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Sound and surface waves in poroelastic media

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Abstract

We consider two problems of propagation of weak discontinuity waves in porous materials. In the first part we present basic properties of bulk waves in fully saturated materials. These materials are modelled by a two-component immiscible mixture. We present general propagation conditions for such a model which yield three modes of propagation: P1-, S-, and P2-waves. Then we discuss the dispersion relation and we show that results are strongly dependent on the way in which waves are excited. In the second part we present some properties of surface waves. We begin with the classical Rayleigh and Love problems and then we extend them on heterogeneous materials important in practical applications. Subsequently we proceed to surface waves in two-component porous materials on the contact surface with vacuum (impermeable boundary) and with a liquid (permeable boundary). We show the existence of different modes of surface waves in the high frequency limit as well as the degeneration of the problem in the low frequency limit.

1 Introduction

We present two problems of weak discontinuity waves in porous materials: acoustic waves in saturated media modelled by a two–component continuum, as well as surface waves in such media and their asymptotic properties.

Propagation of acoustic waves in geophysical porous materials plays a particularly important role in testing porous and granular materials because laboratory measurements on such materials usually differ considerably from *in situ* measurements required in practical applications(e.g. see: [16]). Most of the theoretical results were obtained within the so–called Biot's model (e.g. [3]). They have contributed immensely to the understanding of the subject but simultaneously there are many very controversial issues related to the application of this model. We mention some of them further in this work.

A particular practical bearing have surface waves. Various theoretical and practical aspects of such waves have been investigated for single component continua (e.g. [17], [12], [4]). Very little has been done for two-component materials.

During the last decade the acoustics of porous materials was also developed within a different continuous model derived on the basis of a modern continuum thermodynamics. This model in its linear version is on the one hand side simpler than the Biot's model, in contrast to the Biot's model it does not violate the second law of thermodynamics and the principle of material frame indifference, and on the other hand it describes changes of porosity as an additional microscopical variable. In spite of these differences the number of acoustic modes of propagation and their fundamental properties are the same in both models (e.g. [18]).

The second section contains a review of fundamental properties of P1-, S-, and P2-waves in porous materials. However we emphasize an aspect of such waves which seems to be overlooked in the literature. Namely we demonstrate the dependence of acoustic properties of porous media on the way in which the dynamic disturbance is excited. This way is immaterial for the high frequency (short waves) asymptotics determining the speeds of signals in the medium. However it becomes essential in the limit of low frequencies (long waves) and these are of primary practical importance in soil mechanics and other geophysical applications. As observed by I. Edelman [5] the monochromatic P2-wave as a solution of an initial value problem does not propagate in the case of low wave numbers (long waves). It means that such waves do not exist after some impact excitations (chopping, explosions, etc.). Consequently some surface modes of propagation cannot appear in the range of long waves as well. We return to this problem in the third section where we discuss the propagation of surface waves in two limits: high frequency and low frequency. We present results for a poroelastic materials with the impermeable boundary. Results for both impermeable and permeable boundaries in limits of short and long waves can be found in the papers [7] and [6].

2 Bulk waves in two-component poroelastic media

2.1 Field equations for two-component poroelastic media

We rely on the model of two-component poroelastic saturated media proposed in a fully nonlinear form in the papers [19], [20]. We consider its linear version described by the following fields

partial mass density of the fluid $\rho^F(\mathbf{x},t)$,

velocity of the fluid $\mathbf{v}^F(\mathbf{x},t)$,

velocity of the skeleton $\mathbf{v}^S(\mathbf{x},t)$,

symmetric tensor of small deformations of the skeleton $\mathbf{e}^S(\mathbf{x},t)$, $\quad \left\|\mathbf{e}^S\right\| \ll 1$,[1]

porosity n.

