BEGK3], one shows that

$$\sup_{y\in D\backslash B} \mathbb{E}_y \left[\tau_B | \tau_B \leq \tau_{D^c}\right] = 1/\bar{\lambda}(D\backslash B) \tag{3.33}$$

so that

$$M \leq \frac{1 + C\epsilon^{\alpha/2}}{1 - \bar{\lambda}(D)/\bar{\lambda}(D\backslash B)} \leq (1 + C\epsilon^{\alpha/2})(1 + e^{-\delta/\epsilon}) \tag{3.34}$$

Since by construction $h_{B,D^c}(y) \leq \phi_D(y)$, the assertion of the proposition follows.

It remains to justify the assumption $M < \infty$. However, this is easy. First, ϕ_D is bounded and $C^2(D)$. Thus, $\frac{\phi_D(y)}{h_{B,D^c}(y)}$ may only diverge when $h_{B,D^c}(y) \downarrow 0$. However, since h_{B,D^c}, is harmonic and non-negative on the boundary, it is strictly positive on D by the strong maximum principle. Thus its explosion can occur only at the boundary of D where $h_{B,D^c}(y)$ tends to zero. Moreover, its normal derivative on ∂D is strictly (and since $\bar{D}$ is compact, uniformly) positive (see e.g. Section 5, Proposition 2.2 of [Tay]). Therefore $\frac{\phi_D(y)}{h_{B,D^c}(y)}$ remains bounded also when $y \to \partial D$.

Therefore the proposition is proven if $\bar{D}$ is compact.

In the non-compact case, we can obtain a similar result for the supremum over compact subsets $\Gamma \subset \mathbb{R}^d$, using the rapid decay of the Green's function in regions where $F(y)$ is getting very large. $\diamondsuit$

Eigenfunction and their zeros. We are now ready to derive the crucial a priori estimates on eigenfunctions of L_ϵ (possibly on some domain Ω). Assume that λ is an exponentially small eigenvalue of L_ϵ and let ϕ^λ denote a corresponding eigenfunction. Then we can decompose Ω into open subsets D_i such that either $\phi^\lambda(y) > 0$ for all $y \in D_i$, or $\phi^\lambda(y) < 0$ for all $y \in D_i$, while $\phi^\lambda(y) = 0$, if $y \in \partial D_i$.

Obviously, the restriction of ϕ^λ to D_i is the principal eigenfunction ϕ_{D_i} of D_i, and λ is the principal eigenvalue of any of the sets D_i. This entails that

Lemma 3.4: *Assume that $\lambda \leq e^{-a/\epsilon}$, for some $a \geq a_0$, with $a_0 > 0$ independent of ϵ. Let D_i be the corresponding sets defined above. Then, each set D_i contains at least one minimum x_i of $F(x)$ for which $\mathrm{cap}_{\,\partial B_\epsilon(x_i)}(D_i^c)/\|h_{B_\epsilon(x_i),D_i^c}\|_2^2 \approx \lambda$; for all other minima x_{ij} of F in D_i, $\mathrm{cap}_{\,B_\epsilon(x_{ij})}(D_i^c)/\|h_{B_\epsilon(x_{ij}),D_i^c}\|_2^2 \geq \lambda$. In particular, $\mathrm{dist}(x_i, D_i^c) \geq \rho > 0$, for some ρ independent of ϵ.*

Moreover, if $x_{ij} \in D_i$ is a minimum of F such that $\mathrm{dist}(x_{ij}, D_i^c) \leq 10\sqrt{\epsilon}$, then there exists a constant C such that for all $y \in B_{5\sqrt{\epsilon}}(x_{ij}) \cap D_i$,

$$\phi^\lambda(y) \leq C\epsilon^{-d}\mathrm{cap}_{\,B_\epsilon(x_{ij})}(B_\epsilon(x_i)) \tag{3.35}$$

Proof: The first two assertions follow from Lemma 2.5 and Proposition 3.2 (plus the remark following its proof). The last assertion follows since by Proposition 3.3 and the estimate on the equilibrium potential from Proposition 4.3 of [BEGK3],

$$\phi^\lambda(y) \leq C\frac{\mathrm{cap}_{\,B_\epsilon(y)}(B_\epsilon(x_i))}{\mathrm{cap}_{\,B_\epsilon(y)}(D_i^c)} \tag{3.36}$$

But since $\mathrm{dist}(y, D_i^c) \leq 15\sqrt{\epsilon}$, and y closer than $5\sqrt{\epsilon}$ from a local minimum, $F(z^*(y, D_i^c)) - F(y) \leq C\epsilon$. Thus Proposition 4.7 of [BEGK3] yields

$$\mathrm{cap}_{\,B_\epsilon(y)}(D_i^c) \geq C\epsilon^{d-1/2} \tag{3.37}$$

Finally, $\mathrm{cap}_{\,B_\epsilon(y)}(B_\epsilon(x_i)) \approx \mathrm{cap}_{\,B_\epsilon(x_{ij})}(B_\epsilon(x_i))$ follows e.g. from the explicit formulae obtained in [BEGK3]. This proves the lemma.$\diamondsuit$

Exponentially small eigenvalues and their eigenfunctions. Let us now order all minima x_i of F in such a way that

$$F(z^*(x_{i+1}, \mathcal{M}_i)) - F(x_{i+1}) \leq F(z^*(x_i, \mathcal{M}_{i-1})) - F(x_i) \tag{3.38}$$

for $i = 1, \dots, n-1$, where $\mathcal{M}_i = \{x_1, \dots, x_i\}$. We put moreover $\mathcal{M}_0 \equiv \Omega^c$. We also set $B_i \equiv B_\epsilon(x_i)$ and $\mathcal{S}_i = \cup_{j=1}^i B_i$. Note that considerable simplifications occur when all inequalities in (3.38) are strict, and we will only consider this case here.

