Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

A mathematical model for induction hardening including mechanical effects

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submitted: September 24, 2002

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No. 766
Berlin 2002

1991 *Mathematics Subject Classification.* 74D10,74F05,74F10,77N25.

Key words and phrases. joule heating, thermoviscoelasticity, phase transitions.

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Abstract

We investigate a mathematical model for induction hardening of steel. It accounts for electromagnetic effects that lead to the heating of the workpiece as well as thermomechanical effects that cause the hardening of the workpiece. The new contribution of this paper is that we put a special emphasis on the thermomechanical effects caused by the phase transitions. We take care of effects like transformation strain and transformation plasticity induced by the phase transitions and allow for physical parameters depending on the respective phase volume fractions.

The coupling between the electromagnetic and the thermomechanical part of the model is given through the temperature-dependent electric conductivity on the one hand and through the Joule heating term on the other hand, which appears in the energy balance and leads to the rise in temperature. Owing to the quadratic Joule heat term and a quadratic mechanical dissipation term in the energy balance, we obtain a parabolic equation with L^1 data. We prove existence of a weak solution to the complete system using a truncation argument.

1 Introduction

In most structural components in mechanical engineering, the surface is particularly stressed. Therefore, the aim of surface hardening is to increase the hardness of the boundary layers of a workpiece by rapid heating and subsequent quenching. This heat treatment leads to a change in the microstructure, which produces the desired hardening effect.

Depending on the respective heat source one can distinguish between different surface hardening procedures. The most important ones are flame hardening, laser hardening and induction hardening.

In the latter case the mode of operation relies on the transformer principle. A given current density in the induction coil induces eddy currents inside the workpiece. Because of the Joule effect, these eddy currents lead to an increase in temperature in the boundary layers of the workpiece. Then the current is switched off, and the workpiece is quenched by spray-water cooling.

Induction surface hardening has successfully been applied in industry for more than fifty years. Up to now, the process control and the design of decent induction coils for specific hardening purposes mostly depends on experience.

However, there is a growing demand in industry for a more precise process control, mainly for two reasons. One is the general goal of weight reduction, especially in automotive industry, leading to components made of thinner and thinner steel sheets. The surface hardening of these sheets is a very delicate task, since one must be careful not to harden the complete sheet, which would lead to undesirable fatigue effects. The second reason is the tendency to use high quality steels with only a small carbon content. Since the hardenability of a steel is directly related to its carbon content, already from a metallurgical point of view, the treatment of these steels is extremely difficult. Hence, a precise process control is indispensable for the heat treatment of this kind of steels.

In this paper we try to give a rather complete mathematical treatment of induction hardening. In Section 2, we derive a mathematical model consisting of two components. We employ a vector potential formulation of Maxwell's equations to describe the electromagnetic effects that lead to the heating of the workpiece. The second component is a phenomenological model of the thermomechanical behaviour of the workpiece including the phase transitions that lead to the desired hardening effect.

The interplay between temperature evolution and phase volume fractions has been subject to intensive research by the author during the last years, see e.g. [16], [20]–[22], [25], and is now well understood. In a joint project with industrial partners and metallurgists from the Bergakademie Freiberg related to laser and electron beam hardening, temperature-dependent data functions for a number of important steels have been identified (cf. [25]).

Therefore, we assume in this paper that the relationship between temperature evolution and phase volume fractions is known a priori and concentrate on the thermomechanical effects caused by the phase transitions.

During the last 15 years, the thermomechanical behaviour of steel has been an active research topic of physical metallurgy (cf., e.g., [11], [14], [15] and the references therein). Although it seems that so far there is no unified thermomechanical model at hand that is well accepted and that allows to reproduce all experiments, it is quite clear what the principal effects are that a macroscopic model should account for. We pick up these components and use them to formulate a consistent thermomechanical model. A special feature of this model is that the physical parameters are allowed to depend on the volume fractions of the metallurgical phases by a mixture ansatz.

The resulting system of state equations consists of an elliptic equation for the scalar potential, a degenerate parabolic system for the vector potential, a quasistatic momentum balance coupled with a nonlinear stress-strain relation, and a nonlinear energy balance equation. Owing to the quadratic Joule heat term and a quadratic mechanical dissipation term in the energy balance, we obtain a parabolic equation with L^1 data.

In Sections 3 and 4, we prove existence of a weak solution to the complete system. We truncate the quadratic terms, show existence of a weak solution to the truncated system by a fixed point argument, and finally pass to the limit. Similar arguments have been used, e.g., in [3] in connection with a model for induction heating, and in [12] for a model of resistance welding.

2 The mathematical model

2.1 Process description

Electromagnetic induction provides a method of heating electrically conducting materials. The basic components of an induction heating system are an induction coil (in the sequel called inductor), an alternating current (a.c.) power supply, and the workpiece itself. The inductor, which may take different shapes depending on the required heating pattern, is connected to the power supply. The flow of alternating current through the inductor generates an alternating magnetic field which in turn induces eddy currents in the workpiece that dissipate energy and

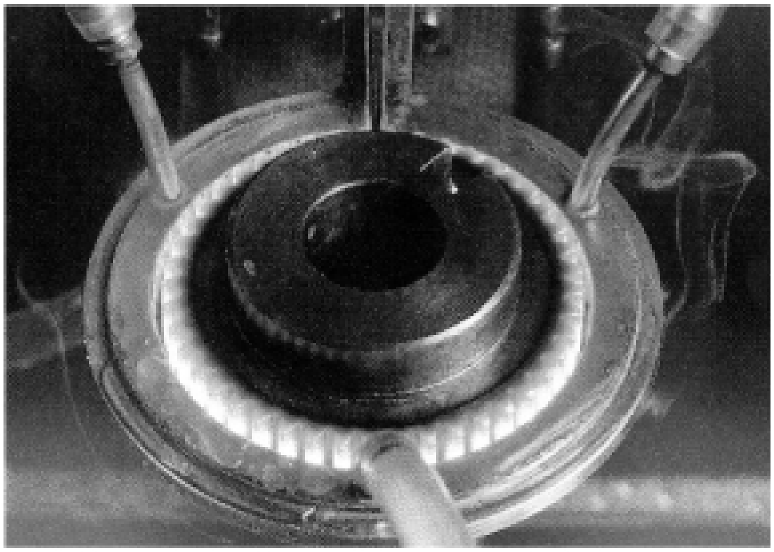

Figure 1: Induction hardening of a gear wheel (by courtesy of Steremat Elektrowärme GmbH, Berlin).

bring about heating.

Since the magnitude of the eddy currents decreases with growing distance from the workpiece surface because of the frequency dependent skin-effect, induction heating is a suitable heat source for surface heat treatments if the current frequency has been chosen big enough. On the other hand, if sufficient time for heat conduction is allowed and the current frequency is not too big, relatively uniform heating patterns can be obtained. Hence induction heating can also be used in heat treatments like annealing. Figure 1 depicts the heating stage in the surface hardening of a gear wheel. The power supply is not visible. After the current has been switched off, the workpiece is quenched by spray-water cooling which leads to the desired hardening effect.

The reason why one can change the hardness of steel by thermal treatment lies in the occurring phase transitions. At room temperature, a hypoeutectoid steel, i.e., a steel with less than 0.8 % carbon content, is a mixture of ferrite, pearlite, bainite, and martensite. Upon heating, these phases are transformed to austenite in the boundary layers of the workpiece. Then, during cooling, austenite is transformed back to a mixture of ferrite, pearlite, bainite and martensite. The actual phase distribution at the end of the heat treatment depends on the cooling strategy. In the case of surface hardening, owing to high cooling rates, most of the austenite is transformed to martensite by a diffusionless phase transition leading to the desired increase of hardness.

Hence a mathematical model for induction surface hardening has to account for the electromagnetic effects that lead to the surface heating as well as for the thermomechanical effects and the phase transitions that are caused by the enormous changes in temperature during the heat treatment.

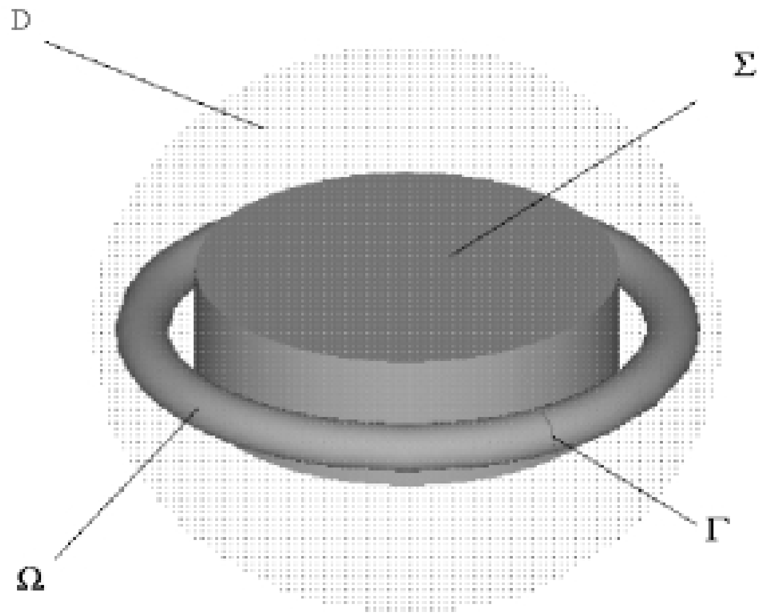

Figure 2: The setting.

2.2 Electromagnetic subproblem

2.2.1 The vector potential formulation of Maxwell's equations

Since we cannot model the hardening machine itself, we restrict ourselves to the following idealized geometric setting (cf. Fig. 2). Let $D \subset \mathbb{R}^3$ be a domain which contains the inductor Ω and the workpiece Σ. We assume

(A1) $\bar{\Omega} \subset D$, $\bar{\Sigma} \subset D, \bar{\Omega} \cap \bar{\Sigma} = \emptyset$, and $\partial\Omega$, $\partial\Sigma$, ∂D are of class $C^{1,1}$.

We call $G = \Omega \cup \Sigma$ the set of conductors and define the space - time cylinder $Q = \Sigma \times (0,T)$. Since we do not consider the hardening machine in our model, we assume that the inductor Ω is a closed tube. Inside we fix a section Γ and model the current density which is generated by the hardening machine by an interface condition on Γ.
In eddy current problems we can neglect displacement currents, hence we consider the following set of Maxwell's equations:

$$\begin{aligned} \text{curl } H &= J, && \text{(2.1a)} \\ \text{curl } E &= -B_t, && \text{(2.1b)} \\ \text{div } B &= 0. && \text{(2.1c)} \end{aligned}$$

Here, E is the electric field, B the magnetic induction, H the magnetic field and J the current density. In addition, we introduce *Ohm's law*

$$J = \kappa E, \quad \text{in } D, \tag{2.2}$$

where κ is the electric conductivity, and assume a linear relation between magnetic induction and magnetic field, i.e.,

$$B = \mu H, \quad \text{in } D, \tag{2.3}$$

with the magnetic permeability μ.
Outside the conductors, we assume zero current density, inside the conductors, the conductivity is positive and may depend on the temperature, i.e.,

$$\kappa(x,t) = \begin{cases} \kappa_0(\theta(x,t)) > 0, & \text{for } x \in \bar{G}, \\ 0, & \text{for } x \in D \setminus \bar{G}. \end{cases} \tag{2.4}$$

In reality, the temperature characteristic will be different in the coil and the workpiece, but to simplify the notation we assume the same temperature dependency in all the conductors (cf. Section 6).
The magnetic permeability may take different values in the workpiece and in the surrounding air, however it is assumed to be independent of temperature. The inductor is usually made of copper which has approximately the same permeability as the air. Hence we assume

$$\mu(x) = \mu_2 \chi_\Sigma + \mu_1 (1 - \chi_\Sigma), \tag{2.5}$$

where μ_1, μ_2 are positive constants with $\mu_1 < \mu_2$ and $\chi_C = 1$ if $x \in C$, and $\chi_C = 0$ is the characteristic function of the set $C \subset \mathbb{R}^n$. In view of (2.1c) we introduce the magnetic vector potential A such that

$$B = \text{ curl } A, \text{ in } D. \tag{2.6}$$

Since A is not uniquely defined by (2.6), we impose the Coulomb gauge

$$\text{div } A = 0, \text{ in } D. \tag{2.7}$$

Using (2.1b) and (2.6), we define the scalar potential ϕ by

$$E + A_t = - \text{ grad } \phi \text{ in } D. \tag{2.8}$$

Combining this with Ohm's law (2.2), we obtain the following expression for the total current density J:

$$J = -\kappa A_t - \kappa \text{ grad } \phi. \tag{2.9}$$

Inserting (2.6) and (2.9) into (2.1a), we obtain

$$\text{curl } \left(\frac{1}{\mu} \text{ curl } A \right) = -\kappa A_t - \kappa \text{ grad } \phi. \tag{2.10}$$

Equation (2.10) is a general model to describe eddy currents. Now we will explain how it looks like precisely in the different domains (cf. [13]). Since the inductor is connected to the hardening machine, the current density in it can be thought of as consisting of two components: an impressed part J_{source} and an induced part J_{eddy}. The impressed part is due to an external source and is defined by the gradient of the scalar potential. The induced part is due to the time-varying field B in the coil itself. Therefore, in the inductor,

$$\text{curl } \left(\frac{1}{\mu} \text{ curl } A \right) + \kappa A_t = J_{source}, \text{ in } \Omega, \tag{2.11}$$

where

$$J_{source} = -\kappa \text{ grad } \phi. \tag{2.12}$$

In the workpiece, there is an induced current density J_{eddy}, but there is no source term, hence we obtain

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} A\right) + \kappa A_t = 0, \text{ in } \Sigma, \tag{2.13}$$

with

$$J_{eddy} = -\kappa A_t. \tag{2.14}$$

There is no electric current in the air ($\kappa = 0$), and so (2.10) can be simplified to

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} A\right) = 0, \text{ in } D \setminus G. \tag{2.15}$$

We assume that the boundary ∂D is a perfect conductor, this means that the tangential component of A vanishes on ∂D. Thus the magnetic vector potential is characterized by the following boundary value problem:

$$\kappa A_t + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} A\right) + \kappa_0 \chi_\Omega \operatorname{grad} \phi = 0, \quad \text{in } D, \tag{2.16a}$$

$$n \times A = 0, \quad \text{on } \partial D. \tag{2.16b}$$

Demanding that the continuity equation holds for the source current J_{source} (cf.(2.12)), we determine the scalar potential ϕ by the elliptic equation

$$-\operatorname{div}\left(\kappa_0 \operatorname{grad} \phi\right) = 0, \quad \text{in } \Omega. \tag{2.17a}$$

On the boundary we assume a homogenous Neumann condition, i.e.,

$$-\kappa_0 \frac{\partial \phi}{\partial \nu} = 0, \text{ in } \partial\Omega. \tag{2.17b}$$

In the section Γ we supply current via an interface condition, i.e.,

$$\left[-\kappa_0 \frac{\partial \phi}{\partial \tilde{\nu}}\right] = j_s, \quad \text{on } \Gamma. \tag{2.17c}$$

Here, j_s is the external source current density, $[f(x)]$ denotes the jump of a function $f(x)$ across the interface Γ, and $\tilde{\nu}$ is a unit normal vector on Γ.