[1] the norm of the deformation tensor is usually defined by means of its eigenvalues $\lambda^i, i = 1,2,3$

$$\left(\mathbf{e}^S - \lambda^i \mathbf{1}\right)\mathbf{k}^i = 0.$$

Namely

$$\left\|\mathbf{e}^S\right\| = \max\left\{\left|\lambda^1\right|, \left|\lambda^2\right|, \left|\lambda^3\right|\right\}.$$

These fields have a purely macroscopic interpretation, and it is not needed in a theoretical analysis to refer to any microscopical quantities related to these macroscopic fields. Certainly in practical applications such a reference may be necessary. For instance it may be useful to estimate macroscopic elastic parameters in terms of true or drained elastic properties of real materials, partial mass densities in terms of true mass densities or a relative velocity in terms of the filter velocity. In order to keep this work in a reasonable size we do not enter this problem in this work (compare: [10]).

For these fields the following field equations hold in the linear model of poroelastic materials

$$\frac{\partial \rho^F}{\partial t} + \rho_0^F \operatorname{div} \mathbf{v}^F = 0, \tag{1}$$

$$\begin{aligned} \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} + \kappa \operatorname{grad} \rho^F + \beta \operatorname{grad} (n - n_0) + \hat{\mathbf{p}} &= 0, \quad \hat{\mathbf{p}} := \pi \left(\mathbf{v}^F - \mathbf{v}^S \right), \\ \left| \frac{\rho^F - \rho_0^F}{\rho_0^F} \right| &\ll 1, \end{aligned} \tag{2}$$

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} - \operatorname{div} \left(\lambda^S \left(\operatorname{tr} \mathbf{e}^S \right) \mathbf{1} + 2\mu \mathbf{e}^S + \beta (n - n_0) \mathbf{1} \right) - \hat{\mathbf{p}} = 0, \tag{3}$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \tag{4}$$

$$\frac{\partial n}{\partial t} + n_0 \operatorname{div} \left(\mathbf{v}^F - \mathbf{v}^S \right) + \frac{n - n_0}{\tau} = 0. \tag{5}$$

In these equations ρ_0^F, ρ_0^S, n_0 denote constant reference values of partial mass densities, and porosity, respectively, and $\kappa, \lambda^S, \mu^S, \beta, \pi, \tau$ are constant material parameters. The first one describes the macroscopic *compressibility* of the fluid component, the next two are macroscopic *elastic constants* of the skeleton, β is the coupling constant, π is the coefficient of *bulk permeability*, and τ is the *relaxation time*. For the purpose of this work we assume $\beta = 0$. Then the problem of evolution of porosity described by equation (5) can be solved separately from the rest of the problem and does not influence the acoustic waves in the medium. Let us mention that the general case has been considered in earlier papers on the subject (e.g. [21], [22], [23]) and it has been shown that coupling effects through β can be neglected in linear models.

2.2 Propagation of acoustic fronts in two-component media

We investigate the propagation of the front carrying weak discontinuities. It is assumed that the front σ_t is given by the relation

$$f(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \sigma_t \subset \mathcal{B}_t, \quad t \in \mathcal{T}, \tag{6}$$

where the function f is assumed to be at least continuously differentiable with respect to both variables. $\mathcal{B}_t$, $\mathcal{T}$ denote the current configuration of the medium, and the time interval, respectively. The surface defined by (6) moves with the normal speed c and possesses a unit normal vector $\mathbf{n}$ given by the relations

$$c := -\frac{\frac{\partial f}{\partial t}}{|\operatorname{grad} f|}, \quad \mathbf{n} := \frac{\operatorname{grad} f}{|\operatorname{grad} f|}. \tag{7}$$

Weak discontinuities of fields introduced in the previous subsection are defined by the following conditions on the surface σ_t oriented by the field $\mathbf{n}(\mathbf{x}, t)$, $\mathbf{x} \in \sigma_t, t \in \mathcal{T}$,

$$\left[\left[\rho^F\right]\right] = 0, \quad \left[\left[\mathbf{v}^F\right]\right] = 0, \quad \left[\left[\mathbf{v}^S\right]\right] = 0, \quad \left[\left[\mathbf{e}^S\right]\right] = 0, \tag{8}$$

where

$$[[\ldots]] := \lim_{\sigma_t^+} (\ldots) - \lim_{\sigma_t^-} (\ldots). \tag{9}$$

