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A filtered no arbitrage model for term structures from noisy data

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Abstract

We consider an affine term structure model of interest rates, where the factors satisfy a linear diffusion equation. We assume that the information available to an agent comes from observing the yields of a finite number of traded bonds and that this information is not sufficient to reconstruct exactly the factors. We derive a method to obtain arbitrage-free prices of illiquid or non traded bonds that are compatible with the available incomplete information. The method is based on an application of the Kalman filter for linear Gaussian systems.

1 Introduction

We study multifactor affine term structure models of interest rates (see e.g. [12, 16]), where the factors $x(t)$ satisfy a linear diffusion equation. The factors may be viewed as representing market fundamentals, but in our context they need not have a specific interpretation and may just be viewed as abstract factors. They are considered as latent variables that are not directly observable, but can be estimated (filtered) from observations of traded bond yields.

The purpose is to derive a consistent pricing system to price illiquid and non traded bonds on the basis of the incomplete information available to agents. We assume that this incomplete/partial information, represented by a subfiltration $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$ of the full filtration $\mathcal{F}_t$, comes from observing the prices $\tilde{p}(t, T_i)$ (or corresponding yields) of a finite number N of traded bonds. The crucial further assumption is that this information is not sufficient to completely reconstruct the factors x_t. More precisely, we assume that each of the N observations comes with additional uncertainty and that the additional uncertainty sources together form a further factor $\xi(t)$ of dimension N. This happens e.g. in the realistic situation when the actually observed term structure does not correspond exactly to a theoretical arbitrage-factor model. We call the thus resulting term structure model the "perturbed model". Assuming a situation of this latter type, we derive a method to obtain arbitrage-free prices $\hat{p}(t, T)$ of non traded (illiquid) bonds that are compatible with the available partial information $\hat{\mathcal{F}}_t$ and we call this the projected price system. Specifically, we obtain the formula

$$\hat{p}(t,T) = \frac{E^Q[\tilde{p}(t,T)/\tilde{M}(t)|\hat{\mathcal{F}}_t]}{E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t]}, \tag{1}$$

where $\tilde{p}(t, T)$ are the bond prices in the perturbed model; $\tilde{M}(t)$ is the corresponding money market account and Q a given risk-neutral (martingale) measure. To this effect we derive some intermediate results justifying formula (1).

Thanks to (1), the computation of the projected price system reduces to the computation of the conditional expectations on the right hand side of (1). It is then shown that these conditional expectations can be computed if one can compute means and covariances of the vector of the original and latent factors $(x(t), \xi(t))$, conditional on $\hat{\mathcal{F}}_t$. This is where stochastic filtering comes in and we show that it reduces to an application of the classical Kalman filter for linear-Gaussian systems. This method extends thus in a nontrivial way a previous related work by two of the authors [21].

Instead of the "economic" definition of the filtered term structure through (1), it is possible to define the filtered forward rates using the filtered factors from the Kalman filter and applying the HJM-no-arbitrage condition. We show that the two definitions are equivalent.

Stochastic filtering techniques have recently found various applications in finance, in particular also in the context of the term structure of interest rates as e.g in [1, 3, 9, 20, 24]. The context of these latter papers is however different from that of the present work.

In the next section 2 we introduce the basic theoretical arbitrage-free affine term structure model. The perturbed model is then described in section 3. In section 4 we show how to derive from the perturbed model the projected pricing system. In section 5 we then show how the projected price system can actually be computed by use of Kalman filtering. In section 6 we show the equivalence of the two alternative definitions of the filtered term structure.

2 Notation and preliminary results

We consider a class of interest rate models which are the output of a time-varying linear Gaussian system. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, assume that we have an n-dimensional diffusion

$$dx(t) = A(t)x(t)dt + B(t)dw(t) \tag{2}$$

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$-matrices, respectively, which depend only on t, w is an m-dimensional Wiener-process and $x(0) = x_0 = 0$ (we will show in the sequel that the latter assumption is not restrictive). The matrices A and B are assumed to be bounded on finite intervals. The forward rates are given by

$$f(t,T) = C(t,T)x(t) + G(t,T), \tag{3}$$

where we assume that the functions $t \mapsto C(t,T)$ and $t \mapsto G(t,T)$ are differentiable.

As usual, $p(t,T) = \exp\{-\int_t^T f(t,s)ds\}$ is the time-t price of the zero-bond maturing at T, $r(t) = f(t,t)$ is the instantaneous short rate, and $M(t) = \int_0^t r(s)ds$ the money market account. Let $f^*(0,T)$ denote the forward rates at time 0.

Setting $C(t) := C(t,t)$ and $G(t) := G(t,t)$, the short rate has the representation

$$\begin{cases} dx(t) = A(t)x(t)dt + B(t)dw(t) \\ r(t) = C(t)x(t) + G(t). \end{cases}$$

What are the conditions on the coefficients C and G to have absence of arbitrage?

Similarly to [22], we define a model to be *arbitrage-free* for a given filtration $\mathcal{F} = \{\mathcal{F}_t\}$, if there exists another probability measure $Q^* \sim Q$, and a numeraire $N(t)$ (a positive process that is bounded away from zero) such that the discounted zero-bond price processes $p(t,T)/N(t)$ are $(Q^*, \mathcal{F})$-martingales for all T. The measure Q^* is called the *risk-neutral probability measure* for the term structure $\{p(t,T)\}_{0\le t\le T}$ w.r.t. the numeraire $N(t)$. (In [22], the numeraire is fixed to the money market account $M(t) = \exp\{\int_0^t r_s ds\}$; in section 4 below, for an information structure represented by a subfiltration $\hat{\mathcal{F}} \subset \mathcal{F}$, we shall consider numeraires different from M.)

How to go from the real-world measure to the risk-neutral measure, which is related to the "market price of risk", is an important question, but this will not be the focus of this paper. As is usual in many other term structure models, we will assume that Q itself is a risk-neutral probability with respect to the money market account as the numeraire. The next proposition gives conditions under which this is the case.

