Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

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submitted: 23 Aug 2002

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No. 764
Berlin 2002

2000 *Mathematics Subject Classification.* 62G10, 62G20.

Key words and phrases. Multichannel signal detection, minimax hypothesis testing, adaptive hypothesis testing, distinguishability conditions.

Edited by
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On multichannel signal detection *†

Yuri Ingster‡ and Oleg Lepski§

Abstract

We consider n-channel signal detection system. Each ith channel could contains (or not contains) a signal. We suppose a signal is a function $f_i(t)$, $t \in (0,1)$ observing in the white Gaussian noise of level $\varepsilon > 0$. Let k be a number of channels which contain the signals. This number could be known or unknown. The functions f_i could be known or unknown as well. If shapes of functions f_i are unknown, then we consider nonparametric case. We suppose that functions f_i belong to the Sobolev ball S^σ where the smoothness parameter $\sigma > 0$ could be known or unknown as well. The cases, when k or σ are unknown, lead to the "adaptive" problems.

We are interested in the following problems:

(1) How large the signals f_i should be in order to detect these signals with vanishing errors, as the number of channels n tends to infinity?

(2) What are the structures of test procedures which provide the detection of signals with the vanishing errors, if it is possible?

We show that there are two main type of results in the problems which, roughly, correspond to the cases either k is "large" (this means $k \gg n^{1/2}$ in the problem) or k is "small" (this means $k \ll n^{1/2}$).

1 Statement of the problem

1.1 Model

The aim of this paper is to study a general properties of the multichannel signal detection problem for the case when a number n of channel is large. There are a lot of applications of problems under considerations: these problems arise in the technical and medical diagnostics, in radio engineerings, in the information transmissions, etc.

In what follows all limits are assumed as $n \to \infty$.

*Key words: multichannel signal detection, minimax hypothesis testing, adaptive hypothesis testing, distinguishability conditions.

†2000 Mathematics Subject Classification: Primary 62G10; secondary 62G20.

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We study the hypothesis testing problem under the white Gaussian noise model. Given n-channel system, we observe n random processes X_i of the form

$$dX_i(t) = \xi_i f_i(t) + \varepsilon dW_i(t),\ t \in (0,1),\ i = 1,\dots,n;\ f_i \in F_n, \tag{1}$$

where $W_i(t)$ are independent Wiener processes, $\varepsilon = \varepsilon_n > 0$ is noise level, $F_n \subset L_2(0,1)$ are the given signal sets and $\xi_i \in \{0,1\}$, that is, $\xi_i = 1$, if ith channel contains a signal and $\xi_i = 0$ otherwise. Let $\bar{f} = (\xi_1 f_1, \dots, \xi_n f_n) \in L_2^n(0,1)$ be the collection of signals in the channels and

$$F_k^n = \{\bar{f}:\ f_i \in F_n,\ i = 1,\dots,n;\quad \sum_{i=1}^n \xi_i = k\} \subset L_2^n(0,1). \tag{2}$$

The quantity $\sum_{i=1}^n \xi_i$ is the number of channels in the system which contain signals.

Let $k = k_n$ be a given quantity, $1 \le k \le n$ (for the case when k is unknown, the statement of "adaptive" problem will be given in Section 3.1). We want to test the null-hypothesis $H_0:\ \bar{f} = 0$ against the alternative $H_1:\ \bar{f} \in F_k^n$.

We study the problem under asymptotic variant of the minimax setting, assuming $n \to \infty$. For the test ψ_n we denote by

$$\gamma_{k,n}(\psi_n) = \gamma(F_k^n, \psi_n) = \alpha_n(\psi_n) + \sup_{\bar{f} \in F_k^n} \beta_n(\psi_n, \bar{f})$$

the sum of the type I error probability $\alpha_n(\psi_n) = E_0(\psi_n)$ and of the maximal type II error probability $\beta_n(\psi_n, \bar{f}) = E_{\bar{f}}(1 - \psi_n)$, where E_0 and $E_{\bar{f}}$ is the expectation over the measure corresponding to the null-hypothesis and to the alternative $\bar{f}$. We also set

$$\gamma_{k,n} = \gamma_n(F_k^n) = \inf_{\psi_n} \gamma_n(F_k^n, \psi_n).$$

Clearly, $0 \le \gamma_{k,n} \le 1$ (see [2]). We are interested in the conditions for minimax distinguishability (i.e., for $\gamma_{k,n} \to 0$) or for minimax nondistinguishability (i.e., for $\gamma_{k,n} \to 1$). Also if $\gamma_{k,n} \to 0$, then we want to construct a test procedure ψ_n which provides distinguishability (i.e., $\gamma_n(F_k^n, \psi_n) \to 0$).

1.2 Results for known shape of signals

Let us formulate the results for the case when the sets $F_n = \{f_n\}$ consists of one known signal $f_n = c_n\phi \in L_2(0,1)$, $c_n = \|f_n\| > 0$, $\|\phi\| = 1$; the function ϕ determines the "shape" of a signal. Let us consider the statistics

$$x_i = \varepsilon^{-1} \int_0^1 \phi(t) dX_i(t),\ i = 1,\dots,n. \tag{3}$$

Clearly, they are the sufficient statistics in the problem. The statistics x_i are i.i.d. standard Gaussian $N(0, 1)$ under the hull-hypothesis. Let $b = b_n = c_n/\varepsilon$ be the signal-to-noise ratio. Then x_i are independent Gaussian $N(\xi_i b, 1)$ under the alternative $\bar{f}_n = (\xi_1 f_n, \dots, \xi_n f_n)$. Set

$$V_k^n = V_k^n(b) = \{v = (\xi_1 b, \dots, \xi_n b),\ \xi_i \in \{0,1\},\ \sum_{i=1}^n \xi_i = k\}. \tag{4}$$

By passing to the statistics (3), we obtain the equivalent hypothesis testing problem on a mean $v \in R^n$ of the n-dimensional Gaussian random vector $x \sim N(v, I_n)$,

$$H_0: \ v = 0, \quad H_1: \ v \in V_k^n.$$

The quantities $\gamma_n(V_{k_n}^n(b_n), \psi_n)$ and $\gamma_n = \gamma_n(V_{k_n}^n(b_n))$ in the problem are defined analogously to the above.

This hypothesis testing problem have been studied in [6] see also [3], [4] where analogous problems were studied; the difference is that it was assumed $\xi_i \in \{0, \pm 1\}$ in refereed papers. The results are of the following form. Suppose $k = k_n \asymp n^{\delta_n}, \ \delta_n \to \delta \in [0, 1]$.

First, let $\delta_n \to \delta \in (1/4, 1]$. Set $u_n^2 = n^{-1} k_n^2 (e^{b_n^2} - 1)$ and consider the tests $\hat{\psi}_{n,b_n} = \hat{\psi}_{n,b_n,H_n} = \max(\psi_{n,b_n,H_n}, \psi_n^{thr})$. Here the tests $\psi_{n,b_n,H_n} = \mathbf{1}_{t_{n,b_n} > H_n}$ are based on the statistics

$$t_{n,b_n} = (n(e^{b_n^2} - 1))^{-1/2} \sum_{i=1}^n \nu(x_i, b_n), \quad \nu(t, b) = (e^{-b^2/2 + tb} - 1)$$

and the tests ψ_n^{thr} are based on the thresholding

$$\psi_n^{thr} = \mathbf{1}_{\mathcal{X}_n}, \quad \mathcal{X}_n = \{x \in R^n : \max_{1 \le i \le n} x_i > T_n\}, \quad T_n = \sqrt{2 \log n} \tag{5}$$

Then

$$\gamma_n(V_{k_n}^n(b_n)) = \gamma_n(V_{k_n}^n(b_n), \hat{\psi}_{n,b_n}) + o(1) = 2\Phi(-u_n/2) + o(1)$$

(this means that the tests $\hat{\psi}_{n,b_n}$ are asymptotically minimax).

Moreover, let $\delta_n \to \delta \in (1/2, 1]$. Consider the tests $\psi_n^{lin} = \mathbf{1}_{t_n > \tilde{u}_n/2}$ based on the linear statistics $t_n = n^{-1/2} \sum_{i=1}^n x_i$ and set $\tilde{u}_n = n^{-1/2} k_n b_n$. Then

$$\gamma_n(V_{k_n}^n(b_n)) = \gamma_n(V_{k_n}^n(b_n), \psi_n^{lin}) + o(1) = 2\Phi(-\tilde{u}_n/2) + o(1).$$

Note that $\tilde{u}_n \le u_n$; $\tilde{u}_n \to \infty$ iff $u_n \to \infty$ and $\tilde{u}_n \sim u_n$ if $b_n = o(1)$. This means that tests ψ_n^{lin} are asymptotically minimax; if $k_n = n$, then the tests ψ_n^{lin} are minimax in the problem (nonasymptotically). Also, if $\delta \in [1/2, 1]$ and $u_n \to \infty$, then $\gamma_n(V_{k_n}^n(b_n), \psi_n^{lin}) \to 0$.

These results describe the sharp asymptotics of the Gaussian type in the problem for the case $\delta_n \to \delta \in (1/2, 1]$. If $\delta_n \to \delta \in (0, 1/4)$, then the sharp asymptotics in the problem are of a special infinite-divisible type (see [3], [4]). However there are simple conditions for distinguishability and for nondistinguishability in the problem. Namely, let $\delta_n \to \delta \in [0, 1/4]$. Set $b_n^*(\delta) = (1 - \delta^{1/2})\sqrt{2 \log n}$ and consider the tests (5). Then:

(a) If $\limsup b_n / b_n^*(\delta_n) < 1$, then $\gamma_n(V_{k_n}^n(b_n)) \to 1$.

(b) If $\liminf b_n / b_n^*(\delta_n) > 1$, then $\gamma_n(V_{k_n}^n(b_n), \psi_n^{thr}) \to 0$. Moreover, this holds for any $\delta_n \to \delta \in [0, 1)$.

These results lead to the conditions for distinguishability and nondistinguishability in the problem (i.e., to the conditions for either $\gamma_n \to 0$ or $\gamma_n \to 1$).

Corollary 1

(1) *Let* $\delta_n \to \delta \in [1/2, 1]$. *Set* $b_n^* = n^{1/2-\delta}$. *Then* $\gamma_n(V_{k_n}^n(b_n)) \to 1$ *iff* $b_n/b_n^* \to 0$ *and* $\gamma_n(V_{k_n}^n(b_n), \psi_n^{lin}) \to 0$ *iff* $b_n/b_n^* \to \infty$.

(2) *Let* $\delta_n \to \delta \in [0, 1/2)$. *Set* $b_n^* = \sqrt{\log n}$. *Then there exist constants* $0 < c_0 \leq c_1$ *(which depend on* δ *and* $c_0 = c_1$ *for* $\delta \in [0, 1/4]$*) such that if* $\limsup b_n/b_n^* < c_0$, *then* $\gamma_n(V_n(b_n, k_n)) \to 1$, *and if* $\limsup b_n/b_n^* > c_1$, *then* $\gamma_n(V_n(b_n, k_n), \psi_n^{thr}) \to 0$.

Thus, tests ψ_n^{lin} and ψ_n^{thr} based on linear statistics and on thresholding, provide the optimal rates for distinguishability in the problem, with loss a constant in the rates for $\delta_n \to \delta \in (1/4, 1/2)$. These tests depend on the shape of the signal f_n but not of this norm.

In the presented paper we study the case when F_n are nonparametric sets. We extend the results of Corollary 1 to this case.

2 Nonparametric signal sets

It is well known (see [2]) that in order to obtain minimax distinguishability in nonparametric problem one needs to remove "small enough" signals and to assume some regularity conditions on signals under consideration. For this reason below we consider signal sets $F_n = F(r_n, \sigma)$ of the form

$$F_n = \{f \in L_2(0,1) : \ \|f\|_p \geq r_n, \ \|f\|_{\sigma,q} \leq B\}, \quad B > 0, \tag{6}$$

where $\|\cdot\|_{\sigma,q}$ is Sobolev norm; to simplify, we set $B = 1$. In this paper we consider the case $\sigma > 0, 1 \leq p \leq 2, \ q \geq p$. So, below we consider the sets F_n^k defined by (2) with the sets F_n defined by (6).

