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# On a nonlocal model of image segmentation

*Dedicated to Klaus Kirchgässner on the occasion of his 70th birthday*

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**Abstract**

We understand an image as binary grey 'alloy' of a black and a white component and use a nonlocal phase separation model to describe image segmentation. The model consists in a degenerate nonlinear parabolic equation with a nonlocal drift term additionally to the familiar Perona-Malik model. We formulate conditions for the model parameters to guarantee global existence of a unique solution that tends exponentially in time to a unique steady state. This steady state is solution of a nonlocal nonlinear elliptic boundary value problem and allows a variational characterization. Numerical examples demonstrate the properties of the model.

# 1. Introduction

Image segmentation means recovering homogeneous image regions and contours or edges. Perona and Malik [16] (comp. also [13, 17] and the literature quoted therein) proposed the initial value problem

$$\frac{\partial u}{\partial t} = \nabla \cdot [f(|\nabla u|)\ \nabla u], \quad u(0,\cdot) = g(\cdot), \tag{1.1}$$

as a model of segmentation of an image represented by the initial value $g$. Here $f$ is a smooth nonincreasing function with $f(0) = 1$, $f(s) \geq 0$, $s \geq 0$, and $f(s) \to 0$ as $s \to \infty$. The idea is that the smoothing process generated by (1.1) is conditional: If $\nabla u(x)$ is large, then the diffusion will be low and the localized edges will be kept; if $\nabla u(x)$ is small, then diffusion tends to smooth still more around $x$.

In this paper we extend (1.1) by

$$\frac{\partial u}{\partial t} - \nabla \cdot [f(|\nabla v|)\ (\nabla u + \frac{\nabla w}{\phi''(u)})] = 0, \quad u(0,\cdot) = g(\cdot), \tag{1.2}$$

where

$$v = \phi'(u) + w, \quad w(t,x) = \int_\Omega \mathcal{K}(|x-y|)(1 - 2\,u(t,y))\,dy. \tag{1.3}$$

Here $\phi$ is a convex function, the kernel $\mathcal{K}$ represents nonlocal attracting forces and $v$ may be interpreted as chemical potential. Evidently, for $\phi = u^2/2$ and $\mathcal{K} = 0$ the system (1.2), (1.3) coincides with the Perona-Malik model (1.1).

The system (1.2), (1.3) has been studied recently in [11] and, for the special case $f = const.$, it was rigorously derived in [9] as a model of isothermal phase segregation. As to the non-isothermal case, we refer to [4].

This phase segregation model can be seen as a nonlocal variant of the Cahn-Hilliard [2] equation which is associated with the local Ginzburg-Landau free energy

$$F_{gl}(u) = \int_\Omega \left\{\phi(u) + \kappa u\,(1-\; u) + \frac{\lambda}{2}\left|\nabla u\right|^2\right\}\,dx. \tag{1.4}$$

It turns out that (1.2), (1.3) has the Lyapunov functional

$$F(u) = \int_\Omega \left\{\phi(u) + u \int_\Omega \mathcal{K}(|x-y|)(1-\; u(y))\,dy\right\}\,dx. \tag{1.5}$$

This free energy may be written in a form more similar to (1.4):

$$F(u) = \int_\Omega \left\{\phi(u) + \kappa u(1-2\,u) + \frac{1}{2}\int_\Omega \mathcal{K}(|x-y|)|u(x)-u(y)|^2\,dy\right\}\,dx, \tag{1.6}$$

where

$$\kappa = \kappa(x) = \int_\Omega \mathcal{K}(|x-y|)\;dy.$$

Since $F(u(t))$ decreases along solutions $(u,v)$ of (1.2), (1.3), it becomes clear that the nonlocal term tends to smooth $u$ outside the edges which have to be enhanced by the function $f$.
In view of the standard Mumford-Shah [14] variational model of image segmentation it seems reasonably to incorporate a third term forcing $u$ to remain close to $g$. So Nordstroem [15] supplemented Perona-Maliks model with the term $u-g$. Following this approach we arrive at our final evolution model of image segmentation:

$$\frac{\partial u}{\partial t} - \nabla\cdot\left[f(|\nabla v|)\Big(\,\nabla u + \frac{\nabla w}{\phi''(u)}\Big)\right] + \beta(u-g) = 0, \quad u(0,\cdot) = g(\cdot), \tag{1.7}$$

$$v = \phi'(u) + w, \quad w(t,x) = \int_\Omega \mathcal{K}(|x-y|)(1-2\,u(t,y))\,dy. \tag{1.8}$$

In terms of $v, w$ (1.7) reads

$$\frac{\partial\phi'^{-1}(v-w)}{\partial t} - \nabla\cdot[\mu(v,w)\nabla v] + \beta(\phi'^{-1}(v-w)-g) = 0, \tag{1.9}$$

$$\phi'^{-1}(v(0,\cdot)-w(0,\cdot)) = g(\cdot),$$

where

$$\mu(v,w) = \varrho(v-w)\;f(|\nabla v|), \quad \varrho = (\phi'^{-1})' = \frac{1}{\phi''\circ\phi'^{-1}}. \tag{1.10}$$

**Remark 1.1.** The system (1.7)-(1.8) may be thermodynamically explained as follows: The segmentation process $u$ minimizes the free energy $F$ under the constraint of mass conservation $\int_\Omega (g-u)\;dx = 0$, or, accordingly to Lagrange's method, the augmented functional $F(u)+\int_\Omega v\;(g-u)\;dx$. The associated Euler-Lagrange equation $v = \partial_u\,F$ coincides with (1.8). Thus the Lagrange multiplier $v$ can be interpreted as chemical potential. Consequently, $-\nabla v$ and $u-g$ should be assumed to be driving forces for the evolution of u towards the desired segmentation of $g$. That leads to (1.7) and closes the explanation.

We complete (1.7) (resp. (1.9)) by homogeneous Neumann boundary conditions. Double degenerated nonlinear parabolic equations like (1.9) with $w = 0$ under Dirichlet conditions have been studied in [8].

In the next section we formulate our assumptions with respect to the image $g$ and the parameter functions $f$, $\phi$, $\mathcal{K}$. Section 3 contains a precise formulation of the problem and a global existence and uniqueness result. The main result, exponential decay in time $t$ of the transient solution $(u(t), v(t), w(t))$ of (1.7), (1.8) to a unique steady state $(u^*, v^*, w^*)$, is proved in Section 4. Section 5 contains a model specification. A discrete version of (1.7),(1.8) is introduced in Section 6. In the final Section 7 some numerical examples are given.