Then according to the Hadamard lemma the following kinematic compatibility conditions hold

$$\begin{aligned} \left[\left[\operatorname{grad} \rho^F\right]\right] &= -\frac{1}{c} R^F \mathbf{n}, \quad \left[\left[\operatorname{grad} \mathbf{e}^S\right]\right] = \frac{1}{2c^2} \left(\mathbf{A}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{A}^S\right) \otimes \mathbf{n}, \\ \left[\left[\operatorname{grad} \mathbf{v}^F\right]\right] &= -\frac{1}{c} \mathbf{A}^F \otimes \mathbf{n}, \quad \left[\left[\operatorname{grad} \mathbf{v}^S\right]\right] = -\frac{1}{c} \mathbf{A}^S \otimes \mathbf{n}, \end{aligned} \tag{10}$$

where

$$R^F := \left[\left[\frac{\partial \rho^F}{\partial t}\right]\right], \quad \mathbf{A}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right], \quad \mathbf{A}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right], \tag{11}$$

are the so–called *amplitudes of discontinuity.*

Substitution in field equations evaluated on both sides of the front σ_t yields the conditions

$$R^F = \frac{\rho_0^F}{c} \mathbf{A}^F \cdot \mathbf{n}, \tag{12}$$

and

$$\begin{aligned} \left(c^2 \mathbf{1} - \frac{\lambda^S + \mu^S}{\rho_0^S} \mathbf{n} \otimes \mathbf{n} - \frac{\mu^S}{\rho_0^S} \mathbf{1}\right) \mathbf{A}^S &= 0, \\ \left(c^2 \mathbf{1} - \kappa \mathbf{n} \otimes \mathbf{n}\right) \mathbf{A}^F &= 0. \end{aligned} \tag{13}$$

Certainly this is an eigenvalue problem which yields three nontrivial solutions:

$$
\begin{aligned}
c_{P1} &: = \sqrt{\frac{\lambda^S + 2\mu^S}{\rho_0^S}}, \quad \mathbf{A}^S \cdot \mathbf{n} \neq 0, \quad \mathbf{A}_\perp^S := \mathbf{A}^S - \left(\mathbf{A}^S \cdot \mathbf{n}\right)\mathbf{n} = 0, \quad \mathbf{A}^F = 0, \\
c_{P2} &: = \sqrt{\kappa}, \quad \mathbf{A}^F \cdot \mathbf{n} \neq 0, \quad \mathbf{A}^S = 0, \quad \mathbf{A}_\perp^F := \mathbf{A}^F - \left(\mathbf{A}^F \cdot \mathbf{n}\right)\mathbf{n} = 0, \\
c_S &: = \sqrt{\frac{\mu^S}{\rho_0^S}}, \quad \mathbf{A}_\perp^S \neq 0, \quad \mathbf{A}^S \cdot \mathbf{n} = 0, \quad \mathbf{A}^F = 0.
\end{aligned} \tag{14}
$$

The first two solutions describe longitudinal P1-, and P2-modes of propagation while the third one is the transversal S–mode in the skeleton. There exists no transversal mode in the fluid: $\mathbf{A}_\perp^F \equiv 0$.

The P2-mode is often called Biot's wave. Its theoretical existence is quite natural in the frame of any two–component continuous model even if both components are fluids (a miscible mixture). However there are problems with the practical observation of its propagation if one of the components is solid. It has been observed for the first time in an artificial porous material made of sintered glass beads by T. J. Plona [11], and in an artificial rock of cemented sand grains by T. Klimentos and C. McCann [8] but *in situ* measurements are extremally difficult to perform. The main reason for those difficulties is a very strong attenuation of P2-waves. We discuss this point in some details further in this section.