Suppose that we want to compute eigenvalues below $\bar{\lambda}(\Omega\backslash\mathcal{S}_k) \equiv \bar{\lambda}_k$. We know that if ϕ^λ is an eigenfunction with $\lambda < \bar{\lambda}_k$, then it can be represented as the solution of the Dirichlet problem

$$\begin{aligned} (L_\epsilon - \lambda)f^\lambda(y) = 0, \quad y \in \Omega\backslash\partial\mathcal{S}_k \\ f^\lambda(y) = \phi^\lambda(y), \quad y \in \partial\mathcal{S}_k \end{aligned} \tag{3.39}$$

Thus, as in the analysis of principle eigenvalues above, the condition on λ will be the existence of a non-trivial ϕ^λ on $\partial\mathcal{S}_k$ such that the surface measure

$$dy e^{-F(y)/\epsilon}(L_\epsilon - \lambda)f^\lambda(y) = e^{-F(y)/\epsilon}(\partial_{n(y)}f^\lambda(y) + \partial_{-n(y)}f^\lambda(y))d\sigma_{\mathcal{S}_k}(y) \tag{3.40}$$

vanishes. A necessary condition for this to happen is of course the vanishing of the total mass on each of the surfaces ∂B_i, $i \leq k$, i.e.

$$\int_{\partial B_i} e^{-F(y)/\epsilon}(\partial_{n(y)}f^\lambda(y) + \partial_{-n(y)}f^\lambda(y))d\sigma_{\mathcal{S}_k}(y) = 0 \tag{3.41}$$

Now if we knew a priori that all minima x_i, $i \leq k$ lie well within the interior of the sets D_i on which ϕ^λ has constant sign, we could use Lemma 3.1 as before in the analysis of principle eigenvalues to show that f^λ is close to the solution of the problem (3.40) where the boundary conditions are replaced by constant values c_i. Unfortunately we do not know this. We know, however, that each connected component D_i contains at least one such minimum, while at those minima that lie close to the boundary of D_i, ϕ^λ is very small (by Lemma 3.4).

In fact we have the following dichotomy: Let $c_i = \inf_{y\in B_i}\phi^\lambda(y)$. Then *either*

(i) $|\sup_{y\in B_i}\phi^\lambda(y)/c_i - 1| \leq C\epsilon^{\alpha/2}$, *or*

(ii) there exists $1 \leq j \leq k$ such that $\sup_{y\in B_i}|\phi^\lambda(y)|/|c_j| \leq C\epsilon^{-d}\text{cap}_{B_i}(B_j)$.

We now consider all possible cases: Let $J \subset \{1,\dots,k\}$ be the set of indices where (i) holds, and let J_j be the subset of indices j where (ii) holds with j. Given such a partition, we set

$$f^\lambda = \sum_{j\in J} c_j\left(h^\lambda_{B_j,\mathcal{S}_k\backslash B_j} + \psi^\lambda_j\right) \tag{3.42}$$

where the $h^\lambda_j \equiv h^\lambda_{B_j,\mathcal{S}_k\backslash B_j}$ are the λ-equilibrium potentials (see [BEGK3], Section 2), i.e. solutions of $(L_\epsilon - \lambda)h_j = O$ with boundary conditions 1 on ∂B_j and 0 on $\partial(\mathcal{S}_k\backslash B_j)$.

Then ψ^λ_j satisfies, for $j \in J$

$$\begin{aligned} (L_\epsilon - \lambda)\psi^\lambda_j(y) &= 0, \quad y \in \Omega\backslash\partial\mathcal{S}_k \\ \psi^\lambda_j(y) &= \phi^\lambda(y)/c_j - 1, \quad y \in \partial B_j \\ \psi^\lambda_j(y) &= \phi^\lambda(y)/c_j, \quad y \in \partial B_l, l \in J_j \\ \psi^\lambda_j(y) &= 0, \quad y \in \partial B_i, i \notin J_j \end{aligned} \tag{3.43}$$

We now proceed as in the analysis of principle eigenvalues, i.e. we write as necessary condition for λ to be an eigenvalue that for all $i = 1, \dots, k$,

$$\begin{aligned} 0 &= \int_{\partial B_i} e^{-F(y)/\epsilon} h_i(y) \left(\partial_{n(y)} f^\lambda(y) + \partial_{-n(y)} f^\lambda(y) \right) d\sigma_{\partial \mathcal{S}_k}(y) \\ &= \int_{\partial \mathcal{S}_k} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) f^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y) \\ &- \frac{\lambda}{\epsilon} \int dy e^{-F(y)/\epsilon} h_i(y) f^\lambda(y) \\ &= \sum_{j\in J} c_j \Bigg[\int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y)(1+\psi_j^\lambda(y)) d\sigma_{\partial \mathcal{S}_k}(y) \\ &\qquad + \sum_{l \in J_j} \int_{\partial B_l} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) \phi^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y) \\ &- \frac{\lambda}{\epsilon} \left(\int dy e^{-F(y)/\epsilon} h_i(y) (h_j^\lambda(y) + \psi_j^\lambda(y)) + \sum_{l\in J_j} dy e^{-F(y)/\epsilon} h_i(y) f^\lambda(y) \right) \Bigg] \end{aligned} \tag{3.44}$$

Note that by the bounds (i) and (ii),

$$\begin{aligned} &\left| \int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) \psi_j^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y) + \sum_{l\in J_j} \int_{\partial B_l} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) \phi^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y) \right| \\ &\le \left| \int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) d\sigma_{\partial \mathcal{S}_k}(y) \right| \\ &\times \left(C\epsilon^{\alpha/2} + \sum_{l\in J_j} C\epsilon^{-d} \text{cap}_{\,B_l}(B_j) \left| \frac{\int_{\partial B_l} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) d\sigma_{\partial \mathcal{S}_k}(y)}{\int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) d\sigma_{\partial \mathcal{S}_k}(y)} \right| \right) \end{aligned} \tag{3.45}$$

At this point it is convenient to realize that Green's first identity and the fact that the h_i are harmonic, implies that

$$\begin{aligned} \left| \int_{\partial B_j} e^{-F(y)/\epsilon} h_j(y) \partial_{n(y)} h_i(y) d\sigma_{B_j}(y) \right| &= e^{-1} \left| \int_{\text{ext}\, \mathcal{S}_k} dy e^{-F(y)/\epsilon} \left(\nabla h_j(y), \nabla h_i(y) \right) \right| \\ &\le \epsilon^{-1} \sqrt{\text{cap}_{\,B_i}(\mathcal{S}_k \backslash B_i) \text{cap}_{\,B_j}(\mathcal{S}_k \backslash B_j)} \end{aligned} \tag{3.46}$$

where the last inequality uses the Cauchy-Schwartz inequality. Noting further that, since

$$\text{cap}_{\,B_l}(B_j) = \text{cap}_{\,B_j}(B_l) \le \text{cap}_{\,B_j}(\mathcal{S}_k \backslash B_j) \tag{3.47}$$

we can bound (3.45) from above by

$$
\begin{aligned}
&\left| \int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) d\sigma_{\partial \mathcal{S}_k}(y) \right| \\
&\times \left(C\epsilon^{\alpha/2} + \sum_{l \in J_j} C\epsilon^{-d} \operatorname{cap}_{B_j}(\mathcal{S}_k \backslash B_j) \left| \frac{\sqrt{\operatorname{cap}_{B_i}(\mathcal{S}_k \backslash B_i) \operatorname{cap}_{B_l}(\mathcal{S}_k \backslash B_l)}}{\int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) d\sigma_{\partial \mathcal{S}_k}(y)} \right| \right)
\end{aligned}
\tag{3.48}
$$