## 2. Assumptions

Let be $\Omega \subset \mathbb{R}^n$ a bounded Lipschitzian domain. Denote by: $L^p = L^p(\Omega)$, $H^{1,p} = H^{1,p}(\Omega)$ , $1 \le p \le \infty$ , the usual function spaces on $\Omega$ , $H^1 = H^{1,2}(\Omega)$ , $\|\cdot\|_2 = \|\cdot\|$ the norm in $L^2$ and $(\cdot,\cdot)$ the pairing between $H^1$ and its dual $(H^1)^*$ ([12],[6]). For a time interval $(0,T)$ , $T > 0$ , and a Banach space $X$ we denote by $L^p(0,T;X)$ the usual spaces ([12, 6]) of Bochner integrable functions with values in $X$ . We set $\mathbb{R}^1_+ = (0,\infty)$ and $Q = (0,T) \times \Omega$ . "Generic" positive constants are denoted by $C$ .

We formulate two groups $(A)$ and $(B)$ of assumptions. Assumptions $(A)$ hold globally throughout the paper, whereas assumptions $(B)$ will be actualized locally.

$(A_1)$ $\quad g \in L^\infty, \quad 0 \le g \le 1, \quad \phi'(g) \in L^1;$

$(A_2)$ $\quad f \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1_+)$ is continuous such that for constants $0 < \delta \le M$

$$(f(s_1)s_1 - f(s_2)s_2)\ (s_1 - s_2) \ge \delta(s_1 - s_2)^2, \quad f(s) \le M, \quad \forall s, s_1, s_2 \in \mathbb{R}^1_+;$$

$(A_3$ $\quad \phi(u) \in C[0,1] \cap C^3(0,1), \quad = \phi'^{-1} \in (\mathbb{R}^1 \mapsto [0,1])\ , \quad \forall s, s_1, s_2 \in (0,1):$

*i)* $\quad \phi''(s) \ge \alpha = const. > 0,$

*ii)*
$$\alpha_2 \left|\frac{1}{\phi''(s_1)} - \frac{1}{\phi''(s_2)}\right|^2 \le \frac{2}{\phi''(\frac{s_1+s_2}{2})} - \frac{1}{\phi''(s_1)} - \frac{1}{\phi''(s_2)} \le \alpha_1(s_1 - s_2)^2,$$

*iii)* $\quad \psi_1(s) = s\phi'(s)$ and $\psi_2(s) = (s-1)\phi'(s)$ satisfy

$$\psi_j(s_1) + \psi_j(s_2) - 2\ \psi_j((s_1 + s_2)/2) \ge \eta(s_1 - s_2)^2, \quad \eta = const. > 0;$$

$(A_4)$
$$\mathcal{K} \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1_+), \quad k_0 := \sup_{x\in\Omega} \int_\Omega \mathcal{K}(|x-y|)dy < \infty,$$

$u \mapsto P\,u = \int_\Omega \mathcal{K}(|x-y|)u(y)\ dy$ satisfies

$$\|Pu\|_{H^{1,p}} \le k_p \|u\|_p, \ 1 \le p \le \infty.$$

**Remark 2.1.** Assumption $(A_3)$ implies that $\log\varrho$ is concave, i. e., that $\frac{\varrho'}{\varrho}$ is nondecreasing, where $\varrho = \frac{1}{\phi''\circ\phi'^{-1}}$ . That is the main structural assumption made in [7, 8] for analyzing equations like (1.9) in the case $w = 0$.

$(B_1)$ $\quad s \mapsto (s-r)\phi'(s)$ is nondecreasing with respect to $s \in [0,1]$ $\forall r \in [0,1]$;

$(B_2)$ $\quad \gamma = \beta\eta - \dfrac{M}{4\alpha}\Big(2\alpha_1 k_\infty^2 + \dfrac{k_2^2}{\alpha_2}\Big) > 0$ ;

$(B_3)$ $\quad F$ from (1.5) is strongly convex such that $\forall u_1, u_2 \in L^2$

$$F(u_1) + F(u_2) - 2F(\frac{u_1+u_2}{2}) \geq a\|u_1 - u_2\|^2, \quad a = const. > 0.$$

**Remark 2.2.** Assumption $(B_3)$ follows from (1.6) and $(A_3 i)$ provided $a = \alpha - 8k_0 > 0$. We give relevant examples for functions $f,\ \phi,\ \mathcal{K}$ satisfying these assumptions :

$$\begin{aligned}
f(s) &= \frac{1+\delta s}{1+s} \text{ fulfills } (A_2) \text{ with } M=1;\\
\phi(s) &= s\log s + (1-s)\log(1-s), \quad \phi'(s) = \log\frac{s}{1-s}, \quad \text{fulfills } (A_3) \text{ and } (B_1) \text{ with}\\
\phi''(s) &= \frac{1}{s(1-s)}, \quad \phi'^{-1}(r) = \frac{1}{1+e^{-r}}, \quad \varrho(r) = \frac{1}{2(1+\cosh r)} \quad \text{and}\\
\alpha &= 4, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{18}, \quad \eta = \frac{27}{16}.
\end{aligned}$$

$(A_4)$ holds for Newton potentials

$$\mathcal{K}(s) = \kappa_n s^{2-n}, \quad n \neq 2; \quad \mathcal{K}(s) = -\kappa_2 \log s, \quad n = 2; \quad \kappa_n = \text{ const. } > 0,$$

Functions like $\mathcal{K}(s) = c\, e^{-\frac{s^q}{\sigma}}$, $1 \leq q < \infty$ and usual mollifiers

$$\mathcal{K}(s) = c\, \exp\big(-\frac{\lambda^2}{\lambda^2 - s^2}\big), \quad \text{if } s < \lambda, \quad \mathcal{K}(s) = 0, \quad \text{if } s \geq \lambda, \quad C,\ \lambda > 0.$$

# 3. Existence and uniqueness of transient solutions

In this section we prove existence and uniqueness of solutions in the sense of

**Definition 3.1.** A triple $(u, v, w)$ is called (weak) solution of (1.7)–(1.8) (completed by homogeneous Neuman boundary conditions) if:

$$u \in C(0,T;L^\infty) \cap L^2(0,T;H^1) \quad , \quad u_t \in L^2(0,T;(H^1)^*), \quad u(0) = g \ ,$$

$$\int_0^T \int_\Omega \mu(v,w)|\nabla v|^2 \, dxdt < \infty \quad , \quad w \in C(0,T;H^{1,\infty}) \ ,$$

$$v = \phi'(u) + w, \quad w(t,x) = \int_\Omega \mathcal{K}(|x-y|)(1 - 2\,u(t,y))\,dy, \tag{3.1}$$

$$\int_0^T \{(u_t, h) + \int_\Omega [\mu(v,w)\nabla v \cdot \nabla h + \beta(u-g)\ h]\ dx\}dt = 0 \ \ \forall h \in L^2(0,T;H^1). \tag{3.2}$$

We define by

$$(A(v,w),h) = \int_\Omega \mu(v,w)\ \nabla v \cdot \nabla h\ dx, \quad \forall h \in H^1, \ \ \mu(v,w) = \frac{f(|\nabla v|)}{\phi''(\phi'^{-1}(v-w))}$$

an operator $A \in (D(A) \mapsto (H^1)^*)$, where

$$D(A) = \{(v,w) : \int_\Omega \mu(v,w)|\nabla v|^2\ dx < \infty,\ w \in H^1\}.$$

The following monotonicity property of $A$ (comp. [3]) is the main tool for proving uniqueness and asymptotic results.