It is known (see [2]) that for the signal sets (6) with $r_n = r_\varepsilon$ the minimax distinguishability in each ith channel is possible if and only if $r_\varepsilon/r_\varepsilon^* \to \infty$, as $\varepsilon \to 0$, where the rates r_ε^* are of the form

$$r_\varepsilon^* = \varepsilon^{4\sigma/(4\sigma+1)}. \tag{7}$$

Moreover, the minimax distinguishability in channels is provided by tests based on χ^2-statistics of the following form.

Taking integer-valued family $m = m_\varepsilon > 1$, let us consider the equispaces partitions of the interval $(0, 1]$ into subintervals

$$\delta_{m,j} = (z_{m,j-1}, z_{m,j}], \ z_{m,j} = j/m, \ j = 1, \ldots, m,$$

and let

$$x_{ijm} = \varepsilon^{-1} m^{1/2}(X_i(z_{m,j}) - X_i(z_{m,j-1}))$$

be normalized increments of the observed processes in the channels on the intervals $\delta_{m,j}$. For each channel let us consider centered and normalized χ^2-statistics

$$\chi_{m,i} = (2m)^{-1/2} \sum_{j=1}^{m} (x_{ijm}^2 - 1), \ i = 1, \ldots, n. \tag{8}$$

Let $r_\varepsilon/r_\varepsilon^* \to \infty$. Take $m = m_\varepsilon(\sigma) \asymp (r_\varepsilon^*)^{-1/\sigma}$. Then there exists thresholds family $T_\varepsilon \to \infty$ such that minimax distinguishability in ith channel is provided by the tests $\psi_{\varepsilon,i} = \mathbf{1}_{\chi_{m,i}>T_\varepsilon}$.

Let us return to the multichannel signal detection problem.

2.1 Large k

First, let us consider the case when $k = k_n$ is "large"; namely let $d_n = nk_n^{-2} = O(1)$; in particular, this holds for $k \asymp n^\delta,\ \delta \in [1/2, 1]$. Set

$$t_{m,n} = n^{-1/2}\sum_{i=1}^{n}\chi_{m,i}, \quad r_n^* = (\varepsilon^4 d_n)^{\sigma/(4\sigma+1)}, \tag{9}$$

where the statistics $\chi_{m,i}$ are defined by (8).

Theorem 1 *Assume $r_n^* \to 0,\ d_n = O(1)$. Then*

(1) *If $r_n/r_n^* \to 0$, then $\gamma_n(F_k^n) \to 1$. If $\limsup r_n/r_n^* < \infty$, then $\liminf \gamma_n(F_k^n) > 0$.*

(2) *Let $r_n/r_n^* \to \infty$. Then $\gamma_n(F_k^n) \to 0$. Moreover, let us take $m = m_n(k_n, \sigma) \asymp (r_n^*)^{-1/\sigma}$ and consider the tests $\psi_{m,n,T} = \mathbf{1}_{t_{m,n}>T}$. There exist a family $T_n \to \infty$ such that $\gamma_n(F_k^n, \psi_{m,n,T_n}) \to 0$.*

Thus, in order to detect a signal in the system with "large" $k = n^\delta,\ \delta \in [1/2, 1]$, one can take tests based on sums of test statistics in all channels. The rates in channels should be decreased by the factor $n^{(2\delta-1)\sigma/(4\sigma+1)}$ with respect to the rates (7) . Note also that we do not assume $\varepsilon = \varepsilon_n \to 0$; it is possible $\varepsilon \to \infty$ for $\delta > 1/2$.

2.2 Small k

Next, let us consider the case when $k = k_n$ is "small"; namely let $k_n = O(n^\delta)$ for some $\delta \in [0, 1/2)$. Set

$$t_n^m = \max_{1\le i\le n}\chi_{m,i}, \quad r_n^* = \begin{cases} (\varepsilon^4 \log n)^{\sigma/(4\sigma+1)}, & \text{if } \varepsilon^2(\log n)^{2\sigma+1} \le 1 \\ \varepsilon\sqrt{\log n}, & \text{if } \varepsilon^2(\log n)^{2\sigma+1} > 1 \end{cases}, \tag{10}$$

i.e., $r_n^* = \max\left((\varepsilon^4 \log n)^{\sigma/(4\sigma+1)}, \varepsilon\sqrt{\log n}\right)$.

Theorem 2 *Assume $r_n^* \to 0$. Then there exist constants $0 < c_0 \le c_1 < \infty$ such that:*

(1) *If $\limsup r_n/r_n^* < c_0$, then $\gamma_n(F_k^n) \to 1$.*

(2) *Let $\liminf r_n/r_n^* > c_1$. Then $\gamma_n(F_k^n) \to 0$. Moreover, taking positive constant C, set*

$$m = m_n(\sigma) = \left[\max\left((\varepsilon^4 \log n)^{-1/(4\sigma+1)},\ C\log n\right)\right] + 1,$$

and $[t]$ is the integer part of $t > 0$. Consider the tests

$$\psi_{n,C} = \mathbf{1}_{t_n^m > \sqrt{C\log n}},$$

where t_n^m *is defined by (10). Then there exist positive* C, c_1 *such that* $\gamma_n(F_k^n, \psi_{n,C}) \to 0$, *as* $\liminf r_n/r_n^* > c_1$.

Thus, in order to detect a signal in the system with "small" $k = n^\delta$, $0 \le \delta < 1/2$, it suffices to increase the thresholds in all channels and to combine the decisions in all channels. If $\varepsilon^2(\log n)^{2\sigma+1} \le 1$, then the rates in the channels should be increased by the factor $(\log n)^{\sigma/(4\sigma+1)}$ with respect to the rates (7); if $\varepsilon^2(\log n)^{2\sigma+1} > 1$, then this factor is $\sqrt{\log n}$ with respect to the "classical" rate $r_\varepsilon^* = \varepsilon$ which corresponds to the case when the shape of a signal is known (see point 2 in Corollary 1).

3 Adaptive problems

3.1 Adaptation on large k

Note that the tests in Theorem 2 do not depend on $\delta \in [0, 1/2)$; moreover these tests provide distinguishability uniformly over k_n such that $1 \le k_n = O(n^{\delta_0})$, $\delta_0 \in [0, 1/2)$.

The situation is different for "large" $k \ge n^\delta$ with $\delta \in [1/2, 1]$, because the "dimensions" m depend essentially on δ. These lead to "adaptive" problem: to construct tests ψ_n which provide distinguishability uniformly over "large" $k = k_n$. The study of analogous estimation problem have been started in [7]–[9]; in hypothesis testing problem adaptive setting was proposed in [10], [11]; see also [5]. Typical results state that often it is impossible to construct adaptive procedures without loss in efficiency; however the losses are small enough.

For our problem let us set

$$\mathcal{K}_n = \mathcal{K}_n(\delta_1, \delta_2) = \{k: \ b_1 n^{\delta_1} < k < b_2 n^{\delta_2}\}; \quad b_1 > 0, \ b_2 > 0, \tag{11}$$

and

$$F^n(\mathcal{K}_n, \sigma) = \bigcup_{k \in \mathcal{K}_n(\delta_1, \delta_2)} F_k^n(r_n(k), \sigma), \tag{12}$$

where $F_k^n(r_n, \sigma)$ are the sets F_k^n in (2) which correspond to the signal set $F(r_n, \sigma)$ in each channel defined by (6); we assume the quantities $r_n = r_n(k)$ could depend on k. We are interested in the asymptotics of the quantities $\gamma_n = \gamma_n(F^n(\mathcal{K}_n, \sigma))$.

Introduce the adaptive rates

$$r_n^*(k) = (\varepsilon^4 n k^{-2} \log\log n)^{\sigma/(4\sigma+1)}.$$

Up to $\log\log$-factor, these rates correspond to the rates $r_n^*(\delta)$ in (9) for $k = n^\delta$, $\delta \ge 1/2$.

Theorem 3 *Let* $1/2 \le \delta_1 < \delta_2 \le 1$. *There exist constants* $0 < c_0 \le c_1 < \infty$ *such that:*

(1) *Assume* $r_n^*(n^\delta) \to 0$ *for some* $\delta \in (\delta_1, \delta_2)$ *and*

$$\limsup \sup_{k \in \mathcal{K}_n} r_n(k)/r_n^*(k) < c_0.$$

Then $\gamma_n(F^n(\mathcal{K}_n,\sigma)) \to 1$.

(2) *Assume* $r_n^*(n^{\delta_1}) \to 0$ *and*

$$\liminf \inf_{k\in\mathcal{K}_n} r_n(k)/r_n^*(k) > c_1. \qquad (13)$$

Then $\gamma_n(F^n(\mathcal{K}_n,\sigma)) \to 0$. *Moreover, one can take an adaptive test procedure of the following form. Take collections* $m_l = 2^l,\ L \le l \le L+M$, *where*

$$m_L \le (r_n^*(n^{\delta_1}))^{-1/\sigma} \le m_{L+1}, \quad m_{L+M-1} \le (r_n^*(n))^{-1/\sigma} \le m_{L+M};$$

this yields $M = M_n \asymp \log n$. *Consider the sets* $X_{n,c}$ *defined by the inequality*

$$\max_{L+1\le l\le L+M} t_{m_l,n} > \sqrt{c\log M},$$

where $t_{m,n}$ *are the statistics defined by (9), and* $c > 2$ *is a constant. Set* $\psi_{n,c} = \mathbf{1}_{X_{n,c}}$. *Then there exists* $c_1 > 0$ *in (13) such that* $\gamma_n(F^n(\mathcal{K}_n,\sigma),\psi_{n,c}) \to 0$.

3.2 Adaptation on σ

The tests presented above depend on σ because these is the dependence on σ of the dimensions m for known k, and of the collections of dimensions for adaptation on k. We would like to construct tests which provide good minimax properties for all $\sigma > 0$. More exactly, we want to obtain the conditions for distinguishability uniformly over $\sigma \in [\sigma_0,\sigma_1]$ for any $0 < \sigma_0 < \sigma_1 < \infty$.

Taking a sequence of functions $r_n = r_n(k,\sigma)$, we set

$$F_k^n(\sigma_0,\sigma_1) = \bigcup_{\sigma\in[\sigma_0,\sigma_1]} F_k^n(r_n(k,\sigma),\sigma),$$

where $F_k^n(r_n,\sigma)$ are the sets F_k^n in (2) which correspond to the signal set $F(r_n,\sigma)$ in each channel defined by (6); we assume the quantities $r_n = r_n(k,\sigma)$ in (6) could depend on k, n, σ. Analogously, taking $0 \le \delta_1 < \delta_2 \le 1$, we set

$$F^n(\mathcal{K}_n,\sigma_0,\sigma_1) = \bigcup_{k\in\mathcal{K}_n(\delta_1,\delta_2)} F_k^n(\sigma_0,\sigma_1),$$

where the sets $\mathcal{K}_n = \mathcal{K}_n(\delta_1,\delta_2)$ are defined by (11). We are interested in the conditions for $\gamma_n \to 0$ and for $\gamma_n \to 1$ for the quantities $\gamma_n = \gamma_n(F_{k_n}^n(\sigma_0,\sigma_1))$ with given sequence k_n, and for $\gamma_n = \gamma_n(F^n(\mathcal{K}_n,\sigma_0,\sigma_1))$ with given sets $\mathcal{K}_n$.