Remark 2.1 *In view of* (2.1a), *the continuity equation should hold for the total current density given in* (2.9). *In the next section we will show (Corollary 3.1) that this is indeed the case, at least in a distributional sense.*

2.2.2 Assumptions and weak formulation

To solve the interface problem (2.17a) – (2.17c) in the coil Ω, we introduce the quotient space $H^1(\Omega)/\mathbb{R}$ with norm

$$\|\bar{\varphi}\|_{H^1(\Omega)/\mathbb{R}} = \inf_{\varphi \in \bar{\varphi}} \|\varphi\|_{H^1(\Omega)}.$$

According to [19, Theorem 1.9], the functional

$$\bar{\varphi} \mapsto \left(\int_\Omega |\nabla \varphi|^2 \, dx\right)^{1/2} \quad \text{for } \varphi \in \bar{\varphi}$$

is an equivalent norm on $H^1(\Omega)/\mathbb{R}$. Hence, there are constants o_*, o^* such that

$$o_*\|\bar{\varphi}\|_{H^1(\Omega)/\mathbb{R}} \le \Big(\int\limits_{\Omega} |\nabla\varphi|^2\,dx\Big)^{1/2} \le o^*\|\bar{\varphi}\|_{H^1(\Omega)/\mathbb{R}}. \tag{2.18}$$

To ensure the solvability of the interface problem we assume

(A2) $j_s \in L^2(0,T;H^{-1/2}(\Gamma))$, such that

$$\int\limits_{\Gamma} j_s(t)\,dx = 0, \qquad \text{a.e. in } (0,T),$$

(A3) $\kappa_0 \in C(\mathbb{R}),\ 0 < \kappa_* \le \kappa_0(x) \le \kappa^* < \infty$ for all $x \in \mathbb{R}$.

Remark 2.2 *(1) The integral in* **(A2)** *has to be understood in the sense of duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.*

(2) Note that for $\bar{u} \in H^1(\Omega)/\mathbb{R}$ and arbitrary $u_{1,2} \in \bar{u}$ we have $\nabla(u_1 - u_2) = 0$ a.e. in Ω and (owing to (A2)) $\int_\Gamma j_s u_1\,dx = \int_\Gamma j_s u_2\,dx$ a.e. in $(0,T)$.

(3) In the sequel we will no longer distinguish between ϕ and $\bar{\phi}$.

Now we dissect Ω once more, producing another interface $\widetilde{\Gamma}$ with $\Gamma \cap \widetilde{\Gamma} = \emptyset$, and two subsets $\Omega_{1,2}$ satisfying $\Omega_1 \cup \Omega_2 = \Omega$, and $\Omega_1 \cap \Omega_2 = \Gamma \cup \widetilde{\Gamma}$. Assuming that the flux of the scalar potential is continuous through the interface $\widetilde{\Gamma}$, we multiply (2.17a) with a test function, integrate by parts in Ω_1 and Ω_2 using the boundary and interface conditions and obtain the following weak formulation of the interface problem in Ω:
Find $\phi \in L^2(0,T;H^1(\Omega)/\mathbb{R})$, such that

$$\int\limits_{\Omega} \kappa_0 \nabla\phi(t)\cdot\nabla\varphi\,dx + \int\limits_{\Gamma} j_s(t)\varphi\,dx = 0 \quad \text{for all } \varphi \in H^1(\Omega)/\mathbb{R} \text{ a.e. in } (0,T). \tag{2.19}$$

Here and in the sequel, $'\cdot'$ denotes the scalar product in $\mathbb{R}^3$. The intergral on Γ again has to be understood in the sense of duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.
Now we turn to the the vector potential A. Denoting $\mathbf{L} = [L]^3$ the vector-valued counterpart of any real-valued Sobolev space L, we introduce the Hilbert space

$$\mathbf{X} = \{v \in H(\text{ curl }, D);\ \text{ div } v = 0 \text{ and } n \times v\Big|_{\partial D} = 0\},$$

where $H(\text{ curl }, D) = \{v \in \mathbf{L}^2(D);\ \text{ curl } v \in \mathbf{L}^2(D)\}$. Since ∂D is of class $C^{1,1}$ (cf. (A1)), $\mathbf{X}$ equipped with the norm

$$\|v\|_{\mathbf{X}} = \|\text{ curl } v\|_{\mathbf{L}^2(D)}$$

is a closed subspace of $\mathbf{H}^1(D)$ (cf. [19, Lemma 3.4]). We recall the Green's formula

$$\int\limits_{\partial D} (n \times f)\cdot g\,dx = \int\limits_{D} \text{ curl } f \cdot g\,dx - \int\limits_{D} f \cdot \text{ curl } g\,dx, \tag{2.20}$$

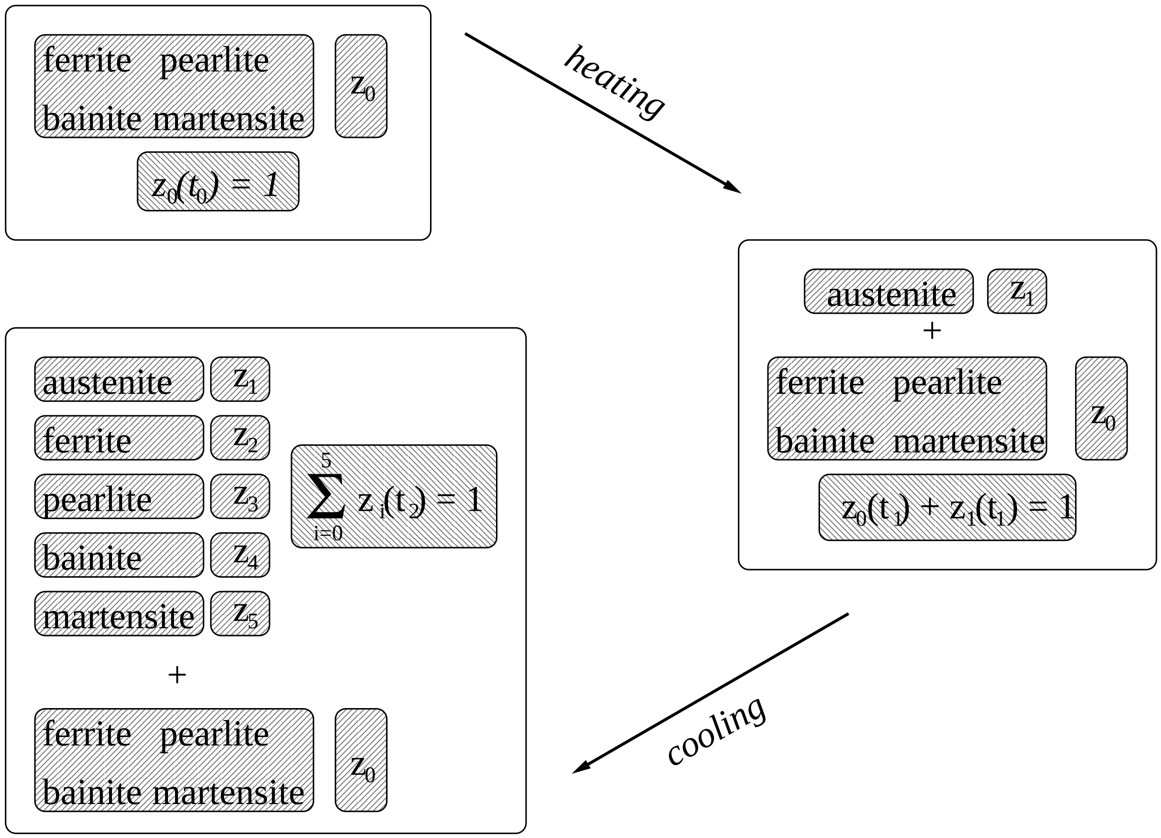

Figure 3: Phase transitions in hypoeutectoid steel during surface hardening (time t_0 : start of heating, t_1 : end of heating, t_2 : end of cooling).

for all $f \in H(\text{ curl }, D)$ and $g \in \mathbf{H}^1(D)$, where the integral on ∂D has to be understood in the sense of duality between $\mathbf{H}^{-1/2}(\partial D)$ and $\mathbf{H}^{1/2}(\partial D)$. With the help of (2.20), we obtain the following weak formulation of (2.16a), (2.16b):
Find $A \in L^2(0,T;\mathbf{X})$, such that $A(0) = A_0$,

$$\int\limits_G \kappa_0 A_t \cdot v\, dx + \int\limits_D \frac{1}{\mu} \text{ curl } A \cdot \text{ curl } v\, dx + \int\limits_\Omega \kappa_0 \nabla\phi \cdot v\, dx \quad = \quad 0, \tag{2.21}$$

for all $v \in \mathbf{X}$ and a.e. in $(0,T)$. For A_0 and μ, we assume

(A4) $A_0 \in \mathbf{X}$,

(A5) $\mu(x) = \mu_2 \chi_{\Sigma} + \mu_1(1 - \chi_{\Sigma})$, with constants $0 < \mu_1 < \mu_2$.

2.3 Phase transitions and thermomechanics

2.3.1 Phase transitions

We do not intend to explain the phenomenology of phase transitions that occur during heat treatments. For this we refer to [16], [20] – [22] and [25]. Instead we confine ourselves to explaining the time evolution of phases during a surface heat treatment, according to Figure 3, for a hypoeutectoid carbon steel.

We distinguish between three characteristic times: the beginning of the heating process t_0, the end of the heating process t_1, and the end of the cooling process t_2. At time t_0 the workpiece which is to be exposed to the heat treatment is assumed to consist of a mixture of ferrite, pearlite, bainite, and possibly already some martensite. Of course, the real phase distribution prior to the heat treatment is unknown. We call the initial phase mixture z_0. Thus we have $z_0(t_0) = 1$ everywhere in the workpiece. At the end of the heating process at time t_1 the outer layers of the workpiece have been transformed to austenite (volume fraction $z_1(t_1)$). Then, upon cooling, this volume fraction is transformed back to a mixture of ferrite (z_2), pearlite (z_3), bainite (z_4) and martensite (z_5). Note that, during the cooling process, the remaining fraction of the initial phase configuration remains unchanged and equal to $z_0(t_1)$. Thus, we can conclude

$$\dot{z}_0(t) \begin{cases} \leq 0 & , \text{ for } t \in [t_0, t_1] \\ = 0 & , \text{ for } t \in [t_1, t_2] \end{cases}, \qquad \dot{z}_1(t) \begin{cases} \geq 0 & , \text{ for } t \in [t_0, t_1] \\ \leq 0 & , \text{ for } t \in [t_1, t_2] \end{cases},$$

and

$$z_1(t_1) = z_1(t_2) + z_2(t_2) + z_3(t_2) + z_4(t_2) + z_5(t_2).$$

In [16], [24], and [25] we show different approaches to obtain these volume fractions as the solution to a system of ordinary differential equations. Here, we assume the existence of an operator $\mathcal{P}$ that assigns to a given temperature evolution θ in the workpiece (recall $Q = \Sigma \times (0,T)$) the vector of volume fractions $z = (z_0, \ldots, z_5)$. More precisely, we assume

(A6) There exists a mapping

$$\mathcal{P} \,:\, L^1(Q) \longrightarrow [W^{1,\infty}(0,T; L^\infty(\Sigma))]^6, \qquad \theta \mapsto z,$$

satisfying:

(i) $z_i(x,t) \in [0,1]$, $i = 0, \ldots 5$, $\quad \sum_{i=0}^{5} z_i(x,t) = 1 \quad$ a.e. in Q,

(ii) $\|z\|_{[W^{1,\infty}(0,T;L^\infty(\Sigma))]^6} \leq z^*$, with a constant z^* independent of θ.