Proposition 2.1 *A necessary and sufficient condition for Q being a risk-neutral probability measure for the term structure model (2), (3) w.r.t. the numeraire M is that the coefficients $C(t,T), G(t,T)$ in (3) satisfy the following:*

$$C(t,T) = C(T)e^{\int_t^T A(s)ds}, \tag{4}$$

where $C(T)$ is a function that is bounded on finite intervals, and

$$G(t,T) = f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds, \tag{5}$$

with

$$\beta(t,T) := \left|\left|\int_t^T C(t,u)B(t)du\right|\right|^2. \tag{6}$$

Proof. Differentiation with respect to t yields

$$\begin{aligned} df(t,T) = {} & C_t(t,T)x(t)dt \\ & + C(t,T)A(t)x(t)dt + C(t,T)B(t)dw(t) + G_t(t,T)dt. \end{aligned} \tag{7}$$

Now, the Heath-Jarrow-Morton drift condition [22] reads

$$\mu(t,T) = C(t,T)B(t)\int_t^T B(t)'C(t,u)'du, \tag{8}$$

where $\mu(t,T)$ is the drift and $\sigma(t,T) = C(t,T)B(t)$ is the diffusion coefficient of $f(t,T)$ in (7). Since $x(t)$ does not appear in (8), its coefficients must vanish in (7); thus we obtain

$$C_t(t,T) + C(t,T)A(t) = 0, \tag{9}$$

which has the solution

$$C(t,T) = C(T)e^{\int_t^T A(s)ds}.$$

The deterministic term must satisfy the equation

$$G_t(t,T) = C(t,T)B(t)B(t)'\int_t^T C'(t,u)du =$$
$$= \frac{1}{2}\frac{\partial}{\partial T}\left\|\int_t^T C(u)e^{\int_t^u A(s)ds}B(t)du\right\|^2 = \frac{1}{2}\beta_T(t,T), \tag{10}$$

where we have used (4). As a consequence of (3) we get $G(0,T) = f^*(0,T)$. Thus, (10) and (5) are equivalent.

This proves that conditions (4) and (5) are equivalent to the HJM drift condition, which is necessary and sufficient for $p(.,T)/M$ being local $(Q,\mathcal{F})$-martingales. Novikov's condition for $p(.,T)/M$ being a martingale on $[0,T]$ is

$$E[\exp(\frac{1}{2}\int_0^T \beta(s,T)ds)] < \infty,$$

which is fulfilled since A, B, and $C(.)$ are bounded on finite intervals. ∎

The moral is that, given the functions f^*, A, B, and $C(.)$, the functions $C(t,T)$ and $G(t,T)$ are completely determined by the no arbitrage assumption.

The quantity $G(t,T)$ can be computed in an almost closed form, as we shall see in Section 5; however, if $A, B, C(.)$ are constant in t and A is invertible, things simplify even further and we have (see [6])

$$G(t,T) = f^*(0,T) + \frac{1}{2}\left\{||CA^{-1}e^{AT}B||^2 - ||CA^{-1}e^{A(T-t)}B||^2\right\}$$
$$+ CA^{-1}\left[e^{A(T-t)} - e^{AT}\right]BB'A'^{-1}C'. \tag{11}$$

We now show that the forward rates $f(t,T)$ are independent of the initial condition x_0. Suppose that in (2) we have an arbitrary initial condition x_0 independent of w and denote by $f^0(t,T)$ the corresponding term structure; then, denoting by $G^0(t,T)$ the correction term, since we want $f(0,T) = f^*(0,T)$ to hold, it must be $G^0(0,T) = -C(0,T)x_0 + f^*(0,T)$, which implies that

$$G^0(t,T) = -C(0,T)x_0 + f^*(t,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds \tag{12}$$

with $\beta(t,T)$ as in (6). Then we have the following lemma:

Lemma 2.2 *Let the forward rates $f^0(t,T)$ be given by*

$$\begin{cases} dx^0(t) = A(t)x^0(t)dt + B(t)dw(t) \\ f^0(t,T) = C(t,T)x(t) + G^0(t,T) \end{cases} \tag{13}$$

with initial condition $x^0(0) = x_0$, and $C(t,T)$ as in (4), and let $G^0(t,T)$ be as in (12). Then the term structure $f^0(t,T)$ is independent of x_0.

Proof. The solution to the first equation in (13) is

$$x^0(t) = e^{\int_0^t A(s)ds}x_0 + \int_0^t e^{\int_s^t A(u)du}B(s)dw(s).$$

In view of (4), $C(0,T) = C(T)e^{\int_0^T A(s)ds}$, which gives

$$\begin{aligned} f^0(t,T) &= C(t,T)x^0(t) - C(0,T)x_0 + f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds \\ &= C(T)e^{\int_t^T A(s)ds}\left[\int_0^t e^{\int_s^t A(u)du}B(s)dw(s) + e^{\int_0^t A(s)ds}x_0\right] \\ &\quad -C(T)e^{\int_0^T A(s)ds}x_0 + f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds \\ &= C(t,T)x(t) + f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds. \end{aligned}$$

where $x(t)$ is the solution to (2) with $x_0 = 0$, as wanted. ∎

Remark 2.3 *If the number N of bonds on the market is greater than the dimension n of the state x, the latter can generally be exactly reconstructed from the knowledge of their yields.*

In fact, let $y(t,T) := \int_t^T f(t,s)ds$ denote the time-t yield of the zero-bond maturing at T and assume these yields are observed for the maturities $T_1 < T_2 < ... < T_n$ with $n \leq N$.

Setting

$$\mathbf{M}(t) = \begin{bmatrix} \int_t^{T_1} C(s)e^{\int_t^s A(u)du}ds \\ \int_t^{T_2} C(s)e^{\int_t^s A(u)du}ds \\ \vdots \\ \int_t^{T_n} C(s)e^{\int_t^s A(u)du}ds \end{bmatrix},$$

we get, from (3) and using (4)

$$\begin{bmatrix} y(t,T_1) \\ y(t,T_2) \\ \vdots \\ y(t,T_n) \end{bmatrix} = \mathbf{M}(t) \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} \int_t^{T_1} G(t,u)du \\ \int_t^{T_2} G(t,u)du \\ \vdots \\ \int_t^{T_n} G(t,u)du \end{bmatrix}, \tag{14}$$

so that we can obtain x explicitly as soon as $\mathbf{M}$ is invertible. Without further assumptions on A, B, C more precise statements are difficult to make; but in the special case when A, B, C are constant, it can be shown that this situation is generic, i.e., the set of maturities $T_1, ..., T_n$, for which $\mathbf{M}$ is rank deficient, is a set contained in an algebraic surface in $\mathbb{R}^n$ (see [6]).

3 The Perturbed Model

Suppose now that we are in a situation where the state cannot be observed directly. This happens e.g. in the realistic situation when a low-dimensional, parsimonious factor model can describe certain long-term, time-series features of the term structure well, but fails to achieve sufficient accuracy in fitting all the current prices. In this context see e.g. [14] [15] in a similar setup. Assume then that the maturities of the actually traded and thus also observed bonds are $T_1, \ldots, T_N$ for some integer N and consider the following perturbed version of (3), namely

$$dx(t) = A(t)x(t)dt + B(t)dw(t) \tag{15}$$

$$d\xi(t) = A_\xi(t)\xi(t)dt + B_\xi(t)dv(t), \tag{16}$$

$$\tilde{f}(t,T) = C(t,T)x(t) + C_\xi(t,T)\xi(t) + \tilde{G}(t,T) \qquad (t \leq T), \tag{17}$$

where v is an $N-$dimensional Wiener process, independent of w and $x(0) = 0$, $\xi(0) = 0$. The function $C_\xi(s,T)$ is, for fixed T, an N-dimensional row vector of functions that are bounded on finite intervals. The function $t \mapsto \tilde{G}(t,T)$ is assumed to be differentiable.