Let us mention in passing that the partial stresses $\mathbf{T}^S, \mathbf{T}^F$ in the skeleton and in the fluid, respectively, which lead to the above used field equations are not coupled if the constant β is equal to zero. Such a coupling, even though of a different – static – nature, is required in the Biot's model commonly used in the wave analysis for porous saturated materials. In the notation of this work such a coupling has the form

$$
\begin{aligned}
\mathbf{T}^S &= \lambda^S \left(\operatorname{tr} \mathbf{e}^S\right) \mathbf{1} + 2\mu^S \mathbf{e}^S - Q\frac{\rho^F - \rho_0^F}{\rho_0^F}\mathbf{1}, \\
\mathbf{T}^F &= -\left(\kappa\left(\rho^F - \rho_0^F\right) - Q \operatorname{tr} \mathbf{e}^S\right)\mathbf{1},
\end{aligned} \tag{15}
$$

where Q is the Biot's *coupling constant*. Such a model is thermodynamically admissible solely in the case of an additional contribution of the gradient of porosity to the momentum balance equations (2), (3)(see: [10])

$$
\hat{\mathbf{p}} = \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) - Q \operatorname{grad} n. \tag{16}
$$

In such a case it can be easily shown that the coefficient Q which would give rise to the off–diagonal terms in the eigenvalue problem (13) has an order of magnitude of the pore pressure, i.e. 10^5 Pa in soils and rocks. This must be compared with elastic constants $\lambda^S, \mu^S, \kappa\rho_0^F$ which are at least of the order 10^8 Pa. Hence, similarly to the assumption that $\beta = 0$, we can leave out this correction in the wave analysis.

The above results do not reveal the attenuation of waves because the behaviour of amplitudes cannot be determined from the properties of field equations on the wave front alone.

In order to see such effects we have to construct solutions of field equations. Relying on the results presented in [24] we proceed to do so for monochromatic waves in infinite domains.

2.3 Monochromatic waves in two-component media

2.3.1 General relations

We seek solutions of the set of equations (1)–(4) in the form of bulk monochromatic waves defined by the following ansatz for harmonic waves

$$\begin{aligned} \rho^F - \rho_0^F &= R^F e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \quad \mathbf{e}^S = \mathbf{E}^S e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \\ \mathbf{v}^F &= \mathbf{V}^F e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \quad \mathbf{v}^S = \mathbf{V}^S e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \end{aligned} \tag{17}$$

where $R^F, \mathbf{E}^S, \mathbf{V}^F, \mathbf{V}^S$ are constant, possibly complex, amplitudes of the disturbance, $\mathbf{n}$ denotes the unit vector in the direction of propagation, k is the wave number, and ω the frequency of the wave. Both k and ω may be complex.

Straightforward calculations lead to the following compatibility relations with field equations

$$R^F = \frac{k\rho_0^F}{\omega}\mathbf{V}^F\cdot\mathbf{n}, \quad \mathbf{E}^S = -\frac{k}{2\omega}\left(\mathbf{V}^S\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{V}^S\right), \tag{18}$$

$$\begin{aligned} \left(\omega^2\mathbf{1} - \frac{\lambda^S+\mu^S}{\rho_0^S}k^2\mathbf{n}\otimes\mathbf{n} - \frac{\mu^S}{\rho_0^S}k^2\mathbf{1} + i\frac{\pi\omega}{\rho_0^S}\mathbf{1}\right)\mathbf{V}^S - i\frac{\pi\omega}{\rho_0^S}\mathbf{V}^F &= 0, \\ -i\frac{\pi\omega}{\rho_0^F}\mathbf{V}^S + \left(\omega^2\mathbf{1} - \kappa k^2\mathbf{n}\otimes\mathbf{n} + i\frac{\pi\omega}{\rho_0^F}\mathbf{1}\right)\mathbf{V}^F &= 0. \end{aligned} \tag{19}$$

Equations (19) form, of course, an eigenvalue problem with a six–dimensional eigenvector $\left(\mathbf{V}^S, \mathbf{V}^F\right)^T$, and ω^2 are the eigenvalues if k is given. We consider further also a modification of this problem with a given ω.