**Lemma 3.2**. *Let*

$$(v_i,w_i) \in D(A), \quad u_i = \phi'^{-1}(w_i - v_i), \quad i=1,2, \quad u_m = \frac{u_1+u_2}{2}, \quad w_m = \frac{w_1+w_2}{2}.$$

*Then*

$$\begin{aligned} (i)\ d_1 :&= \sum_{i=1,2} (A(v_i,w_i), v_i - \phi'(u_m) - w_m) \\ &\geq -\frac{M}{4\alpha}(2\alpha_1 k_\infty^2 \|u_1-u_2\|^2 + \frac{1}{\alpha_2}\|\nabla(w_1-w_2)\|^2), \\ (ii)\ \ d_2 :&= \sum_{i=1,2} (u_i - g, v_i - \phi'(u_m) - w_m) \geq \eta \|u_1-u_2\|^2. \end{aligned}$$

*Proof.* Set $\phi_i' = \phi'(u_i)$, $\phi_i'' = \phi''(u_i)$, $\varrho_i = \frac{1}{\phi_i''}$, $f_i = f(|\nabla v_i|)$, $\mu_i = \varrho_i f_i$, $i=1,2,m$, $K = 2\varrho_m - \varrho_1 - \varrho_2$.
$(i)$ we have

$$\begin{aligned} d_1 &= \int_\Omega \sum_{i=1,2} \mu_i \nabla v_i \cdot \nabla(v_i - \phi_m' - w_m)\ dx \\ &= \int_\Omega \sum_{i=1,2} \Big\{\mu_i \nabla v_i \cdot [\nabla(v_i - w_m) - \frac{\phi_m''}{2}\sum_{j=1,2}\varrho_j \nabla(v_j - w_j)]\Big\}\ dx \\ &= \int_\Omega \sum_{i=1,2} \Big\{\mu_i \nabla v_i \cdot [\nabla v_i - \frac{\phi_m''}{2}\sum_{j=1,2}(\varrho_j \nabla v_j - (\varrho_j - \varrho_m)\nabla w_j)]\Big\}\ dx \\ &= \int_\Omega \frac{\phi_m''}{2}\Big\{K\sum_{i=1,2}\mu_i|\nabla v_i|^2 + \varrho_1\varrho_2(f_1\nabla v_1 - f_2 \nabla v_2)\cdot\nabla(v_1-v_2) \\ &+ \sum_{i=1,2}(\varrho_i - \varrho_m)\nabla w_i \cdot \sum_{i=1,2}\mu_i \nabla v_i\Big\}\ dx \quad \geq \quad \int_\Omega \frac{\phi_m''}{2}\Big\{K\sum_{i=1,2}\mu_i|\nabla v_i|^2 \\ &+ \frac{1}{2}[(\varrho_1-\varrho_2)\nabla(w_1-w_2) + K\nabla(w_1+w_2)]\cdot \sum_{i=1,2}\mu_i\nabla v_i\Big\}dx \\ &\geq \int_\Omega \frac{\phi_m''}{2}\Big\{K\sum_{i=1,2}\mu_i|\nabla v_i|^2 - \frac{1}{2}\sum_{i=1,2}\mu_i[\alpha_2|\varrho_1-\varrho_2|^2|\nabla v_i|^2 + \frac{1}{4\alpha_2}|\nabla(w_1-w_2)|^2] \\ &+ -\frac{K}{2}\sum_{i=1,2}\mu_i[|\nabla v_i|^2 + \frac{|\nabla(w_1+w_2)|^2}{4}]\Big\}dx \\ &\geq -\frac{M}{4\alpha}\Big(2\alpha_1 k_\infty^2\|u_1-u_2\|^2 + \frac{1}{\alpha_2}\|\nabla(w_1-w_2)\|^2\Big). \end{aligned}$$

$(ii)$ Denoting by $z_+ = \frac{1}{2}(|z|+z)$ the positive part of a function $z$, we get

$$\begin{aligned} d_2 &= \int_\Omega \sum_{i=1,2} [\psi_1(u_i) - \psi_1(u_m) - g(\phi_i' - \phi_m')] \; dx \\ &\geq \int_\Omega \Big\{ \sum_{i=1,2} [\psi_1(u_i) - \psi_1(u_m)] - g[\phi_1' + \phi_2' - 2\phi_m']_+ \Big\} \; dx \\ &\geq \int_\Omega \Big\{ \sum_{i=1,2} [\psi_1(u_i) - \psi_1(u_m)] - [\phi_1' + \phi_2' - 2\phi_m']_+ \Big\} \; dx \\ &\geq \int_\Omega \min_{j=1,2} \Big\{ \sum_{i=1,2} [\psi_j(u_i) - \psi_j(u_m)] \Big\} \; dx \geq \eta \|u_1 - u_2\|^2. \quad \square \end{aligned}$$

The key for proving existence is the Lyapunov property (decreasing in time) of the functional

$$F(t,u) = F(u) + \beta \int_0^t (u-g, v) \; ds \; ,$$

where $F(u)$ is the free energy functional given by (1.5).

**Lemma 3.3**. *Let $(u,v,w)$ be a solution to (1.7)–(1.8). Then*

$$(i) \qquad \frac{dF(t,u)}{dt} = -\int_\Omega \mu(v,w) \; |\nabla v|^2 \; dx < 0,$$

$$(ii) \qquad \alpha\delta \|\nabla u\|^2_{L^2(Q)} \leq \int_0^T \int_\Omega \mu(v,w) \; |\nabla v|^2 \; dx \leq C(T).$$

*Proof.* $(i)$ Using (3.1), (3.2) we obtain

$$\frac{dF(t,u)}{dt} = (u_t, \phi'(u) + w) + \beta(u-g, v) = -\int_\Omega \mu(v,w) \; |\nabla v|^2 \; dx.$$

$(ii)$ We have

$$F(u) \geq \int_\Omega [u(1-\frac{1}{u}) + (1-u)(1-\frac{1}{1-u})] \; dx = -|\Omega|$$

and by the monotonicity of $\phi'$

$$\begin{aligned} \int_0^t (u-g,v) \; ds &= \int_0^t (u-g, \phi'(u) - \phi'(g) + \phi'(g) + w) \; ds \\ &\geq \int_0^t (u-g, \phi'(g) + w) \; ds \geq -t(\|\phi'(g)\|_1 + k_0|\Omega|). \end{aligned}$$

Hence, using assumption $(A_1)$, we get

$$\begin{aligned} \alpha\delta\|\nabla u\|^2_{L^2(Q)} &\leq \int_0^T \int_\Omega \phi''(u) f(|\nabla v|) \; |\nabla u|^2 \; dx \; dt = \int_0^T \int_\Omega \mu(v,w) \; |\nabla v|^2 \; dx \; dt = \\ -\int_0^T d \; F &= -F(T, u(T)) + F(g) \leq |\Omega|(1 + \beta T k_0) + \beta T \|\phi'(g)\|_1 + F(g). \quad \square \end{aligned}$$

**Proposition 3.4**. *The problem* (1.7), (1.8) *has a unique solution* $(u, v, w)$.

*Proof.* In [11] existence and uniqueness of a solution to problem (1.7), (1.8) was proved for the case that $\phi(s) = s\log s + (1-s)\log(1-s)$ and the term $\beta(u-g)$ is cancelled. But Lemma 3.2 and Lemma 3.4 allow us to carry over that proof to our situation. □

To end this section we prove a regularity result for the $v$ component of the solution to (1.7), (1.8).