In particular, in the case when $i = j$, this simplifies to

$$
\operatorname{cap}_{B_j}(\mathcal{S}_k \backslash B_j) \left(C\epsilon^{\alpha/2} + \sum_{l \in J_j} C\epsilon^{-d} \sqrt{\operatorname{cap}_{B_i}(\mathcal{S}_k \backslash B_i) \operatorname{cap}_{B_l}(\mathcal{S}_k \backslash B_l)} \right) \tag{3.49}
$$

For the terms in the last line of (3.44) we obtain in complete analogy to the derivation of the bounds (3.20) and (3.21) that

$$
\begin{aligned}
&\int dy e^{-F(y)/\epsilon} h_i(y)(h_j^\lambda(y) - h_j(y) + \psi_j^\lambda) + \sum_{l \in J_j} \int dy e^{-F(y)/\epsilon} h_i(y) f^\lambda(y) \\
&= O(\epsilon^{\alpha/2})(1 + O(e^{-\delta/\epsilon})) \int dy e^{-F(y)/\epsilon} h_i(y) h_j(y) \\
&+ O\left(C\epsilon^{-d} \sum_{l \in J_j} \operatorname{cap}_{B_l}(B_j) \int dy e^{-F(y)/\epsilon} h_i(y) h_j(y) (1 + e^{-\delta/\epsilon}) \right)
\end{aligned}
\tag{3.50}
$$

But note that

$$
\begin{aligned}
\operatorname{cap}_{B_l}(B_j) \int dy e^{-F(y)/\epsilon} h_i(y) h_j(y) &\leq \operatorname{cap}_{B_l}(B_j) \|h_i\|_2 \|h_l\|_2 \\
&\leq \frac{\|h_l\|_2}{\|h_j\|_2} \operatorname{cap}_{B_l}(B_j) \|h_i\|_2 \|h_j\|_2 \\
&\leq e^{-\delta/\epsilon} \|h_i\|_2 \|h_j\|_2
\end{aligned}
\tag{3.51}
$$

Let us define the classical capacity matrix[1] $\mathcal{C}$ with elements

$$
\mathcal{C}_{ij} \equiv \mathcal{C}_{ij}^{(k)} \equiv \epsilon \int_{\partial B_j} e^{-F(y)/\epsilon} h_j(y) \partial_{n(y)} h_i(y) d\sigma_{B_j}(y)
$$

and its normalized version

$$
\mathcal{K}_{ij} \equiv \mathcal{K}_{ij}^{(k)} \equiv \frac{\mathcal{C}_{ij}^{(k)}}{\|h_i\|_2 \|h_j\|_2} \tag{3.52}
$$

[1] The matrix $\mathcal{C}$ is a classical object in electrostatics, the diagonal elements being called capacities, and the off-diagonal ones coefficients of induction [Jack]. The off-diagonal coefficients represent the charge induced in the i-th ball when the j-th has potential one and all others are at potential zero.

Note that this matrix is symmetric[2]. If we introduce the matrices

$$A_{ij} \equiv \frac{\epsilon \int_{\partial B_j} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) \psi_j^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y) + \sum_{l \in J_j} \epsilon \int_{\partial B_l} e^{-F(y)/\epsilon} \partial_{n(y)} h_i(y) \phi^\lambda(y) d\sigma_{\partial \mathcal{S}_k}(y)}{\|h_i\|_2 \|h_j\|_2} \tag{3.53}$$

$$B_{ij} \equiv \mathbb{1}_{i \neq j} \frac{\int dy e^{-F(y)/\epsilon} h_i(y)(h_j^\lambda(y) + \psi_j^\lambda) + \sum_{l \in J_j} dy e^{-F(y)/\epsilon} h_i(y) \psi_j^\lambda(y)}{\|h_i\|_2 \|h_j\|_2} \tag{3.54}$$

and

$$D_{jj} \equiv \frac{\int dy e^{-F(y)/\epsilon} h_j(y)(h_j^\lambda(y) - h_j(y) + \psi_j^\lambda) + \sum_{l \in J_j} dy e^{-F(y)/\epsilon} h_j(y) \phi_j^\lambda(y)}{\|h_j\|_2^2} \tag{3.55}$$

Then the conditions (3.44) for λ can be written as

$$0 = \sum_{j \in J} \hat{c}_j \left(\mathcal{K}_{ij} - \lambda \delta_{ij} + A_{ij} - \lambda (D_{jj} + B_{ij}) \right) \tag{3.56}$$

where $\hat{c}_j = \|h_j\|_2 c_j$. To show that all the off-diagonal terms in B_{ij} are small, we still need to show that the normalized functions h_i and h_j are almost orthogonal.

Lemma 3.5: *There is a constant $C < \infty$ such that*

$$\begin{aligned} \max_{i \neq j} \frac{\int dy e^{-F(y)/\epsilon} h_j(y) h_i(y)}{\|h_i\|_2 \|h_j\|_2} &\leq C \epsilon^{-(d+1)/2} \max_i e^{-[F(z^*(x_i, \mathcal{S}_k \backslash B_i)) - F(x_i)]/\epsilon} \\ &\leq e^{-\theta/\epsilon} \end{aligned} \tag{3.57}$$

for some $\theta > 0$.