We can easily separate the components in the direction of the vector $\mathbf{n}$, and in the direction perpendicular to this vector. We consider these problems in the subsequent two subsections.

2.3.2 Longitudinal modes of propagation

Scalar multiplication of equations (19) by the vector $\mathbf{n}$ yields

$$\begin{pmatrix} \omega^2 - \frac{\lambda^S+2\mu^S}{\rho_0^S}k^2 + i\frac{\pi\omega}{\rho_0^S} & -i\frac{\pi\omega}{\rho_0^S} \\ -i\frac{\pi\omega}{\rho_0^F} & \omega^2 - \kappa k^2 + i\frac{\pi\omega}{\rho_0^F} \end{pmatrix} \begin{pmatrix} \mathbf{V}^S\cdot\mathbf{n} \\ \mathbf{V}^F\cdot\mathbf{n} \end{pmatrix} = 0. \tag{20}$$

This two–dimensional eigenvalue problem yields immediately the following *dispersion relation*

$$\left(\omega^2 - c_{P1}k^2 + i\frac{\pi\omega}{\rho_0^S}\right)\left(\omega^2 - c_{P2}^2k^2 + i\frac{\pi\omega}{\rho_0^F}\right) + \frac{\pi^2\omega^2}{\rho_0^S\rho_0^F} = 0. \tag{21}$$

We consider two cases.

1. The frequency ω is real and given. This corresponds to the problem of a harmonic excitation with a given frequency ("boundary value problem").
2. The wave number k is real and given. This corresponds to an external impact ("initial value problem", chopping, explosion).

In the first case the equation (21) can be easily solved for k and we obtain

$$k^2 = \frac{1}{2}\left[\frac{1}{c_{P1}^2}\left(\omega^2 + i\frac{\pi\omega}{\rho_0^S}\right) + \frac{1}{c_{P2}^2}\left(\omega^2 + i\frac{\pi\omega}{\rho_0^F}\right) \pm \sqrt{D}\right], \tag{22}$$

$$D := \left[\frac{1}{c_{P1}^2}\left(\omega^2 + i\frac{\pi\omega}{\rho_0^S}\right) - \frac{1}{c_{P2}^2}\left(\omega^2 + i\frac{\pi\omega}{\rho_0^F}\right)\right]^2 - \frac{4}{c_{P1}^2c_{P2}^2}\frac{\pi^2\omega^2}{\rho_0^S\rho_0^F}.$$

The solution with the plus sign corresponds to the longitudinal P1-wave. Correspondingly, the minus sign yields the relation for P2-wave. In the limit of low and high frequencies one obtains easily

$$P1: \quad \lim_{\omega\to 0}\frac{\omega}{\operatorname{Re} k} = \sqrt{\frac{\lambda^S + 2\mu^S + \kappa\rho_0^F}{\rho_0^S + \rho_0^F}} =: c_{oP1}, \quad \lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} = \sqrt{\frac{\lambda^S + 2\mu^S}{\rho_0^S}} \equiv c_{P1}, \tag{23}$$

$$P2: \quad \lim_{\omega\to 0}\frac{\omega}{\operatorname{Re} k} = 0, \quad \lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} = \sqrt{\kappa} \equiv c_{P2}.$$

These limits were obtained also for the Biot's model. Obviously the additional coupling appearing in this model does not influence the result.

In the next two Figures we illustrate these results for the following numerical data

$$c_{P1} = 2500\frac{m}{s}, \quad c_{P2} = 1000\frac{m}{s}, \quad \rho_0^S = 2500\frac{kg}{m^3}, \quad \rho_0^F = 250\frac{kg}{m^3}. \tag{24}$$

In Figure 1 we plot the phase velocity $c_{ph} = \dfrac{\omega}{Rek}$ of both longitudinal modes, and in Figure 2 the attenuation $\gamma = Imk$.