Let then

$$\tilde{p}(t,T) := \exp\left[-\int_t^T \tilde{f}(t,u)du\right] \tag{18}$$

and consider as numeraire

$$\tilde{M}(t) := \exp\left[\int_0^t \tilde{r}(s)ds\right] \qquad \text{with } \tilde{r}(t) = \tilde{f}(t,t). \tag{19}$$

Equation (17) together with the dynamics of the extended state

$$\tilde{x}(t) := \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \tag{20}$$

can be written in the same form as the unperturbed system (2)-(3):

$$d\tilde{x}(t) = \tilde{A}(t)\tilde{x}(t)dt + \tilde{B}(t)d\tilde{w}(t) \tag{21}$$

$$\tilde{f}(t,T) = \tilde{C}(t,T)\tilde{x}(t) + \tilde{G}(t,T) \tag{22}$$

with

$$\begin{aligned} \tilde{A}(t) &:= \begin{bmatrix} A(t) & 0 \\ 0 & A_\xi(t) \end{bmatrix}, & \tilde{B}(t) &:= \begin{bmatrix} B(t) & 0 \\ 0 & B_\xi(t) \end{bmatrix}, \\ \tilde{C}(t,T) &:= [C(t,T), C_\xi(t,T)], & \tilde{w}(t) &:= \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \end{aligned} \tag{23}$$

We will assume, without loss of generality, that $\tilde{B}$ has full column-rank. Thus applying Proposition (2.1) to the new system (21)-(22) leads to

Proposition 3.1 *A necessary and sufficient condition for Q being a risk-neutral probability measure for the term structure $(\tilde{p}(t,T))_{0\le t\le T<\infty}$ with respect to the numeraire $\tilde{M}_t$ is that $\tilde{C}(t,T)$ and $\tilde{G}(t,T)$ in (22) satisfy the following two conditions corresponding to (4) and (5)*

$$\tilde{C}(t,T) = \tilde{C}(T)e^{\int_t^T \tilde{A}(s)ds}, \tag{24}$$

where $\tilde{C}(T)$ is bounded on finite intervals, and

$$\tilde{G}(t,T) = \tilde{f}^*(0,T) + \frac{1}{2}\int_0^t \tilde{\beta}_T(s,T)ds, \tag{25}$$

where

$$\tilde{\beta}(t,T) := \left|\left|\int_t^T \tilde{C}(t,u)\tilde{B}(t)du\right|\right|^2 = \beta(t,T) + \left|\left|\int_t^T C_\xi(t,u)B_\xi(t)du\right|\right|^2. \tag{26}$$

Remark 3.2 *Corresponding to Remark 2.3 notice now that in our perturbed term structure model, reformulated as (21) and (22), we shall never have enough bonds to reconstruct the (augmented) state $\tilde{x}$ exactly. In fact, since the dimension of $\tilde{x}$ is $n+N$, it is impossible to derive an invertible matrix $\mathbf{M}$ as in (14).*

Remark 3.3 *Notice that (16) and (24) yield*

$$C_\xi(t,T)\xi(t) = C_\xi(T)\int_0^t e^{\int_s^T A_\xi(\tau)d\tau}B_\xi(s)dv(s).$$

Taking $A_\xi = 0$, $B_\xi = I$ and $C_\xi^i(T) := \chi_{(T_{i-1},T_i]}$, we get the special case discussed in [21].

In what follows we shall therefore suppose that we are in a situation where the state cannot be observed directly and that our (partial) information corresponds to a subfiltration $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$. Typically, and this will be the setting in section 5 below, $\hat{\mathcal{F}}$ results from the observations of the traded bond prices (or their yields), but for the time being, in particular for the next section 4, we shall consider a generic subfiltration $\hat{\mathcal{F}}_t \subset \mathcal{F}$ containing the σ-algebra generated by the set of prices $(\tilde{p}(t, T_i))_{i=1,\dots,N}$.

4 The Projected Price System

In a previous paper [21] two of the authors have studied the problem of constructing a consistent price system under partial information in a similar setting. It relies, however, on the assumption that the perturbed money market account $\tilde{M}_t$ is observed ($\hat{\mathcal{F}}$-adapted) and liquidly traded, which may be unrealistic. Think of the term structure of defaultable bonds from a specific issuer, for example. In the following we present a way of defining an arbitrage-free term structure $\hat{p}$ that is $\hat{\mathcal{F}}$-adapted, in the case when the money market account $\tilde{M}$ is not observed.

We start by recalling the price system defined by the triple $(Q, \tilde{M}, \mathcal{F})$, which is

$$\Pi_{t,T}(X; Q, \tilde{M}, \mathcal{F}) := \tilde{M}(t) E^Q[X/\tilde{M}(T) \mid \mathcal{F}_t],$$

where $\Pi_{t,T}(X; Q, \tilde{M}, \mathcal{F})$ is the price of a time-T claim $X \in \mathcal{F}_T$, contracted at time t.

If $\tilde{M}_t$ is not observed (nor traded), it seems prudent to take a numeraire that is actually traded and observed. The same price system Π can alternatively be represented by a risk-neutral probability measure Q^* w.r.t. another numeraire M^*, if M^* is of the form

$$M^*(t) = L(t)\tilde{M}(t),$$

where L is a positive $(Q, \mathcal{F})$-martingale with $L(0) = 1$. L is then a valid Radon-Nikodym derivative and $d(Q^*|\mathcal{F}_T) := L(T)d(Q|\mathcal{F}_T)$ defines another probability measure on $\mathcal{F}_T$. The general formula for the change of measure in conditional expectations,

$$E^{Q^*}[X|\mathcal{G}] = \frac{E^Q[L(T)X|\mathcal{G}]}{E^Q[L(T)|\mathcal{G}]} \quad \forall\, \mathcal{G} \subseteq \mathcal{F}_T \text{ and } X \text{ bounded, } \mathcal{F}_T\text{-measurable,} \tag{27}$$

together with $L(t) = E^Q[L(T)|\mathcal{F}_t]$ imply that $(Q, \tilde{M}, \mathcal{F})$ and $(Q^*, M^*, \mathcal{F})$ define the same price system.