**Proposition 3.5**. *Let assumption* $(B_1)$ *be satisfied and let*

$$\phi'(g) \in L^\infty, \quad |\varrho'(s)| \le \varrho_0 |\varrho''(s)|, \quad s \in \mathbb{R}_1. \tag{3.3}$$

*Then* $v \in C(0;T;L^\infty) \cap L^2(0,T;H^1)$.

*Proof.* Note that by (3.1), $(A_4)$ and (3.3)

$$\|w\|_{L^\infty(Q)} \le k_0, \quad v(0) = \phi'(g) + w(0) \in L^\infty. \tag{3.4}$$

So, by Remark 2.1, we can choose a $m \ge 0$ such that $\varrho'(v-w) < 0$ if $v > m$. We test (3.2) with

$$h = \phi''(u)\varphi^r, \quad r \ge 1, \quad \varphi = [v-m]_+ = \max(0, v-m).$$

Then, proceeding as in [11], we get the estimate

$$\begin{aligned} \frac{1}{(r+1)}\frac{d}{dt}\int_{v\ge w}\varphi^{r+1}dx \;&\le\; -\frac{2\delta r}{(r+1)^2}\int_\Omega |\nabla(\varphi^{\frac{r+1}{2}})|^2 dx \\ &+\; \frac{Mr_\infty^2}{4}\int_\Omega \left\{2r\varphi^{r-1} + \varrho_0\varphi^r\right\}dx + \beta\int_\Omega \phi''(u)(g-u)\varphi^r dx. \end{aligned}$$

By $(B_1)$ we have

$$\phi''(u)(g-u) = \phi''(\phi'^{-1}(v-w))\,(g - \phi'^{-1}(v-w)) \le \phi''(\phi'^{-1}(0))\,(g-\phi'^{-1}(0)) \le C.$$

and therefore by Young's inequality

$$\frac{d}{dt}\int_\Omega \varphi^{r+1}dx + \delta\int_\Omega |\nabla(\varphi^{\frac{r+1}{2}})|^2 dx \le Cr\Big(1 + r\int_\Omega \varphi^{r+1}dx\Big). \tag{3.5}$$

Hence for $r = 2$ we conclude by (3.3) and Gronwall's lemma

$$\|\varphi\|_{L^\infty(0,T;L^2)} \le C(T).$$

Using this and (3.5) we can apply Alikakos' technique (comp. [1]) to prove that

$$\|\varphi\|_{L^\infty(Q)} \le C(T),$$

which gives an upper bound for $v$. Analogously, from (3.2) with the test function

$$h = \phi''(u)\psi^r, \quad r \geq 1, \quad \psi = -\min(0, v-m), \quad m \text{ such that } \varrho'(v-w) > 0 \text{ if } v < m,$$

and (3.4) we get a lower bound for $v$. Now the assertion follows from (1.8) and Lemma 3.2. □

**Remark 3.1.** For the reference convex function $\phi(s) = s \log s + (1-s)\log(1-s)$ the second part of condition (3.3) holds with $\varrho_0 = 1$.

# 4. Global behaviour in time

In this section we study the asymptotic behavior of the solution $(u(t), v(t), w(t))$ to (1.7), (1.8) as $t \to \infty$.

**Definition 4.1.** A triple $(u^*, v^*, w^*)$ is called steady state (of (1.7)–(1.8)) if:

$$u^*, v^* \in H^1 \cap L^\infty, \quad w^* \in H^{1,\infty}, \tag{4.1}$$

$$v^* = \phi'(u^*) + w^*, \quad w^*(x) = \int_\Omega \mathcal{K}(|x-y|)(1 - 2\,u^*(y))\,dy, \tag{4.2}$$

$$\int_\Omega \Big\{\mu(v^*, w^*)\nabla v^* \cdot \nabla h + \beta(u^* - g)\ h]\ dx\Big\}dt = 0 \quad \forall h \in H^1. \tag{4.3}$$

**Proposition 4.2**. *Suppose* $(B_1, B_2)$. *Then a unique steady state exists.*

*Proof.* **(Uniqueness:)** Let $(u_i^*, v_i^*, w_i^*)$, $i = 1, 2$, be steady states. Then, using (4.3) and Lemma 3.2, we get

$$\gamma\|u_1^* - u_2^*\|^2 \leq \sum_{i=1,2} \Big(A(v_i^*, w_i^*) + u_i^* - g, v_i^* - \phi'(\frac{u_1^* + u_2^*}{2}) - \frac{w_1^* + w_2^*}{2}\Big) = 0.$$

Thus $(B_2)$ implies $u_1^* = u_2^*$. By (4.2) this means $w_1^* = w_2^*$ and $v_1^* = v_2^*$.
**(Existence:)** Let $u \in L^\infty$ be given, set $w = w(u) = P\ (1-2u) \in H^{1,\infty}$ (comp. $A_4$). Let further $v(u) \in H^1 \cap L^\infty$ be the solution of the equation

$$A(v, w) + \phi'^{-1}(v - w) = g.$$

(Existence and uniqueness of $v(u)$ can be proved analogously as in [7], where the case $w = 0$ has been considered. The key is a counterpart to the $\|v\|_\infty$ estimate given by Theorem 2 in [7], which can be deduced via Moser iteration (comp. [10]) by using the test functions

$$h = \frac{[v-m]_+^r}{\varrho(v-w)} \int_m^{v-w} \varrho(s)\ ds, \quad r \geq 0,$$

where $m \geq 0$ is chosen such that $\varrho'(v-w) < 0$ if $v > m$.)
Now we define the operator $u \mapsto S\ u = \phi'^{-1}(v(u) - w)$. Then for $u_i \in L^\infty,\ i = 1, 2,$ Lemma 3.2 implies

$$
\begin{aligned}
0 &= \sum_{i=1,2} \left\{ A(v_i, w_i) + Su_i - g, v_i - \phi'(\frac{u_1 + u_2}{2})) \right\} \\
&\geq -\frac{M}{4\alpha}(2\alpha_1 k_\infty^2 \|Su_1 - Su_2\|^2 + \frac{1}{\alpha_2}\|\nabla(w_1 - w_2)\|^2) + \beta\eta\|Su_1 - Su_2\|^2.
\end{aligned}
$$