Proof: Note first that the terms in the denominator in (3.57) are bounded via

$$\begin{aligned} \int dy e^{-F(y)/\epsilon} h_j^2(y) &\geq \int_{B_{\sqrt{\epsilon}}(x_j)} dy e^{-F(y)/\epsilon} \left(1 - C \epsilon^{-1/2} e^{-[F(z^*(x_j, \mathcal{S}_k \backslash B_j))]/\epsilon} \right)^2 \\ &= C \epsilon^{d/2} e^{-F(x_j)/\epsilon} \end{aligned} \tag{3.58}$$

On the other hand, for $i \neq j$,

$$\begin{aligned} \int dy e^{-F(y)/\epsilon} h_j(y) h_i(y) &= \int_{y : F(y) \leq \max(F(z^*(x_i, \mathcal{S}_k \backslash B_i)), F(z^*(x_j, \mathcal{S}_k \backslash B_j)))} dy e^{-F(y)/\epsilon} h_j(y) h_i(y) \\ &+ \int_{y : F(y) > \max(F(z^*(x_i, \mathcal{S}_k \backslash B_i)), F(z^*(x_j, \mathcal{S}_k \backslash B_j)))} dy e^{-F(y)/\epsilon} h_j(y) h_i(y) \end{aligned} \tag{3.59}$$

[2] One could also introduce a matrix $\widehat{\mathcal{K}}_{ij} = \mathcal{C}_{ij} / \|h_i\|_2^2$ which then would be a stochastic matrix (resp. sub-stochastic, if Dirichlet boundary conditions are imposed on Ω^c).

In the second integral we just use that $h_i(y) \leq 1$; by our general assumptions on F, this gives a bound $Ce^{-\max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)),F(z^*(x_j,\mathcal{S}_k\backslash B_j)))/\epsilon}$. In the first integral we use the bounds on the equilibrium potential from Corollary 4.8 of [BEGK3]. Note that for any y, at most one of the factors $h_i(y)$ or $h_j(y)$ can be close to one. Thus even the roughest estimate yields that[3]

$$
\begin{aligned}
&\int_{y:F(y)\leq\max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)),F(z^*(x_j,\mathcal{S}_k\backslash B_j)))} dy e^{-F(y)/\epsilon} h_j(y) h_i(y) \\
&\leq \int_{y:F(y)\leq\max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)),F(z^*(x_j,\mathcal{S}_k\backslash B_j)))} dy e^{-F(y)/\epsilon} \\
&\qquad \times C\epsilon^{-1/2} e^{-[\max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)),F(z^*(x_j,\mathcal{S}_k\backslash B_j)))-F(y)]/\epsilon} \\
&\leq C\epsilon^{-1/2} |\{y : F(y) \leq \max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)), F(z^*(x_j,\mathcal{S}_k\backslash B_j)))\}| \\
&\qquad \times e^{-\max(F(z^*(x_i,\mathcal{S}_k\backslash B_i)),F(z^*(x_j,\mathcal{S}_k\backslash B_j)))/\epsilon}
\end{aligned}
\tag{3.60}
$$

Combining this upper bound with the lower bound we arrive at the assertion of the lemma.$\diamondsuit$

We can now collect the estimates on these matrix elements:

$$|B_{ij}| \leq e^{-\theta/\epsilon} \tag{3.61}$$

$$|D_{jj}| \leq C\epsilon^{\alpha/2} \tag{3.62}$$

and for all i, j,

$$|A_{ij}| \leq |\mathcal{K}_{ij}|C\epsilon^{\alpha/2} + e^{-\delta/\epsilon}\sqrt{\mathcal{K}_{ii}\mathcal{K}_{jj}} \tag{3.63}$$

where the last bound uses (3.45) together with (3.46) and (3.47) and the fact that $\sqrt{\mathrm{cap}_{B_j}(\mathcal{S}_k\backslash B_j)\mathrm{cap}_{B_l}(\mathcal{S}_k\backslash B_l)} \leq e^{-\delta/\epsilon}$ for some $\delta > 0$.

We collect the results obtained so far as

Theorem 3.6: *Let $\mathcal{S}_k \equiv \cup_{i=1}^k B_\epsilon(x_i)$ and let $\bar\lambda_k$ denote the principal eigenvalue of the operator L_ϵ with Dirichlet conditions on $\mathcal{S}_k$ (and possibly an additional set Ω). Then a number $\lambda < \bar\lambda_k$ may be an eigenvalue of the operator L_ϵ, if there exists a nonempty set $J \subset \{1, \ldots, k\}$ such that, if $\mathcal{G}(\lambda)$ denotes the $|J| \times |J|$ matrix with elements*

$$\mathcal{G}_{ij}(\lambda) \equiv \mathcal{K}_{ij} + A_{ij} - \lambda(\delta_{ij} + D_{ii} + B_{ij}) \tag{3.64}$$

$i, j \in J$,

$$\det(\mathcal{G}(\lambda)) = 0 \tag{3.65}$$

[3] See the proof of (3.79) for more details.

Remark: Note that the matrix $\mathcal{G}$ depends on the constants c_i; however, this will not bother us: in fact, we will only use Theorem 3.6 to derive conditions on λ uniformly in all symmetric matrices A, B, D satisfying the bounds(3.61), (3.62) and (3.63).

The usefulness of this theorem arises from the fact that we can control the eigenvalues to a very good precision in terms of the *capacity matrix.*

Theorem 3.7: *Under the same hypothesis as in the preceding theorem, if $\lambda < \bar{\lambda}_k$ is an eigenvalue of L_ϵ, then for some nonempty set $J \subset \{1, \dots, k\}$ there exists exists an eigenvalue μ of the $|J| \times |J|$-matrix $\mathcal{K}^J + A^J$ (where by A^J we understand the matrix made of the indices A_{ij}, $i, j \in J$, etc.) such that $\lambda = \mu\left(1 + O\left(e^{-D/\epsilon}, \lambda/(\bar{\lambda}_k - \lambda)\right)\right)$.*

Proof: The proof will rely on Theorem 7.1 that we prove in the appendix. Since

$$\mathcal{G}(\lambda) = \mathcal{K}^J + A^J - \lambda(\mathbb{1} + (B^J + D^J)) \tag{3.66)}$$

to apply Theorem 7.1 requires us to bound the norm of $B^J + D^J$. As a consequence of estimates (3.54) and (3.66), the preceding Lemmata, we see that the matrix $B^J + D^J$ is indeed bounded in norm by

$$\|B^J + D^J\| \leq C\epsilon^{\alpha/2} + e^{-\delta/\epsilon} \tag{3.67}$$

for some $\delta > 0$. The theorem follows now from Theorem 7.1 of the appendix.$\diamondsuit$

It remains to estimate the eigenvalues of the matrix $\mathcal{K}^J + \mathcal{A}^J$. We will do this only in the non-degenerate situation when all "depths" of the valleys x_i are distinct, i.e. when for all $i < k$ the inequalities (3.38) are strict.