First, let us consider the case of "large" k. Namely, let the sets $\mathcal{K}_n = \mathcal{K}_n(\delta_1,\delta_2)$ correspond to $1/2 < \delta_1 < \delta_2 \le 1$. Introduce adaptive rates. Set

$$\rho_n(k) = \varepsilon^4 nk^{-2}, \quad w_n(k) = \rho_n(k)\log\rho_n^{-1}(k), \quad r_n^*(k,\sigma) = (w_n(k))^{\sigma/(4\sigma+1)}.$$

Theorem 4

(1) *Let* $k_n \in \mathcal{K}_n(\delta_1,\delta_2)$ *and*

$$\rho_n(k_n) \to 0, \quad d_n = nk_n^{-2}\log w_n^{-1}(k_n) \to 0 \qquad (14)$$

(the first relation in (14) yields $w_n(k_n) \to 0$ *and* $r_n^*(k,\sigma) \to 0\ \forall \sigma > 0$*). Set*

$$g_n^+ = \sup_{\sigma\in[\sigma_0,\sigma_1]} r_n(k_n,\sigma)/r_n^*(k_n,\sigma).$$

If $g_n^+ = o(1)$*, then* $\gamma_n = \gamma_n(F_{k_n}^n(\sigma_0,\sigma_1)) \to 1$*. If* $g_n^+ = O(1)$*, then* $\liminf \gamma_n > 0$*.*

(2) *Assume*

$$\exists\ \delta_0 < \delta_1\ :\ \rho_n(n^{\delta_0}) = \varepsilon^4 n^{1-2\delta_0} \to 0 \tag{15}$$

(the assumption (15) yields $\sup_{k\in\mathcal{K}_n(\delta_1,\delta_2)} r_n^*(k,\sigma) \to 0$*) and let*

$$g_n^- = \inf_{\sigma\in[\sigma_0,\sigma_1],k\in\mathcal{K}_n} r_n(k,\sigma)/r_n^*(k,\sigma) \to \infty.$$

Then $\gamma_n(F^n(\mathcal{K}_n,\sigma_0,\sigma_1)) \to 0$*. Moreover, one can take an adaptive test procedure of the following form. Set*

$$G_n = \max\Big(\log n,\ \log(\rho_n^{-1}(n^{\delta_0}))\Big).$$

Let $m_l = 2^l,\ l = 1,2,\dots,$ *and* $a_{n,l} = c_n^{-1/2}2^{-(l-1)/G_n}$*, where*

$$c_n = \sum_{l=1}^{\infty} 2^{-2(l-1)/G_n} = (1-2^{-2/G_n})^{-1} \asymp G_n,$$

that is, we take $a_{n,l}$ *in order to obtain*

$$a_{n,l} \asymp G_n^{-1/2}2^{-(l-1)/G_n}, \quad \sum_{l=1}^{\infty} a_{n,l}^2 = 1. \tag{16}$$

Consider the statistics

$$t_n = \sum_{l=1}^{\infty} a_{n,l} t_{m_l,n}, \tag{17}$$

where $t_{m,n}$ *are the statistics defined by (9). Set* $\psi_{n,T} = \mathbf{1}_{t_n>T}$*. Then there exist* $T = T_n \to \infty$ *such that*

$$\gamma_n(F^n(\mathcal{K}_n(\delta_1,\delta_2),\sigma_0,\sigma_1),\psi_{n,T_n}) \to 0.$$

Remark 3.1 *Under the assumption (15) we get*

$$\log(r_n^*(k,\sigma))^{-1} \asymp (\log \rho_n^{-1}(k)) \asymp G_n. \tag{18}$$

Therefore, uniformly over $k \in \mathcal{K}_n(\delta_1,\delta_2),\ \sigma \in (\sigma_0,\sigma_1)$*, one has*

$$r_n^*(k,\sigma) \asymp (\varepsilon^4 n k^{-2} G_n)^{\sigma/(4\sigma+1)}.$$

Next, let us consider the case of "small" k. Namely, let the sets $\mathcal{K}_n = \mathcal{K}_n(\delta_1,\delta_2)$ correspond to $0 \le \delta_1 < \delta_2 < 1/2$. Introduce adaptive rates

$$r_n^*(\sigma) = \begin{cases} (\varepsilon^4 \log(n\log\varepsilon^{-1}))^{\sigma/(4\sigma+1)}, & \text{if } \varepsilon^2(\log n)^{2\sigma+1} \le 1 \\ \varepsilon\sqrt{\log n}, & \text{if } \varepsilon^2(\log n)^{2\sigma+1} > 1 \end{cases}; \tag{19}$$

these rates are asymptotically the same as

$$r_n^*(\sigma) \sim \max\Big(\varepsilon^4\log(n\log\varepsilon^{-1}))^{\sigma/(4\sigma+1)},\ \varepsilon\sqrt{\log n}\Big).$$

Theorem 5 *Assume*

$$\sup_{\sigma\in[\sigma_0,\sigma_1]} r_n^*(\sigma)\to 0.$$

Then there exist $0<c_0\le c_1<\infty$ *such that:*

(1) *Let* $k_n n^{-\delta}=O(1)$ *for some* $0<\delta<1/2$. *Set*

$$g_n^+=\sup_{\sigma\in[\sigma_0,\sigma_1]} r_n(k_n,\sigma)/r_n^*(\sigma).$$

If $\limsup_{n\to\infty} g_n^+<c_0$, *then* $\gamma_n(F_{k_n}^n(\sigma_0,\sigma_1))\to 1$.

(2) *Let*

$$g_n^-=\inf_{\sigma\in[\sigma_0,\sigma_1],k\in\mathcal{K}_n} r_n(k,\sigma)/r_n^*(\sigma),\quad \liminf_{n\to\infty} g_n^->c_1. \tag{20}$$

Then $\gamma_n(F^n(\mathcal{K}_n,\sigma_0,\sigma_1))\to 0$. *Moreover, one can use an adaptive test procedure of the following form. Taking a constant* $C>0$, *set*

$$L_n=2\log_2 C+\log_2\log n,\quad m_l=2^l,\quad l=1,2,\dots,$$

$$T_{n,l}=\begin{cases} C^2 2^{-l/2}\log n, & \text{if } 1\le l\le L_n,\\ C\sqrt{\log(nl)}, & \text{if } l>L_n.\end{cases}$$

Let the set $\mathcal{X}_{n,C}$ *be defined by the relation* $\sup_{l\ge 1} t_n^{m_l}/T_{n,l}>1$, *where* t_n^m *are the statistics defined by (10). Introduce the tests* $\psi_{n,C}=\mathbf{1}_{\mathcal{X}_{n,C}}$. *Then there exist constants* $C>1,\ c_1>0$ *such that under (20) one has* $\gamma_n(F^n(\mathcal{K}_n,\sigma_0,\sigma_1),\psi_{n,C})\to 0$.

Note that if

$$\log\log n=o(\log\varepsilon^{-1}),\quad \log\log\varepsilon^{-1}=O(\log n), \tag{21}$$

then adaptive rates (19) are of the same order that nonadaptive rates in (10). Therefore, under the assumptions (21) the test procedure from Theorem 5 provides (up to a constant factor in the rates) the best distinguishability for all $\sigma>0$.

4 Proofs of Theorems 1, 2

The proofs are based on combinations of the methods developed in [2], [4], [6].

4.1 Lower bounds

First, let us consider Theorem 2, the case $\varepsilon^2(\log n)^{2\sigma+1}>1$ and $r_n^*=\varepsilon\sqrt{\log n}\to 0$. Let us take σ-smooth function $\phi(t)$ supported on $(0,1)$, $\|\phi\|_2=1$ and let $f_i(t)=r_n\phi(t),\ i=1,\dots,n,\ r_n<c_0r_n^*$. Clearly, $f_i(t)\in F(r_n,\sigma)$, and the lower bounds for the problem follows from Corollary 1. Therefore below we consider the lower bounds of Theorem 2 for the case

$$\varepsilon^2(\log n)^{2\sigma+1}\le 1,\quad r_n^*=(\varepsilon^4\log n)^{\sigma/(1+4\sigma)}. \tag{22}$$

Let us pass to finite-dimensional problems. Let us take orthonormal collections $\bar{\phi}_m = \phi_{m,1}, \dots, \phi_{m,m}$ in $L_2(0,1)$ which have disjoint support. Namely, by taking regular enough function

$$\phi(t), \quad t \in R^1; \quad \|\phi\|_2 = 1, \quad \phi(t) = 0, \quad t \notin (0,1),$$

we set $\phi_{m,j} = \sqrt{m}\phi(mt - j + 1), \ j = 1, \dots, m$. We consider the functions of the form

$$f(t, \boldsymbol{\theta}) = \varepsilon \sum_{j=1}^{m} \theta_j \phi_{m,j}(t), \quad \boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in R^m. \tag{23}$$

It is easily seen (compare with [2], Sect. 4.2) that the relation $f(\cdot, \boldsymbol{\theta}) \in F(r_n, \sigma)$ follows from the relation $\boldsymbol{\theta} \in \Theta_{n,m}$, where the set $\Theta_{n,m} = \Theta_{n,m}(r_n, \sigma)$ is determined by the inequalities

$$m^{1/2-1/p}|\boldsymbol{\theta}|_p \geq c_1 r_n/\varepsilon, \quad m^{\sigma+1/2-1/q}|\boldsymbol{\theta}|_q \leq c_2/\varepsilon; \tag{24}$$

here $|\boldsymbol{\theta}|_s = (\sum_{j=1}^m |\theta_j|^s)^{1/s}$, $c_1 = \|\phi\|_p^{-1}$ and $c_2 > 0$ depend on q and on the norms of derivatives of ϕ.

Analogously to (3), let us consider random variables

$$x_{ij} = \varepsilon^{-1} \int_0^1 \phi_{m,j} dX_i, \quad i = 1, \dots, n, \quad j = 1, \dots, m. \tag{25}$$

Note that these variables are independent Gaussian with the unit variances and with the means $\xi_i \theta_{ij} = \varepsilon^{-1}(\xi_i f_i, \phi_{m,j})$, where $\xi_i f_i$ is a signal in ith channel. For the functions $f_i(t) = f(t, \boldsymbol{\theta}_i), \ i = 1, \dots, n$ of the form (23) the model (1) is equivalent to the mn-dimensional Gaussian model for the observations

$$\bar{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in R^{nm}, \quad \mathbf{x}_i = (x_{i1}, \dots, x_{im}) \in R^m$$

of the form (25) with the Gaussian measure $P_{\bar{\boldsymbol{\theta}}}$ on R^{nm} with the mean vector $\bar{\boldsymbol{\theta}}$ and unit covariation matrix. Recall that the likelihood ratio is of the form

$$dP_{\bar{\boldsymbol{\theta}}}/dP_0 = \exp(-|\bar{\boldsymbol{\theta}}|^2/2 + (\bar{\boldsymbol{\theta}}, \bar{\mathbf{x}})).$$

Therefore it suffices to obtain the lower bounds for the hypothesis testing problem

$$H_0: \ \bar{\boldsymbol{\theta}} = 0 \quad \text{against} \quad H_1: \ \bar{\boldsymbol{\theta}} \in \ \Theta^n_{n,m,k}, \tag{26}$$

where the sets $\Theta^n_{n,m,k} \subset R^{nm}$ are defined analogously to (2),

$$\Theta^n_{n,m,k} = \{\bar{\boldsymbol{\theta}} = (\xi_1 \boldsymbol{\theta}_1, \dots, \xi_n \boldsymbol{\theta}_n), \ \boldsymbol{\theta}_i \in \Theta_{n,m}; \ \xi_i \in \{0,1\}, \ \sum_{i=1}^n \xi_i = k\}. \tag{27}$$

Let $\Theta^n_{n,m,k+}$ be the set defined analogously to (27) but the equality $\sum_{i=1}^n \xi_i = k$ is replaced to the inequality $\sum_{i=1}^n \xi_i \geq k$.

Proposition 1 $\quad \gamma(\Theta^n_{n,m,k}) = \gamma(\Theta^n_{n,m,k+})$.

Proof of Proposition is based on the symmetry and the convexity of admissible set of the minimax test for the problem (26) and on the Anderson lemma. We omit details here.

It follows from [2] that on order to prove the lower bounds of Theorem 1, it suffices choose a sequence $m = m_n$ and to construct a sequence of probability measures (priors) π^n on the space R^{nm} such that

$$\pi^n(\Theta^n_{n,m,k+}) \to 1; \quad \begin{cases} E_0(dP_{\pi^n}/dP_0 - 1)^2 = O(1), & \text{if } r_n = O(r^*_n), \\ E_0(dP_{\pi^n}/dP_0 - 1)^2 = o(1), & \text{if } r_n = o(r^*_n). \end{cases} \tag{28}$$

Here

$$P_{\pi^n}(A) = \int P_{\bar{\boldsymbol{\theta}}}(A)\,\pi^n(d\bar{\boldsymbol{\theta}})$$

is the mixture of the Gaussian measures $P_{\bar{\boldsymbol{\theta}}}$ on R^{nm}.