(iii) Let $\{\theta_k\} \subset L^1(Q)$ with $\theta_k \to \theta$ strongly in $L^1(Q)$, then

$$z_k \longrightarrow z \quad \text{strongly in } W^{1,p}(0,T; L^p(\Sigma)) \quad \text{for all } p \in [1, \infty),$$

where $z = \mathcal{P}[\theta]$.

Remark 2.3 *Assumption (A6) is satisfied by most of the phase transition models used in practice. However, a recent result (cf. [25]) seems to indicate that taking into account the time derivative of the temperature evolution can be necessary in order to reproduce nonisothermal measurements. This case is not covered by our analysis and will require further research.*

2.3.2 Thermomechanical modeling

We only consider a weak coupling of thermomechanical and electromagnetic effects. We admit a temperature dependency of the electric conductivity (cf. (A3)) and assume that Joule heating is the only electromechanical effect responsible for the rise in temperature. Doing so, we neglect the Lorentz force in the momentum balance and do not account for the Thomson and Peltier effects[1]. While the latter are of particular importance in semiconductors, the Lorentz force can play a role in induction hardening, especially in the case of a moving inductor. This case is not covered by our model. For details about a more involved coupling of thermomechanical and electromagnetic effects, we refer to the monograph [30]. Since phase transitions only occur in the workpiece Σ made of steel, we will consider the complete thermomechanical model only in Σ and neglect mechanical effects in the inductor. Figure 4 shows the complex interdependence of the relevant physical quantities. The interplay between temperature θ and volume fraction z is well understood and has been subject to intensive research by the author during the last years, see e.g. [16], [20]–[22], [25]. While the volume fractions z can be computed from the temperature evolution (cf. (A6)), the phase transitions lead to a release or a consumption of latent heat and thus influence the temperature. The new feature is that we account also for mechanical effects. During the last 15 years, the thermomechanical modeling of phase transitions in steel has been an active research topic of physical metallurgy (cf., eg., [11], [14], [15] and the references therein). Although it seems that so far there is no unified thermomechanical model at hand that is well accepted and that allows to reproduce all experiments, it is quite clear what the principle effects are that a macroscopic model should account for (cf. Figure 4):

- The metallurgical phases z_i have material parameters with different thermal characteristics, hence their effective values have to be computed by a mixture ansatz.

- The different densities of the metallurgical phases result in a different thermal expansion. This thermal and transformation strain is the major contribution to the evolution of internal stresses during heat treatments.

- Experiments with phase transformations under applied loading show an additional irreversible deformation even when the equivalent stress corresponding to the load is far below the normal yield stress. This effect is called *transformation-induced plasticity*.

- The irreversible deformation leads to a mechanical dissipation that acts as a source term in the energy balance.

- The internal stresses influence the transformation kinetics. This effect will be neglected in our model. In line with (A6) we assume that the transformation kinetics only depend on the time evolution.

In the following, we combine these ingredients to form a consistent model and work out the inherent mathematical features. Assuming right from the beginning small deformations, we formulate the balance laws in the undeformed domain.
To determine the displacement u (or the velocity $v = u_t$, respectively), the stress tensor σ, and the temperature θ, we evaluate the quasistatic balance law of momentum and the balance law

[1] Roughly speaking, the Thomson effect means that a temperature gradient can produce an electric current in the absence of an electric field. The fact that heat can be generated by an elastic field in a spatially uniform temperature field is called Peltier effect (cf. [30]).

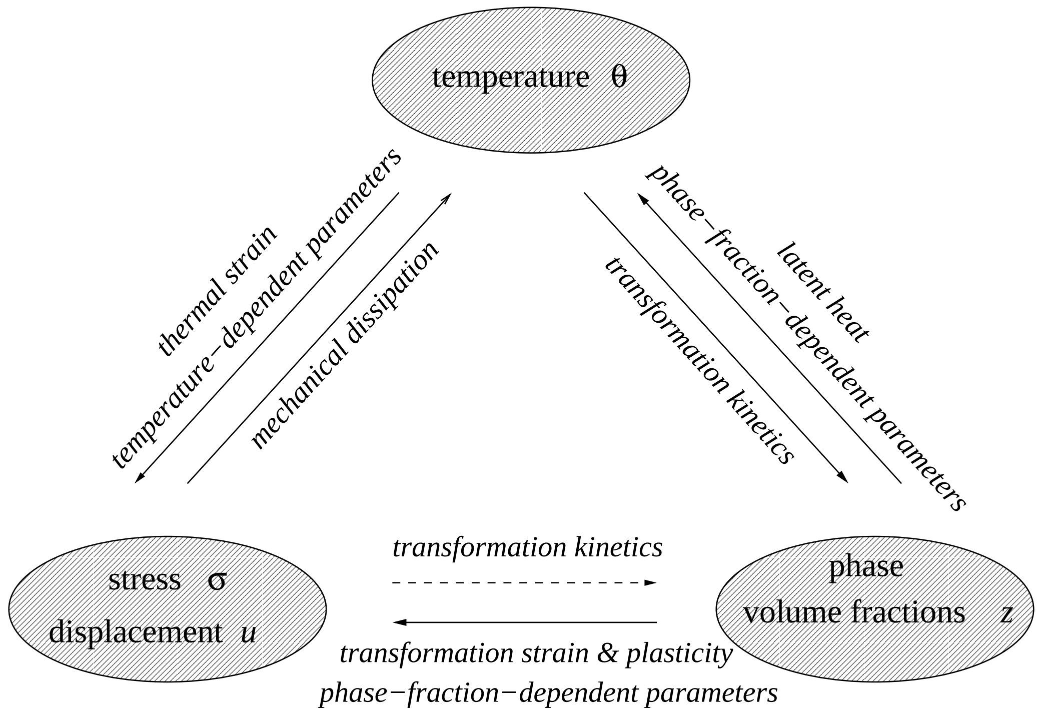

Figure 4: Phase transitions with thermomechanics - interdependence of physical quantities.

of internal energy:

$$
\begin{aligned}
-\operatorname{div} \sigma &= f, &\quad (2.22a)\\
\rho e_t + \operatorname{div} q &= \sigma : \varepsilon(v) + h. &\quad (2.22b)
\end{aligned}
$$

Here ρ is the mass density, f an external force, q is the heat flux, e the specific internal energy and

$$\varepsilon(v) = \frac{1}{2}(\mathcal{D}v + \mathcal{D}^T v) \tag{2.23}$$

the symmetric part of the strain rate tensor. The scalar product in $\mathbb{R}^{3\times 3}$ is denoted by $' : '$ and the corresponding norm by $|\,.\,|$. The external heat source h in our case only consists of the contribution of the Joule heat, i.e.,

$$h = \frac{1}{\kappa_0}|J|^2, \tag{2.24}$$

where the current density J is defined in (2.9).
We employ the laws of Fourier and Hooke, respectively,

$$
\begin{aligned}
q &= -k \operatorname{grad} \theta, &\quad (2.25)\\
\sigma &= K\varepsilon^{el}. &\quad (2.26)
\end{aligned}
$$

Here, k is the thermal conductivity, ε^{el} the elastic strain, and $K = \{K_{ijkl}\}$ the isotropic stiffness tensor. Moreover, we assume that the total strain $\varepsilon(u)$ can be additively decomposed in an elastic part, a thermal part ε^{th}, and a nonelastic part induced by the phase transitions, which we refer to as ε^{trip}, where *trip* is short for *transformation-induced plasticity*, i.e.,

$$\varepsilon(u) = \varepsilon^{el} + \varepsilon^{th} + \varepsilon^{trip}. \tag{2.27}$$

In linearized thermoelasticity one usually assumes a linear relation between temperature difference and thermal strain, i.e., $\varepsilon^{th} = \widetilde{\beta}(\theta, z)I$ with

$$\widetilde{\beta}(\theta, z) = \widetilde{\alpha}(\theta - \theta_0),$$

where $\widetilde{\alpha}$ is the thermal expansion coefficient, I the identity matrix, and θ_0 a reference temperature. This approach has at least two disadvantages for our purpose. Firstly, strictly speaking, the above relation only holds for small temperature variations around θ_0, which is definitely not the case in surface hardening, where the temperature usually varies between room temperature and some 1000 degrees centigrade. Moreover, dealing with phase transitions, we have to take into account also volume changes due to a change in the volume fraction of the constituting phases. Hence, sometimes (cf., e.g., [11], [35]), an additional transformation strain is introduced.

However, it seems to be more natural to describe the thermal and transformation strain in a unified manner through changes in the density, as it has been done in [31]. To this end we make a mixture ansatz for the density,

$$\rho(\theta, z) = \sum_{i=0}^{5} z_i \rho_i(\theta). \tag{2.28}$$

Here, $\rho_i(\theta)$ is the measured homogenous temperature-dependent density of the phase z_i. Then we describe the thermal strain by

$$\varepsilon^{th} = \left(\left(\frac{\hat{\rho}}{\rho} \right)^{\frac{1}{3}} - 1 \right) I, \tag{2.29}$$

where $\hat{\rho}$ is the homogenous, measured density of the initial phase configuration $z_0(t_0)$ at the initial temperature $\theta(t_0)$.

Diffusonal and diffusionless phase transformations under applied loading exhibit an irreversible deformation even when the equivalent stress corresponding to the load is far below the normal yield stress.

For $i \geq 2$, let ε_i^{trip} be this *transformation-induced plasticity* (*trip*) strain contribution of the formation of the phase z_i from austenite during cooling (cf. Figure 3). The notation has been chosen to be compatible with the engineering literature. However, note that ε^{trip} is not rate independent and therefore a viscoelastic rather than a plastic strain. A general model to describe the *trip* strain rate is

$$\varepsilon_{i,t}^{trip} = \Lambda_1^i(\theta) \frac{\partial \Lambda_2^i(z_i)}{\partial z_i} z_{i,t} S, \tag{2.30}$$

where the deviator (i.e., the trace-free part of the stress tensor) is defined by

$$S = \sigma - \frac{1}{3} \operatorname{tr} \sigma I. \tag{2.31}$$

In the engineering literature, various formulas of this structure can be found to describe the *trip* strain. In most cases, they only differ in the choice of the functions $\Lambda_{1,2}^i$. For a review of

these models and micromechanical considerations to derive (2.30), we refer the reader to [14]. However, a rigorous derivation of this kind of models is not yet available. A promising new approach is described in [15]. The basic idea is to describe the usual plastic strain and the *trip* strain in a unified manner.
In the sequel, we will use the following ansatz to describe the *trip* strain increment:

$$\varepsilon_t^{trip} = \big(\mathrm{grad}_z \Lambda(\theta, z)\big) \cdot z_t\, S. \tag{2.32}$$

If we put

$$\Lambda(\theta, z) = \sum_{i=0}^{5} \Lambda_1^i(\theta)\Lambda_2^i(z_i),$$

we recover (2.30). Since we only account for contributions to the *trip* strain of phases that grow from the austenite, which itself has been formed during the heating process, we should have $\Lambda_2^1 = \Lambda_2^2 = 0$. From the mathematical point of view, this is irrelevant, and we will only prescribe a certain regularity for Λ in the next subsection.

Remark 2.4 *To simplify the derivation of constitutive relations for the internal energy, we assume that the temperature dependency of the density ρ, the stiffness matrix K, and the function Λ in* (2.32) *can be neglected, i.e.,*

$$\rho_{,\theta} = 0, \quad K_{,\theta} = 0, \quad \Lambda_{,\theta} = 0, \tag{2.33}$$

where $f_{,\theta}$ is short for $\frac{\partial f}{\partial \theta}$. In this way we avoid dissipation terms in the energy balance that are usually neglected in engineering literature. Moreover, they would cause severe difficulties in the mathematical treatment. In Section 6 we will come back to this assumption and review the mathematical difficulties that would arise without making assumption (2.33).