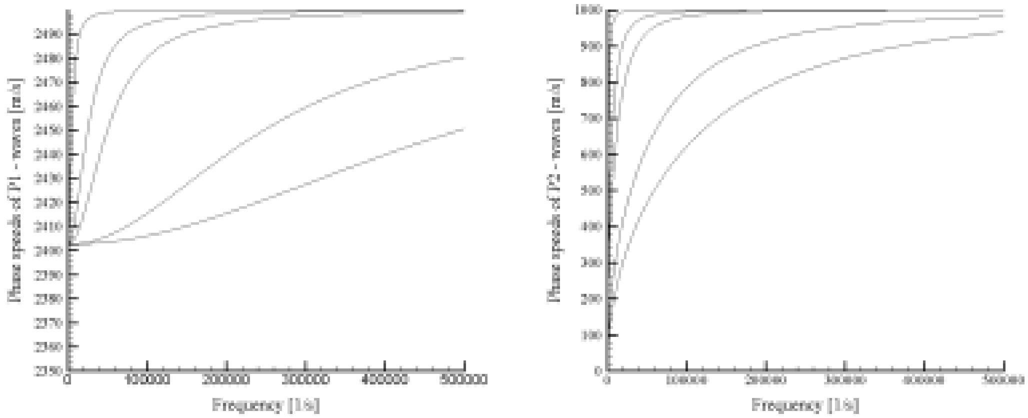

Figure 1: *Phase speed of P1- (left), and P2-waves (right) as functions of frequency ω. The curves correspond to the permeabilities π (from top to bottom):* $10^6, 5*10^6, 10^7, 5*10^7, 10^8$ $[\frac{kg}{m^3 s}]$.

Inspection of Figure 1 shows that both modes of propagation exist for any frequency of the excitation. The phase speed of P1-waves grows a little from its initial value to the asymptotic speed c_{P1} for $\omega \to \infty$. On the other hand the phase speed of P2-waves is equal to zero for $\omega = 0$ and grows asymptotically to the limit c_{P2} for $\omega \to \infty$. For both modes the growth becomes slower for larger permeability coefficients π which we demonstrate for practically reasonable values of this coefficient.

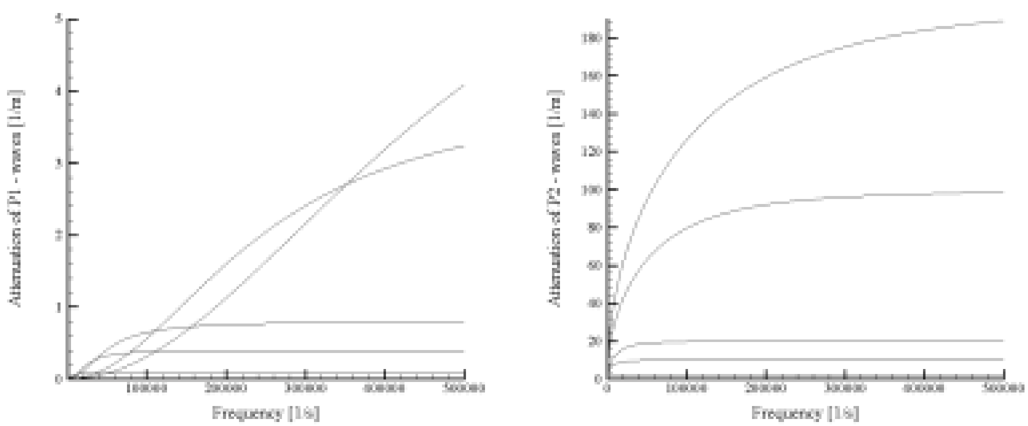

Figure 2: *Attenuation of P1- (left), and P2-waves (right) as functions of frequency ω. The same values of permeability π as in Fig. 1 growing from the bottom to the top.*

It is clear from Figure 2 that the attenuation of P2-waves is much stronger than this of P1-waves. This observation justifies the remark made in the Introduction that the strong attenuation of P2-waves causes difficulties in their *in situ* measurements.