For Theorem 2 we need to change the second relation in (28) into the following

$$E_0(dP_{\pi^n}/dP_0 - 1)^2 = o(1) \quad \text{if} \quad \limsup r_n/r^*_n < c_0 \tag{29}$$

Let us take the prior π^n in the form

$$\pi^n(d\bar{\boldsymbol{\theta}}) = \prod_{i=1}^{n} \mu_{m,h,z}(d\boldsymbol{\theta}_i), \quad \mu_{m,h,z} = (1-h)\delta_{\bar{0}} + h\pi_{m,z}, \tag{30}$$

$$\pi_{m,z}(d\boldsymbol{\theta}) = \prod_{j=1}^{m} \pi_z(d\theta_j), \quad \pi_z = (\delta_z + \delta_{-z})/2, \tag{31}$$

where $\delta_{\bar{0}}$ is the mass at the point $\bar{0} \in R^m$ and δ_z is the mass at the point $z \in R^1$. The prior (30) corresponds to the random vector

$$\bar{\boldsymbol{\theta}} = (\xi_1\boldsymbol{\theta}_1, \ldots, \xi_n\boldsymbol{\theta}_n), \quad \boldsymbol{\theta}_i = (z\xi_{i1,}, \ldots, z\xi_{im,}),$$

where the random vectors $\{\xi_i\}$, $\{\xi_{ij}\}$ and their components are independent,

$$P(\xi_j = 1) = h, \quad P(\xi_j = 0) = 1 - h; \quad P(\xi_{ij} = 1) = P(\xi_{ij} = -1) = 1/2.$$

Clearly $|\boldsymbol{\theta}_i|_p = zm^{1/p}$, $|\boldsymbol{\theta}_i|_q = zm^{1/q}$, and in order to satisfy (24) it suffices

$$\varepsilon^2 m z^2 \geq c_1^2 r_n^2, \quad \varepsilon^2 m^{1+2\sigma} z^2 \leq c_2^2. \tag{32}$$

One can take an integer-valued sequences $m = m_n$ and a positive sequence $z = z_n$ such that (32) hold. These yield the measure $\pi_{m,z}$ is concentrated on the set $\Theta_{n,m}$, i.e., $\pi_{m,z}(\Theta_{n,m}) = 1$, and

$$m \asymp r_n^{-1/\sigma}, \quad z \asymp \varepsilon^{-1} r_n^{1+1/2\sigma}. \tag{33}$$

It follows from (33) that

$$mz^4 \asymp \varepsilon^{-4} r_n^{4+1/\sigma}. \tag{34}$$

Under the assumptions of the theorems we have $r_n \to 0$ which yields $m_n \to \infty$. Under the assumptions of Theorem 1 we have $d_n = nk_n^{-2} = O(1),\ c_n = (r_n/r_n^*)^{4+1/\sigma} = O(1)$. By (34) these yield

$$mz^4 \asymp d_n c_n = O(1), \quad z_n \to 0. \tag{35}$$

Under the assumptions of Theorem 2 it suffices to assume $c_n \asymp 1,\ \delta > 0,\ k_n \to \infty$. For Theorem 2 also we have

$$z_n \asymp \left(\varepsilon^2(\log n)^{2\sigma+1}\right)^{1/(8\sigma+2)} = O(1)$$

by the assumption (22). Using (34) we see that

$$mz^4 \asymp c_n \log n \to \infty. \tag{36}$$

To take the quantities h_n in (30) for Theorems 1, 2, given a positive sequence $\eta_n \to 0$ such that $\eta_n^2 k_n \to \infty$, we set

$$h = h_n = \min\{1,\ k_n(1+\eta_n)/n\}. \tag{37}$$

Since the integers $\xi_1, \dots, \xi_n$ are i.i.d. Bernoulli random variables, we have

$$\pi^n(\Theta^n_{n,m,k_n+}) = P(\sum_{i=1}^n \xi_i \ge k_n) \to 1. \tag{38}$$

In fact, if $h_n = 1$, then the relation (38) is evident. If $h_n < 1$, then it easily follows from the Chebyshev inequality that

$$P(\sum_{i=1}^n \xi_i < k_n) = P(nh_n - \sum_{i=1}^n \xi_i > \eta_n k_n) < nh_n/\eta_n^2 k_n^2 \sim (\eta_n^2 k_n)^{-1} \to 0.$$

To complicate the proof it suffices to obtain the second relations in (28) or (29). Using some simple calculations and the inequality $1 + x \le e^x$, we have

$$E_0(dP_{\pi_{m,z}}/dP_0 - 1)^2 = E_0(dP_{\pi_{m,z}}/dP_0)^2 - 1 = \left(E_0(dP_{\pi_z}/dP_0)^2\right)^m - 1 =$$
$$(1 + 2\sinh^2(z^2/2))^m - 1 \le \exp(2m\sinh^2(z^2/2)) - 1;$$
$$dP_{\mu_{m,h,z}}/dP_0 = 1 + h(dP_{\pi_{m,z}}/dP_0 - 1);$$

here we use the same notation P_0 for the standard Gaussian measure on the spaces R^{nm}, R^m, R^1 and E_0 stands for expectation with respect to this measure. Using these relations analogously to the above, we get

$$E_0(dP_{\pi^n}/dP_0 - 1)^2 = E_0(dP_{\pi^n}/dP_0)^2 - 1 = \left(E_0(dP_{\mu_{m,h,z}}/dP_0)^2\right)^n - 1 =$$
$$\left(1 + E_0(dP_{\mu_{m,h,z}}/dP_0 - 1)^2\right)^n - 1 = \left(1 + h^2 E_0(dP_{\pi_{m,z}}/dP_0 - 1)^2\right)^n - 1 \le$$
$$\left(1 + h^2(\exp(2m\sinh^2(z^2/2)) - 1)\right)^n - 1 \le$$
$$\exp\left(nh^2(\exp(2m\sinh^2(z^2/2)) - 1)\right) - 1 = \exp(u_n^2) - 1, \tag{39}$$

where we set

$$u_n^2 = nh^2(\exp(2m\sinh^2(z^2/2)) - 1).$$

Under the assumptions of Theorem 1 we have $mz^4 \asymp d_n c_n = O(1)$. Since $z \to 0,\ m \to \infty$, using the asymptotics $\sinh^2(z^2/2) \asymp z^4$ and $e^x - 1 \asymp x$ for $x = O(1)$, we get $u_n^2 \asymp d_n^{-1} mz^4 \asymp c_n$, and the second relations (28) follows from (35), (39).

Let us turn to Theorem 2 where $nh^2 \sim n^{-1}k_n^2 = O(n^{2\delta-1}) \to 0$. It suffices to verify that if r_n/r_n^* is small enough (it is the same that c_n is small enough), then $u_n \to 0$. Since $z = O(1)$ and $2\sinh^2(z^2/2) \asymp z^4$, we have, for some $B_1 > 0,\ B_2 > 0,\ B_3 > 0$,

$$u_n^2 \le n^{-1}k_n^2 \exp(B_1 mz^4) \le B_3 n^{2\delta-1}\exp(B_2 c_n \log n) = Bn^{2\delta-1+B_2c_n} \to 0,$$

if $B_2 \limsup c_n < 1 - 2\delta$. □

4.2 Upper bounds

First, let us consider the distributions of statistics (8) and (9).

Under the null-hypothesis the statistics x_{ijm} are i.i.d. standard Gaussian $N(0,1)$, the statistics $y_{ij} = (x_{ijm}^2 - 1)/\sqrt{2}$ are i.i.d. and $E_0 y_{ij} = 0,\ E_0 y_{ij}^2 = 1,\ E_0 y_{ij}^4 = 15$. Therefore,

$$E_0\chi_{m,i} = E_0 t_{m,n} = 0, \quad E_0\chi_{m,i}^2 = E_0 t_{m,n}^2 = 1,$$

these statistics are asymptotically standard Gaussian $N(0,1)$, as $m \to \infty$ or $mn \to \infty$. Moreover, observe the large deviation inequality for the statistics of the form (8) and (9).

Lemma 4.1 *Let $\eta_1, \dots, \eta_m$ be i.i.d. standard Gaussian random variables and*

$$\chi_m = (2m)^{-1/2}\sum_{i=1}^{m}(\eta_i^2 - 1).$$

Then there exist functions $b_k = b_k(a) > 0,\ k = 1,2$, which are continuous in $a > 0$ and such that

$$P(\chi_m > T) \le \begin{cases} \exp(-b_1(a)T^2), & \text{if } 0 < T^2 \le am, \\ \exp(-b_2(a)T\sqrt{m}), & \text{if } T^2 \ge am. \end{cases}$$

Moreover, $b_1(a) \to 1/2$ as $a \to 0$, and $b_2(a) \to 2^{-1/2}$ as $a \to \infty$.

Proof. For $\eta \sim N(0,1)$ set $y = \eta^2 - 1$. Simple calculation yields

$$E\exp(by) = e^{-b}(1-2b)^{-1/2}, \quad \text{for } 2b < 1,$$

which implies the equality

$$E\exp(b\chi_m) = e^{-bd}(1 - b/d)^{-d^2}, \quad \text{for } b < d = (m/2)^{1/2}.$$

Using this relation and the Markov inequality, for any $0 < b < d,\ T > 0$, we get,

$$P(\chi_m > T) \le P(\exp(b\chi_m) > \exp(Tb)) \le \exp(-Tb) E \exp(b\chi_m) = \\ \exp(-U(b,T)); \quad U(b,T) = b(d+T) + d^2 \log(1 - b/d).$$

By maximizing the concave function $U(b,T)$ over b, we obtain the inequality

$$P(\chi_m > T) \le \exp(-L(d,T)); \quad L(d,T) = Td + d^2 \log(d/(T+d)). \tag{40}$$

Setting $t = T/d$, note that if $T^2 \le am$ or $T^2 \ge am$, then $t^2 \le 2a,\ t^2 \ge 2a$ respectively. We can write

$$L(d,T) = Td - d^2 \log(1 + T/d) = T^2 \varphi_1(t) = Td\varphi_2(t),$$

where

$$\varphi_1(t) = t^{-2}(t - \log(1+t)), \quad \varphi_2(t) = 1 - t^{-1}\log(1+t)).$$

We have

$$\varphi_1(t) \sim 1/2, \text{ as } t \to 0; \quad b_1(a) = \inf_{t \in (0,(2a)^{1/2}]} \varphi(t) > 0,$$

$$\varphi_2(t) \sim 1, \text{ as } t \to \infty; \quad b_2(a) = 2^{-1/2} \inf_{t \in [(2a)^{1/2},\infty)} \varphi_2(t) > 0.$$

These yield the statement of Lemma. □

Corollary 2 *There exists a function $b(a) > 0$ which is continuous in $a > 0$ and such that*

$$P_0(\chi_{m,i} > T) \le \begin{cases} \exp(-b(a)T^2), & \text{if } T^2 \le am, \\ \exp(-b(a)T\sqrt{m}), & \text{if } T^2 \ge am, \end{cases}$$

$$P_0(t_{m,n} > T) \le \begin{cases} \exp(-b(a)T^2), & \text{if } T^2 \le anm, \\ \exp(-b(a)T\sqrt{nm}), & \text{if } T^2 \ge anm. \end{cases}$$

Under the alternative $\bar{f} = (\xi_1 f_1, \ldots, \xi_n f_n)$ the statistics x_{ijm} are independent and Gaussian $N(v_{ijm}, 1)$, with

$$v_{ijm} = \xi_i \varepsilon^{-1} p_j(f_i), \quad p_j(f) = m^{1/2} \int_{\delta_{m,j}} f(t)\,dt,$$

the statistics $y_{ij} = (x_{ijm}^2 - 1)/\sqrt{2}$ are independent and

$$E_{\bar{f}}(y_{ij}) = v_{ijm}^2/\sqrt{2}, \quad \mathrm{Var}_{n,\bar{f}}(y_{ij}) = 1 + 2v_{ijm}^2.$$

Since $\chi_{m,i} = m^{-1/2} \sum_{j=1}^m y_{ij}$, we get

$$E_{\bar{f}}(\chi_{m,i}) = (2m)^{-1/2} \sum_{j=1}^m v_{ijm}^2 = \xi_i \|Pr_m(f_i)\|_2^2/\varepsilon^2\sqrt{2m}, \tag{41}$$

$$\mathrm{Var}_{n,\bar{f}}(\chi_{m,i}) = 1 + (2/m) \sum_{j=1}^m v_{ijm}^2 = 1 + o(E_{\bar{f}}(\chi_{m,i})), \quad \text{as } m \to \infty. \tag{42}$$

Here $Pr_m(f)$ is the projection of the function f into m-dimensional space which consists of the step functions corresponding to the partition of the interval $(0,1]$ into m subintervals $\delta_{m,j}$:

$$Pr_m(f) = m^{1/2}\sum_{j=1}^{m} p_j(f)\mathbf{1}_{\delta_{m,j}}; \quad \|Pr_m(f)\|_2^2 = \sum_{j=1}^{m} p_j^2(f).$$

Analogously,

$$E_{\bar f}(t_{m,n}) = \varepsilon^{-2}(2mn)^{-1/2}\sum_{i=1}^{n}\xi\|Pr_m(f_i)\|_2^2, \quad \mathrm{Var}_{\bar f}(t_{m,n}) = 1 + o(E_{\bar f}t_{m,n}). \tag{43}$$

To prove the upper bounds of Theorems it suffices to consider the case $q = p$, since $q \geq p$ and the case $p = q$ corresponds to the “widest” alternative.