According to (2.26) and (2.27), Hooke's law is given by

$$\sigma = K(\varepsilon(u) - \varepsilon^{th} - \varepsilon^{trip}), \tag{2.34}$$

thus an immediate consequence of (2.33) is

$$\sigma_{,\theta} = 0. \tag{2.35}$$

Moreover, we can conclude

$$\varepsilon_t^{th} = -\frac{1}{3}\frac{\hat{\rho}^{1/3}}{\rho^{-4/3}}(\ \mathrm{grad}\ \rho) \cdot z_t\, I \tag{2.36}$$

and we can write (2.32) in the compact form

$$\varepsilon_t^{trip} = \Lambda(z)_t\, S. \tag{2.37}$$

To derive a constitutive relation for the internal energy e, we proceed as in [29, Section 2.4.2]. To this end we introduce the *Helmholtz free energy* ψ and the entropy s, which are related by the thermodynamic identity

$$e = \psi + \theta s. \tag{2.38}$$

We assume that there exists a twice continuously differentiable material function $\hat{\psi}$ such that

$$\psi = \hat{\psi}(\varepsilon^{el}, \theta, z_1). \tag{2.39}$$

The dependency on ε^{el} and θ is standard. Moreover, we have chosen the austenite volume fraction z_1 as an internal variable since, during the heating, austenite is formed, and then, during the cooling process, the other phases form at the expense of austenite. From (2.39), we obtain immediately

$$\psi_t = \frac{\partial \hat{\psi}}{\partial \varepsilon^{el}} : \varepsilon_t^{el} + \frac{\partial \hat{\psi}}{\partial \theta}\, \theta_t + \frac{\partial \hat{\psi}}{\partial z_1}\, z_{1,t}. \tag{2.40}$$

To obtain a thermodynamically consistent model, we demand that the Clausius-Duhem inequality is satisfied for all solutions to the field equations. For small deformations, it reads

$$\sigma : \varepsilon(v) - \rho(\psi_t + s\theta_t) - \frac{1}{\theta}\, q \cdot \text{ grad } \theta \geq 0. \tag{2.41}$$

Inserting (2.25), (2.27), and (2.40), we obtain

$$\Big(\sigma - \rho \frac{\partial \hat{\psi}}{\partial \varepsilon^{el}}\Big) : \varepsilon_t^{el} + \sigma : (\varepsilon(v) - \varepsilon_t^{el}) - \rho\Big(s + \frac{\partial \hat{\psi}}{\partial \theta}\Big)\theta_t - \rho \frac{\partial \hat{\psi}}{\partial z_1}\, z_{1,t} + \frac{1}{\theta}\Big| \text{ grad } \theta\Big|^2 \geq 0. \tag{2.42}$$

Since this inequality holds for all solutions to the field equations, we first consider an elastic deformation at constant and uniform temperature (i.e., $\theta_t = 0$ and $\text{grad } \theta = 0$), moreover, we assume that neither the inelastic strain nor the internal variable z_1 are altered. Since the Clausius-Duhem inequality has to be satisfied for all elastic strain rates ε_t^{el}, we can infer

$$\sigma = \rho \frac{\partial \hat{\psi}}{\partial \varepsilon^{el}}. \tag{2.43a}$$

Now we consider a purely thermal deformation, again uniformly in space without change in inelastic strain and internal variable. Since (2.42) has to be satisfied for every θ_t, we can conclude that

$$s = -\frac{\partial \hat{\psi}}{\partial \theta}. \tag{2.43b}$$

We define the thermodynamic force associated with the internal variable z_1 by

$$L = \frac{\partial \hat{\psi}}{\partial z_1}. \tag{2.43c}$$

To be compatible with the terminology in papers dealing with phase transition models without mechanics, we call L the latent heat. Owing to Fourier's law of heat conduction, the last term in (2.41) is always nonnegative. Thus, we end up with the inequality

$$\sigma : (\varepsilon_t^{th} + \varepsilon_t^{trip}) - \rho L z_{1,t} \geq 0. \tag{2.44}$$

It reflects the fact that the intrinsic dissipation is necessarily positive. In principle, it could be used to derive further constitutive relations for the latent heat L, the function Λ in (2.32), as well as for an evolution equation for the volume fractions z.
Let us consider an isothermal, slow transformation from austenite to some other phase, then we may assume that the mecahnical dissipation terms in (2.44) can be neglected. Moreover, the new phase grows at the expense of austenite (i.e., $z_{1,t} < 0$), Thus we can conclude

$$L > 0$$

and we say that latent heat is released. During the heating process, austenite is formed (i.e., $z_{1,t} > 0$). Thus we have $-\rho L z_{1,t} < 0$, i.e., latent heat is consumed. This has to be compensated by a positive contribution of the mechanical dissipation.
We will not go further into this. We come back to our goal of deriving a constitutive relation for the internal energy instead. Invoking (2.38) and (2.40), we obtain

$$\rho e_t = \sigma : \varepsilon_t^{el} + \rho\theta s_t + \rho L z_{1,t}. \tag{2.45}$$

Moreover, recall that

$$s = -\frac{\partial\hat{\psi}(\varepsilon^{el}, \theta, z_1)}{\partial\theta}.$$

Since $\hat{\psi}$ is assumed to be twice continuously differentiable, in view of (2.33), (2.43a), (2.43b), and (2.45), we can infer

$$\begin{aligned} s_t &= -\frac{\partial^2\hat{\psi}}{\partial\varepsilon^{el}\partial\theta} : \varepsilon_t^{el} - \frac{\partial^2\hat{\psi}}{(\partial\theta)^2}\theta_t - \frac{\partial^2\hat{\psi}}{\partial z_1\partial\theta} z_{1,t} \\ &= -\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\sigma\right) : \varepsilon_t^{el} + \frac{\partial s}{\partial\theta}\theta_t - \frac{\partial L}{\partial\theta} z_{1,t} \\ &= \frac{\partial s}{\partial\theta}\theta_t - \frac{\partial L}{\partial\theta} z_{1,t}. \end{aligned}$$

As in [29], we define the specific heat capacity at constant strain

$$c_\varepsilon = \theta\frac{\partial s}{\partial\theta}. \tag{2.46}$$

Moreover, we assume

$$\frac{\partial L}{\partial\theta} = 0 \tag{2.47}$$

and obtain finally for the internal energy

$$\rho e_t = \rho c_\varepsilon\theta_t + \sigma : \varepsilon_t^{el} + \rho L z_{1,t}.$$

Inserting the above expression in the balance of internal energy (2.22b) using (2.25), we obtain

$$\rho c_\varepsilon\theta_t - \operatorname{div}\Big(k \operatorname{grad}\theta\Big) = -\rho L z_{1,t} + \sigma : (\varepsilon_t^{th} + \varepsilon_t^{trip}) + h. \tag{2.48}$$

The mechanical dissipation is given by (cf. (2.36), (2.37))

$$\sigma : (\varepsilon_t^{th} + \varepsilon_t^{trip}) = -\frac{1}{3}\frac{\hat{\rho}^{1/3}}{\rho^{-4/3}}(\operatorname{grad}\rho)\cdot z_t \operatorname{tr}\sigma + \Lambda(z)_t|S|^2. \tag{2.49}$$

For later use, we define the abbreviation

$$F(z, z_t, \sigma) = -\rho L z_{1,t} - \frac{1}{3}\frac{\hat{\rho}^{1/3}}{\rho^{-4/3}}(\operatorname{grad}\rho)\cdot z_t \operatorname{tr}\sigma \tag{2.50}$$

	physical field		domain of definition
A	magnetic vector potential	D	the big domain
ϕ	the scalar potential ϕ	Ω	the inductor
θ	temperature	$G = \Omega \cup \Sigma$	the set of conductors
σ	stress tensor	Σ	the workpiece
u	displacement	Σ	the workpiece
z	the phase fractions	Σ	the workpiece

Table 1: Domain of definition of the different physical fields.

So far in this section we did not care about the domain of definition of the different unknowns (cf. Table 1). Since we want to emphasize the interplay between phase transitions and mechanics, we restrict the domain of definition for stress and displacement to the workpiece Σ made of steel. Since the electric conductivity κ_0 is assumed to depend on temperature, and the inductor Ω is also heated up during the process, we have to consider the temperature θ in the workpiece and in the inductor.

We allow for different values of density and specific heat in the different steel phases, hence we define:

$$(\rho c_\varepsilon)(x,t) = \begin{cases} \bar{\rho}(z(x,t))\bar{c}_\varepsilon(z(x,t)), & \text{for } x \in \Sigma, \\ \rho^\Omega c_\varepsilon^\Omega, & \text{for } x \in \Omega. \end{cases} \tag{2.51}$$

The effective material parameters $\bar{\rho}$ and $\bar{c}_\varepsilon$ can be computed from a mixture rule like (2.28), however, we will not specify a special mixture rule and only assume a certain regularity of the effective parameters with respect to the volume fractions.

To conclude this section we have to specify boundary conditions for u, σ, and θ. For the temperature we neglect a possible radiative heat transfer between inductor and workpiece and assume

$$-k\frac{\partial \theta}{\partial \nu} = \alpha(\theta - \theta_e), \tag{2.52}$$

where ν is the unit normal vector on $\partial\Omega$ as well as on $\partial\Sigma$ and θ_e is the temperature of the spray water. The initial temperature is given by

$$\theta(0) = \theta_0$$

and the heat transfer coefficient by

$$\alpha(x,t) = \begin{cases} 0, & \text{on } \partial\Omega \times (0,T), \\ \alpha_0(t), & \text{on } \partial\Sigma \times (0,T). \end{cases} \tag{2.53}$$

Hence, there is no heat flux across the inductor. For the workpiece, the heat transfer coefficient $\alpha_0(t)$ is zero during the heating time. After the current has been switched off, the workpiece is quenched by spray-water cooling and $\alpha_0(t)$ is positive.

The workpiece boundary $\partial\Sigma = \partial\Sigma_g \cup \partial\Sigma_u$ is dissected into a part $\partial\Sigma_g$ where a pressure g is applied, and a part $\partial\Sigma_u$ where the workpiece is fixed and for which we assume meas $\partial\Sigma_u > 0$. Using Einstein's summation convention (which will also be applied in the sequel without remarking it explicitely), we have

$$\sigma_{ij}\nu_j = g, \quad \text{on } \partial\Sigma_g \times (0,T), \tag{2.54a}$$

$$u = 0, \quad \text{on } \partial\Sigma_u \times (0,T). \tag{2.54b}$$

2.3.3 Assumptions and weak formulation

First, we introduce the solution spaces for the stress tensor and the displacement field,

$$\mathcal{M} = \{(\tau_{ij}) \Big| \tau_{ij} = \tau_{ji}, \text{ and } \tau_{ij} \in L^2(\Sigma) \text{ for } 1 \le i,j \le 3\}, \tag{2.55}$$

$$\mathbf{U} = \{v \in \mathbf{H}^1(\Sigma) \Big| v = 0 \text{ on } \partial\Sigma_u\}. \tag{2.56}$$

Moreover, we define

$$\mathcal{M}(\text{ div }) = \{\tau \in \mathcal{M} \Big| \text{ div } \tau \in \mathbf{L}^2(\Sigma)\}. \tag{2.57}$$

with the usual notation $\text{div } \tau = \{\frac{\partial \tau_{ij}}{\partial x_j}\}$. According to [18, Proposition 5], the trace map $\sigma \mapsto \sigma_{ij}\nu_j$ is a linear and continuous operator from $\mathcal{M}(\text{ div })$ to $[H^{-1/2}(\partial\Sigma)]^3$ and we have the Green's formula

$$\int_\Sigma \sigma_{ij}\varepsilon_{ij}(v)\,dx + \int_\Sigma \sigma_{ij,j}v_i\,dx = \int_{\partial\Sigma} \sigma_{ij}\nu_j v_i\,dx, \tag{2.58}$$

for all $\sigma \in \mathcal{M}(\text{ div })$ and $v \in \mathbf{H}^1(\Sigma)$, where $\varepsilon(v)$ is defined in (2.23). Moreover, $\sigma_{ij,j}$ is short for $\frac{\partial}{\partial x_j}\sigma_{ij}$. For later use, we also introduce

$$\widetilde{\mathcal{M}} = \{\tau \in \mathcal{M} \Big| \text{ div } \tau = 0 \text{ in } \Sigma \text{ and } \tau_{ij}\nu_j = 0 \text{ on } \partial\Sigma_g\}. \tag{2.59}$$

Remark 2.5 *The last integral in* (2.58) *and the boundary condition in* (2.59) *have to be understood in the sense of duality between* $H^{1/2}(\partial\Sigma)$ *and* $H^{-1/2}(\partial\Sigma)$.
In the next section, we will make use of the following result (cf. [18, Corollary 2]):
Lemma 2.1 *Let* $\tilde{e} \in \mathcal{M}$. *Then the following statements are equivalent:*

(1)

$$\tilde{e} = \varepsilon(u) \quad \textit{for some } u \in \mathbf{U}.$$

(2)

$$\int_\Sigma \sigma : \tilde{e}\,dx = 0 \quad \textit{for all } \sigma \in \widetilde{\mathcal{M}}.$$

Using (2.58), we introduce the weak formulation of the quasi-static mechanical subproblem:

Find $\sigma(t) \in \mathcal{M}$ and $u(t) \in \mathbf{U}$ such that

$$\int_\Sigma \sigma(t) : \varepsilon(w)\,dx = \int_\Sigma f(t)\cdot w\,dx + \int_{\partial\Sigma} g(t)\cdot w\,dx,$$
$$\text{for all } w \in \mathbf{U} \text{ and a.e. } t \in (0,T), \tag{2.60a}$$

$$\varepsilon(u) = C(z)\sigma + \varepsilon^{th} + \int_0^t \Lambda(z)_\xi S(\xi)\,d\xi, \qquad \text{a.e. in } Q, \tag{2.60b}$$

where the thermal and transformation strain ε^{th} is defined in (2.29) and $C = K^{-1}$ is allowed to depend on the phase volume fractions z.