The above described properties of monochromatic waves have been discussed in details in earlier works on this model of poroelastic materials (e.g. [19], [20], [21], [22], [23]).

We proceed to present properties of the second case – external impact (initial value problem, chopping). In this case the wave number k is given and real, and the frequency ω is complex. It follows as the solution of the dispersion relation (21). This solution cannot be obtained analytically. Asymptotic analysis for high and low wave numbers has been performed by I. Edelman [5]. We present here a few typical numerical examples. We use the data (24).

In contrast to the above discussed boundary value problem P2-waves may not exist in the case of the initial value problem. For any chosen real wave number k solutions of the dispersion relation (21) consist of four complex ω symmetric with respect to zero. Consequently there are two essential real parts of ω which determine the P1-, and P2-mode. In Figure 3 we show the real part of ω corresponding to the P2-mode for different values of the permeability coefficient π.

It is seen that for sufficiently low wave numbers k (i.e. long waves) the real part of ω is constant and equal to zero. Consequently in these ranges the P2-modes contain only damping and they cannot propagate as waves. The extent of the plateau of the constant real part of frequency changes approximately in linear way with π and, for instance, for $\pi = 10^9[\frac{kg}{m^3s}]$ (the right figure) it reaches the value $k \approx 2050[\frac{1}{m}]$, which corresponds to the wave length $0.05cm$. Obviously from the physical point of view the P2-wave does not exist any more because the wave length would have to be smaller than the characteristic dimension of the microstructure. However the minimum length of the wave for smaller permeabilities lies in the physically reasonable range. For instance for $\pi = 10^7[\frac{kg}{m^3s}]$ it is app. 5 cm (see the left figure).

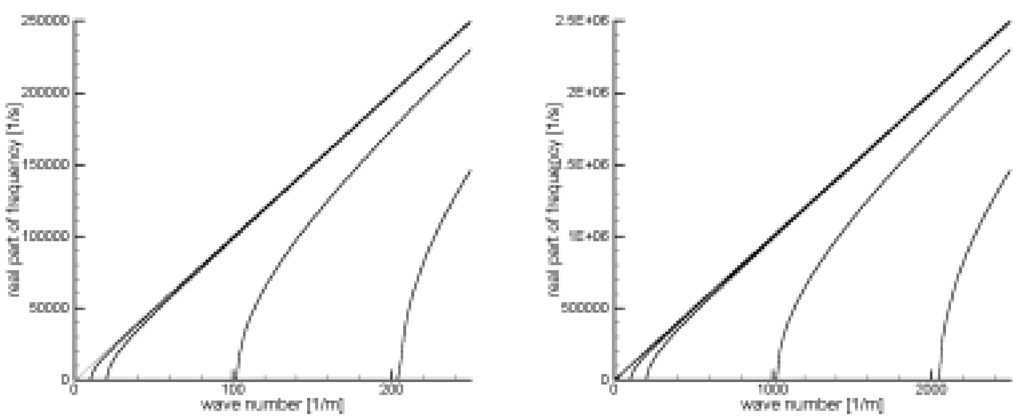

Figure 3: *Real part of the frequency as a function of the wave number for P2-waves. The left hand side is the magnification of the figure on the right hand side for the following values of permeability* π: $10^6, 5*10^6, 10^7, 5*10^7, 10^8$ $[\frac{kg}{m^3s}]$ *growing from the left to the right. On the right figure the curves for* $\pi = 5*10^8$ *and* 10^9 $[\frac{kg}{m^3s}]$ *are shown in addition.*

The problem of existence of propagation does not concern the P1-mode. These waves behave in a way similar to these of the boundary value problem. In Figure 4 we show their phase speeds for the data (24). The speed grows a little and reaches the limit value c_{P1} for $k \to \infty$.