Hence, using $(A_4)$, we get

$$
(\beta\eta - \frac{M\alpha_1 k_\infty^2}{2\alpha})\|Su_1 - Su_2\|^2 \leq \frac{Mk_2^2}{4\alpha\alpha_2}\|u_1 - u_2\|^2,
$$

and by $(B_2)$

$$
\|Su_1 - Su_2\|^2 \leq k\|u_1 - u_2\|^2, \quad k = \frac{k_2^2 M}{2\alpha_2(2\alpha\beta\eta - M\alpha_1)k_\infty^2} < 1. \tag{4.4}
$$

So $S \in (L^2 \mapsto L^2)$ is strictly contractive and has by Banach's theorem a fixed point $u^*$, which can be completed by $w^*$ and $v^*$ according to (4.2) to a steady state. □

**Remark 4.1.** Since operator $S$ is contractive by (4.4), the sequence $(u_i, v_i, w_i)$ defined by

$$
\begin{aligned}
w_i &= P\ (1 - 2u_{i-1}), \\
\int_\Omega \{\varrho(v_i - w_i) f(|\nabla v_i|)\nabla v_i \cdot \nabla h + \beta(\phi'^{-1}(v_i - w_i) - g)h\}\ dx &= 0\ \forall h \in H^1, \\
u_i &= \phi'^{-1}(v_i - w_i), \quad i = 1, 2, \ldots, \ u_0 = g,
\end{aligned} \tag{4.5}
$$

converges to the steady state $(u^*, v^*, w^*)$.

The steady state allows a variational characterization:

**Proposition 4.3**. *$(u^*, v^*, w^*)$ with (4.1) is steady state if and only if $u^*$ is minimizer of*

$$
\min_u \int_\Omega \left\{ \phi(u) + u \int_\Omega \mathcal{K}(|x - y|)(1 - u(y))\ dy + (g - u)\ v^* \right\}\ dx, \tag{4.6}
$$

*and, simultaneously, $v^*$ is minimizer of*

$$
\min_v \int_\Omega \left\{ \frac{1}{2}\phi''(u^*) \int_0^{|\nabla v|^2} f(\sqrt{s})\ ds + \beta(u^* - g)\ v \right\}\ dx. \tag{4.7}
$$

*Proof.* It is easy to check, that (4.2), (4.3) is the Euler-Lagrange system to (4.6), (4.7). Hence the assertion follows from the monotonicity of the functions $\phi'$ and $s \mapsto f(s)s$ (comp. $(A2), (A_3 i)$). □

**Remark 4.2.** Proposition 4.3 suggests the following iteration procedure for approximating the steady state:

$$\int_\Omega \Big\{\phi''(u_{i-1})f(|\nabla v_i|)\nabla v_i \cdot \nabla h + (u_{i-1} - g)\, h\Big\}\, dx = 0 \quad \forall h \in H^1, \tag{4.8}$$

$$\phi'(u_i) + P\ (1 - 2u_i) = v_i, \quad i = 1, 2, \ldots, \quad u_0 = g. \tag{4.9}$$

Note that the steps (4.8), (4.9) correspond to the minimum problems (4.7), (4.6), respectively. This procedure can be seen as natural alternative to (4.5). But, unfortunately, we cannot prove its convergence to $(u^*, v^*, w^*)$.

Now we are ready to prove our main result, convergence of an 'evolutionary' procedure for constructing the steady state.

**Theorem 4.4**. *Suppose* $(B_1 - B_3)$. *Let be* $(u, v, w)$ *the solution to* (1.7), (1.8) *and let be* $(u^*, v^*, w^*)$ *the steady state. Then*

$$\|u(t) - u^*\|^2 \leq \frac{e^{-\frac{\gamma t}{a}}}{\gamma}\Big(F(g) + F(u^*) - 2F(\frac{g + u*}{2})\Big). \tag{4.10}$$

*Proof.* Lemma 3.2 with $u_1 = u(t)$ and $u_2 = u^*$ yields

$$\frac{d\Big(F(u) + F(u^*) - 2F(\frac{g+u^*}{2})\Big)}{dt} \quad = \quad \Big(u_t, \phi'(u) + w - \phi'(\frac{u + u^*}{2}) - \frac{w + w^*}{2}\Big)$$

$$\begin{aligned}
&= -\Big(A(v, w) + u - g, v - \phi'(\frac{u + u^*}{2}) - \frac{w + w^*}{2}\Big)\\
&= -\Big(A(v, w) + u - g, v - \phi'(\frac{u + u^*}{2}) - \frac{w + w^*}{2}\Big)\\
&\quad -\Big(A(v^*, w^*) + u^* - g, v^* - \phi'(\frac{u + u^*}{2}) - \frac{w + w^*}{2}\Big)\\
&\leq \frac{M}{4\alpha}(2\alpha_1\|u - u^*\|^2 + \frac{1}{\alpha_2}\|\nabla(w - w^*)\|^2) - \beta\eta\|u - u^*\|^2 \leq -\gamma\|u - u^*\|^2.
\end{aligned}$$

Thus by $(B_3)$ we get

$$a\|u(t) - u^*\|^2 + \gamma \int_0^t \|u(s) - u^*\|^2\, ds \leq F(g) + F(u^*) - 2F(\frac{g + u^*}{2}).$$

This implies (4.10). □

# 5. Specification of the nonlocal operator $P$ and identification of its parameters

In the following we use

$$\phi(s) = s\log s + (1-s)\log(1-s),$$

$$f(s) = f_0\frac{1+\delta s/s_0}{1+s/s_0}, \quad s_0 = \frac{||\nabla w||_1}{|\Omega|},$$

and specify the operator $P$ as

$$\varrho \to P\varrho = w, \tag{5.1}$$

where $w$ is the solution of the boundary value problem

$$F_1(u,w) = -\sigma_0^2\nabla\cdot\nabla w + w = m_0\varrho, \quad \nu\cdot\nabla w = 0 \ \text{ on } \ \partial\Omega. \tag{5.2}$$

**Remark 5.1.** In the case that $\dim\Omega = 1$ the operator $P$ is related to the kernel

$$\mathcal{K}(|x-y|) = \frac{m_0}{2}e^{-\frac{|x-y|}{\sigma_0}}.$$

Thus $m_0 > 0$ can be interpreted as an attractive force with a range of interaction $\sigma_0$.