Potential candidates for such alternative numeraires are the traded and observed zero-bonds $M^i(t) := \frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)}$, since $L^i(t) := \frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)\tilde{M}(t)}$ is a positive $(Q, \mathcal{F})$-martingale with $L(0) = 1$ for any i. The measure Q^i is defined on $\mathcal{F}_T$ $(T \le T_i)$ by $d(Q^i|\mathcal{F}_T) := L^i(T)d(Q|\mathcal{F}_T)$ (see e.g. [19] or [5]).

If the actual set of information up to time t is $\hat{\mathcal{F}}_t$, it is natural to consider the *projected price systems*

$$\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) = M^i(t) E^{Q^i}[X/M^i(T) \,|\, \hat{\mathcal{F}}_t], \quad (t \leq T \leq T_i) \tag{28}$$

The crucial question is how this projection of the original price system onto the (smaller) filtration $\hat{\mathcal{F}}$ depends on the choice of the representation $(Q^i, M^i, \hat{\mathcal{F}})$.

Proposition 4.1 *Let $(Q_1, M_1, \mathcal{F})$ and $(Q_2, M_2, \mathcal{F})$ be triples that define the same price system and let $\hat{\mathcal{F}}$ be a subfiltration of $\mathcal{F}$. If M_1 and M_2 are $\hat{\mathcal{F}}$-adapted, then*

$$\Pi_{t,T}(X; Q_1, M_1, \hat{\mathcal{F}}) = \Pi_{t,T}(X; Q_2, M_2, \hat{\mathcal{F}})$$

holds for all bounded, $\mathcal{F}_T$-measurable X. In other words, the projected price system (28) is invariant under changes of the numeraire, as long as both numeraires are observed.

Proof. W.l.o.g. assume $M_1(0) = M_2(0) = 1$. Let T be arbitrary and X be a bounded, $\mathcal{F}_T$-measurable random variable. Both representations define the same price system, i.e.,

$$M_1(t) E^{Q_1}[X/M_1(T) | \mathcal{F}_t] = M_2(t) E^{Q_2}[X/M_2(T) | \mathcal{F}_t]. \quad (t \leq T) \tag{29}$$

Define $L(t) := \frac{dQ_2|\mathcal{F}_t}{dQ_1|\mathcal{F}_t}$. Setting $t = 0$ and using the definition of L, (29) gives

$$E^{Q_1}[X/M_1(T)] = E^{Q_1}[L(T) X/M_2(T)],$$

which implies

$$L(T) = \frac{M_2(T)}{M_1(T)} \quad (Q_1|\mathcal{F}_T) - \text{almost surely.} \tag{30}$$

Thus, L is in fact $\hat{\mathcal{F}}$-adapted and $L(t) = \frac{dQ_2|\hat{\mathcal{F}}_t}{dQ_1|\hat{\mathcal{F}}_t} = E^{Q_1}[L(T)|\hat{\mathcal{F}}_t]$ (for $t \leq T$) holds as well. Now

$$\Pi_{t,T}(X; Q_2, M_2, \hat{\mathcal{F}}) = M_2(t) E^{Q_2}[X/M_2(T) | \hat{\mathcal{F}}_t],$$

becomes, using (27),

$$= \frac{M_2(t)}{L(t)} E^{Q_1}[L(T) X/M_2(T) | \hat{\mathcal{F}}_t],$$

and, using (30),

$$= M_1(t) E^{Q_1}[X/M_1(T) | \hat{\mathcal{F}}_t].$$

∎

Note that this proposition applies to any measure change in a filtering setting, not just to term structures of interest rates.

As a consequence of Proposition 4.1, $\Pi_{t,T}(1; Q^i, M^i, \hat{\mathcal{F}})$ and $\Pi_{t,T}(1; Q^j, M^j, \hat{\mathcal{F}})$ are equal for $t \leq T \leq \min(T_i, T_j)$. The projected zero-bond prices in (28) are thus independent of which of the traded zero-bonds is chosen as numeraire, except that longer maturity bonds allow for a larger domain of definition. Due to the restriction $t \leq T \leq T_i$, we choose as numeraire the bond with largest maturity T_N to temporarily define here the *projected zero-bond prices* for $t \leq T \leq T_N$ by

$$\hat{p}(t,T) := \Pi_{t,T}(1; Q^i, M^i, \hat{\mathcal{F}}), \tag{31}$$

Below (see Corollary 4.3) we shall extend this definition also beyond T_N.

Proposition 4.2 *The system of bond prices $\hat{p}$ as defined in (31) is arbitrage-free in the sense of section 2, more precisely, for each i*

$$\frac{\hat{p}(t,T)}{\tilde{p}(t,T_i)} \text{ is a } (Q^i, \hat{\mathcal{F}})\text{-martingale} \quad (\forall T \leq T_i).$$

Furthermore, letting

$$M_0(t) := 1/E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t] \tag{32}$$

one has

$$\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) = \Pi_{t,T}(X; Q, M_0, \hat{\mathcal{F}}), \quad (T \leq T_i)$$

for all bounded, $\hat{\mathcal{F}}_T$-measurable X. In other words, the triple $(Q, M_0, \hat{\mathcal{F}})$ is yet another way to represent the price system defined by either of the triples $(Q^i, M^i, \hat{\mathcal{F}})$, but only for $\hat{\mathcal{F}}_T$-claims.

Proof. Using the definition of M^i, the price of a bounded, $\mathcal{F}_T$-measurable random variable X under the projected price system is

$$\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) = E^{Q^i}\left[\frac{\tilde{p}(t,T_i)}{\tilde{p}(T,T_i)} X \,\middle|\, \hat{\mathcal{F}}_t\right], \quad (T \leq T_i). \tag{33}$$

Since $\tilde{p}(t,T_i)$ is observed ($\hat{\mathcal{F}}$-measurable), this equation (with $X = 1$) implies that $\hat{p}(t,T)/\tilde{p}(t,T_i)$ is a $(Q^i, \hat{\mathcal{F}})$-martingale, thus showing the first statement.