Let us first consider the case $J = \{1, \dots, n\}$.

Lemma 3.8: *Let $\mathcal{K}_{ij}$ be the normalized capacity matrix and assume that*

$$\max_{i<k} \mathcal{K}_{ii} \leq e^{-\delta/\epsilon} \mathcal{K}_{kk} \tag{3.68}$$

Then the largest eigenvalue, μ_k, of $\mathcal{K} + A$ satisfies

$$\mu_k = \mathcal{K}_{kk}(1 + O(e^{-\delta/2\epsilon}, \epsilon^{\alpha/2})) \tag{3.69}$$

while all other eigenvalues are smaller than $Ce^{-\delta/\epsilon}\lambda_k$. Moreover, the eigenvector, $v = (v_1, \dots, v_k)$, corresponding to the largest eigenvalues normalized s.t. $v_k = 1$ satisfies $v_i \leq Ce^{-\delta/\epsilon}$, for $i < k$.

Proof: This is a simple perturbation argument. Note that we can write

$$\mathcal{K} = \hat{\mathcal{K}} + \check{\mathcal{K}} \tag{3.70}$$

where $\hat{\mathcal{K}}_{ij} = \mathcal{K}_{kk}\delta_{jk}\delta_{ik}$. In the same way we decompose $A = \hat{A} + \check{A}$. Now we estimate the norm of $\check{\mathcal{K}}$ as in the proof of Lemma 3.5.

Now recall that

$$|\mathcal{K}_{ji}| \leq \mathcal{K}_{ii}\mathcal{K}_{jj} \tag{3.71}$$

Whence by assumption (3.68),

$$\|\check{\mathcal{K}}\| \leq \mathcal{K}_{kk}\sqrt{\delta k + \delta^2 k^2} \tag{3.72}$$

By the estimate (3.63), we also get that

$$|\check{A}_{ij}| \leq O(e^{-\delta/\epsilon})\mathcal{K}_{kk} \tag{3.73}$$

for $(i,j) \neq (k,k)$, and $|A_{kk}| \leq C\epsilon^{\alpha/2}\mathcal{K}_{kk}$. Since obviously $\hat{\mathcal{K}} + \hat{A}$ has one eigenvalue $\mathcal{K}_{kk} + A_{kk}$ with the obvious eigenvector and all other eigenvalues are zero, the announced result follows from standard perturbation theory. $\diamondsuit$

Since $\mathcal{K}_{kk} = \frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2^2} \approx \bar{\lambda}_{k-1}$, this is precisely the value we expect. Now if we consider a smaller set J, then, if $k \in J$, we will simply find that the largest eigenvalue of $\mathcal{K}^J + A^J$ is the same as before, while if $k \notin J$, we would only get a smaller candidate for an eigenvalue; in fact, we would produce all diagonal elements of $\mathcal{K}$ as candidates for approximate eigenvectors, but only $\mathcal{K}_{kk}$ is larger than $\bar{\lambda}_{k-1}$, so we are not interested in these at present.

Corollary 3.9: *If there exists an eigenvalue λ_k of L_ϵ in the interval $(\bar{\lambda}_k, \bar{\lambda}_{k-1}]$, then*

(i)

$$\lambda_k = \operatorname{cap}_{B_k}(\mathcal{S}_{k-1})/\|h_k\|_2^2\left(1 + O(\epsilon^{\alpha/2}, e^{-\delta/\epsilon})\right) \tag{3.74}$$

(ii) The eigenvalue λ_k is simple and the corresponding eigenfunction f_k^λ can be written as

$$\phi_k^\lambda(y) = \frac{h_k(y)}{\|h_k\|_2}(1 + O(\epsilon^{\alpha/2})) + \sum_{j=1}^{k-1} d_j(y)\frac{h_j(y)}{\|h_j\|_2} \tag{3.75}$$

where $|d_j(y)| \leq e^{-\delta/\epsilon}$ for some $\delta > 0$ (uniformly on compact subsets if Ω is unbounded) .

Proof: It is easy to see that if $k \in J$, then Lemma 3.8 together with the bounds on B and D implies that a solution of (3.56) with $\hat{c}_k = 1$ must satisfy $|\hat{c}_j| \leq e^{-\delta/\epsilon}$ for all $j \neq k$. By (3.42), this implies that

$$\phi_k^\lambda(y) = \frac{h_k^\lambda(y) + \phi_k^\lambda(y)}{\|h_k\|_2} + \sum_{j \in J \setminus k} \hat{c}_j \frac{h_j^\lambda(y) + \phi_k^\lambda(y)}{\|h_j\|_2} \tag{3.76}$$

Using the same arguments as in the proof of Proposition 3.3, and the bounds on $\phi^\lambda - c_J$ on the boundaries ∂B_j, we get that for $j \in J$

$$\frac{|\phi_j^\lambda(y) - h_j(y)|}{\|h_j\|_2} \leq C\epsilon^{\alpha/2} \frac{h_j(y)}{\|h_j\|_2} + \sum_{l \in J_j} \frac{\operatorname{cap}_{B_l}(B_j)\|h_l\|_2}{\|h_j\|_2} \frac{h_l(y)}{\|h_l\|_2} \leq C\epsilon^{\alpha/2} \frac{h_j(y)}{\|h_j\|_2} + \sum_{l \in J_j} e^{-\delta/\epsilon} \frac{h_l(y)}{\|h_l\|_2} \tag{3.77}$$

Combining these estimates we arrive at (3.75). Note that this final estimate does not actually depend on the choice of J. Since it is impossible that two functions satisfying (3.75) are orthogonal, it follows that λ_k is a simple eigenvalue. $\diamondsuit$

Now we can further explore the eigenvalues below $\bar{\lambda}_{k-1}$, etc., with the same results. Thus at the end of the procedure we arrive at the conclusion that L_ϵ can have at most the n simple eigenvalues given by the values of the preceding corollary below the values $C\epsilon^{d-1}$. But since we know that there must be n such eigenvalues, we conclude that all these candidates are in fact eigenvalues. This yields the following proposition:

Proposition 3.10: *Assume that all inequalities (3.38) are strict for all $i = 1, \ldots, n$. Then the spectrum of L_ϵ below ϵ^{d-1} consists of n simple eigenvalues that satisfy:*

$$\begin{aligned}
\lambda_k &= \frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2^2}\left(1 + O(\epsilon^{\alpha/2} + e^{-\delta/\epsilon})\right) \quad, k = 1, \ldots, n \\
&= \operatorname{cap}_{B_k}(\mathcal{S}_{k-1}) \frac{\sqrt{\det(\nabla^2 F(x_k))}}{\sqrt{2\pi\epsilon}^d} e^{F(x_k)/\epsilon} \left(1 + O\left(\epsilon^{1/2}|\ln\epsilon|, \epsilon^{\alpha/2}, e^{-\delta/\epsilon}\right)\right) \\
&= \frac{1}{\mathbb{E}_{x_k} \tau_{\mathcal{S}_{k-1}}}\left(1 + O(\epsilon^{\alpha/2} + e^{-\delta/\epsilon})\right)
\end{aligned} \tag{3.78}$$

The corresponding eigenfunctions satisfy (3.75).

Proof: We have seen in fact that $\lambda_k = \mathcal{K}_{kk}^{(k)}\left(1 + O(e^{-\theta/\epsilon}, \epsilon^{\alpha/2})\right)$, which provides the first assertion of Proposition 3.10. It remains to identify the eigenvalues with the inverse mean times. This follows from Proposition 6.1 in [BEGK3], provided we can show that

$$\int dy e^{-F(y)/\epsilon} h_k^2(y) \sim \int dy e^{-F(y)/\epsilon} h_k(y) \tag{3.79}$$

In fact, we will show more, namely that both sides of (3.79) are asymptotically equal to

$$e^{-F(x_k)/\epsilon}\frac{\sqrt{2\pi\epsilon}^d}{\sqrt{\det(\nabla^2 F(x_k))}} \tag{3.80}$$

We must show that the main contribution of the integrals comes from a small neighborhood of x_k, which yields the contribution (3.80). It is clear that all contributions from the set $y : F(y) > F(x_k) + \epsilon|\ln\epsilon|$ give only sub-leading corrections. To treat the complement of this set, we use the bounds on the equilibrium potential of Eq. (4.27) in [BEGK3]. Up to polynomial factors in ϵ, it implies that the integrand on the right-hand side of (3.79) (and a fortiori on the left-hand side) in the connected components of this level set that do not contain x_k is smaller than

$$e^{-[F(y)+F(z^*(y,B_k))-F(z^*(y,\mathcal{S}_{k-1}))]/\epsilon} \tag{3.81}$$

If y is in the component of the level set that contains the minimum x_j, and $j < k$, we see that this is equal to

$$e^{-F(z^*(x_j,B_k))/\epsilon} \tag{3.82}$$

which is exponentially smaller than $\exp(-F(x_k)/\epsilon)$, independent of y. If $j > k$, we still get the same result if $F(y) \geq F(z^*(x_j,\mathcal{S}_{k-1}))$. Otherwise, we can write (3.81) as

$$e^{-[F(y)-F(x_j)]/\epsilon}e^{-[F(x_k)+(F(z^*(x_j,B_k))-F(x_k))-(F(z^*(x_j,\mathcal{S}_{k-1}))-F(x_j))]/\epsilon} \tag{3.83}$$

We will argue that

$$F(z^*(x_j,B_k)) - F(x_k) > F(z^*(x_j,\mathcal{S}_{k-1})) - F(x_j) \tag{3.84}$$

Assume the contrary. Note that trivially

$$F(z^*(x_j,\mathcal{S}_{k-1})) \geq F(z^*(x_j,\mathcal{S}_{j-1})) \tag{3.85}$$

while

$$F(z^*(x_j,B_k)) = F(z^*(x_k,B_j)) \leq F(z^*(x_k,\mathcal{S}_j\backslash B_k)) \tag{3.86}$$

Therefore, our assumption implies that

$$F(z^*(x_j,\mathcal{S}_{j-1})) - F(x_j) \leq F(z^*(x_k,\mathcal{S}_j\backslash B_k)) - F(x_k) \tag{3.87}$$

which a moments reflection shows to be in contradiction with the conditions (3.38) at stage j. In other words, if our assumption was true, then the set B_k would have had to yields the largest eigenvalue at stage j, i.e. it would have had to be labelled B_j. Thus (3.84) must hold.

Since by assumption the inequalities are strict (which is more than we need), it follows that indeed

$$\int dy e^{-F(y)/\epsilon} h_k(y) = e^{-F(x_k)/\epsilon} \frac{\sqrt{2\pi\epsilon}^d}{\sqrt{\det(\nabla^2 F(x_k))}} \left(1 + O\left(\epsilon^{1/2} |\ln \epsilon|\right)\right) \tag{3.88}$$

and of course the same bound holds when h_k is replaced by h_k^2. This concludes the proof of the theorem.$\diamondsuit$

Improved error estimates. To conclude the proofs of Theorems 1.1 and 1.2 we only need to improve the error estimates. In the proofs of this section we have produced error terms from two sources: the exponentially small errors resulting from the perturbation around $\lambda = 0$ and the not perfect orthogonality of the functions h_i, and the much larger errors of order $\epsilon^{\alpha/2}$ that resulted from the a priori control on the regularity of the eigenfunctions obtained from the Hölder estimate of Lemma 3.1. In the light of the estimates obtained on the eigenfunctions these can now be improved successively (as in the proof of Theorem 3.1 of [BEGK3]). Notice first that the eigenfunction corresponding to the minimum x_k is small enough at all the minima x_l, $l < k$ that we can actually take $J = \{k\}$ and $J_k = \{1, \dots, k-1\}$ in (3.43), (3.44). Then we know from Corollary 3.9 that

$$\operatorname{osc}_{y \in B_{4\sqrt{\epsilon}}(x_k)} \phi_k(y) \leq C\epsilon^{\alpha/2} \sup_{y \in B_{4\sqrt{\epsilon}}(x_k)} \phi_k(y) \tag{3.89}$$

which improves the a priori estimate (3.5). Then the Hölder estimate Lemma 4.1 in [BEGK3] gives the improvement

$$\operatorname{osc}_{y \in B_\epsilon(x_k)} \phi_k(y) \leq C\epsilon^{\alpha/2} \left(C\epsilon^{\alpha/2} + \lambda_k \epsilon^{(d+1)/2}\right) \sup_{y \in B_{4\sqrt{\epsilon}}(x_k)} \phi_k(y) \leq C\epsilon^{\alpha} \sup_{y \in B_{4\sqrt{\epsilon}}(x_k)} \phi_k(y) \tag{3.90}$$

over the estimate (3.3). This allows to replace all errors of order $\epsilon^{\alpha/2}$ by errors of order ϵ^{α}. This procedure can be iterated m times to get errors of order $\epsilon^{m\alpha/2}$ until these are as small as the exponentially small errors.