For segmentation we choose $\beta = 0$. For image reconstruction we fix $\beta$ accordingly to the estimate ($B_2$) and Remark 2.2 as

$$\beta = \frac{M}{4\alpha\eta}\Big(2\alpha_1 k_\infty^2 + \frac{k_2^2}{\alpha_2}\Big),$$

where in case of Remark 5.1 $k_2 = k_\infty = m_0/(2\sigma_0)$ holds.
A proper choice of the parameters $m_0$, $\sigma_0$ should detect the amount of noise in $g$ and ensure desirable properties of $u$ as smoothness, low entropy $(-\phi(u))$ and small distance to $g$. We try to meet these requirements by chosing $(m_0,\sigma_0)$ to be minimizers of the functional

$$F(m,\sigma) = f_0 R(u) + \frac{\lambda_1}{R(g)}\frac{||u-g||^2}{||u-\bar{g}||^2}. \tag{5.3}$$

Here

$$R(u) = \frac{||\nabla u||^2}{||u-\bar{g}||^2}, \quad u = \phi'^{-1}(v-w) = \frac{1}{1+e^{w-v}},$$

$w$ solves

$$-\sigma^2\Delta w + w = m(1-2g) \ \text{ in } \ \Omega, \ \ \nu\cdot\nabla w = 0 \ \text{ on } \ \partial\Omega,$$

and the constant $v$ is determined such that $\bar{u} = \bar{g}$. Finally $\lambda_1$ is the first (nontrivial) eigenvalue of

$$-\Delta z = \lambda z \ \text{ in } \ \Omega, \ \ \bar{z} = 0, \ \ \nu\cdot\nabla z = 0 \ \text{ on } \ \partial\Omega.$$

**Remark 5.2.** For the special choice $m = 2$, $\sigma = 0$ the function $u = u(m, \sigma) = u(2, 0)$ is near to $g$ in the sense that

$$|u(x) - g(x)| \leq \frac{1}{6}, \quad x \in \Omega. \tag{5.4}$$

Indeed, since the Fermi–function $\varphi(s) = \frac{1}{1+e^{-s}}$ satisfies

$$\begin{aligned} 0 \leq \varphi'(s) &= \varphi(s)(1-\varphi(s)) \leq \varphi'(0) = \frac{1}{4}, \\ \varphi''(s) &= \varphi(1-\varphi)(1-2\varphi), \quad \varphi''(0) = 0, \\ -\frac{1}{8} = \varphi'''(0) \leq \varphi'''(s) &= \varphi(1-\varphi)(1-6\varphi(1-\varphi)) \leq \varphi'''\Big(\varphi^{-1}\big(\frac{1}{2} \pm \sqrt{3/20}\big)\Big) = \frac{1}{25}, \end{aligned}$$

its Taylor expansion yields for a suitable $\theta \in [0,1]$

$$\varphi(s) = \frac{1}{2} + \frac{1}{4}s + \frac{1}{6}\varphi'''(\theta s)s^3.$$

Setting $s = v - w$, $v = 0$, $w = m(1-2g) = 2(1-2g)$, we find

$$u = \varphi(-w) = g - \frac{1}{6}\varphi'''(\theta s)w^3.$$

Because of $0 \leq g \leq 1$ this implies (5.4).
Moreover, in the special case $\bar{g} = \frac{1}{2}$ we have

$$u - \bar{g} = \Big(g - \frac{1}{2}\Big)\Big(1 + \frac{4}{6}\varphi'''(\theta s)w^2\Big) \geq \frac{2}{3}\Big(g - \frac{1}{2}\Big) = \frac{2}{3}\big(g - \bar{g}\big)$$

and hence

$$\begin{aligned} F(2,0) &= \frac{1}{||u-\bar{g}||^2}\Big[f_0||\varphi(1-\varphi)\nabla w||^2 + \frac{\lambda_1}{R_g}||u-g||^2\Big] \\ &\leq \frac{1}{||u-\bar{g}||^2}\Big[f_0||\nabla g||^2 + \frac{\lambda_1|\Omega|}{36R(g)}\Big] \leq \frac{3}{2}\Big[f_0 R(g) + \frac{\lambda_1|\Omega|}{36||\nabla g||^2}\Big]. \end{aligned}$$

In view of (5.4) the minimization process can be considered as an image preprocessing step.

Accordingly we replace $g$ by

$$g_\lambda = \lambda g + (1-\lambda)u(m_0, \sigma_0), \quad \lambda = \min\Big(1, \frac{F(m_0,\sigma_0)}{f_0 R(g)}\Big),$$

where $u = u(m_0, \sigma_0)$ corresponds to a minimizer of (5.3).

## 6. Discretization

With the specification (5.1) the original system (1.7), (1.8) reads

$$F_1(u,w) = -\sigma_0^2\nabla\cdot\nabla w + w + m_0(2u-1) = 0, \quad \nu\cdot\nabla w \;\text{ on }\; \partial\Omega, \tag{6.1}$$

$$F_2(u,w,t) = \frac{\partial u}{\partial t} - \nabla\cdot f(|\nabla v|)(\nabla u + u(1-u)\nabla w) + \beta(u-g_\lambda) = 0. \tag{6.2}$$

For time discretization Euler's backward scheme is adequate. Regarding space, $F_1$ is discretized in standard way (finite volume scheme, for notations see [5]) but in $F_2$ both $f(s)$ and the term $u(1-u)$ degenerate. These singularities must be preserved during any averaging procedure (i.e. the symmetry with respect to zero and one) applied in the discretization process. For instance the following three options are seen:

a) Scharfetter-Gummel discretization (constant total current along an edge, yields a Ricatti equation per edge);

b) $u(1-u)|_{ij} = e^{ln(u(1-u))}|_{ij} \approx e^{\frac{1}{2}(ln(u_i(1-u_i))+ln(u_j(1-u_j)))} = (u_i(1-u_i)u_j(1-u_j))^{\frac{1}{2}} := d_{ij,b}$, here $\psi(u)|_{ij}$ denotes an intermediate value of $\psi$ with respect to $u(x_i)$ and $u(x_j)$;

c) inverse averaging $u(1-u)$ (analogously to any diffusion coefficient) $d_{ij,c} := 2d_id_j/(d_i+d_j)$, $d_i = u_i(1-u_i)$.

$$-\nabla\cdot\nabla \approx G^TG,$$
$$G = [\sigma_\Delta]\tilde{G}_\Delta,$$
$$\tilde{G}_\Delta = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Here $[\cdot]$ denotes a diagonal matrix, especially $[\sigma_\Delta]^2$ is the diagonal matrix of Voronoi surfaces divided by the edge lengthes of the simplex. $\tilde{G}$ maps vertices to edges. $\tilde{d}_{ij}$ denotes the edge ($k$) averaged (accordingly to b), c)) function

$$\tilde{d}_{ij} = \begin{cases} d_{ij} & \text{if } \; u_l(1-u_l) > 0, l = i \;\text{ and }\; l = j, \\ 0 & \text{if } \; u_l(1-u_l) \le 0, l = i \;\text{ or }\; l = j. \end{cases}$$

The part of interest for the discretization of $F_2$ ($\hat{F}_2$) reads on a simplex in discrete form:

$$\hat{F}_2 = G^T\tilde{f}(s)(G\mathbf{u} + [\tilde{d}]G\mathbf{w}).$$

$\tilde{f}(s)$ is a scalar function on each simplex (defined above, but with $1/(u(1-u))$ replaced by $[\tilde{d}_\epsilon]^{-1}$, $\tilde{d}_{k,\epsilon} := \tilde{d}_k + \epsilon$):

$$s^2 = \frac{1}{V_\Delta}([\tilde{d}_\epsilon]^{-1}G\mathbf{u} + G\mathbf{w})^T([\tilde{d}_\epsilon]^{-1}G\mathbf{u} + G\mathbf{w}),$$

where $V_\Delta$ is the simplex volume.