Re-expressing now (33) as an expectation under Q, using formula (27) and the definition of L^i gives

$$\begin{aligned}
\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) &= \frac{E^Q\left[L^i(T)\tilde{p}(t,T_i)X/\tilde{p}(T,T_i) \,\middle|\, \hat{\mathcal{F}}_t\right]}{E^Q\left[L^i(t) \,\middle|\, \hat{\mathcal{F}}_t\right]} \\
&= \frac{E^Q\left[\frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)\tilde{M}(T)} X \,\middle|\, \hat{\mathcal{F}}_t\right]}{E^Q\left[\frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)\tilde{M}(t)} \,\middle|\, \hat{\mathcal{F}}_t\right]}.
\end{aligned}$$

Using the fact that $\tilde{p}(t, T_i)$ is $\hat{\mathcal{F}}_t$-measurable, this reduces to

$$\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) = \frac{E^Q[X/\tilde{M}(T)|\hat{\mathcal{F}}_t]}{E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t]}, \tag{34}$$

i.e., by definition of M_0 to

$$\Pi_{t,T}(X; Q^i, M^i, \hat{\mathcal{F}}) = \Pi_{t,T}(X; Q, M_0, \hat{\mathcal{F}})$$

for all bounded, $\hat{\mathcal{F}}_T$-measurable X. ∎

Since $E^Q[1/\tilde{M}(T)|\mathcal{F}_t] = \tilde{p}(t,T)/\tilde{M}(t)$, equation (34) in the above proof leads to the following corollary

Corollary 4.3 *The system of bond prices $\hat{p}$, defined in (31) for $T \leq T_N$, admits the representation*

$$\hat{p}(t,T) = \frac{E^Q[\tilde{p}(t,T)/\tilde{M}(t)|\hat{\mathcal{F}}_t]}{E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t]}. \tag{35}$$

and this justifies (35) as definition for the projected zero-bond prices also for $0 \leq t \leq T < \infty$.

Furthermore, if the money market account $\tilde{M}$ is observable, then $M_0 = \tilde{M}$ and formula (35) reduces to

$$\hat{p}(t,T) = E^Q[\tilde{p}(t,T)|\hat{\mathcal{F}}_t],$$

thereby recovering the special case of [21].

5 Computation of the Projected Prices by Kalman Filtering

The purpose of this section is to show that the projected price system $\hat{p}$ of the previous section 4 (see (35)) can actually be computed by the use of Kalman filtering, if the subfiltration $\hat{\mathcal{F}}_t$ is generated by the N prices $(\tilde{p}(t,T_i))_{i=1,\dots,N}$, or equivalently, the *cumulative* yields $(\tilde{y}(t,T_i))_{i=1,\dots,N}$ defined by

$$\tilde{y}(t,T) := -\log(\tilde{p}(t,T)) = \int_t^T \tilde{f}(t,s)ds. \tag{36}$$

Lemma 5.1 *Let $\hat{\mathcal{F}}$ be the filtration that is generated by the N yields $(\tilde{y}(t,T_i))_{i=1,\dots,N}$. Then we have*

$$\frac{E^Q[\tilde{p}(t,T)/\tilde{M}(t)|\hat{\mathcal{F}}_t]}{E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t]} = \exp\left\{-\hat{y}(t,T) + \frac{1}{2}\Gamma_1(t,T) + \Gamma_2(t,T)\right\}, \tag{37}$$

with

$$\hat{y}(t,T) := E^Q\left[\tilde{y}(t,T)\,\middle|\hat{\mathcal{F}}_t\right], \tag{38}$$

$$\Gamma_1(t,T) := \mathrm{var}^Q\left[\tilde{y}(t,T)\,\middle|\hat{\mathcal{F}}_t\right], \text{ and} \tag{39}$$

$$\Gamma_2(t,T) := \mathrm{cov}^Q\left[\tilde{y}(t,T), \int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right]. \tag{40}$$

$\Gamma_1(t,T)$ and $\Gamma_2(t,T)$ are constant as a function of ω, i.e., they are deterministic.

Proof. From the moment generating function of the normal distribution, we have

$$E[e^Y|\mathcal{F}] = e^{E[Y|\mathcal{F}]+\frac{1}{2}\mathrm{var}[Y|\mathcal{F}]}, \tag{41}$$

whenever the conditional distribution of some random variable Y under some σ-algebra $\mathcal{F}$ is Gaussian. (The second term in the exponent is the variance of the conditional distribution of Y given $\mathcal{F}$, $\mathrm{var}[Y|\mathcal{F}] = E[(Y - E[Y|\mathcal{F}])^2|\mathcal{F}]$.)

Thus, in view of (41), we can write

$$\begin{aligned}\frac{E^Q[\tilde{p}(t,T)/\tilde{M}(t)|\hat{\mathcal{F}}_t]}{E^Q[1/\tilde{M}(t)|\hat{\mathcal{F}}_t]} &= \frac{E^Q\left[\exp\left\{-\tilde{y}(t,T) - \int_0^t \tilde{f}(s,s)ds\right\}\middle|\hat{\mathcal{F}}_t\right]}{E^Q\left[\exp\left\{-\int_0^t \tilde{f}(s,s)ds\right\}\middle|\hat{\mathcal{F}}_t\right]} \\ &= \frac{\exp\left\{E^Q\left[-\tilde{y}(t,T) - \int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right] + \frac{1}{2}\Sigma_1\right\}}{\exp\left\{E^Q\left[-\int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right] + \frac{1}{2}\Sigma_2\right\}}\end{aligned} \tag{42}$$

where

$$\begin{aligned}\Sigma_1 &= \mathrm{var}^Q\left[-\tilde{y}(t,T) - \int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right] \\ &= \mathrm{var}^Q\left[\tilde{y}(t,T)\,\middle|\hat{\mathcal{F}}_t\right] + \mathrm{var}^Q\left[\int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right] \\ &\quad + 2\,\mathrm{cov}^Q\left[\tilde{y}(t,T), \int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right]\end{aligned} \tag{43}$$

and

$$\Sigma_2 = \mathrm{var}^Q\left[\int_0^t \tilde{f}(s,s)ds\middle|\hat{\mathcal{F}}_t\right]. \tag{44}$$

Putting (43) and (44) into (42) and canceling terms, gives (37).

Given random variables X, Y, Z that are joint normally distributed, X and $(Y - E[Y|X])(Z - E[Z|X])$ are independent, since X and $Y - E[Y|X]$ as well as X and $Z - E[Z|X]$ are uncorrelated. Thus the conditional covariance

$$\mathrm{cov}[Y,Z|X] = E\big[(Y - E[Y|X])(Z - E[Z|X])\big|X\big]$$

is actually the constant

$$= E\big[(Y - E[Y|X])(Z - E[Z|X])\big].$$

This applies to Γ_1 and Γ_2 since all forward rates $\tilde{f}(t,T)$ and yields $\tilde{y}(t,T)$ are joint normally distributed. ∎

As a consequence of the lemma we see that our goal is achieved if we are able to compute explicitly the conditional means and variances in (38)-(40).