Note the inequality (see [2], the relation (5.16)) which is of importance for below: there exist constants $C_1 > 0$, $C_2 > 0$ such that for all $f \in L_2$, $\|f\|_{\sigma,p} \leq 1$ one has

$$\|Pr_m(f)\|_p \geq C_1\|f\|_p - C_2 m^{-\sigma}. \tag{44}$$

Let us turn to Theorem 1. It suffices to show that for all $\alpha \in (0,1)$ one can take a nonrandom sequence $T_{n,\alpha}$ such that

$$P_0(t_{m,n} \geq T_{n,\alpha}) \leq \alpha + o(1), \quad \sup_{\bar f \in F_n^k} P_{\bar f}(t_{m,n} < T_{n,\alpha}) = o(1). \tag{45}$$

In fact, if (45) holds true for all $\alpha \in (0,1)$, then one can take a sequence $\alpha = \alpha_n \to 0$ such that (45) holds true as well. Then one can take $T_n = T_{n,\alpha_n}$.

Let us take $T_{n,\alpha} = T_\alpha$, this yields $T_n = T_{n,\alpha_n} \to \infty$ for any $\alpha_n \to 0$. Then the first relation follows from asymptotic normality of the statistics $t_{m,n}$ under the null-hypothesis. To obtain the second relation in (45) it suffices to verify that

$$\inf_{\bar f \in F_n^k} E_{\bar f}(t_{m,n}) \to \infty. \tag{46}$$

In fact, using the Chebyshev inequality and the second relation in (43), we get

$$\begin{aligned} &P_{\bar f}(t_{m,n} < T_\alpha) = P_{\bar f}(E_{\bar f}(t_{m,n}) - t_{m,n} > E_{\bar f}(t_{m,n}) - T_\alpha) \leq \\ &\mathrm{Var}_{\bar f}(t_{m,n})/(E_{\bar f}(t_{m,n}) - T_\alpha)^2 \to 0, \quad \text{as } E_{\bar f}(t_{m,n}) \to \infty. \end{aligned} \tag{47}$$

By the first relation (43), relation (46) is equivalent to

$$\varepsilon^{-2}(2mn)^{-1/2}\inf_{\bar f \in F_n^k}\sum_{i=1}^{n}\xi_i\|Pr_m(f_i)\|_2^2 \to \infty. \tag{48}$$

Using inequality (44) and by $m^{-\sigma} \asymp r_n^* = o(r_n)$, $p \leq 2$, we get

$$\|Pr_m(f)\|_2 \geq \|Pr_m(f)\|_p \geq C_1 r_n(1 + o(1)) \text{ uniformly over } f \in F_n. \tag{49}$$

By the definition of set F_n^k this yields

$$\inf_{\bar f\in F_n^k}\sum_{i=1}^n \xi_i\|Pr_m(f_i)\|_2^2 \ge C_1^2 k r_n^2(1+o(1)).$$

The last relation yields (48) because

$$k r_n^2/\varepsilon^2\sqrt{nm} \asymp (r_n/r_n^*)^2 \to \infty.$$

Let us turn to Theorem 2. Take $C>1/b,\ C>1$, where $b=b(1)>0$ is from Corollary 2. By the choice $T,\ m$ we have $T^2 = C\log n < m$. Therefore Corollary 2 yields

$$\alpha_n(\psi_{n,C}) = P_0(\max_{1\le i\le n}\chi_{m,i} > T) \le n\exp(-bT^2) = n^{1-bC} \to 0.$$

Let us verify the relation

$$\sup_{\bar f\in F_n^k} P_{\bar f}(\max_{1\le i\le n}\chi_{m,i} < T) \to 0.$$

Let $\bar f\in F_n^k$. To specify, suppose that the first channel contains a signal $f\in F_n$, i.e., $\xi_1=1$. Then, analogously to above, we have

$$\begin{gathered} P_{\bar f}(\max\nolimits_{1\le i\le n}\chi_{m,i} < T) \le P_{\bar f}(\chi_{m,1} < T) = \\ P_{\bar f}(E_{\bar f}(\chi_{m,1}) - \chi_{m,1} > E_{\bar f}(\chi_{m,1}) - T) \le \mathrm{Var}_{\bar f}(\chi_{m,1})/(E_{\bar f}(\chi_{m,1}) - T)^2. \end{gathered} \tag{50}$$

Taking into account relation (42) it suffices to verify that

$$\liminf \sup_{\bar f\in F_n^k} E_{\bar f}(\chi_{m,1})/T > 1. \tag{51}$$

First, let us consider the case

$$\varepsilon^2(\log n)^{2\sigma+1} \le 1, \quad r_n^* = (\varepsilon^4 \log n)^{\sigma/(1+4\sigma)} \ge \varepsilon\sqrt{\log n}. \tag{52}$$

Since $m^{-\sigma} < r_n^*$, analogously to above using (44), uniformly over $\bar f\in F_n^k$, we have

$$\|Pr_m(f)\|_2 \ge \|Pr_m(f)\|_p \ge C_1 r_n - C_2 r_n^* \ge C_3 r_n \tag{53}$$

where $C_3 = C_1 - C_2/c_1 > 0$ for large enough c_1. Under (52) we have

$$(\varepsilon^4 \log n)^{-1/(1+4\sigma)} \ge \log n.$$

Therefore $m \le C(r_n^*)^{-1/\sigma} + 1$ and using relations (41) and (53) we get

$$E_{\bar f}(\chi_{m,1})/T \ge C_3^2 r_n^2/\varepsilon^2\sqrt{2Cm\log n} \ge C_4(r_n/r_n^*)^2(1+o(1)) > C_5 > 1 \tag{54}$$

for large enough c_1; here $C_4 = C_3^2/\sqrt{2}C$, $C_5 = C_4 c_1^2$.

Next, let

$$\varepsilon^2(\log n)^{2\sigma+1} > 1, \quad r_n^* = \varepsilon\sqrt{\log n} > (\varepsilon^4 \log n)^{\sigma/(1+4\sigma)}. \tag{55}$$

Under (55) we have

$$(\varepsilon^4 \log n)^{-1/(1+4\sigma)} < \log n.$$

This yields $C\log n < m \le C\log n + 1$. Setting $C_6 = C^{-\sigma}$, we have

$$m^{-\sigma} \le (C\log n)^{-\sigma} < C_6\varepsilon\sqrt{\log n} = C_6 r_n^*,$$

which yields (53) with $C_3 = C_1 - C_6C_2/c_1 > 0$ for large enough c_1. Analogously to (54), uniformly over $\bar f \in F_n^k$, we get

$$E_{\bar f}(\chi_{m,1})/T \ge C_3^2 r_n^2/\varepsilon^2\sqrt{2Cm\log n} \ge C_4(r_n/r_n^*)^2(1+o(1)) > C_5 > 1. \qquad \square$$

5 Proof of Theorems 3 - 5

5.1 Lower bounds

Fix $\sigma_0 > 0$ and take σ_0-regular orthonormal wavelet basis $\psi_{jl},\ l \ge l_0 \ge 0,\ j \in J_l$, where

$$J_{l_0} = \{1, \dots, k\}, \quad J_l = \{1, \dots, 2^l\}, \ \text{ for } \ l > l_0,$$

such that, for any $f_{\boldsymbol\theta} = \sum_{l\ge l_0}\sum_{j\in J_l}\theta_{jl}\psi_{jl}$ and any $0 \le \sigma < \sigma_0,\ 1 \le h, q \le \infty$, one has the inequality

$$B_0|\boldsymbol\theta|_{sqh} \le \|f_{\boldsymbol\theta}\|_{\sigma,q,h} \le B_1|\boldsymbol\theta|_{sqh}, \quad s = \sigma + 1/2 - 1/q, \tag{56}$$

where $\boldsymbol\theta = \{\theta_{jl},\ l \ge l_0,\ j \in J_l\}$, $\|\cdot\|_{\sigma,q,h}$ is Besov norm in the functional space $L_2(0,1)$, and $|\cdot|_{sqh}$ is Besov norm in the sequence space: if $1 \le h, q < \infty$, then

$$|\boldsymbol\theta|_{sqh} = \left(\Big(\sum_{j=1}^{k}|\theta_{jl_0}|^q\Big)^{h/q} + \sum_{l>l_0}2^{lsh}\Big(\sum_{j}|\theta_{jl}|^q\Big)^{h/q}\right)^{1/h}, \tag{57}$$

with the natural extension for $h, q = \infty$, (see [1]) where B_0, B_1 are positive constants. Recall the relations between Sobolev and Besov norms:

$$B_2\|f\|_{\sigma,q,\infty} \le \|f\|_{\sigma,q} \le B_3\|f\|_{\sigma,q,1}; \quad B_2 > 0, \quad B_3 > 0. \tag{58}$$

For $l > l_0$ we denote $m_l = 2^l$, $\mathcal{J} = \{(jl) : l \ge l_0,\ j \in J_l\}$. Set

$$l^2_{\mathcal{J}} = \{\theta_{jl}, \quad (jl) \in \mathcal{J}, \quad \sum_{(jl)\in\mathcal{J}}\theta_{jl}^2 < \infty\},$$

and let

$$L_l = \{f(t,\boldsymbol\theta_l) = \varepsilon\sum_{j=1}^{m_l}\theta_j\psi_{jl}(t), \quad \boldsymbol\theta_l \in R^{m_l}\}$$

be the linear subspace of dimension m_l generated by the lth resolution. It follows from (56), (58) that, for any $f(\cdot,\boldsymbol{\theta}_l)\in L_l$, one has

$$C_0\varepsilon m_l^{\sigma+1/2-1/q}|\boldsymbol{\theta}_l|_q \le \|f(\cdot,\boldsymbol{\theta}_l)\|_{\sigma,q} \le C_1\varepsilon m_l^{\sigma+1/2-1/q}|\boldsymbol{\theta}_l|_q;$$
$$C_0\varepsilon m_l^{1/2-1/p}|\boldsymbol{\theta}_l|_p \le \|f(\cdot,\boldsymbol{\theta}_l)\|_p \le C_1\varepsilon m_l^{1/2-1/p}|\boldsymbol{\theta}_l|_p.$$

Therefore, the relation $\boldsymbol{\theta}_l\in\Theta_{n,m_l}(r_n,\sigma)$ is sufficient for the function $f(\cdot,\boldsymbol{\theta}_l)\in L_l$ belongs to the set $F(r_n,\sigma)$; here the set $\Theta_{n,m_l}(r_n,\sigma)$ is determined by the constraint analogous to (24),

$$m_l^{1/2-1/p}|\boldsymbol{\theta}_l|_p \ge c_3 r_n/\varepsilon, \quad m_l^{\sigma+1/2-1/q}|\boldsymbol{\theta}_l|_q \le c_4/\varepsilon, \tag{59}$$

with some different positive constants c_3, c_4. Analogously to Section 4, we can pass to random variables and parameters

$$x_{ijl}=\varepsilon^{-1}\int_0^1 \psi_{jl}(t)dX_i(t)$$

which are independent in i,j,l and are Gaussian $N(\theta_{ijl},1)$ with $\theta_{ijl}=\varepsilon^{-1}(f_i,\psi_{jl})$. Setting

$$\boldsymbol{\theta}_{i,l}=(\theta_{i1l},\dots,\theta_{im_l l}),\quad \bar{\boldsymbol{\theta}}_l=(\xi_1\boldsymbol{\theta}_{1,l},\dots,\xi_n\boldsymbol{\theta}_{n,l}),\ \xi_i\in\{0,1\},\quad \bar{\boldsymbol{\theta}}=\{\bar{\boldsymbol{\theta}}_l,\ l\ge l_0\},$$

we can identify the vectors $\boldsymbol{\theta}_{i,l},\ \bar{\boldsymbol{\theta}}_l,\ \bar{\boldsymbol{\theta}}$ with the sequences either in $l^2_{\mathcal{J}}$ or in $(l^2_{\mathcal{J}})^n$ or in $(l^2)^n$. Analogously to (27), introduce the sets

$$\Theta^n_{n,m_l,k}(r_n,\sigma)=\{\bar{\boldsymbol{\theta}}_l:\quad \boldsymbol{\theta}_{i,l}\in\Theta_{n,m_l}(r_n,\sigma);\ \sum_{i=1}^n\xi_i=k\}\subset(l^2_{\mathcal{J}})^n; \tag{60}$$
$$\Theta^n_{n,k}(r_n,\sigma)=\bigcup_{l>l_0}\Theta^n_{n,m_l,k}(r_n,\sigma);. \tag{61}$$