A close inspection of the dissipative terms in the energy balance (2.48) shows that we can only expect L^1 - regularity for them. To deal with this difficulty, we introduce the space of test functions

$$Y_{q'} = \{y \in L^{q'}(0,T;W^{1,q'}(G)) \cap C([0,T];L^\infty(G)) \,\Big|\, y_t \in L^{q'}(0,t;L^{q'}(\Sigma)),\ y(T)=0\} \tag{2.61}$$

with $\frac{1}{q} + \frac{1}{q'} = 1$ and q to be fixed later. Invoking (2.49), (2.50) and (2.52), we consider the following weak formulation for (2.48):

$$\begin{aligned} -\int_0^T \int_G \theta(\rho c_\varepsilon y)_t dx\ dt + \int_0^T \int_G k(\theta)\nabla\theta\nabla y\, dx\, dt + \int_{\partial\Sigma} \alpha_0(\theta-\theta_e) y\, dx \\ = \int_0^T \int_\Sigma F(z,z_t,\sigma) y\, dx\, dt + \int_0^T \int_\Sigma \Lambda(z)_t\ |S|^2\, dx\, dt \\ + \int_0^T \int_G \kappa_0(\theta)|\chi_\Omega \nabla\widetilde{\phi} + A_t|^2 y\, dx\ dt + \int_G \rho c_\varepsilon \theta_0 y(0)\, dx, \end{aligned} \tag{2.62}$$

for all $y \in Y_{q'}$. [2]

Remark 2.6 *The domain of definition for the scalar potential ϕ is restricted to the inductor Ω. Since $\partial\Omega$ is of class $C^{1,1}$, there exists a well-defined extension $\widetilde{\phi}$ onto G. In the sequel we will not distinguish between ϕ and $\widetilde{\phi}$ as long as this will not lead to any confusion.*

To complete the mathematical setting, we introduce the following assumptions on the data of the thermomechanical subproblem:

(A7) $\theta_0 \in L^1(G)$,

(A8) $k \in C(\mathbb{R}),\ 0 < k_* \le k(x) \le k^* < \infty$ for all $x \in \mathbb{R}$,

(A9) $\alpha_0 \in C([0,T])$, $\alpha_0(\xi) \ge 0$ for all $\xi \in [0,T]$,

(A10) $\bar{c}_\varepsilon \in C^1([0,1]^6)$, $c_{\varepsilon *} \le \bar{c}_\varepsilon(z) \le c_\varepsilon^*$ for all $z \in [0,1]^6$,

(A11) $\hat{\rho}$, ρ^Ω, c_ε^Ω, L, θ_e are positive constants, and $\rho_* \le \rho^\Omega \le \rho^*$, $c_{\varepsilon *} \le c_\varepsilon^\Omega \le c_\varepsilon^*$,

(A12) $\bar{\rho} \in C^1([0,1]^6)$, $\rho_* \le \bar{\rho}(z) \le \rho^*$ for all $z \in [0,1]^6$,

(A13) $\Lambda \in C^1([0,1]^6)$, $|\operatorname{grad}\Lambda(z)| \le \Lambda^* < \infty$ for all $z \in [0,1]^6$,

(A14) $C_{ijkl} \in C([0,1]^6)$, $C_{ijkl} = C_{jikl} = C_{klij}$, $C_{ijkl}(z) \le C^*$, $C_{ijkl}\xi_{ij}\xi_{kl} \ge C_*|\xi|^2$ for all $z \in [0,1]^6$ and $\xi \in \mathbb{R}^{3\times 3}$,

[2]To simplify the exposition, we assume that the thermal conductivity k only depends on temperature. However, for the mathematical analysis we could also allow for an additional dependency on z (cf. Section 6).

(A15) $f \in L^\infty(0,T;\mathbf{L}^2(\Sigma))$, $g \in L^\infty(0,T;H^{-1/2}(\Sigma))$ and there exists a tensor $\sigma_0 \in L^\infty(0,T;\mathcal{M})$, satisfying

$$\begin{aligned} -\operatorname{div}\sigma_0 &= f, && \text{a.e. in } Q, \\ \int_{\partial\Sigma} \Big((\sigma_0)_{ij}\nu_j - g_i\Big)\Phi_i\,dx &= 0, && \text{for all } \Phi \in \mathbf{H}^{1/2}(\partial\Sigma). \end{aligned}$$

The assumptions are quite standard. (A10) and (A12) reflect the fact that we will have to integrate by parts in the energy balance with respect to time. (A14) is the usual assumption for linear elastic materials. For the case of an isotropic material, we would have

$$C_{ijkl}(z) = \frac{1}{2\mu(z)}\delta_{ik}\delta_{jl} - \frac{1}{2\mu(z)}\frac{\lambda(z)}{3\lambda(z)+2\mu(z)}\delta_{ij}\delta_{kl},$$

with Lamé-coefficients λ, μ allowed to depend on the phase volume fractions z.
The tensor σ_0 in (A15) can for instance be thought of as the solution to the quasistatic linear elastic problem corresponding to the constitutive law $C\sigma_0 = \varepsilon(u)$.

3 Main result

Let us recall the complete electro-magneto-thermomechanical model of induction hardening we have derived in the previous section.
(P$_1$) Find $A \in L^2(0,T;\mathbf{X})$, $\phi \in L^2(0,T;H^1(\Omega)/\mathbb{R})$, $\sigma \in L^2(0,T;\mathcal{M})$, $u \in L^2(0,T;\mathbf{U})$ and $\theta \in L^q(0,T;W^{1,q}(G))$ such that

$$A(0) = A_0, \qquad \text{in } D, \tag{3.1a}$$

$$\int_G \kappa_0(\theta)A_t\cdot v\,dx + \int_D \frac{1}{\mu}\operatorname{curl} A\cdot\operatorname{curl} v\,dx + \int_\Omega \kappa_0(\theta)\nabla\phi\cdot v\,dx = 0, \quad \text{for all } v \in \mathbf{X} \text{ a.e. in } (0,T), \tag{3.1b}$$

$$\int_\Omega \kappa_0(\theta)\nabla\phi(t)\cdot\nabla\varphi\,dx + \int_\Gamma j_s(t)\varphi\,dx = 0, \quad \text{for all } \varphi \in H^1(\Omega)/\mathbb{R} \text{ a.e. in } (0,T), \tag{3.1c}$$

$$\int_\Sigma \sigma(t) : \varepsilon(w)\,dx = \int_\Sigma f(t)\cdot w\,dx + \int_{\partial\Sigma} g(t)\cdot w\,dx, \quad \text{for all } w \in \mathbf{U} \text{ a.e. in } (0,T), \tag{3.1d}$$

$$\varepsilon(u) = C(z)\sigma + \varepsilon^{th} + \int_0^t \Lambda(z)_\xi S(\xi)\,d\xi, \qquad \text{a.e. in } Q,[3] \tag{3.1e}$$

[3] Recall that $\Lambda(z)_\xi$ is short for $\frac{\partial\Lambda(z(\xi))}{\partial\xi}$.

$$
\begin{aligned}
-\int_0^T \int_G \theta(\rho c_\varepsilon y)_t \, dx \; dt + \int_0^T \int_G k(\theta)\nabla\theta\nabla y \, dx \, dt + \int_0^T \int_{\partial\Sigma} \alpha_0(\theta-\theta_e) y \, dx \, dt \\
= \int_0^T \int_\Sigma F(z, z_t, \sigma) y \, dx \, dt + \int_0^T \int_\Sigma \Lambda(z)_t \; |S|^2 y \, dx \, dt \\
+ \int_0^T \int_G \kappa_0(\theta)|\chi_\Omega \nabla\phi + A_t|^2 y \, dx \; dt + \int_G \rho c_\varepsilon \theta_0 y(0) \, dx, \qquad (3.1\text{f})
\end{aligned}
$$

for all $y \in Y_{q'}$.

Note that F has been defined in (2.50).
The coupling between the equations is given through the Joule heating and the mechanical dissipation in the energy balance on the one hand. On the other hand it is given through the temperature dependent electric conductivity, which appears in (3.1b) and (3.1c), and the temperature dependent volume fractions (cf. (A6)), which appear in (3.1e) and (3.1f).

Remark 3.1 *Owing to* (2.51), *ρ and c_ε depend on the volume fractions $z = \mathcal{P}[\theta]$ in Σ. In view of (A6) and (A10)–(A12) we can infer that there exists a constant $\tilde{M} > 0$ independent of θ such that*

$$\|(\rho c_\varepsilon)\|_{W^{1,\infty}(0,T;L^\infty(G))} \leq \tilde{M}.$$

Our main result is

Theorem 3.1 *Assume* (A1) – (A15) *and let $q \in (1, \frac{5}{4})$, then* (P_1) *has a solution. Moreover, there exists a constant $M > 0$ depending on the physical constants and on j_s, such that*

$$
\begin{aligned}
\|A\|_{L^\infty(0,T;\mathbf{X})} + \|A\|_{H^1(0,T;L^2(G))} + \|\phi\|_{L^2(0,T;H^1(\Omega)/\mathbb{R})} \\
+ \|\theta\|_{L^q(0,T;W^{1,q}(G))} + \|\sigma\|_{L^\infty(0,T;\mathcal{M})} + \|u\|_{L^\infty(0,T;\mathbf{U})} \leq M.
\end{aligned}
$$

A particular consequence of Theorem 3.1 is that the current density J as defined in (2.9) satisfies the continuity equation, at least in a distributional sense, i.e.,

Corollary 3.1 *div $J = 0$ in $[C_0^\infty(G)]'$.*

Proof. Let $w \in C_0^\infty(G)$. Using (2.9), (2.10), and (2.20), we obtain

$$
\begin{aligned}
< \text{ div } J, w > &= -\int_G J \cdot \nabla \; w \; dx \\
&= -\int_D \text{ curl } \left(\frac{1}{\mu} \text{ curl } A\right) \cdot \nabla w \; dx \\
&= -\int_D \frac{1}{\mu} \text{ curl } A \cdot \text{ curl } (\nabla w) dx = 0.
\end{aligned}
$$

□

The main difficulty in proving Theorem 3.1 lies in the quadratic terms on the right-hand side of (3.1f). To prove the existence of a weak solution to (P_1), we first truncate these terms with a cut-off function and show that this auxiliary problem has a weak solution. Then we investigate the original problem. The delicate task is to obtain an a priori estimate for θ which is uniform in the truncation parameter. To this end we use an estimate developed in [2] for parabolic equations with L^1- data.

Remark 3.2 *Note that similar arguments have been used, e.g., in [3], where a simplified induction heating problem has been considered, and in [12], for a mathematical model of resistance welding.*

To pass to the limit in the state equations, we utilize the following compactness result due to Simon [34, Theorem 5]:

Lemma 3.1 *Let X, B, Y be Banach spaces satisfying $X \subset B \subset Y$, such that the embedding $X \subset B$ is compact, and let $r \in [1, \infty)$.*
If $\mathcal{K} \subset L^r(0,T;X)$ satisfies

(i) $\mathcal{K}$ is bounded in $L^r(0,T;X)$ and

(ii) $\|f(t+h) - f(t)\|_{L^r(0,T-h;Y)} \longrightarrow 0$ as $h \to 0$ uniformly for $f \in \mathcal{K}$,

then $\mathcal{K}$ is relatively compact in $L^r(0,T;B)$.

4 An auxiliary problem

For $\delta > 0$ we define a truncation function by

$$\mathcal{T}_\delta(x) = \begin{cases} 1/\delta & , \quad x > 1/\delta, \\ x & , \quad |x| \leq 1/\delta, \\ -1/\delta & , \quad x < -1/\delta, \end{cases}$$

and consider the following approximate problem:

($\mathbf{P}_1^\delta$) Find $A^\delta \in L^2(0,T;\mathbf{X})$, $\bar{\phi}^\delta \in L^2(0,T;H^1(\Omega)/\mathbb{R})$, $\sigma^\delta \in L^2(0,T;\mathcal{M})$, $u^\delta \in L^2(0,T;\mathbf{U})$, and $\theta^\delta \in L^2(0,T;H^1(G))$ such that

$$A^\delta(0) = A_0, \qquad \text{in } D, \tag{4.1a}$$

$$\int_G \kappa_0(\theta^\delta) A_t^\delta \cdot v \, dx + \int_D \frac{1}{\mu} \text{ curl } A^\delta \cdot \text{ curl } v \, dx + \int_\Omega \kappa_0(\theta^\delta) \nabla \phi^\delta \cdot v \, dx = 0, \tag{4.1b}$$

for all $v \in \mathbf{X}$ a.e. in $(0,T)$,

$$\int_\Omega \kappa_0(\theta^\delta) \nabla \phi^\delta(t) \cdot \nabla \varphi \, dx + \int_\Gamma j_s(t) \varphi \, dx = 0, \tag{4.1c}$$

for all $\varphi \in H^1(\Omega)/\mathbb{R}$ a.e. in $(0,T)$,

$$\int_\Sigma \sigma^\delta(t) : \varepsilon(w) \, dx = \int_\Sigma f(t) \cdot w \, dx + \int_{\partial\Sigma} g(t) \cdot w \, dx, \tag{4.1d}$$

for all $w \in \mathbf{U}$ a.e. in $(0,T)$,

$$\varepsilon(u^\delta) = C(z^\delta)\sigma^\delta + (\varepsilon^{th})^\delta + \int_0^t \Lambda(z^\delta)_\xi \, S^\delta(\xi) \, d\xi, \qquad \text{a.e. in } Q, \tag{4.1e}$$

$$\theta_\delta(0) = \mathcal{T}_\delta(\theta_0), \tag{4.1f}$$

$$\begin{aligned}\int\limits_G \rho^\delta c_\varepsilon^\delta \theta_t^\delta y\, dx + \int\limits_G k(\theta^\delta)\nabla\theta^\delta \cdot \nabla y\, dx + \int\limits_{\partial\Sigma} \alpha_0(\theta - \theta_e) y\, dx \\ = \int\limits_\Sigma F(z^\delta, z_t^\delta, \sigma^\delta) y\, dx + \int\limits_\Sigma \Lambda(z^\delta)_t\ \mathcal{T}_\delta\Big(|S^\delta|^2\Big) y\, dx \\ + \int_G \kappa_0(\theta^\delta)\mathcal{T}_\delta\Big(|\chi_\Omega \nabla\phi^\delta + A_t^\delta|^2\Big) y\, dx,\end{aligned} \tag{4.1g}$$

for all $y \in H^1(\Sigma)$ a.e. in $(0,T)$.