The conditional mean (38) in the exponent (37) can be computed by means of a Kalman filter and this is what we are going to derive now. In order to make the partially observed system more compact, define

$$\tilde{z}(t) := \begin{bmatrix} \tilde{y}(t,T_1) - \int_t^{T_1} \tilde{G}(t,u)du \\ \tilde{y}(t,T_2) - \int_t^{T_2} \tilde{G}(t,u)du \\ \vdots \\ \tilde{y}(t,T_N) - \int_t^{T_N} \tilde{G}(t,u)du \end{bmatrix}. \tag{45}$$

Taking into account (22), (24), (36), and putting $\tilde{C}(t) := \tilde{C}(t,t)$, $\tilde{G}(t) := \tilde{G}(t,t)$, we obtain the yield dynamics

$$\begin{aligned} d\tilde{y}(t,T) =& -\tilde{f}(t,t)dt + \int_t^T d\tilde{f}(t,s)ds \\ =& -\tilde{C}(t)\tilde{x}(t)dt - \tilde{G}(t)dt + \\ &+ \left(\int_t^T \tilde{C}(t,u)du\tilde{B}(t)\right) d\tilde{w}(t) + \left(\int_t^T \tilde{G}_t(t,u)du\right) dt, \end{aligned} \tag{46}$$

giving

$$d\tilde{z}(t) = -\begin{bmatrix} \tilde{C}(t) \\ \tilde{C}(t) \\ \vdots \\ \tilde{C}(t) \end{bmatrix} \tilde{x}(t)dt + \begin{bmatrix} \int_t^{T_1} \tilde{C}(t,u)du\tilde{B}(t) \\ \int_t^{T_2} \tilde{C}(t,u)du\tilde{B}(t) \\ \vdots \\ \int_t^{T_N} \tilde{C}(t,u)du\tilde{B}(t) \end{bmatrix} d\tilde{w}(t). \tag{47}$$

The partially observed system can now be written as

$$\begin{cases} d\tilde{x}(t) &= \tilde{A}(t)\tilde{x}(t)dt + \tilde{B}(t)d\tilde{w}(t) \\ d\tilde{z}(t) &= C_e(t)\tilde{x}(t)dt + N(t)d\tilde{w}(t) \end{cases} \tag{48}$$

with $C_e(t)$ and $N(t)$ being the terms in brackets in the equation (47). It is a classical linear-Gaussian system, to which one can apply the Kalman filter, where $\tilde{x}(t)$ is the unobservable component and $\tilde{z}(t)$ is the observable one. Clearly,

$$\hat{\mathcal{F}}_t = \sigma\{\tilde{z}(s), s \leq t\} \tag{49}$$

and the following proposition follows from standard Kalman filtering theory (see e.g. [26, Theorem 10.3, p.396]).

Proposition 5.2 *Let the system $(\tilde{x}(t), \tilde{z}(t))$ satisfy (48) and $\hat{\mathcal{F}}_t$ be given by (49). Then the conditional distribution of $\tilde{x}(t)$, given $\hat{\mathcal{F}}_t$, is Gaussian with mean*

$$\hat{x}(t) := E^Q\left[\tilde{x}(t)\,\middle|\,\hat{\mathcal{F}}_t\right] \tag{50}$$

and covariance matrix

$$\overline{P}(t) := \mathrm{var}^Q\left[\tilde{x}(t)\,\middle|\,\hat{\mathcal{F}}_t\right], \tag{51}$$

which is deterministic

$$= E^Q\big[(\tilde{x}(t) - \hat{x}(t))(\tilde{x}(t) - \hat{x}(t))'\big]. \tag{52}$$

Assuming that the matrix

$$D(t) := [N(t)N(t)']^{1/2} \tag{53}$$

is invertible, the conditional mean has the dynamics

$$d\hat{x}(t) = \tilde{A}(t)\hat{x}(t)dt + \hat{B}(t)d\hat{w}(t), \tag{54}$$

with $\hat{x}_0 = 0$,

$$\hat{B}(t) = \Big(\tilde{B}(t)N(t)' + \overline{P}(t)C_e(t)'\Big)\,[D(t)']^{-1} \tag{55}$$

and $\hat{w}(t)$ is the innovations process

$$d\hat{w}(t) = D(t)^{-1}[d\tilde{z}(t) - C_e(t)\hat{x}(t)dt]. \tag{56}$$

Furthermore, $\overline{P}(t)$ is the solution of the Riccati equation

$$\frac{d\overline{P}}{dt} = \tilde{A}\overline{P} + \overline{P}\tilde{A}' - [\tilde{B}N' + \overline{P}C_e'](DD')^{-1}[\tilde{B}N' + \overline{P}C_e']' + \tilde{B}\tilde{B}' \tag{57}$$

with initial condition $\overline{P}(0) = 0$.

We have used the symbol D' although D is symmetric, to follow the standard notation for the Kalman filter. It should be noted that the term appearing on the right-hand side of (57) for $t = 0, P(0) = 0$ is

$$\tilde{B}(I - N'(NN')^{-1}N)\tilde{B}'. \tag{58}$$

Now, $N'(NN')^{-1}N$ is the projector on the column-space (image) of $N'(t)$ in $\mathbb{R}^{m+N}$. Since $N(t)$ has dimensions $N \times (m+N)$, it cannot have full rank; and since we assume $\tilde{B}$ to have full column rank, (58) cannot be zero, and thus the solution to (57) does not vanish identically.

Proposition 5.2 yields the means to compute the conditional mean $\hat{y}(t,T)$ as

$$\hat{y}(t,T) = E^Q[\tilde{y}(t,T)|\hat{\mathcal{F}}_t] = \int_t^T \tilde{C}(t,u)du\,\hat{x}(t) + \int_t^T \tilde{G}(t,u)du. \tag{59}$$

The conditional variance of $\tilde{y}(t,T)$ can be computed similarly:

Lemma 5.3 *Suppose $\tilde{f}(t,T)$ has dynamics as in (22) and $\hat{\mathcal{F}}_t$ is as in (49) and let $\overline{P}$ be the solution to (57). Then the functions Γ_1 and Γ_2 in (37) are given by*

$$\Gamma_1(t,T) = \left\{ \left[\int_t^T \tilde{C}(t,u)du \right] \overline{P}(t) \left[\int_t^T \tilde{C}'(t,u)du \right] \right\}$$

and

$$\Gamma_2(t,T) = \left\{ \int_0^t \tilde{C}(u,u)\overline{P}(u)e^{\int_u^t A'(\tau)d\tau}du \right\} \int_t^T \tilde{C}'(t,u)du.$$

Proof. we can write

$$\begin{aligned}
\Gamma_1(t,T) &= \operatorname{var}^Q \left[\tilde{y}(t,T) \,\middle|\, \hat{\mathcal{F}}_t \right] = E^Q\left[(\tilde{y}(t,T) - \hat{y}(t,T))^2 \right] \\
&= E^Q \left[\left(\int_t^T \tilde{C}(t,u)du\ (\tilde{x}(t) - \hat{x}(t)) \right)^2 \right] \\
&= E^Q \left[\int_t^T \tilde{C}(t,u)du\ (\tilde{x}(t) - \hat{x}(t))(\tilde{x}(t)' - \hat{x}'(t)) \int_t^T \tilde{C}'(t,u)du \right] \\
&= \left\{ \int_t^T \tilde{C}(t,u)du \right\} \overline{P}(t) \left\{ \int_t^T \tilde{C}'(t,u)du \right\}. \qquad (60)
\end{aligned}$$