Set analogously (12)

$$\Theta^n_n(\mathcal{K}_n,\sigma)=\bigcup_{k\in\mathcal{K}_n}\Theta^n_{n,k}(r_n(k),\sigma), \tag{62}$$
$$\Theta^n_n(k;\sigma_0,\sigma_1)=\bigcup_{\sigma\in[\sigma_0,\sigma_1]}\Theta^n_{n,k}(r_n(k),\sigma),, \tag{63}$$

where the set $\mathcal{K}_n=\mathcal{K}_n(\delta_1,\delta_2)$ is defined by (11). It follows from the constructions, that $\bar{f}(\cdot,\bar{\boldsymbol{\theta}})=(\xi_1 f(\cdot,\boldsymbol{\theta}_1),\dots,\xi_n f(\cdot,\boldsymbol{\theta}_n))\in F^n(\mathcal{K}_n,\sigma)$ when $\bar{\boldsymbol{\theta}}\in\Theta^n_n(\mathcal{K}_n\sigma)$ analogous relations hold true for the sets $\Theta^n_n(k;\sigma_0,\sigma_1)$ and $F^n_k(\sigma_0,\sigma_1)$.

5.1.1 Lower bounds for Theorem 3

Let us consider hypothesis testing problem analogous to (26): using the observation $\bar{\mathbf{x}}=\{x_{ijl};\ (jl)\in\mathcal{J},\ i=1,\dots,n\}$ which correspond to the Gaussian measure $P_{\bar{\boldsymbol{\theta}}},\ \bar{\boldsymbol{\theta}}\in(l^2_{\mathcal{J}})^n$ to test

$$H_0:\ \bar{\boldsymbol{\theta}}=0\quad\text{against}\quad H_1:\ \bar{\boldsymbol{\theta}}\in\Theta^n_n(\mathcal{K}_n,\sigma)$$

To obtain the lower bounds of Theorem it suffices to verify that

$$\gamma(\Theta_n^n(\mathcal{K}_n, \sigma)) \to 1.$$

Let $\Theta^n_{n,m_l,k+}(r_n(k),\sigma)$ be the set defined analogously to (60), but the equality $\sum_i \xi_i = k$ is changed to the inequality $\sum_i \xi_i \geq k$, and the sets $\Theta^n_{n,k+}(r_n,\sigma)$, $\Theta^n_{n+}(\mathcal{K}_n,\sigma)$ are defined analogously to (61), (62) with the change of the sets $\Theta^n_{n,m_l,k}(r_n,\sigma)$ to $\Theta^n_{n,m_l,k+}(r_n,\sigma)$. Analogously to Proposition 1 we have

$$\gamma(\Theta_n^n(\mathcal{K}_n, \sigma)) = \gamma(\Theta_{n+}^n(\mathcal{K}_n, \sigma)).$$

It suffices to consider the case $r_n(k) = cr_n^*(k)$ for small enough $c \in (0,1)$. Let us take a collection

$$m_l = 2^l,\ l = L,\ L+1,\ \dots,\ L+M,\ L \to \infty,\ M \to \infty, \tag{64}$$

and corresponding collections k_l, z_l such that k_l are integers, $n^\delta \leq k_l \leq b_1 n^{\delta_3}$ where $\delta \in (\delta_1,\delta_2)$, $\delta < \delta_3 < \delta_2$ is the parameter form Theorem, and the relations analogous to (33)–(35) are fulfilled uniformly on l,

$$m_l \sim d_1(r_n(k_l))^{-1/\sigma} \to \infty, \quad z_l \sim d_2\varepsilon^{-1}(r_n(k_l))^{1+1/2\sigma}; \tag{65}$$

$$m_l z_l^4 \sim D\varepsilon^{-4}(r_n(k_l))^{4+1/\sigma} \sim c_1 D n k_l^{-2} \log\log n \to 0; \tag{66}$$

the last relation holds since $\delta > 1/2$, and $nk_l^{-2} = O(n^{1-2\delta}) = o(\log\log n)$, where $d_1 > 0, d_2 > 0$ are constants such that inequalities (59) holds for $r_n = r_n(k_l)$, $l = L+1,\dots,L+M$; $c_1 = c^{4+1/\sigma}$ and $D = D(d_1,d_2,\sigma)$ does not depends on c, l, n. Relations (65), (66) yield $z_l \to 0$ uniformly on l. Under assumptions of Theorem 3 we can take L and M in such way that

$$L \sim \log_2(r_n^{-1}(n^\delta))/\sigma \to \infty, \quad M \sim \frac{2(\delta_3-\delta)}{4\sigma+1}\log_2 n \asymp \log_2 n.$$

For each i, let us construct the collections of product priors $\pi_{m_l,z_l}(d\boldsymbol{\theta}_{i,l})$ according to (31) with $m = m_l$, $z = z_l$ which are concentrated on the set $\Theta_{n,m_l}(r_n(k_l),\sigma)$, and set

$$\pi_l^n(d\bar{\boldsymbol{\theta}}_l) = \prod_{i=1}^n \Big((1-h_{n,l})\delta_0 + h_{n,l}\pi_{m_l,z_l}(d\boldsymbol{\theta}_{i,l})\Big),$$

where $h_{n,l} \sim k_l/n$ are taken according to (37). Analogously to (38), we have

$$\pi_l^n(\Theta^n_{n,m_l,k_l+}(r_n(k_l),\sigma)) \to 1. \tag{67}$$

At last, set

$$\pi^n(d\bar{\boldsymbol{\theta}}) = M^{-1}\sum_{l=L+1}^{L+M}\pi_l^n(d\bar{\boldsymbol{\theta}}_l).$$

It follows from (67) that

$$\pi^n(\Theta^n_{n+}(\mathcal{K}_n,\sigma)) \to 1, \tag{68}$$

and to obtain the lower bounds of Theorem it suffices to control that the relation analogous to (29) holds true. Since the priors π_l^n, $l = L+1,\dots,L+M$ correspond to different resolutions, the likelihood ratios $dP_{\pi_l^n}/dP_0$ are independent statistics and, analogously to (39), we get

$$E_0\Big(\frac{dP_{\pi^n}}{dP_0}-1\Big)^2 = M^{-2}\sum_{l=L+1}^{L+M}\Big(\frac{dP_{\pi_l^n}}{dP_0}-1\Big)^2 \le$$
$$M^{-2}\sum_{l=L+1}^{L+M}\Big(\exp\Big(nh_{n,l}^2(\exp(2m_l\sinh^2(z_l^2/2))-1)\Big)-1\Big) =$$
$$M^{-2}\sum_{l=L+1}^{L+M}\Big(\exp(u_{n,l}^2)-1\Big);\quad u_{n,l}^2 = nh_{n,l}^2(\exp(2m_l\sinh^2(z_l^2/2))-1).$$

To obtain (29) it suffices to verify that

$$\limsup \max_{L+1\le l\le L+M} u_{n,l}^2/\log M < 1. \tag{69}$$

It was noted above that $z_l \to 0$. Jointly with (66), this yields

$$m_l\sinh^2(z_l^2/2)\sim m_l z_l^4/4\to 0;\quad u_{n,l}^2\sim nh_{n,l}^2 m_l z_l^4/2\sim c_1 D(\log\log n)/2 \tag{70}$$

and since $\log M\sim\log\log n$, these yield that one can take $r_n(k) = cr_n^*(k)$ with small enough $c>0$ such that (69) is fulfilled. □

5.1.2 Lower bounds for Theorem 4

Let the wavelet basis be σ^*-regular with $\sigma^*>\sigma_1$. Assume without loss of generality that $k_n = n^{\delta_n}$, $1/2<\delta_1\le\delta_n\le\delta_2<1$, and

$$r_n(k_n,\sigma) = g_n^+ r_n^*(k_n,\sigma),\quad g_n^+ = O(1),\quad \log(g_n^+)^{-1} = o(\log(w_n(k_n)^{-1}).$$

Analogously to Section 5.1.1 it suffices to obtain the lower bounds in the hypothesis testing problem

$$H_0:\ \bar{\boldsymbol\theta}=0\quad\text{against}\quad H_1:\ \bar{\boldsymbol\theta}\in\Theta_n^n(k_n,\sigma_0,\sigma_1).$$

Introduce the sets $\Theta_{n+}^n(k_n,\sigma_0,\sigma_1)$ and note, analogously to Proposition 1, that

$$\gamma(\Theta_n^n(k_n,\sigma_0,\sigma_1)) = \gamma(\Theta_{n+}^n(k_n,\sigma_0,\sigma_1)).$$

Denote

$$G_n = \sup_{\sigma\in[\sigma_0,\sigma_1]}(r_n(k_n,\sigma)/r_n^*(k_n,\sigma))^{4+1/\sigma}, \tag{71}$$

clearly $G_n = O(1)$ by $g_n^+ = O(1)$ and $G_n\to 0$ iff $g_n^+\to 0$.

Take the collections m_l of the form (64) and corresponding collections $\sigma_l\in(\sigma_0,\sigma_1)$, $z_l>0$, $l = L+1,\dots,M$ such that

$$m_l\sim d_1(r_n(k_n,\sigma_l))^{-1/\sigma_l}\to\infty,\quad z_l\sim d_2\varepsilon^{-1}(r_n(k_n,\sigma_l))^{1+1/2\sigma_l}; \tag{72}$$
$$m_l z_l^4\asymp\varepsilon^{-4}(r_n(k_n,\sigma_l))^{4+1/\sigma_l} = O(G_n nk_n^{-2}\log w_n^{-1}(k_n)) = O(G_n d_n)\to 0, \tag{73}$$

where $d_1 > 0, d_2 > 0$ are constants such that inequalities (59) holds for $r_n = r_n(k_n, \sigma_l)$, $l = L+1, \dots, L+M$. These yield that the priors π_{m_l,z_l} of the form (31) are concentrated on the sets $\Theta_{n,m_l}(r_n, \sigma_l)$. Under assumptions of Theorem 4 we can take

$$L \geq \sigma_1^{-1} \log_2(r_n^{-1}(k_n, \sigma_1)) + O(1) \to \infty,$$
$$M \sim \frac{4(\sigma_1 - \sigma_0)\log_2(w_n^{-1}(k_n))}{(4\sigma_0+1)(4\sigma_1+1)} \asymp \log w_n^{-1}(k_n); \quad \max_l z_l = o(1), \tag{74}$$

where the last relation in (74) follows from (72), (73). Set

$$\pi_M = M^{-1} \sum_{l=L+1}^{M} \pi_{m_l,z_l}, \quad \mu_{M,h} = (1-h)\delta_{\bar 0} + h\pi_M; \quad \pi_M^n(d\bar{\boldsymbol\theta}) = \prod_{i=1}^{n} \mu_{M,h}(d\boldsymbol\theta_i), \tag{75}$$

where $h = h_n \sim k_n/n$ is taking according to (37).