Finally we would like to improve the precision with which we relate the eigenvalues to the inverse mean exit times. This precision is so far limited by the precision with which

$$\mathbb{E}_{x_k} \tau_{\mathcal{S}_{k-1}} \approx \frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2} \tag{3.91}$$

holds. From Proposition 6.1 of [BEGK3] we know that this precision is limited only by the variation of $\mathbb{E}_x \tau_{\mathcal{S}_{k-1}}$ on B_k. To improve this, we need to control the

$$\frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2} - \frac{\operatorname{cap}_{B_\epsilon(x)}(\mathcal{S}_{k-1})}{\|h_{B_\epsilon(x),\mathcal{S}_{k-1}}\|_2} \tag{3.92}$$

Now it is very simple so see that if $x \in B_{\sqrt{\epsilon}}(x_k)$, then

$$|h_{B_\epsilon(x),\mathcal{S}_{k-1}}(y) - h_k(y)| \leq e^{-\delta/\epsilon} h_k(y) \tag{3.93}$$

Namely,

$$\begin{aligned} &|h_{B_\epsilon(x),\mathcal{S}_{k-1}}(y) - h_k(y)| \\ &\leq \mathbb{P}_y \left[\{\tau_{B_k} < \tau_{\mathcal{S}_{k-1}}\} \cap \{\tau_{\mathcal{S}_{k-1}} < \tau_{B_\epsilon(x)}\} \right] + \mathbb{P}_y \left[\{\tau_{B_\epsilon(x)} < \tau_{\mathcal{S}_{k-1}}\} \cap \{\tau_{\mathcal{S}_{k-1}} < \tau_{B_k} \tau_{B_\epsilon(x)}\} \right] \end{aligned} \tag{3.94}$$

But by the Markov property

$$\begin{aligned} &\mathbb{P}_y \left[\{\tau_{B_k} < \tau_{\mathcal{S}_{k-1}}\} \cap \{\tau_{\mathcal{S}_{k-1}} < \tau_{B_\epsilon(x)}\} \right] \\ &\leq \mathbb{P}_y \left[\tau_{B_k} < \tau_{\mathcal{S}_{k-1}} \right] \max_{z \in B_k} \mathbb{P} \left[\tau_{\mathcal{S}_{k-1}} < \tau_{B_\epsilon(x)} \right] \leq e^{-\delta/\epsilon} \mathbb{P}_y \left[\tau_{B_k} < \tau_{\mathcal{S}_{k-1}} \right] \end{aligned} \tag{3.95}$$

The second summand in (3.94) is bounded in the same way.

This implies of course that

$$\|h_{B_\epsilon(x),\mathcal{S}_{k-1}}\|_2 - \|h_k\|_2 \leq e^{-\delta/\epsilon} \|h_k\|_2 \tag{3.96}$$

We only need a similar estimate for capacities. While this may appear more difficult at first sight, we can take advantage of the fact that as long as $\bar{\lambda}((B_\epsilon(x) \cup \mathcal{S}_{k-1})^c) \gg \lambda_k$, we can replace B_k in the proof of Proposition 3.10 without further changes by $B_\epsilon(x)$. Thus

$$\lambda_k = \frac{\operatorname{cap}_{B_\epsilon(x)}(\mathcal{S}_{k-1})}{\|h_{B_\epsilon(x),\mathcal{S}_{k-1}}\|_2^2} \left(1 + O(e^{-\delta/\epsilon})\right) = \frac{\operatorname{cap}_{B_k}(\mathcal{S}_{k-1})}{\|h_k\|_2^2} \left(1 + O(e^{-\delta/\epsilon})\right) \tag{3.97}$$

which implies together with (3.96) that

$$|\operatorname{cap}_{B_\epsilon(x)}(\mathcal{S}_{k-1}) - \operatorname{cap}_{B_k}(\mathcal{S}_{k-1})| \leq e^{-\delta/\epsilon} \operatorname{cap}_{B_k}(\mathcal{S}_{k-1}) \tag{3.98}$$

Based on (3.98) and (3.95), one can improve Proposition 6.1 of [BEGK3] iteratively as above to yield

$$\mathbb{E}_{x_k} \tau_{\mathcal{S}_k} = \frac{\operatorname{cap}_{B_k} \mathcal{S}_{k-1}}{\|h_k\|_2} \left(1 + O(e^{-\delta/\epsilon})\right) \tag{3.99}$$

which implies the first equality in Theorem 1.1. Thus all error terms of order $\epsilon^{\alpha/2}$ can be removed in (3.78) and (3.75), completing the proofs of Theorems 1.1 and Theorem 1.2. $\diamondsuit\diamondsuit$

Exponential distribution of exit times. We conclude this chapter with a result that will imply Theorem 1.3 on the exponential distribution of exit times. Let L_ϵ^D denote the Dirichlet operator with Dirichlet conditions in D. To avoid confusion, we assume that $D = \mathcal{S}_{k-1}$. Note that Proposition 3.10 (and its impovement) also applies to the operator L_ϵ^D, and if we denote by $\bar\lambda_k^i$ the $i-th$ eigenvalue of L_ϵ^D, we see that within our usual errors,

$$\bar\lambda_k^i \sim \lambda_{k+i} \tag{3.100}$$

for $i=1,\dots,n-k$, and the corresponding eigenfunction $\bar\phi_k^i$ satisfies

$$\phi_k^i(y) = \frac{h_{k+i}(y)}{\|h_{k+i}\|_2}(1+O(e^{-\delta/\epsilon})) + \sum_{j=k}^n d_j(y)h_j(y)\|h_j\|_2 \tag{3.101}$$

with $|d_j(y)| \le e^{-\delta/\epsilon}$. Let us denote henceforth by $\bar\phi_k^i$ the corresponding normalized eigenfunctions (e.g. $\|\bar\phi_k^i\|_2 = 1$). Note that $\bar\phi_k^i = \phi_k^i(1+O(e^{-\delta/\epsilon}))$, so in fact they can be represented in the same way as (3.101) with redefined d_j satisfying the same bounds.