As for the conditional covariance term (40), we get

$$\begin{aligned}
\Gamma_2(t,T) &= \operatorname{cov}^Q \left[\int_0^t \tilde{f}(u,u)du,\ \tilde{y}(t,T) \,\middle|\, \hat{\mathcal{F}}_t \right] \\
&= E^Q \left[\left(\int_0^t \tilde{f}(u,u)du - E^Q \left[\int_0^t \tilde{f}(u,u)du \,\middle|\, \hat{\mathcal{F}}_t \right] \right) (\tilde{y}(t,T) - \hat{y}(t,T))' \right].
\end{aligned}$$

Plugging in the expressions for $\tilde{f}$, $\tilde{y}$, and $\hat{y}$ and using Fubini's theorem twice gives

$$\begin{aligned}
\Gamma_2(t,T) &= E^Q \left[\int_0^t \tilde{C}(u,u) \left(\tilde{x}(u) - E^Q \left[\tilde{x}(u) \,\middle|\, \hat{\mathcal{F}}_t \right] \right) du (\tilde{x}(t)' - \hat{x}(t)') \int_t^T \tilde{C}(t,u)'du \right] \\
&= \int_0^t \tilde{C}(u,u) E^Q \left[\tilde{x}(u)(\tilde{x}(t) - \hat{x}(t))' \right] du \int_t^T \tilde{C}(t,u)'du \\
&\quad - \int_0^t \tilde{C}(u,u) E^Q \left[E^Q \left[\tilde{x}(u) \,\middle|\, \hat{\mathcal{F}}_t \right] (\tilde{x}(t) - \hat{x}(t))' \right] du \int_t^T \tilde{C}(t,u)'du.
\end{aligned}$$

Since $\tilde{x}(t) - \hat{x}(t)$ is, by definition, orthogonal to any element which is measurable with respect to $\hat{\mathcal{F}}_t$, the term in the last line is 0. Similarly, since $\hat{x}(u)$ is orthogonal to $\tilde{x}(t) - \hat{x}(t)$, we can write

$$\Gamma_2(t,T) = \int_0^t \tilde{C}(u,u) E^Q \left[(\tilde{x}(u) - \hat{x}(u))(\tilde{x}(t) - \hat{x}(t))' \right] du \left(\int_t^T \tilde{C}'(t,u)du \right). \qquad (61)$$

It is easily verified that the SDE (21) has the solution

$$\tilde{x}(t) = e^{\int_u^t \tilde{A}(\tau)d\tau}\tilde{x}(u) + \int_u^t e^{\int_s^t \tilde{A}(\tau)d\tau}\tilde{B}(s)d\tilde{w}(s) \tag{62}$$

for $t \geq u$. The process $\hat{x}(t)$ follows the analogous SDE (54) with the substitutions $\tilde{A} \to \hat{A}$, $\tilde{B} \to \hat{B}$, and $\tilde{w} \to \hat{w}$. Since $\hat{w}$ is a Wiener process with respect to the filtration $\hat{\mathcal{F}}$ ([26]), the analogous equation to (62) holds.

Therefore,

$$\begin{aligned}
&E^Q\{(\tilde{x}(u) - \hat{x}(u))(\tilde{x}(t)' - \hat{x}'(t))\\
=&E^Q\{(\tilde{x}(u) - \hat{x}(u))(\tilde{x}(u)' - \hat{x}'(u))e^{\int_u^t \tilde{A}'(\tau)d\tau}\}\\
=&\overline{P}(u)e^{\int_u^t \tilde{A}'(\tau)d\tau}.
\end{aligned} \tag{63}$$

Now, substitution of (63) in (61) yields

$$\Gamma_2(t,T) = \left(\int_0^t \tilde{C}(u,u)\overline{P}(u)e^{\int_u^t \tilde{A}'(\tau)d\tau}du\right)\int_t^T \tilde{C}'(t,u)du.$$

■

In conclusion, putting together Lemma 5.1, relation (59), and Lemma 5.3, we have the following:

Theorem 5.4 *If $\tilde{f}(t,T)$ has dynamics as in (22) and that $\hat{\mathcal{F}}_t$ is as in (49), then the projected prices $\hat{p}(t,T)$ (35) are given by*

$$\begin{aligned}
\hat{p}(t,T) =& \exp\left\{-\left(\int_t^T \tilde{C}(t,u)du\right)\hat{x}(t) - \int_t^T \tilde{G}(t,u)du\right\}\\
&\cdot\exp\left\{\frac{1}{2}\left(\int_t^T \tilde{C}(t,u)du\right)\overline{P}(t)\left(\int_t^T \tilde{C}'(t,u)du\right)\right\}\\
&\cdot\exp\left\{\left(\int_0^t \tilde{C}(u,u)\overline{P}(u)e^{\int_u^t \tilde{A}'(\tau)d\tau}du\right)\int_t^T \tilde{C}'(t,u)du\right\},
\end{aligned} \tag{64}$$

where $\hat{x}(t)$ and $\overline{P}(t)$ are computed by using the Kalman filter as in Proposition 5.2 with initial conditions $\hat{x}(0) = \hat{x}_0 = 0$ and $\overline{P}(0) = 0$.

6 Filtered Forward Rates

In this section, we define forward rates $\hat{f}$, based on the filtered state $\hat{x}$. We show that this term structure $\hat{f}$ is induced by the quintuple $(\tilde{f}^*, \tilde{A}, \hat{B}, \tilde{C}, \hat{w})$ in the same way as $\tilde{f}$ is induced by $(\tilde{f}^*, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{w})$ and f is induced by (f^*, A, B, C, w).

Moreover, we show that the forward rates $\hat{f}$ are indeed those associated to $\hat{p}$ defined earlier.

In fact, in complete analogy to Propositions 2.1 and 3.1, we can define forward rates processes as

$$\hat{f}(t,T) := \tilde{C}(t,T)\hat{x}(t) + \hat{G}(t,T) \tag{65}$$

with $\hat{G}(t,T)$ given by

$$\hat{G}(t,T) \;:=\; \tilde{f}^*(0,T) + \frac{1}{2}\int_0^t \hat{\beta}_T(s,T)ds \tag{66}$$

and

$$\hat{\beta}(t,T) := \left|\left|\int_t^T \tilde{C}(t,u)\hat{B}(t)du\right|\right|^2 . \tag{67}$$

It is not immediately obvious that the forward rates $\hat{f}$ thus defined are indeed those associated to $\hat{p}$. It turns out, though, that this indeed the case:

Theorem 6.1 *Let $\hat{p}(t,T)$ be defined by* (35) *and $\hat{f}(t,T)$ by* (65) - (67)*. Then*

$$\hat{p}(t,T) = \exp\left\{-\int_t^T \hat{f}(t,u)du\right\}. \tag{68}$$

Before we prove this theorem, we need some intermediate results.