Let denote in the sequel $\|x\|_p := (\sum_i V_i|x_i|^p)^{1/p}$ the discrete Voronoi volume weighted

$p$-norm and $\|\cdot\| = \|\cdot\|_2$, $\mathbf{x}^t$ the quantity $\mathbf{x}$ at time step $t$, $\sum_\Delta$ the sum over all elements, $[V]$ the diagonal matrix of the Voronoi cell volumes, $\mathbf{x}|_\Delta$ the restriction of $\mathbf{x}$ on the element $\Delta$, and $\sum_\Delta G^TG$ the global coefficient matrix. The discrete version of (6.2) reads with the initial value $\mathbf{u}^0 = \mathbf{g}_\lambda$ and $\tau = t^{t+1} - t^t$:

$$\frac{[V]}{\tau}(\mathbf{u}^{t+1} - \mathbf{u}^t) + \sum_\Delta G^T \tilde{f}(s^{t+1})(G\mathbf{u}^{t+1} + [\tilde{d}^{t+1}]G\mathbf{w}^{\mathbf{t+1}}) + \beta[V](\mathbf{u}^{t+1} - \mathbf{u}^0) = \mathbf{0}. \quad (6.3)$$

**Remark 6.1.** (6.3) and the spatially discretized version of (1.8) preserve the mass $\mathbf{V}^T\mathbf{u}^t = \mathbf{V}^T\mathbf{u}^0 = \bar{u}^0|V|$.

**Proposition 6.1**. *Let be $\mathbf{u}^t$ a discrete solution at time step $t$, such that $\|w\| < \infty$. Let $u_i^0 \in [0,1]$. Than the a priori bounds $0 \le u_i^t \le 1$ hold.*

*Proof.* By induction (the initial value fulfills the assumption).

Suppose $u_j^t > 1$, $j = 1, \dots, l$, $h_i = \max(0, u_i^t - 1) \ge 0$ $\forall i \in \Omega$. Testing (6.3) by $\mathbf{h}^T$ yields (because $\mathbf{h}^T \sum_\Delta (G^T \tilde{f}^t [\tilde{d}^t] G)\mathbf{w}^t = \sum_\Delta (\mathbf{h}^T|_\Delta G^T \tilde{f}^t [\tilde{d}^t] G\mathbf{w}^t|_\Delta)$ and on each edge either $\mathbf{h}^T|_\Delta G^T$ or $[\tilde{d}^t]$ is zero)

$$\mathbf{h}^T \sum_\Delta G^T \tilde{f} G\mathbf{u}^t = \beta \mathbf{h}^T[V](\mathbf{u}^0 - \mathbf{u}^t) + \frac{1}{\tau}\mathbf{h}^T[V](\mathbf{u}^{t-1} - \mathbf{u}^t).$$

Since $\mathbf{h}^T|_\Delta G^T$ and $G\mathbf{u}^t|_\Delta$ are either zero or have the same sign on each edge, we have $\sum_\Delta(\mathbf{h}^T|_\Delta G^T \tilde{f} G\mathbf{u}^t|_\Delta) > 0$. On the other hand hold $\mathbf{h}^T[V](\mathbf{u}^0 - \mathbf{u}^t) < 0$, and $\mathbf{h}^T[V](\mathbf{u}^{t-1} - \mathbf{u}^t) < 0$, hence a contradiction, thus requires $0 \le u_i^t \le 1$ (what is true for models without the term $\mathbf{u}^0 - \mathbf{u}^t$, too). $u_i^t \ge 0$ follows analogously. ☐

The discrete equation related to (6.1) and (6.3) are solved by Newtons and direct sparse methods (with $LU = A_k$ used as preconditioner for a CGS iteration with $A_{k+l}$, $l > 0$ as long as the iteration time at step $k + l$ is smaller than the factorization time). The Newton linearization of $\hat{F}_2$ can be easily obtained by using the above given formulas on elements:

$$\hat{F}_{2w}\delta\mathbf{w} = (\tilde{f} G^T [\tilde{d}_\epsilon] G + G^T [\tilde{d}_\epsilon] G\mathbf{w} \tilde{f}' \mathbf{s_w}^T)\delta\mathbf{w}, \quad (6.4)$$

$$\hat{F}_{2u}\delta\mathbf{u} = (\tilde{f} G^T G + \tilde{f} G^T [G\mathbf{w}][\tilde{d}_\epsilon]_u + G^T(G\mathbf{u} + [\tilde{d}_\epsilon]G\mathbf{w})\tilde{f}'\mathbf{s_u}^T)\delta\mathbf{u}. \quad (6.5)$$

With

$$\tilde{f}'(s) = \frac{f_0}{s_0}\frac{\delta - 1}{(1 + s/s_0)^2},$$

and the transposed vectors

$$\mathbf{s_w}^T = \frac{1}{sV_\Delta}([\tilde{d}_\epsilon]^{-1} G\mathbf{u} + G\mathbf{w})^T G,$$

$$\mathbf{s_u}^T = \frac{1}{sV_\Delta}([\tilde{d}_\epsilon]^{-1} G\mathbf{u} + G\mathbf{w})^T ([\tilde{d}_\epsilon]^{-1} G - [G\mathbf{u}][\tilde{d}_\epsilon]^{-2}[\tilde{d}_\epsilon]_u),$$

$$[\tilde{d}_\epsilon]_u = \begin{pmatrix} 0 & d_{23,u_2} & d_{23,u_3} \\ d_{31,u_1} & 0 & d_{31,u_3} \\ d_{12,u_1} & d_{12,u_2} & 0 \end{pmatrix},$$

and

$$d_{ij,u_k} = 2(u_i(1-u_i) + u_j(1-u_j))^{-2} \begin{cases} u_j^2(1-u_j)^2(1-2u_i) & \text{if } k = i, \\ u_i^2(1-u_i)^2(1-2u_j) & \text{if } k = j, \\ 0 & \text{else} \end{cases}$$

in case c). Hence the contributions to the Jacobian on the simplex have either the same structure as the discrete Laplace operator or constitute tensor products of vectors defined on edges, mapped back to vertices by $G^T$.