Denote by P_{k-i} the projector on the subspace generated by ϕ_{k-1}^i and by $P_\perp$ the projector to the subspace orthogonal to $\mathrm{span}(\bar\phi_{k-1}^1,\dots,\bar\phi_{k-1}^{n-k})$. Note that

$$\begin{aligned}
\mathbb{P}_{x_k}\left[\tau_D > T\right] &= \left(\delta_{x_k}, e^{-TL_\epsilon^D}\mathbb{1}_{D^c}\right)\\
&= \sum_{i=1}^{n-k}\left(\delta_{x_k}, e^{-TL_\epsilon^D}P_{k-i}\mathbb{1}_{D^c}\right) + \left(\delta_{x_k}, e^{-TL_\epsilon^D}P_\perp\mathbb{1}_{D^c}\right)\\
&= \sum_{i=1}^{n-k} e^{-\bar\lambda_{k-1}^i T}\bar\phi_{k-1}^i(x_k)\int_{D^c} dy\, e^{-F(y)/\epsilon}\bar\phi_{k-1}^i(y) + O\left(e^{-T\bar\lambda_n}\right)
\end{aligned} \tag{3.102}$$

Given the precise control on the eigenfunctions, it is not difficult to obtain that

$$\begin{aligned}
\mathbb{P}_{x_k}\left[\tau_D > T\right] &= \left(\delta_{x_k}, e^{-TL_\epsilon^D}\mathbb{1}_{D^c}\right)\\
&= \sum_{i=1}^{n-k} e^{-\bar\lambda_{k-1}^i T}\bar\phi_{k-1}^i(x_k)\int_{D^c} dy\, e^{-F(y)/\epsilon}\bar\phi_{k-1}^i(y) + O\left(e^{-T\bar\lambda_n}\right)
\end{aligned} \tag{3.103}$$

Now using (3.79), (3.80), we get

$$\begin{aligned}
&\bar\phi_{k-1}^i(x_k)\int_{D^c} dy\, e^{-F(y)/\epsilon}\bar\phi_{k-1}^i(y)\\
&= h_{k-1+i}(x_k)\frac{\int_{D^c} dy\, e^{-F(y)/\epsilon}h_{k-1+i}(y)}{\|h_{k-1+i}\|_2^2} + \sum_{(j,j')\neq(k-1+i,k-1+i)} d_j d_{j'} h_j(x_k)\frac{\int_{D^c} dy\, e^{-F(y)/\epsilon}h_{j'}(y)}{\|h_j\|_2\|h_j\|_2}\\
&= h_{k-1+i}(x_k)(1+O(e^{-\delta/\epsilon})) + \sum_{(j,j')\neq(k-1+i,k-1+i)} d_j d_{j'} h_j(x_k) c_{j,j'} e^{-[F(x_j')-F(x_j)]/2\epsilon}
\end{aligned} \tag{3.104}$$

Now if $j = k$, the term in the last sum is

$$d_k d_{j'} c_{k,j'} e^{-[F(x_j')-F(x_k)]/2\epsilon} \leq e^{-\delta/\epsilon} \tag{3.105}$$

since $F(x_{j'}) > F(x_k)$ for $j'' > k$; in all other cases,

$$h_j(x_k) \approx e^{-[F(z^*(x_j, \mathcal{M}_{k+i} \backslash x_j) - F(x_k))]/\epsilon} < e^{-[F(x_j)-F(x_k)]/\epsilon} \tag{3.106}$$

so that

$$h_j(x_k) e^{-[F(x_j')-F(x_j)]/2\epsilon} < e^{-[F(x_j)-F(x_k)]/2\epsilon} e^{-[F(x_j')-F(x_k)]/2\epsilon} < \epsilon^{-\delta/\epsilon} \tag{3.107}$$

This shows that

$$\mathbb{P}_{x_k}\left[\tau_D > T\right] = e^{-\bar\lambda_{k-1} T}(1 + O(e^{-\delta/\epsilon})) + \sum_{i=2}^{n-k} e^{-\bar\lambda_{k-1}^i} O(e^{-\delta/\epsilon}) + O(1) e^{-T \bar\lambda_{k-1}^{n-k}} \tag{3.108}$$

This proves Theorem 1.3. $\diamondsuit\diamondsuit$

A. Appendix

In this appendix we prove a general perturbation estimate that is needed in Section 5. This should be well-known, but we have not been able to find a precise references.

Theorem 7.1: *Let A be a self-adjoint operator in some (finite-dimensional) Hilbert space $L^2(\Omega, \mu)$. Let $B(\lambda)$ a continous family of bounded operators on the same space that satisfies the bound $\|B(\lambda)\| \leq \delta + \lambda C$ for $0 \leq \delta \ll 1$, and $0 \leq C < \infty$. Assume that A has k eigenvalues $\lambda_1, \dots, \lambda_k$ in an interval $[0, a]$ with $a < 1/C$. Then the equatation*

$$\det(A - \lambda(\mathbb{1} + B(\lambda))) = 0 \tag{7.1}$$

has at most k solutions $\lambda_1', \dots, \lambda_k'$ and each solution satisfies $|\lambda_i' - \lambda_i| \leq 4\delta\lambda_i$.

Proof: If (7.1) holds, then there exist a non-zero vector c such that

$$(A - \lambda)c = \lambda B(\lambda) c \tag{7.2}$$

or

$$c = \frac{\lambda}{A - \lambda} B(\lambda) c \tag{7.3}$$

Thus

$$\|c\|_2 \leq \lambda \|(A - \lambda)^{-1}\| \|B(\lambda)\| \|c\|_2 \tag{7.4}$$

Since c is non-zero, this means that

$$\lambda \|(A-\lambda)^{-1}\| \|B(\lambda)\| \geq 1 \tag{7.5}$$

Now since A is symmetric, we have that

$$\|(A-\lambda)^{-1}\| \leq \max_i \frac{1}{|\lambda_i - \lambda|} \wedge \frac{1}{a-\lambda} \tag{7.6}$$

and so

$$\min_{i=1}^{k} |\lambda_i - \lambda| \leq \lambda \|B(\lambda)\| \leq \lambda(\delta + C\lambda) \tag{7.7}$$

which implies the claimed result. $\diamondsuit$

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