Here, $z^\delta = \mathcal{P}[\theta^\delta]$ (cf. (A6)) and ρ^δ and c_ε^δ depend on δ according to their definition in (2.51).

Theorem 4.1 *Assume* (A1)–(A15)*, then* (P_1^δ) *has a solution.*

In the sequel we will drop the δ-dependency, whenever this does not lead to confusion.

For the proof, we apply the Schauder fixed-point theorem. We begin with four preparatory lemmas.

Lemma 4.1 *Assume* (A1) – (A3) *and let* $\hat\theta \in L^2(0,T;L^2(\Omega))$*. Then there exists a unique solution* ϕ *to*

$$\int_\Omega \kappa_0(\hat\theta)\nabla\phi\cdot\nabla\varphi\ dx + \int_\Gamma j_s\varphi dx = 0,\ \textit{for all}\ \varphi \in H^1(\Omega)/\mathbb{R} \tag{4.2}$$

satisfying

$$\|\nabla\phi\|_{L^2(0,T;\mathbf{L}^2\Omega))} \le M_1, \tag{4.3}$$

with a constant M_1 *depending on* j_s *but independent of* $\hat\theta$*.*

Proof. In view of (2.18), the existence and uniqueness proof is a standard application of the Lax-Milgram lemma. To obtain the a priori estimate, we insert $\varphi = \phi$ into (4.2) and use (A2), (A3), (2.18), and Young's inequality to obtain

$$o_*^2\kappa_*\|\phi\|^2_{H^1(\Omega)/\mathbb{R}} \le \kappa_* \int\limits_\Omega |\nabla\phi|^2\, dx \le \frac{1}{4\gamma}\|j_s\|^2_{H^{-1/2}(\Gamma)} + \gamma\|\phi\|^2_{H^{1/2}(\Gamma)} \qquad \text{a.e. in } (0,T).$$

Invoking the embedding $H^1(\Omega) \subset H^{1/2}(\Gamma)$ and again (2.18), we obtain (for γ small enough)

$$\int\limits_0^T \|\phi\|^2_{H^1(\Omega)/\mathbb{R}} \le c_1 \int\limits_0^T \|j_s\|^2_{H^{-1/2}(\Gamma)}$$

with a constant c_1 independent of $\hat\theta$. □

Lemma 4.2 *Assume* (A1)–(A4)*, let* $\hat\theta \in L^2(0,T;L^2(G))$*, and let* ϕ *be the solution to* (4.2)*. Then the solution to*

$$A(0) = A_0, \qquad \textit{in } D,$$

$$\int_G \kappa_0(\hat{\theta}) A_t \cdot v dx + \int_D \frac{1}{\mu} \; curl\; A \cdot \; curl\; v\; dx + \int_\Omega \kappa_0(\hat{\theta}) \cdot \nabla\phi v\; dx = 0 \textit{ for all } v \in \mathbf{X} \tag{4.4}$$

is uniquely defined and satisfies the estimate

$$\|A\|_{L^\infty(0,T;\mathbf{X})} + \|A\|_{H^1(0,T;L^2(G))} < M_2, \tag{4.5}$$

with a constant M_2, depending on j_s, T, and A_0, but independent of $\hat{\theta}$.

Proof. The a priori estimate (4.5) can be obtained formally by inserting $v = A_t$ into (4.4). To prove that this linear degenerate system has a unique solution, one can use, e.g., Rothe's method as described in the monograph [26]. □

Lemma 4.3 *Assume* (A12)–(A15)*, let $\hat{\theta} \in L^2(0,T;L^2(\Sigma))$, and $z = \mathcal{P}[\hat{\theta}]$ (cf. (A6)). Then, there exists a unique solution (σ, u) to*

$$\begin{aligned}
\int_\Sigma \sigma(t) : \varepsilon(w)\, dx &= \int_\Sigma f(t) \cdot w\, dx + \int_{\partial\Sigma} g(t) \cdot w\, dx, \\
& \qquad\qquad \textit{for all } w \in \mathbf{U} \textit{ a.e. in } (0,T) \qquad (4.6) \\
\varepsilon(u) &= C(z)\sigma + \varepsilon^{th}(z) + \int_0^t \Lambda(z)_\xi S(\xi)\, d\xi, \qquad \textit{a.e. in } Q, \qquad (4.7)
\end{aligned}$$

such that

$$\|\sigma\|_{L^\infty(0,T;\mathcal{M})} + \|u\|_{L^\infty(0,T;\mathbf{U})} \le M_3, \tag{4.8}$$

where M_3 is independent of $\hat{\theta}$.

Proof. Let us first recall that the deviator S, i.e., the trace-free part of σ is defined by

$$S = \sigma - \frac{1}{3} tr(\sigma) I,$$

where I is the identity matrix and $tr(\sigma)$ is the trace of σ. Moreover, we have

$$|S|^2 = (\sigma - \frac{1}{3} tr(\sigma) I) : (\sigma - \frac{1}{3} tr(\sigma) I) = |\sigma|^2 - \frac{1}{3}(tr(\sigma))^2 \le |\sigma|^2.$$

We proceed in 3 steps.

Step 1:

First we introduce

$$N = -C\sigma_0 - \int_0^t \Lambda(z)_\xi S_0(\xi)\, d\xi - \varepsilon^{th}(z), \tag{4.9}$$

where S_0 is the deviator of σ_0, defined in (A15). Owing to (A13)–(A15), there exist constants c_1 and c_2 such that

$$\|N\|_{L^\infty(0,T;\mathcal{M})} \le c_1 + c_2 \|\sigma_0\|_{L^\infty(0,T;\mathcal{M})}. \tag{4.10}$$

Now, we consider the variational equation

$$\int\limits_{\Sigma} C(z)\hat{\sigma} : \tau\, dx + \int\limits_{\Sigma} \left(\int\limits_0^t \Lambda(z)_\xi \hat{S}(\xi)\, d\xi \right) : \tau\, dx = \int\limits_{\Sigma} N : \tau\, dx, \tag{4.11}$$

for all $\tau \in \widetilde{\mathcal{M}}$ and a.e. $t \in (0,T)$. Using a straightforward fixed-point argument (which will be omitted for reasons of space) one can prove that (4.11) has a unique solution

$$\hat{\sigma} \in L^\infty(0,T;\widetilde{\mathcal{M}}).$$

To obtain an a priori estimate we insert $\tau = \hat{\sigma}$ in (4.11). Applying the inequalities of Cauchy-Schwarz and Young, as well as (A6), (A13), and (A14),we obtain

$$\begin{aligned} \frac{C_*}{2} \int\limits_{\Sigma} |\hat{\sigma}(t)|^2\, dx &\le \frac{1}{C_*} \int\limits_{\Sigma} |N(t)|^2\, dx + \frac{1}{C_*} \int\limits_{\Sigma} \left| \int\limits_0^t \Lambda(z)_\xi \hat{S}(\xi)\, d\xi \right|^2 dx \\ &\le \frac{1}{C_*} \int\limits_{\Sigma} |N(t)|^2\, dx + \frac{\Lambda^*\, z^*\, t}{C_*} \int\limits_0^t \int\limits_{\Sigma} |\hat{\sigma}(\xi)|^2\, dx\, dt. \end{aligned} \tag{4.12}$$

Using Gronwall's lemma, we infer

$$\int\limits_{\Sigma} |\hat{\sigma}(t)|^2\, dx \le \frac{2}{C_*^2} \exp\left(\frac{2\Lambda^*\, z^*\, T^2}{C_*^2} \right) \|N\|^2_{L^\infty(0,T;\mathcal{M})}. \tag{4.13}$$

Step 2:

We define

$$e(t) = C(z)\hat{\sigma} + \int\limits_0^t \Big(\operatorname{grad} \Lambda \Big) \cdot z_\xi \hat{S}(\xi)\, d\xi - N(t). \tag{4.14}$$

Since $\hat{\sigma}$ is the solution to (4.11), we have

$$\int\limits_{\Sigma} e(t) : \tau\, dx = 0, \text{ for all } \tau \in \widetilde{\mathcal{M}}, \text{ and a.e. } t \in (0,T).$$

Thus we can apply Lemma 2.1 to conclude that there exists $u(t) \in \mathbf{U}$, such that $e(t) = \varepsilon(u(t))$, for a.e. $t \in (0,T)$. Owing to Korn's inequality (cf., e.g., [4]), there exists a positive constant c_3 such that

$$c_3 \|u(t)\|_{\mathbf{U}} \le \|\varepsilon(u(t))\|_{\mathcal{M}}, \quad \text{for a.e. } t \in (0,T).$$

Moreover, in view of (4.10), (4.13), and (4.14), there exists a constant c_4, depending on the constants defined in (A12)–(A14) and on σ_0, such that

$$\|\varepsilon(u(t))\|_{\mathcal{M}} \le c_4, \quad \text{for a.e. } t \in (0,T).$$

Step 3:

We define $\sigma = \hat{\sigma} + \sigma_0$. Then (σ, u) is a solution to (4.6), (4.7), and satisfies (4.8). To show that the solution is unique, we assume that there exist two solutions (σ^i, u^i), i=1,2. We take the difference of (4.6) for σ^1 and σ^2, and test it with $\bar{u} = u^1 - u^2$. Then we take the difference of (4.7) for (σ^1, u^1) and (σ^2, u^2), and test it with $\bar{\sigma} = \sigma^1 - \sigma^2$. Altogether, we obtain

$$\int_\Sigma C(z)\bar{\sigma}(t) : \bar{\sigma}\, dx \le \int_\Sigma \Big(\int_0^t \Lambda(z)_\xi \bar{S}(\xi)\, d\xi \Big) : \bar{\sigma}\, dx.$$

Proceeding as in the proof of (4.13), we obtain $\bar{\sigma} = 0$ a.e. in Q. Using again Korn's inequality, we also obtain $\bar{u} = 0$ a.e. in Q. □

Lemma 4.4 *Assume* (A1)–(A15), *let* $\hat{\theta} \in L^2(0,T; L^2(G))$, $z = \mathcal{P}[\hat{\theta}]$, *and let* (ϕ, A, σ, u) *be the solution to* (4.2), (4.4), (4.6), *and* (4.7), *respectively. Then the solution to*

$$\theta_\delta(0) = \mathcal{T}_\delta(\theta_0), \tag{4.15}$$

$$\begin{aligned} \int_G \rho c_\varepsilon \theta_t y\, dx + \int_G k(\hat{\theta}) \nabla\theta \cdot \nabla y\, dx + \int_{\partial\Sigma} \alpha_0(\theta - \theta_e) y\, dx \\ = \int_\Sigma F(z, z_t, \sigma) y\, dx + \int_\Sigma \Lambda(z)_t\ \mathcal{T}_\delta\Big(|S|^2\Big) y\, dx \\ + \int_G \kappa_0(\theta) \mathcal{T}_\delta\Big(|\chi_\Omega \nabla\phi + A_t|^2\Big) y\, dx\, dt, \end{aligned} \tag{4.16}$$

for all $y \in H^1(G)$ *a.e. in* $(0,T)$, *is uniquely defined and satisfies the estimates*

$$\begin{aligned} \|\theta\|_{L^\infty(0,T;L^2(G)) \cap L^2(0,T;H^1(G))} &\le M_4, && (4.17) \\ \|\theta(t+h) - \theta(t)\|_{L^2(0,T-h;(H^1(G))^*)} &\le h\, M_5, && (4.18) \end{aligned}$$

for $h \ge 0$, *where the constants* $M_{4,5}$ *depend on* δ *but are independent of* $\hat{\theta}$.

Proof. We can apply standard results of the theory of parabolic equations (cf., e.g., [27]) to infer the existence of a unique solution to (4.15), (4.16) satisfying the a priori estimate

$$\|\theta\|_{L^\infty(0,T;L^2(G)) \cap L^2(0,T;H^1(G))} \le c_1$$

with a positive constant c_1 independent of $\hat{\theta}$, but dependent on δ. By comparison in (4.16), we obtain the estimate

$$\|(\rho c_\varepsilon \theta)_t\|_{L^2(0,T;(H^1(G))^*)} \le c_2$$

with a constant c_2 again independent of $\hat{\theta}$, but dependent on δ. Taking into account Remark 3.1, we can apply [34, Lemma 4] to obtain $\rho c_\varepsilon \theta \in C([0,T]; (H^1(G))^*)$ and

$$\Big\| \Big(\rho c_\varepsilon \theta\Big)(t+h) - \Big(\rho c_\varepsilon \theta\Big)(t) \Big\|_{L^2(0,T;(H^1(G))^*)} \le h \|(\rho c_\varepsilon \theta)_t\|_{L^2(0,T-h;(H^1(G))^*)} \le h\, c_2. \tag{4.19}$$

Since $\rho c_\varepsilon \geq \rho_* c_{\varepsilon *} > 0$ a.e. in $G \times (0,T)$, we have the identity

$$\theta(t+h) - \theta(t) = \frac{1}{\Big(\rho c_\varepsilon\Big)(t+h)} \Big(\Big(\rho c_\varepsilon \theta\Big)(t+h) - \Big(\rho c_\varepsilon \theta\Big)(t) \Big) \\ - h \frac{\theta(t)}{\Big(\rho c_\varepsilon\Big)(t+h)} \left(\frac{\Big(\rho c_\varepsilon\Big)(t+h) - \Big(\rho c_\varepsilon\Big)(t)}{h} \right). \tag{4.20}$$

Hence $\theta \in C([0,T];(H^1(G))^*)$, and the initial condition (4.15) is satisfied. Using (4.17), (4.19), and Remark 3.1, we obtain (4.18). $\square$

Now we can prove Theorem 4.1 using the Schauder fixed-point theorem. We take a function $\hat{\theta} \in L^2(0,T;L^2(G))$ and define $z = \mathcal{P}[\hat{\theta}]$, the vector of phase volume fractions, in line with (A6). Then, we obtain consecutively ϕ, A, σ, u as solutions to (4.2), (4.4), (4.6), and (4.7). Finally we obtain a new function $\theta \in L^2(0,T;H^1(G))$ as the solution to (4.15) and (4.16).