It is well-known in system theory that the covariance $P(t) = E[x(t)x(t)']$ of the process $x(t)$ defined by (2) (with $x_0 = 0$) satisfies the Lyapunov equation

$$\frac{dP}{dt}(t) = A(t)P(t) + P(t)A'(t) + B(t)B'(t) \tag{69}$$

with the initial condition $P(0) = 0$. Analogously, the covariance $\tilde{P}$ of $\tilde{x}(t)$ satisfies the Lyapunov equation for the pair $(\tilde{A}, \tilde{B})$ and the covariance $\hat{P}$ of $\hat{x}(t)$ satisfies the Lyapunov equation for the pair $(\tilde{A}, \hat{B})$.

Notice next that, since $\hat{x}(t)$ and $\tilde{x}(t) - \hat{x}(t)$ are orthogonal, we have

$$\tilde{P} = E[\tilde{x}(t)\tilde{x}'(t)] = E[(\tilde{x}(t) - \hat{x}(t))(\tilde{x}(t) - \hat{x}(t))'] + E[(\hat{x}(t))\hat{x}'(t)] = \overline{P} + \hat{P}.$$

Lemma 6.2 *Let $x(t)$ be the solution to* (2)*, $C(t,T)$ be as in* (4) *and $P(t)$ be the covariance of $x(t)$. Then $G(t,T)$ in* (5) *can alternatively be written as*

$$\begin{aligned} G(t,T) \;&=\; f^*(0,T) + C(t,T)P(t)\int_t^T C'(t,u)du \qquad (70)\\ &\quad + \int_0^t C(u,u)P(u)e^{\int_u^t A'(s)ds}du\; C'(t,T) \\ &=: \; G(t,T,A,B,C). \qquad (71)\end{aligned}$$

Proof. Observe first that

$$C_t(t,T) = -C(T)e^{\int_t^T A(s)ds}A(t) = -C(t,T)A(t).$$

Then, since the two expressions (5) and (70) of $G(t,T)$ coincide for $t=0$, we just need to show that the partial derivatives in t are equal. Thus, from (70),

$$\begin{aligned}
G_t(t,T) \;\; = \;\; & -C(t,T)A(t)P(t)\int_t^T C'(t,u)du + C(t,T)\frac{dP}{dt}(t)\int_t^T C'(t,u)du \\
& -C(t,T)P(t)A'(t)\int_t^T C'(t,u)du \\
& -C(t,T)P(t)C'(t,t) + C(t,t)P(t)C'(t,T) \\
& +\int_0^t C(u,u)P(u)e^{\int_u^t A'(s)ds}A'(t)du\; C'(t,T) \\
& -\int_0^t C(u,u)P(u)e^{\int_u^t A'(s)ds}du\; A'(t)C'(t,T) \\
& = C(t,T)[-A(t)P(t) + \frac{dP}{dt}(t) - P(t)A'(t)]\int_t^T C'(t,u)du \\
& = C(t,T)B(t)B'(t)\int_t^T C'(t,u)du,
\end{aligned}$$

which is (10), as wanted. ∎

In a completely similar manner, we have that

$$\tilde{G}(t,T) = G(t,T,\tilde{A},\tilde{B},\tilde{C}) \tag{72}$$

and

$$\hat{G}(t,T) = G(t,T,\tilde{A},\hat{B},\tilde{C}).$$

Proof of Theorem 6.1. Since (68) obviously holds for $t=T$, it suffices to show

$$-\frac{\partial}{\partial T}\log\hat{p}(t,T) - \tilde{C}(t,T)\hat{x}(t) = \hat{G}(t,T). \tag{73}$$

Using (64), we can write:

$$\begin{aligned}
-\frac{\partial}{\partial T}\log\hat{p}(t,T) - \tilde{C}(t,T)\hat{x}(t) = \tilde{G}(t,T) - \tilde{C}(t,T)\,\overline{P}(t)\int_t^T \tilde{C}'(t,u)du \\
- \left\{\int_0^t \tilde{C}(u,u)\overline{P}(u)e^{\int_u^t \tilde{A}'(s)ds}du\right\}\tilde{C}'(t,T).
\end{aligned}$$

Plugging in (72) yields

$$
\begin{aligned}
&-\frac{\partial}{\partial T}\log \hat{p}(t,T) - \tilde{C}(t,T)\hat{x}(t) = \tilde{f}^*(0,T) \\
&\quad + \tilde{C}(t,T)\tilde{P}(t)\int_t^T \tilde{C}'(t,u)du + \left\{\int_0^t \tilde{C}(u,u)\tilde{P}(u)e^{\int_u^t \tilde{A}'(s)ds}du\right\}\ \tilde{C}'(t,T) \\
&\quad - \tilde{C}(t,T)\ \overline{P}(t)\int_t^T \tilde{C}'(t,u)du - \left\{\int_0^t \tilde{C}(u,u)\overline{P}(u)e^{\int_u^t \tilde{A}'(s)ds}du\right\}\tilde{C}'(t,T),
\end{aligned}
$$

and using the fact that $\tilde{P}(t) - \overline{P}(t) = \hat{P}(t)$,

$$
\begin{aligned}
&-\frac{\partial}{\partial T}\log \hat{p}(t,T) - \tilde{C}(t,T)\hat{x}(t) = \tilde{f}^*(0,T) \\
&\quad + \tilde{C}(t,T)\hat{P}(t)\int_t^T \tilde{C}'(t,u)du + \left\{\int_0^t \tilde{C}(u,u)\hat{P}(u)e^{\int_u^t \tilde{A}'(s)ds}du\right\}\ \tilde{C}'(t,T) \\
&= G(t,T,\tilde{A},\hat{B},\tilde{C}) = \hat{G}(t,T),
\end{aligned}
$$

which completes the proof. ∎

Conclusion

We showed that it is possible to define a filtered term structure in a general, model-free way, when the usual numeraire, the bank account, is not observed (section 4).

Although not themselves linear-Gaussian, the filtered prices can be computed by application of the standard Kalman-filter in the specific linear-Gaussian setting (section 5).

There is a complete analogy between the term structures f, $\tilde{f}$ and $\hat{f}$. The filtered prices could, instead of the "economic" definition of section 4, alternatively be defined by mathematical analogy according to (65). It turns out – but is not obvious – that both definitions are equivalent (section 6).

We conjecture that these three ideas carry over to a general HJM-Ito-process setting, except for the explicit computations in section 5.

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