# 7. Examples

We demonstrate our approach on three examples based on the well known 'Lena' image. At first the image segmentation is illustrated (Figure 7.1) using the parameters (given in natural length units of the problem, the edge length of one (square) pixel [pxl])

$$m_0 = 128, \quad \sigma_0^2 = 32\text{pxl}^2, \quad s_0 = 0.5171\ 1/\text{pxl}, \quad \lambda = 1, \quad \beta = 0, \quad f_0 = 32\text{pxl}^2/\text{T},$$

The remaining two examples are selected to illustrate the possibility to interpret image segmentation with respect to noise reduction and contrast enhancement. We fixed $\delta = 10^{-6}$, $f_0 = 10^{-4}\text{pxl}^2/\text{T}$, and $\sigma_0$, $m_0$, are determined by the procedure stated above. Because the discretization cases b) and c) introduce only minor technical differences both have been tested and resulted in slightly different solutions. A simple arithmetic averaging of $u(1-u)$ does not allow to prove $0 \leq u_i^t \leq 1$ and violated the restriction pretty soon in a practical test.

To illustrate the influence of the minimization process, the second example (Figure 7.2) shows a 157 by 124 pixel section of 'Lena' perturbed with 20 added sequences of random numbers in the interval $(-0.5, 0.5)$. The values of the parameters are given by

$$m_0 = 44.6883, \quad \sigma_0 = 4.0388\text{pxl}, \quad s_0 = 0.40367, \quad \lambda = 0.26208, \quad \beta = 8.61528{\cdot}10^{-3}1/\text{T}.$$

The last example (Figure 7.3) uses the same initial picture but the perturbation procedure is applied only once and the amplitude of the random numbers is reduced by $10^{-7}$: The values of the parameters are

$$m_0 = 2.4055, \quad \sigma_0 = 0.15093\text{pxl}, \quad s_0 = 0.24242, \quad \lambda = 1.0, \quad \beta = 1.7876 \cdot 10^{-2}1/\text{T}.$$

These parameters suppress diffusion effectively — hence the image remains essentially unchanged and the presented identification procedure seems to determine reasonable initial values for the parameters.

Figure 7.1: Segmentation of the 'Lena' grey scale image (512 by 512 pixel), $t = 0$ (u.l.); dark areas are moved towards black, small granules start to form, $t = 0.05s$ (u.r.); grey areas surrounded by mainly black ones show deficits of black particles in 'central' positions, $t = 0.1s$ (l.l.); most small features are removed resulting in a 'xylography', $t = 1.0s$ (l.r.).

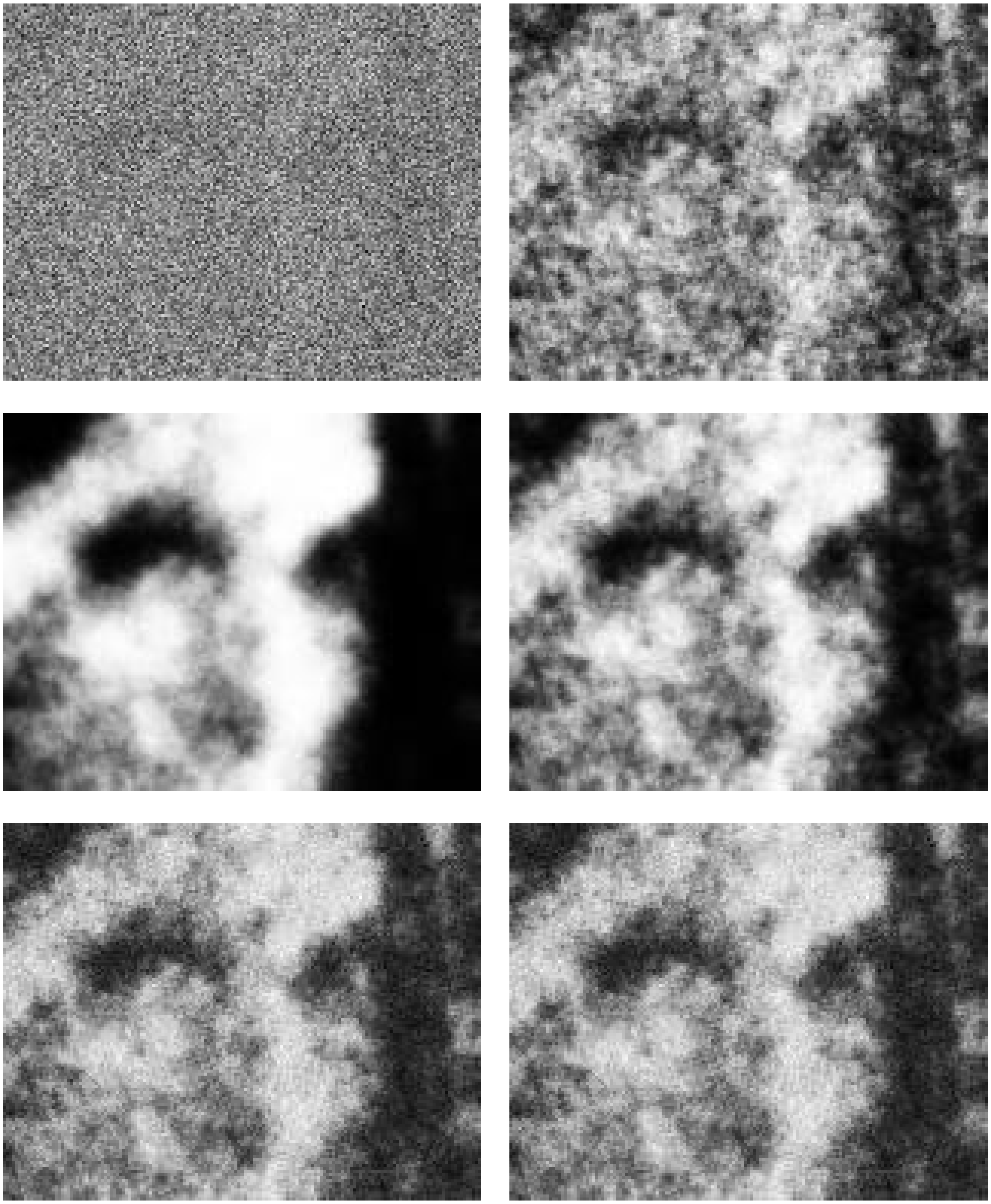

Figure 7.2: noisy picture (u.l.), first optimization step (u.r.), intermediate optimization step (m.l.), final optimization result (m.r.), initial convex combination (l.l.), steady state (l.r.).

Figure 7.3: noisy picture (u.l.), first optimization step (u.r.), intermediate optimization step (m.l.), final optimization result (m.r.), initial convex combination (l.l.), steady state (l.r.).

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