Thus we have defined a mapping

$$\mathcal{F} \,:\, L^2(0,T;L^2(G)) \to L^2(0,T;L^2(G)), \;\; \hat{\theta} \;\mapsto\; \theta. \tag{4.21}$$

Owing to Lemmas 4.1–4.4, the operator is well defined. Thanks to the a priori estimate (4.17), which is uniform in $\hat{\theta}$, $\mathcal{F}$ is a self-mapping on

$$K = \{y \in L^2(0,T;L^2(G)) \,\Big|\, \|y\|_{L^2(0,T;H^1(G))} \leq M\},$$

provided the constant M has been chosen large enough.

Moreover, in view of (4.18), we can employ Lemma 3.1 with $X = H^1(G)$, $B = L^2(G)$, $Y = (H^1(G))^*$, and $r = 2$ to conclude that

$$\mathcal{K} = \mathcal{F}[K] \quad \text{is relatively compact in } \; L^2(0,T;L^2(G)). \tag{4.22}$$

Hence, if the operator $\mathcal{F}$ is also continuous, we can apply the Schauder theorem to conclude that $\mathcal{F}$ has a fixed point, which then is a solution to (P_1^δ).

So, it remains to show:

Lemma 4.5 *The operator $\mathcal{F}$ as defined in* (4.21) *is continuous.*

Proof. Let $\hat{\theta}^k \to \hat{\theta}$ strongly in $L^2(G)$ and $\theta^k = \mathcal{F}[\hat{\theta}^k]$. Invoking (A6), we obtain immediately

$$z^k = \mathcal{P}[\hat{\theta}^k] \to z = \mathcal{P}[\hat{\theta}], \;\text{ strongly in } [W^{1,p}(0,T;L^p(\Sigma)]^6, \;\text{for } p \in [1,\infty). \tag{4.23}$$

Owing to the assumptions on the data and using Lebesgue's lemma and (4.23), we have

$$\kappa_0(\hat{\theta}^k) \;\to\; \kappa_0(\hat{\theta}), \;\text{ strongly in } L^p(0,T;L^p(\Omega))), \;\text{for } p \in [1,\infty), \tag{4.24}$$

$$k(\hat{\theta}^k) \;\to\; k(\hat{\theta}), \qquad \rho^k \;\to\; \rho, \qquad c_\varepsilon^k \;\to\; c_\varepsilon, \tag{4.25}$$

strongly in $L^p(0,T;L^p(G))$, for $p \in [1,\infty)$, and

$$C(z^k) \;\;\to\;\; C(z), \qquad \Lambda(z^k) \;\;\to\;\; \Lambda(z) \qquad \Lambda(z^k)_t \;\;\to\;\; \Lambda(z)_t. \tag{4.26}$$

strongly in $L^p(0,T;L^p(\Sigma))$, for $p \in [1,\infty)$. Note that ρ^k and ρ are the extensions of $\bar{\rho}(\hat{\theta}^k)$ and $\bar{\rho}(\hat{\theta})$ onto G according to (2.51) and that mutatis mutandis the same holds true for c_ε^k and c_ε.
We begin with the equation for the scalar potential (4.2). Owing to (4.3) and (4.24), there exists a subsequence (still indexed with k), such that

$$\nabla\phi^k \to \nabla\phi, \text{ weakly in } L^\infty(0,T;\mathbf{L}^2(\Omega)). \tag{4.27}$$

Since we can pass to the limit in (4.2), and $\nabla\phi$ is uniquely determined, (4.27) holds for the whole sequence.
Now consider $\hat{\phi}^k := \phi^k - \phi$, then $\hat{\phi}^k$ solves

$$\int_0^T \int_\Omega^\sigma \kappa_0(\hat{\theta}^k)\nabla\hat{\phi}^k\nabla u \; dx\, dt = -\int_0^T \int_\Gamma j_s u \; dx \; dt - \int_0^T \int_\Omega \kappa_0(\hat{\theta}^k)\nabla\phi\nabla u \; dx \; dt$$

for all $u \in H^1(\Omega)/\mathbb{R}$. Inserting $u = \hat{\phi}^k$ we obtain

$$\kappa_* \int_0^T \int_\Omega |\nabla\hat{\phi}^k|^2 \leq -\int_0^T \int_\Omega \kappa_0(\hat{\theta}^k)\nabla\phi\nabla\hat{\phi}^k dx \; dt - \int_0^T \int_\Gamma j_s\hat{\theta}^k dx \; dt \xrightarrow[k\to\infty]{} 0,$$

hence, we have

$$\nabla\phi^k \xrightarrow[k\to\infty]{} \nabla\phi, \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Omega)). \tag{4.28}$$

Next we consider (4.4). Owing to (4.5) we have

$$A^k \to A, \qquad \begin{array}{l} \text{weakly* in } L^\infty(0,T;\mathbf{X}), \\ \text{weakly in } H^1(0,T;\mathbf{L}^2(G)). \end{array} \tag{4.29}$$

Using (4.27) and (4.24), we can pass to the limit in (4.4) and again conclude that (4.29) is valid for the whole sequence.
Now, we consider $\bar{A}^k := A^k - A$, which satisfies

$$\begin{aligned} &\bar{A}^k(0) = 0 \text{ a.e. in } D, \\ &\int_G \kappa_0(\hat{\theta}^k)\bar{A}_t^k \cdot v \; dx + \int_D \frac{1}{\mu} \text{ curl } \bar{A}^k \cdot \text{ curl } v \, dx = -\int_\Omega \kappa_0(\hat{\theta}^k)\nabla\phi^k \cdot v \; dx \\ &+ \int_\Omega \kappa_0(\hat{\theta})\nabla\phi \cdot v \; dx + \int_G (\kappa_0(\hat{\theta}) - \kappa_0(\hat{\theta}^k))A_t \cdot v \; dx. \end{aligned}$$

Inserting $v = \bar{A}_{k,t}$, and integrating in time, we obtain

$$\begin{aligned} \kappa_* \int_0^T \int_G |\bar{A}_t^k|^2 dx \; dt \;\; &\leq \;\; \int_0^T \int_G \kappa_0(\hat{\theta}^k)|\bar{A}_t^k|^2 dx \; dt + \int_D \frac{1}{\mu}| \text{ curl } \bar{A}^k(t)|^2 dx \\ &= \;\; -\int_0^T \int_\Omega \kappa_0(\hat{\theta}^k)\nabla\phi^k \cdot \bar{A}_t^k dx \; dt + \int_0^T \int_\Omega \kappa_0(\hat{\theta})\nabla\phi \cdot \bar{A}_t^k dx \; dt \\ &\quad + \int_0^T \int_G (\kappa_0(\hat{\theta}) - \kappa_0(\hat{\theta}^k))A_t \cdot \bar{A}_t^k dx \; dt \\ &\quad \xrightarrow[k\to\infty]{} 0. \end{aligned} \tag{4.30}$$

Thus, we have

$$A_t^k \to A_t, \text{ strongly in } L^2(0,T;\mathbf{L}^2(G)). \tag{4.31}$$

Note that the derivation of (4.30) is only formal. Since $v = \bar{A}_{k,t} \notin \mathbf{X}$, one should use $v = \frac{1}{h}(\bar{A}^k(t) - \bar{A}^k(t-h))$. Passing to the limit with $h \to 0$ would yield the same result.
Now we turn to the mechanical subproblem. Invoking (4.8), there exists a subsequence (σ^k, u^k) satisfying

$$\begin{aligned} \sigma_{ij}^k,\, S_{ij}^k \to \sigma_{ij},\, S_{ij}, \qquad & \text{weakly in } L^2(0,T;L^2(\Sigma)), \\ u^k \to u, \qquad & \text{weakly in } L^2(0,T;\mathbf{U}). \end{aligned} \tag{4.32}$$

Hence, applying Lebesgue's lemma, (A6), and (A12) – (A14), we can pass to the limit in

$$\int_\Sigma \sigma^k(t) : \varepsilon(w)\, dx = \int_\Sigma f(t)\cdot w\, dx + \int_{\partial\Sigma} g(t)\cdot w\, dx,$$

$$\text{for all } w \in \mathbf{U} \text{ a.e. in } (0,T), \tag{4.33}$$

$$\varepsilon(u^k) = C(z^k)\sigma^k + \varepsilon^{th}(z^k) + \int_0^t \Lambda(z^k)_\xi\, S^k(\xi)\, d\xi, \text{ a.e. in } Q. \tag{4.34}$$

Owing to the unique solvability of (4.6), (4.7), we can again conclude that the convergence holds for the whole sequence.
Now we define $\bar\sigma^k = \sigma^k - \sigma$, $\bar S^k = S^k - S$, and $\bar u^k = u^k - u$. We test the difference of (4.33) and (4.6) with $\bar u^k$, and the difference of (4.34) and (4.7) with $\bar\sigma^k$, leading to

$$\begin{aligned} \int_\Sigma C(z^k)\bar\sigma^k : \bar\sigma^k\, dx &= \int_\Sigma (C(z) - C(z^k)\sigma : \bar\sigma^k\, dx \\ &\quad - \int_\Sigma \Big[\int_0^t \Lambda(z^k)_\xi\, \bar S^k\, d\xi\Big] : \bar\sigma^k\, dx \\ &\quad + \int_\Sigma \Big[\int_0^t \Big(\Lambda(z)_\xi - \Lambda(z^k)_\xi\Big) S\, d\xi\Big] : \bar\sigma^k\, dx. \end{aligned}$$

Integrating in time, applying Young's inequality, Lebesgue's lemma and Gronwall's inequality, we can infer

$$\begin{aligned} \sigma_{ij}^k \to \sigma_{ij}, \qquad & \text{strongly in } L^2(Q)), \\ |S^k|^2 \to |S|^2, \qquad & \text{strongly in } L^1(Q)), \\ \mathcal{T}_\delta(|S^k|^2) \to \mathcal{T}_\delta(|S|^2), \qquad & \text{strongly in } L^p(Q)) \text{ for } p \in (1,\infty). \end{aligned} \tag{4.35}$$

Finally we consider (4.16). Thanks to (4.17) and (4.22) we obtain

$$\begin{aligned} \theta^k \to \theta, \qquad & \text{weakly* in } L^\infty(0,T;L^2(G)) \\ & \text{weakly in } L^2(0,T;H^1(G)) \\ & \text{strongly in } L^2(0,T;L^2(G)). \end{aligned} \tag{4.36}$$

Using Lebesgue's lemma and (4.28), (4.31), possibly extracting a further subsequence, we have

$$\mathcal{T}_\delta(|\chi_\Omega \nabla\phi^k + A_t^k|^2) \xrightarrow{k\to\infty} \mathcal{T}_\delta(|\chi_\Omega \nabla\phi + A_t|^2) \tag{4.37}$$

strongly in $L^p(0,T;L^p(G))$, for $p \in [1,\infty)$. Similarly, we obtain for F defined in (2.49)

$$F(z^k, z_t^k, \sigma^k) \xrightarrow{k\to\infty} F(z, z_t, \sigma) \tag{4.38}$$

strongly in $L^2(Q)$.
As usual, now we can multiply (4.16) with a testfunction $\psi \in H^1(0,T)$, such that $\psi(T) = 0$ and integrate in time. Integrating the first term by parts, we can pass to the limit in (4.16), and thanks to the unique solvability of (4.16), we conclude that the convergences (4.36) hold for the whole sequence $\{\theta^k\}$.
Thus we have proved the continuity of the operator

$$\mathcal{F} : K \subset L^2(0,T;L^2(G)) \to L^2(0,T;L^2(G)).$$

$\square$

5 Proof of Theorem 3.1

The bounds $M_1 - M_3$ for $\nabla\phi^\delta$, A^δ, σ^δ, and u^δ in (4.3) (4.5), and (4.8), respectively are independent of $\hat{\theta}$ and hence uniform in δ. So the main issue of the proof is to obtain uniform a priori estimates for θ^δ.
For the last two terms on the right-hand side of (4.1g) we have (for $y \equiv 1$)

$$\begin{aligned}
&\int_0^T \int_\Sigma \Lambda(z^\delta)_t\, \mathcal{T}_\delta\Big(|S^\delta|^2\Big)\, dx\, dt + \int_0^T \int_G \kappa_0(\theta^\delta)\mathcal{T}_\delta(|\chi_\Omega \nabla\phi^\delta + A_t^\delta|^2)\, dx\, dt \\
&\quad\le \Lambda^* z^* \int_0^T \int_\Sigma |S^\delta|^2\, dx\, dt + 2\kappa^* \int_0^T \int_\Omega |\nabla\phi^\delta|^2 dx\ dt + 2\kappa^* \int_0^T \int_G |A_t|^2 dx\ ds \\
&\quad\le \Lambda^* z^* T M_3^2 + 2\kappa^*(M_1^2 + M_2^2).
\end{aligned} \tag{5.1}$$