Analogously to (38), (39) we get, $\pi_M^n(\Theta_{n+}^n(k_n, \sigma_0, \sigma_1)) \to 1$,

$$E_0\left(dP_{\pi_M^n}/dP_0 - 1\right)^2 \leq \exp\left(nh^2 E_0\left(dP_{\pi_M}/dP_0 - 1\right)^2\right) - 1, \tag{76}$$

and it suffices to verify that

$$nh^2 E_0\left(dP_{\pi_M}/dP_0 - 1\right)^2 = \begin{cases} O(1) & \text{if } G_n = O(1) \\ o(1) & \text{if } G_n = o(1) \end{cases}. \tag{77}$$

We get

$$E_0\left(dP_{\pi_M}/dP_0 - 1\right)^2 = M^{-2} \sum_{l=L+1}^{M} E_0\left(dP_{\pi_{m_l,z_l}}/dP_0 - 1\right)^2 \leq$$
$$M^{-2} \sum_{l=L+1}^{M} \left(\exp\left(2m_l \sinh^2(z_l^2/2)\right) - 1\right) = M^{-2} \sum_{l=L+1}^{M} \left(\exp(u_{n,l}^2) - 1\right), \tag{78}$$

where $u_{n,l}^2 = 2m_l \sinh^2(z_l^2/2)$. It follows from (73), (74) that, for some $B > 0$,

$$\exp(u_{n,l}^2) - 1 \sim u_{n,l}^2 \sim m_l z_l^4/2 \to 0; \quad E_0\left(dP_{\pi_M}/dP_0 - 1\right)^2 \leq$$
$$M^{-2} \sum_{l=L+1}^{M} u_{n,l}^2(1 + o(1)) \leq BM^{-1} n k_n^{-2} \log w_n^{-1}(k_n) G_n \leq BG_n M^{-1} d_n.$$

Since $d_n = nk_n^{-2} \log w_n^{-1}(k_n)$, $nh^2 \sim n^{-1}k_n^2$, it follows from (74) and the last relation that

$$nh^2 E_0\left(dP_{\pi_M}/dP_0 - 1\right)^2 \leq BG_n n^{-1} k_n^2 M^{-1} d_n(1 + o(1)) \asymp G_n,$$

which yields (77). □

5.1.3 Lower bounds for Theorem 5

It suffices to assume $r_n(k_n,\sigma) = cr_n^*(\sigma)$ for small enough $c \in (0,1)$. First, let there exist $\sigma \in (\sigma_0,\sigma_1)$ such that $\varepsilon^2(\log n)^{2\sigma+1} > 1$, i.e., $r_n^*(\sigma) \sim \varepsilon\sqrt{\log n} \to 0$. Since $F_{k_n}^n = F_{k_n}^n(r_n(k_n,\sigma),\sigma) \subset F_{k_n}^n(\sigma_0,\sigma_1)$, the lower bounds of Theorem 5 follow from Theorem 2 in this case. Therefore below we assume

$$\varepsilon^2(\log n)^{2\sigma_1+1} \le 1, \quad r_n^*(\sigma) = (\varepsilon^4 \log(n\log\varepsilon^{-1}))^{\sigma/(4\sigma+1)}. \tag{79}$$

We repeat the constructions of Section 5.1.2. We take the collections m_l of the form (64) and corresponding collections $\sigma_l \in (\sigma_0,\sigma_1)$, z_l such that (72) holds and relation (73) is changed by

$$m_l z_l^4 \asymp \varepsilon^{-4}(r_n(k_n,\sigma_l))^{4+1/\sigma_l} \sim C_l \log(n\log\varepsilon^{-1}) \to \infty,\ \max_{l>L} z_l = O(1), \tag{80}$$

where $C_l = c^{4+1/\sigma_l} \le c^4$, and the last relation follows from relation (79). Also under (79) we have

$$2\log\varepsilon^{-1} \ge (2\sigma_1+1+o(1))\log\log(n\log\varepsilon^{-1}) > \log\log(n\log\varepsilon^{-1}),$$

which yields

$$M \sim B(4\log\varepsilon^{-1} - \log\log n(\log\varepsilon^{-1})) \asymp \log\varepsilon^{-1}.$$

Taking priors (75), we repeat estimations (76), (78). However it follows from (74), (80) that, for some $B > 0$,

$$\exp(u_{n,l}^2) - 1 \asymp u_{n,l}^2 \asymp m_l z_l^4 \to \infty;\ E_0\Big(dP_{\pi_M}/dP_0 - 1\Big)^2 \le$$
$$M^{-2}\sum_{l=L+1}^{M}\exp(u_{n,l}^2) \le M^{-1}\exp\Big(Bc^4\log(n\log\varepsilon^{-1})\Big) = M^{-1}(n\log\varepsilon^{-1})^{Bc^4},$$

and to obtain the relation $E_0\Big(dP_{\pi_M^n}/dP_0 - 1\Big) \to 0$ it suffices

$$n^{-1}k_n^2 M^{-1}(n\log\varepsilon^{-1})^{Bc^4} \to 0.$$

Since $n^{-1}k_n^2 = O(n^{2\delta-1})$, $\delta < 1/2$, the last relation holds for $Bc^4 < 1-2\delta$. $\square$

5.2 Upper bounds

Recall that to prove the upper bounds of Theorems we can assume $p = q$ which corresponds to the widest alternative.

5.2.1 Upper bounds for Theorem 3

We need to verify the following relations: for every $c > 2$,

$$\alpha_n(\psi_{n,c}) = P_0(\max_{L+1\le l\le L+M} t_{m_l,n} > \sqrt{c\log M}) \to 0, \tag{81}$$

and there exists $c_1 > 0$ such that if $r_n(k)/r^*(k) > c_1$, then

$$\beta_n(\psi_{n,c}) = \sup_{\bar f \in F^n(\mathcal{K}_n,\sigma)} P_{\bar f}\big(\max_{L+1\le l\le L+M} t_{m_l,n} \le \sqrt{c\log M}\big) \to 0. \qquad (82)$$

By the choose $m_l,\ M$, we have $\log M \sim \log\log n = o(m_l n)$, and using Lemma 4.1 for all $c > 2$ we have (81), since

$$\alpha_n(\psi_{n,c}) \le \sum_{l=L+1}^{L+M} P_0(t_{m_l,n} > \sqrt{c\log M}) = O(M^{1-c(1+o(1))/2}) \to 0.$$

Let $\bar f = (\xi_i f_i, \dots, \xi_n f_n) \in F^n(k_n,\sigma),\ k_n \in \mathcal{K}_n$. Clearly,

$$P_{\bar f}\big(\max_{L+1\le l\le L+M} t_{m_l,n} \le \sqrt{c\log M}\big) \le \min_{L+1\le l\le L+M} P_{\bar f}(t_{m_l,n} \le \sqrt{c\log M}).$$

Using relation (43) and estimation analogous to (47) we see that relation (82) follows from the inequality (compare with (48)): for all $k_n \in \mathcal{K}_n$, large enough n and some $B > 1$ one has

$$\varepsilon^{-2} \max_{L+1\le l\le L+M} (2m_l n)^{-1/2} \inf_{\bar f\in F^n(k_n,\sigma)} \sum_{i=1}^n \xi_i \|Pr_{m_l}(f_i)\|_2^2 > \sqrt{Bc\log M}. \qquad (83)$$

To verify (83), let us take the integer $l,\ L+1 \le l \le L+M$ such that

$$m_{l-1} = 2^{l-1} \le (r_n^*(k_n))^{-1/\sigma} \le m_l = 2^l, \qquad (84)$$

(this is possible under the choice L and M in Theorem). It follows from (44) that one can take c_1 in relation $r_n(k_n) > c_1 r_n^*(k_n)$, and $d > 0$ in such way that, uniformly over $f \in F_n$,

$$\|Pr_{m_l}(f)\|_2 \ge \|Pr_{m_l}(f)\|_p \ge d r_n(k_n),$$

compare with (49); the set F_n is defined by (6) with $r_n = r_n(k_n)$. Therefore,

$$\inf_{\bar f\in F^n(k_n,\sigma)} \sum_{i=1}^n \xi_i \|Pr_{m_l}(f_i)\|_2^2 \ge k_n d^2 r_n^2(k_n),$$

and the left-hand side of (83) is not smaller than $d^2\varepsilon^{-2}(2m_l n)^{-1/2} k_n r_n^2(k_n)$. By the choice $r_n^*(k)$ and m_l, this quantity is not smaller than $(dc_1)^2\sqrt{\log\log n}/2$ which yields (83) for $(dc_1)^4 > 4Bc$. $\square$.

5.2.2 Upper bounds for Theorem 4

It suffices to show that, for any $\alpha \in (0,1)$, one can take H_α such that

$$P_0(t_n \ge H_\alpha) \le \alpha; \quad \sup_{\bar f\in F^n(\mathcal{K}_n,\sigma_0,\sigma_1)} P_{\bar f}(t_n < H_\alpha) = o(1), \qquad (85)$$

where the statistics t_n are defined by (16), (17), (9).

To evaluate the first relation in (85), note that $E_0 t_n = 0$. Let us evaluate the variance of the statistics t_n in P_0-probability. We have

$$\mathrm{Var}_0 t_n = \sum_{l=1}^{\infty} a_{n,l}^2 \mathrm{Var}_0 t_{n,m_l} + 2 \sum_{1 \le k < l < \infty} a_{n,k} a_{n,m_l} \mathrm{Cov}_0 t_{n,m_k} t_{n,m_l};$$

$$\mathrm{Var}_0 t_{n,m_l} = n^{-1} \sum_{i=1}^{n} \mathrm{Var}_0 \chi_{m_l,i} = 1;$$

$$\mathrm{Cov}_0 t_{n,m_k} t_{n,m_l} = n^{-1} \sum_{i=1}^{n} \mathrm{Cov}_0 \chi_{m_k,i} \chi_{m_l,i}.$$

Since the items in the last sum do not depend on i, omitting index i, we have, for $m_k = 2^k,\ m_l = 2^l,\ 1 \le k < l$,

$$\mathrm{Cov}_0 \chi_{m_k} \chi_{m_l} = 2^{-(l+k)/2-1} \sum_{s=1}^{2^k} \sum_{t=1}^{2^l} \mathrm{Cov}_0 x_{sm_k}^2 x_{tm_l}^2$$

$$= 2^{-(l+k)/2-1} \sum_{s=1}^{2^k} \sum_{t\,:\,\delta_{m_l,t} \subset \delta_{m_k,s}} \mathrm{Cov}_0 x_{sm_k}^2 x_{tm_l}^2;$$

the last equality holds since items x_{sm_k} and x_{tm_l} are independent, if the interval $\delta_{m_k,s}$ does not contain the interval $\delta_{m_l,t}$. Since $\#\{t\ :\ \delta_{m_l,t} \subset \delta_{m_k,s}\} = 2^{l-k}$, by making a renumbering, we have the equality

$$x_{sm_k} = 2^{(k-l)/2} \sum_{\gamma=1}^{2^{l-k}} x_{\gamma m_l}.$$

Since the items $x_{\gamma m_l}$ are i.i.d. standard Gaussian, we have the equality

$$\mathrm{Cov}_0 x_{sm_k}^2 x_{tm_l}^2 = 2^{k-l} \mathrm{Cov}_0 \left(\Big(\sum_{\gamma=1}^{2^{l-k}} x_{\gamma m_l} \Big)^2 x_{tm_l}^2 \right) = 2^{k-l+1}.$$

Since $\sum_{l=1}^{\infty} a_{n,l}^2 = 1$ and $\sum_{k=1}^{\infty} a_{n,k} a_{n,k+s} \le 1$ for all $s \ge 1$, these yield the equality $\mathrm{Cov}_0 \chi_{m_k} \chi_{m_l} = 2^{(k-l)/2}$ and the inequality

$$\mathrm{Var}_0 t_n = \sum_{l=1}^{\infty} a_{n,l}^2 + 2 \sum_{1 \le k < l < \infty} 2^{(k-l)/2} a_{n,k} a_{n,l} = 1 + 2 \sum_{s=1}^{\infty} 2^{-s/2} \sum_{k=1}^{\infty} a_{n,k} a_{n,k+s} \le C$$

with $C = 3 + 2\sqrt{2}$. Therefore, setting $H_\alpha = \sqrt{C/\alpha}$ and using the Chebyshev inequality, we obtain the first relation in (85)

$$P_0(t_n \ge H_\alpha) \le \mathrm{Var}_0 t_n / H_\alpha^2 \le C/H_\alpha^2 = \alpha.$$

To verify the second relation in (85), let us evaluate means and variances of the statistics t_n in $P_{\bar{f}}$-probability, $\bar{f}_n = (\xi_1 f_1, \ldots, \xi_1 f_1)$. Using the notations from

Section 4.2 and the relations (41), (42), we get

$$E_{\bar f}t_n = n^{-1/2}\sum_{l=1}^{\infty} a_{n,l}\sum_{i=1}^{n}\chi_{m_l,i} = n^{-1/2}\sum_{l=1}^{\infty} a_{n,l}\sum_{i=1}^{n}\sum_{s=1}^{2^l} 2^{-(l+1)/2}v^2_{ism_l}$$
$$= \varepsilon^{-2}n^{-1/2}\sum_{i=1}^{n}\xi_i\sum_{l=1}^{\infty} a_{n,l}2^{-(l+1)/2}\|Pr_{m_l}f_i\|^2; \quad Var_{\bar f}\chi_{m_l,i} = 1 + 2^{-l+1}\xi_i\|Pr_{m_l}f_i\|^2.$$