Thus, the right-hand side of (4.1g) is bounded in $L^1(Q)$. Now, we define a function

$$\psi(s) = \begin{cases} 1, & \text{for } s > 1, \\ -1, & \text{for } s < -1, \\ s, & \text{for } -1 \le s \le 1, \end{cases} \tag{5.2}$$

and a set

$$B(\theta) = \{(x,t) \in G \times (0,T) \,\Big|\, |\theta(x,t)| \le 1\}.$$

Note that $\psi(\theta)$ is constant on $G \times (0,T) \setminus B(\theta)$. Moreover, defining an antiderivative of ψ by

$$\Psi(s) = \int_0^s \psi(\xi)\, d\xi = \begin{cases} \frac{1}{2}s^2, & \text{for } |s| < 1, \\ |s| - \frac{1}{2}, & \text{for } |s| \ge 1, \end{cases}$$

we have

$$|\Psi(s)| \le s, \ \text{ for all } s \in \mathbb{R}, \ \text{ and } \ \Psi(s) \ge \frac{1}{2}|s|, \text{ for all } |s| \ge 1. \tag{5.3}$$

With these preparations, assuming without restriction $\theta_e \equiv 0$, we test (4.1g) with $\psi(\theta^\delta)$, and integrate by parts with respect to time to obtain

$$\int_G \rho^\delta c_\varepsilon^\delta \Psi(\theta^\delta(t))\,dx + \int_0^t \int_{B(\theta^\delta)} k(\theta^\delta)|\nabla\theta^\delta|^2\,dx\,dt + \int_0^t \int_{\partial\Sigma} \alpha_0(\xi)\theta^\delta\psi(\theta^\delta)\,dx\,ds$$

$$\le \int_G \rho^\delta c_\varepsilon^\delta \Psi(\theta^\delta(0))\,dx + c_1 \int_0^t \int_G |\theta^\delta|\,dx\,ds + c_2. \tag{5.4}$$

Here, we have also used (5.1), (5.3), and (A10)–(A13). Since the last two terms on the left-hand side of (5.4) are positive (cf. (A8) and (A9)), we can use Gronwall's lemma and (A7) to infer that there is a constant M_6, independent of δ, such that

$$\|\theta^\delta\|_{L^\infty(0,T;L^1(G))} \le M_6. \tag{5.5}$$

Unfortunately this estimate is not enough to pass to the limit in the state equations. Therefore, we apply a result by Boccardo and Galluët (cf. [2, Theorem 4]). It says that the solution θ^δ of (4.1g) satisfies the additional a priori estimate

$$\|\theta^\delta\|_{L^q(0,T;W^{1,q}(G))} \le M_7, \ \text{ for } q \in [1, 5/4], \tag{5.6}$$

where the constant M_7 is independent of δ.

Remark 5.1

(1) The bound on q depends of course on the space dimension.

(2) In the original paper by Boccardo and Galluët, the result has been proved for homogeneous Dirichlet data and a constant coefficient in front of the time derivative. Later, it has been shown by Clain [5] that the result also holds in the case of homogeneous Neumann data, which we are considering on $\partial\Omega$. To derive estimate (5.6), test functions $\tilde\psi(\theta)$ are used, where $\tilde\psi$ is a cut-off function similar to ψ in (5.2). Hence, as in (5.4), the additional term which stems from the Newton cooling law assumed on $\partial\Sigma$ is positive. Moreover, thanks to (A6), the additional term ρc_ε in front of the time derivative poses no additional difficulty and can be treated as in (5.4).

Now, we insert $y \in Y_{q'}$ in (4.1g) integrate in time and the first term by parts to obtain

$$-\int_0^T \int_G \rho^\delta c_\varepsilon^\delta \theta^\delta y_t\,dx\ dt + \int_0^T \int_G k(\theta^\delta)\nabla\theta^\delta\nabla y\,dx\,dt + \int_0^T \int_{\partial\Sigma} \alpha_0(\theta^\delta - \theta_e)y\,dx\,dt$$

$$= \int_0^T \int_\Sigma F(z^\delta, z_t^\delta, \sigma)y\,dx\,dt + \int_0^T \int_\Sigma \Lambda(z^\delta)_t\ \mathcal{T}_\delta\Big(|S^\delta|^2\Big)y\,dx\,dt$$

$$+ \int_0^T \int_G \kappa_0(\theta^\delta)\mathcal{T}_\delta\Big(|\chi_\Omega\nabla\phi^\delta + A_t^\delta|^2\Big)y\,dx\ dt + \int_G \rho^\delta c_\varepsilon^\delta \mathcal{T}_\delta(\theta_0)y(0)\,dx$$

$$+ \int_0^T \int_G \theta^\delta(\rho^\delta c_\varepsilon^\delta)_t y\,dx\ dt. \tag{5.7}$$

Using (5.1) and (5.6), as well as (A7) and Remark 3.1, by comparison in (5.7) we obtain

$$\|(\rho^\delta c_\varepsilon^\delta \theta^\delta)_t\|_{L^q(0,T;[W^{1,q}(G)]^*)+L^1(0,T;L^1(G))} \le M_7,$$

where M_7 is independent of δ. Since $q \in [1, 5/4)$ and $L^1(G) \subset (W^{1,p}(G))^*$, for $p > 3$, we have the continuous embedding

$$L^q(0,T;[W^{1,q}(G)]^*) + L^1(0,T;L^1(G)) \subset L^1(0,T;(W^{1,p}(G))^*), \qquad \text{for } p > 3.$$

Thus we can infer

$$\|(\rho^\delta c_\varepsilon^\delta \theta^\delta)_t\|_{L^1(0,T;(W^{1,p}(G))^*)} \le M_8, \quad \text{for } p > 3, \tag{5.8}$$

where M_8 again is independent of δ. Now we can proceed as in the proof of Lemma 4.4. We apply [34, Lemma 4] to obtain $\rho c_\varepsilon \theta \in C([0,T];(W^{1,p}(G))^*)$ and

$$\left\| \left(\rho^\delta c_\varepsilon^\delta \theta^\delta\right)(t+h) - \left(\rho^\delta c_\varepsilon^\delta \theta^\delta\right)(t) \right\|_{L^1(0,T;(W^{1,p}(G))^*)} \le h \|(\rho^\delta c_\varepsilon^\delta \theta^\delta)_t\|_{L^1(0,T-h;(W^{1,p}(G))^*)} \le h\, M_9.$$

Invoking the identity (4.20) and Remark 3.1 we see that θ^δ is in $C([0,T];(W^{1,p}(G))^*)$, and with the additional help of (5.6), we obtain

$$\|\theta^\delta(t+h) - \theta^\delta(t)\|_{L^1(0,T;(W^{1,p}(G))^*)} \le h\, M_{10},$$

with some constant M_{10}, independent of δ. Then we can employ Lemma 3.1 with $X = W^{1,q}(G)$, $B = L^1(G)$, $Y = (W^{1,p}(G))^*$, and $r = 1$ to conclude that $\{\theta^\delta\}$ is relatively compact in $L^1(0,T;L^1(G))$.

Hence, we have the following convergences for a subsequence still denoted by δ:

$$\begin{aligned} \theta^\delta \to \theta \qquad & \text{weakly in } L^q(0,T;W^{1,q}(G)),\ q < \frac{5}{4}, \\ & \text{strongly in } L^1((0,T);L^1(G)) \text{ and a.e. in } G \times (0,T). \end{aligned} \tag{5.9}$$

Invoking Lebesgue's lemma, we obtain as in (4.23) – (4.26)

$$z^\delta = \mathcal{P}[\theta^\delta] \to z = \mathcal{P}[\theta], \text{ strongly in } [W^{1,p}(0,T;L^p(\Sigma)]^6, \text{ for } p \in [1,\infty), \tag{5.10}$$

$$\kappa_0(\theta^\delta) \to \kappa_0(\theta), \text{ strongly in } L^p(0,T;L^p(\Omega))), \text{ for } p \in [1,\infty), \tag{5.11}$$

$$k(\theta^\delta) \to k(\theta), \qquad \rho^\delta \to \rho, \qquad c_\varepsilon^\delta \to c_\varepsilon, \tag{5.12}$$

strongly in $L^p(0,T;L^p(G))$, for $p \in [1,\infty)$ and

$$C(z^\delta) \to C(z), \qquad \Lambda(z^\delta) \to \Lambda(z), \qquad \Lambda(z^\delta)_t \to \Lambda(z)_t, \tag{5.13}$$

strongly in $L^p(0,T;L^p(\Sigma))$, for $p \in [1,\infty)$.

Moreover, as in the proof of Lemma 4.5 we can conclude

$$\begin{aligned} \nabla\phi^\delta &\to \nabla\phi, && \text{strongly in } L^2(0,T;\mathbf{L}^2(\Omega)), && (5.14)\\ A^\delta &\to A, && \text{weakly* in } L^\infty(0,T;\mathbf{X}), && (5.15)\\ A_t^\delta &\to A_t, && \text{strongly in } L^2(0,T;L^2(G)), && (5.16)\\ u^\delta &\to u, && \text{weakly* in } L^\infty(0,T;\mathbf{U}) && (5.17)\\ \sigma_{ij}^\delta &\to \sigma_{ij}, && \text{strongly in } L^2(Q)). && (5.18) \end{aligned}$$

Using these convergences, we can pass to the limit in (4.1b) – (4.1e).
In view of (5.14), (5.16), and (5.18), using again Lebesgue's lemma, we obtain

$$\kappa_0(\theta^\delta)\mathcal{T}_\delta(|\chi_\Omega \nabla\phi^\delta + A_t^\delta|^2) \to \kappa_0(\theta)|\chi_\Omega \nabla\phi + A_t|^2, \tag{5.19}$$

strongly in $L^1(0,T;L^1(G))$,

$$\Lambda(z^\delta)_t \; \mathcal{T}_\delta\left(|S^\delta|^2\right) \to \Lambda(z)_t \; |S|^2, \tag{5.20}$$

strongly in $L^1(Q)$, and

$$\mathcal{T}_\delta(\theta_0) \to \theta_0, \tag{5.21}$$

strongly in $L^1(G)$.
Similarly, we have

$$k(\theta^\delta)\nabla\theta^\delta \to k(\theta)\nabla\theta \text{ weakly in } L^q(Q), q < \frac{5}{4}. \tag{5.22}$$

Using (5.19)–(5.22), we can also pass to the limit in (5.7) and obtain (3.1f).
Thus we have proved that $(A, \phi, \theta, \sigma, u)$ is a solution to (P_1).

6 Extensions and open problems

Let us first consider the model as it has been developed in Section 2. Since the inductor Ω is usually made of copper while the workpiece is made of steel, it is natural to expect that the electric and the thermal conductivities, κ_0 and k, respectively, have different thermal characteristics in Ω and Σ, i.e.,

$$\kappa_0 = \begin{cases} \kappa^\Omega(\theta), & \text{in } \Omega, \\ \sum_{i=0}^{5} z_i\, \kappa_i^\Sigma(\theta), & \text{in } \Sigma, \end{cases} \qquad \text{and} \qquad k = \begin{cases} k^\Omega(\theta), & \text{in } \Omega, \\ \sum_{i=0}^{5} z_i\, k_i^\Sigma(\theta), & \text{in } \Sigma. \end{cases}$$

It is easy to see that Theorem 3.1 still holds in this case, under suitable assumptions on κ^Ω, k^Ω, κ_i^Ω, and k_i^Ω, similar to (A3) and (A8).
In the same way the result can be extended to the case of a volume fraction and temperature-dependent stiffness matrix C^{-1}.
However, the case of temperature- and phase-fraction-dependent density and heat capacity is not covered by Theorem 3.1. If these coefficients only depended on temperature, we could use the Kirchhoff transformation

$$u = \int_{\theta_0}^{\theta} (\rho c_\varepsilon)(x)\, dx$$

to get rid of the coefficient in front θ_t. In turn we would obtain a parabolic equation with a nonlinear but monotone boundary condition. Since it has been shown that the Boccardo-Gallüët estimate also holds for this case (cf. [7]), we can again proceed as in the proof of Theorem 3.1.
The case of a mere phase fraction dependency has been treated in the Theorem 3.1. In the case of temperature- and phase-fraction-dependent density and heat capacity we cannot apply the

Kirchhoff transform. Since our proof strongly relies on the differentiability of ρc_ε with respect to time, our result does not cover this case, since we cannot control θ_t.
In the vector potential equation, we could easily assume a mixture ansatz for the permeability. This would be covered by our analysis as long as the permeabilities of the constituent phases do not depend on temperature. However, we should remark that especially in the case of ferromagnetic steels, the permeability μ strongly depends on temperature. Since we have to control also A_t, this situation is not covered by our analysis and remains an open problem.
Now let us come back to the model restrictions that we have imposed in Remark 2.4. If we admitted a temperature-dependent density ρ, we would obtain terms that either contain the velocity $v = u_t$ or the stress rate σ_t. However, they are coupled through Hooke's law (2.34) with the thermal strain rate ε_t^{th}, from which we would get a factor θ_t in that case. Hence, to control v or σ_t we have to control θ_t. But owing to the quadratic dissipation terms in the energy balance, we do not get an estimate for θ_t.
Note that this is a structural problem which already appears in nonlinear thermoviscoelasticity (cf. [12], [23]).

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