By repeating the scheme of calculations above, we get

$$\mathrm{Var}_{\bar f}t_n = n^{-1}\sum_{i=1}^{n}\left(\sum_{l=1}^{\infty} a^2_{n,l}Var_{\bar f}\chi_{m_l,i} + 2\sum_{1\le k<l<\infty} a_{n,k}a_{n,m_l}\mathrm{Cov}_{\bar f}\chi_{m_k,i}\chi_{m_l,i}\right);$$
$$\mathrm{Cov}_{\bar f}\chi_{m_k,i}\chi_{m_l,i} = 2^{-(l+k)/2-1}\sum_{s=1}^{2^k}\sum_{t:\delta_{m_l,t}\subset\delta_{m_k,s}}\mathrm{Cov}_{\bar f}x^2_{ism_k}x^2_{itm_l}$$

(we suppose $k < l$ in the last relation). For t such that $\delta_{m_l,t}\subset\delta_{m_k,s}$, we have the equality

$$x_{ism_k} = 2^{(k-l)/2}\sum_{\gamma=1}^{2^{l-k}}(v_{itm_l}+\eta_t), \;\; x_{itm_l} = v_{itm_l}+\eta_t$$

in $P_{\bar f}$-probability, where $\eta_t \sim N(0,1)$ are i.i.d. This yields

$$\mathrm{Cov}_{\bar f}x^2_{ism_k}x^2_{itm_l} = 2^{(k-l)/2}\mathrm{Cov}_{\bar f}\left(\Big(\sum_{\gamma=1}^{2^{l-k}} v_{itm_l}+\eta_t\Big)^2\Big(v_{itm_l}+\eta_t\Big)^2\right)$$
$$= 2^{k-l}\mathrm{Var}_{\bar f}\Big(v_{itm_l}+\eta_t\Big)^2 = 2^{k-l+1}(1+2v^2_{itm_l}).$$

Therefore,

$$\mathrm{Cov}_{\bar f}\chi_{m_k,i}\chi_{m_l,i} = 2^{(k-l)/2}\Big(1+2^{-l+1}\sum_{t=1}^{2^l}v^2_{itm_l}\Big) = 2^{(k-l)/2}\Big(1+2^{-l+1}\varepsilon^{-2}\xi_i\|Pr_{m_l}f_i\|^2\Big).$$

Using these relations we get

$$\mathrm{Var}_{\bar f}t_n = n^{-1}\sum_{i=1}^{n}\Big(\sum_{l=1}^{\infty}a^2_{n,l}\Big(1+2^{-l+1}\xi_i\varepsilon^{-2}\|Pr_{m_l}f_i\|^2\Big)$$
$$+2\sum_{1\le k<l<\infty}a_{n,k}a_{n,m_l}2^{(k-l)/2}\Big((1+2^{-l+1}\xi_i\varepsilon^{-2}\|Pr_{m_l}f_i\|^2\Big)\Big)$$
$$= \mathrm{Var}_0 t_n + n^{-1}\varepsilon^{-2}\sum_{i=1}^{n}\xi_i\sum_{l=1}^{\infty}a_{n,l}2^{-l+1}\|Pr_{m_l}f_i\|^2\Big(a_{n,l}+2\sum_{k=1}^{l-1}a_{n,k}\Big),$$

which yields, uniformly over $\bar f$, $\mathrm{Var}_{\bar f}t_n \le C + o(E_{\bar f}t_n)$.

It follows from evaluations above and from the Chebyshev inequality, that, to obtain the second relation (85), it suffices to verify the relation

$$\inf_{\bar f\in F^n(\mathcal{K}_n,\sigma_0,\sigma_1)} E_{\bar f}t_n = \varepsilon^{-2}n^{-1/2}\inf_{\bar f\in F^n(\mathcal{K}_n,\sigma_0,\sigma_1)}\sum_{i=1}^{n}\xi_i\sum_{l=1}^{\infty}a_{n,l}2^{-(l+1)/2}\|Pr_{m_l}f_i\|^2 \to \infty. \tag{86}$$

Let us consider sequences $\sigma_n \in (\sigma_0, \sigma_1),\ k_n \in \mathcal{K}_n$ and $\bar{f}_n = (\xi_1 f_{1,n}, \dots, \xi_1 f_{1,n}) \in F^n(\mathcal{K}_n, \sigma_0, \sigma_1)$ such that

$$\|f_{i,n}\|_p \geq r_n(k_n, \sigma_n),\ \|f_{i,n}\|_{\sigma_n,q} \leq 1,\ i = 1, \dots, n;\ \sum_{i=1}^n \xi_i = k_n \geq b_1 n^{\delta_1},\ \delta_1 > 1/2,$$

and $r_n(k_n, \sigma_n)/r_n^*(k_n, \sigma_n) \geq g_n^- \to \infty$. Let us take an integer $l = l_n$ such that (84) holds for $\sigma = \sigma_n,\ r_n^*(k_n) = r_n^*(k_n, \sigma_n)$. Using (16), (18) we have

$$l_n \asymp \log(r_n^*(k_n, \sigma_n))^{-1} \asymp G_n,\ a_{n,l_n} \asymp G_n^{-1/2}.$$

Using (44), for any $d \in (0,1)$ and large enough n one has

$$\|Pr_{m_l}(f_{i,n})\|_2 \geq \|Pr_{m_l}(f_{i,n})\|_p \geq d r_n(k_n, \sigma_n);\ i = 1, \dots, n$$

(compare with (49)). Using these relations and Remark 3.1 we get, as $g_n^- \to \infty$,

$$\begin{aligned}&\varepsilon^{-2} n^{-1/2} \sum_{i=1}^n \xi_i \sum_{l=1}^\infty a_{n,l} 2^{-(l+1)/2} \|Pr_{m_l} f_{i,n}\|^2 \\ &\geq d\varepsilon^{-2} n^{-1/2} k_n a_{n,l_n} 2^{-(l_n+1)/2} r_n^2(k_n, \sigma_n) \asymp (r_n(k_n, \sigma_n)/r_n^*(k_n, \sigma_n))^2 \to \infty,\end{aligned}$$

uniformly over $\bar{f} \in F^n(\mathcal{K}_n, \sigma_0, \sigma_1)$. These yield (86). □

5.2.3 Upper bounds for Theorem 5

By construction of $T_{n,l}$ we have

$$T_{n,l}^2 \geq m_l \quad \text{for } l \leq L_n, \quad T_{n,l}^2 \leq m_l(1 + o(1)) \quad \text{for } l > L_n.$$

Take constants $C > \sqrt{2}$ such that $A = C^2 b > 1$, where $b = b(1)$ is from Corollary 2. Using Corollary 2 we get, for some $A_1 \in (1, A)$ and large enough n,

$$\begin{aligned}&\alpha_n(\psi_{n,C}) \leq \sum_{l=1}^\infty \sum_{i=1}^n P_0(\chi_{m_l,i} > T_{n,l}) \\ &\leq n \sum_{1 \leq l \leq L_n} \exp(-A \log n) + n \sum_{L_n \leq l} \exp(-A_1 \log(nl)) \\ &\leq L_n n^{1-A} + n^{1-A_1} \sum_{L_n \leq l} l^{-A_1} \to 0.\end{aligned}$$

Let us consider sequences $\sigma_n \in (\sigma_0, \sigma_1),\ k_n \in \mathcal{K}_n$ and $\bar{f}_n = (\xi_1 f_{1,n}, \dots, \xi_1 f_{1,n}) \in F^n(\mathcal{K}_n, \sigma_0, \sigma_1)$ such that

$$\|f_{i,n}\|_p \geq r_n(k_n, \sigma_n), \quad \|f_{i,n}\|_{\sigma_n,q} \leq 1, \quad i = 1, \dots, n; \quad \sum_{i=1}^n \xi_i = k_n \geq 1,$$

and $r_n(k_n, \sigma_n)/r_n^*(\sigma_n) \geq g_n^- > c_1$. Next evaluations are uniform over k_n, σ_n. To simplify, suppose that the first cannel contains a signal, i.e., $\xi_1 = 1$. Clearly,

$$\beta_n(\psi_{n,C}, \bar{f}) \leq P_{\bar{f}}(\sup_{l>L} \chi_{m_l,1} > T_{n,l}) \leq \inf_{l>L} P_{\bar{f}}(\chi_{m_l,1} > T_{n,l}).$$

Using the relation (43) and estimation analogous to (50), we see that it suffices to verify the inequality: for large enough n and some $B > 1$, one has

$$\varepsilon^{-2} \max_{l>L} (2m_l)^{-1/2} T_{n,l}^{-1} \|Pr_{m_l}(f_1)\|_2^2 > B, \tag{87}$$

compare with (48), (51). Let us take an integer $l = l_n$ such that

$$m_{l_n-1} = 2^{l_n-1} \leq (r_n^*(\sigma_n))^{-1/\sigma_n} \leq m_{l_n} = 2^{l_n},$$

compare with (84). Using (44) one can take $d > 0$ such that

$$\|Pr_{m_{l_n}}(f_1)\|_2 \geq \|Pr_{m_{l_n}}(f_1)\|_p \geq dr_n(k_n, \sigma_n) \tag{88}$$

for large enough n, compare with (49).

First, suppose

$$r_n^*(\sigma_n) = \varepsilon\sqrt{\log n}, \quad \varepsilon^2(\log n)^{2\sigma_n+1} > 1. \tag{89}$$

It follows from (89) that

$$\log n > (r_n^*(\sigma_n))^{-1/\sigma_n} \geq m_l/2 \tag{90}$$

which yields $l_n \leq L_n,\ T_{n,l_n} = C^2 m_l^{-1/2} \log n$. Since $\varepsilon^{-2}(r_n^*(k_n, \sigma_n))^2 = \log n$, using (88) we get

$$\varepsilon^{-2}(2m_{l_n})^{-1/2} T_{n,l}^{-1} \|Pr_{m_{l_n}}(f_1)\|_2^2$$
$$\geq d^2 \varepsilon^{-2}(r_n^*(k_n, \sigma_n))^2 (\log n)^{-1} 2^{-1/2} (r_n(k_n, \sigma_n)/r_n^*(k_n, \sigma_n)C)^2 \geq (dc_1/C)^2 2^{-1/2},$$

which yields (87) for $(dc_1)^2 > C^2 B 2^{-1/2}$.

Next, let

$$r_n^*(\sigma_n) = \varepsilon^4 (\log(n \log \varepsilon^{-1}))^{\sigma_n/(4\sigma_n+1)}, \quad \varepsilon^2(\log n)^{2\sigma_n+1} \leq 1,$$

which yields

$$\varepsilon^2(\log(n \log \varepsilon^{-1}))^{2\sigma_n+1} \leq 1 + o(1), \quad \log n \leq (r_n^*(\sigma_n))^{-1/\sigma_n} \leq 2^{l_n}.$$

We have

$$l_n \geq L_{n,1} - 2\log_2 C, \quad l_n \asymp \log(r_n^*(\sigma_n))^{-1} \asymp \log \varepsilon^{-4} - \log\log(n \log \varepsilon^{-1}) \leq 4 \log \varepsilon^{-1},$$

which yields

$$T_{n,l}^2 \leq B_0 \log(nl_n) \leq B_1 \log(n \log \varepsilon^{-1}) \tag{91}$$

for some $B_0 > 0,\ B_1 > 0$ and large enough n. Using (88) we get

$$\varepsilon^{-2}(2m_{l_n})^{-1/2} T_{n,l}^{-1} \|Pr_{m_{l_n}}(f_1)\|_2^2 \geq d^2 \varepsilon^{-2} (2m_{l_n} C^{-1} \log(nl_n))^{-1/2} r_n^2(k_n, \sigma_n). \tag{92}$$

By the choice of $r_n^*(k)$ and m_{l_n} and taking into account (91), we see that the right-hand side of (92) is not smaller then $(dc_1)^2/C(2B_1)^{1/2}$. This yields (87) for $(dc_1)^2 > (2B_1)^{1/2} BC$. $\square$.

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