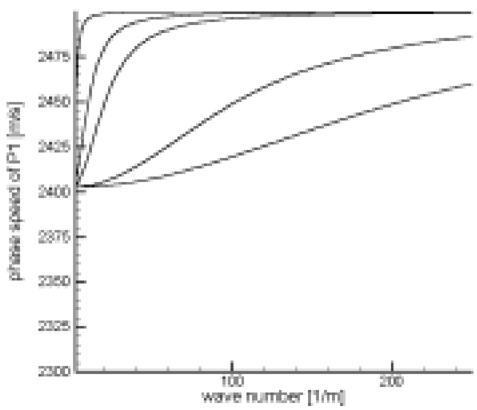

Figure 4: *Phase velocity of P1-waves for permeabilities* π *(from the left to the right)*: $10^6, 5 * 10^6, 10^7, 5 * 10^7, 10^8$ $[\frac{kg}{m^3 s}]$

As indicated above the P2-waves do not propagate below a critical value of k which changes with π. We show this behaviour in Figure 5. In the range of large values of k the P2-modes propagate and reach the limit value c_{P2} for $k \to \infty$.

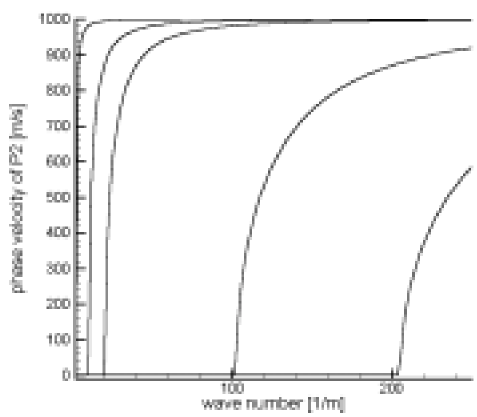

Figure 5: *Phase velocity of P2-waves for permeabilities* π *(from the left to the right)*: $10^6, 5 * 10^6, 10^7, 5 * 10^7, 10^8$ $[\frac{kg}{m^3 s}]$

Imaginary parts of the frequency ω determine the damping of waves. This attenuation in time behaves differently from the attenuation in space discussed in the first case. In the case of P1-waves (Figure 6) it grows with the growth of the wave number k (i.e. with the decay of the wave length). However in the range of long waves the damping in media with a larger permeability π is smaller than this for media with a smaller permeability. Most likely this is related to the fact that the energy of the wave created by the impact remains longer in the vicinity of the impact if the value of π is larger which, as seen in Figure 4 yields a lower speed of propagation.

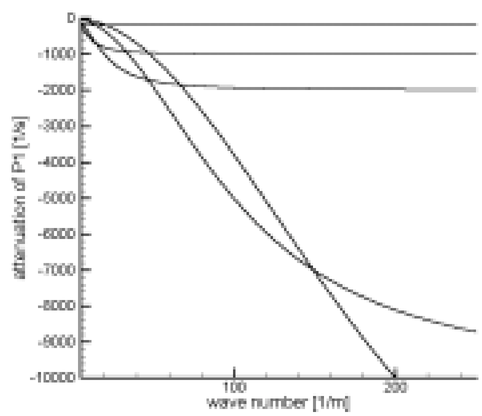

Figure 6: *Attenuation of P1-waves for permeabilities* π: 10^6 *(the smallest attenuation),* $5*10^6, 10^7, 5*10^7, 10^8$ *(the largest attenuation)* $[\frac{kg}{m^3 s}]$.

The behaviour of the P2-modes is entirely different due to the existence of plateaus. The ranges of these plateaus are visible also in Figure 7 which illustrates the attenuation of P2-modes. For any value of permeability π the range of small values of k contains solely damping – the frequency ω is imaginary. For larger values of k we see the attenuation of P2-waves. As in the case of the boundary value problem it is much stronger than in the case of P1-waves.

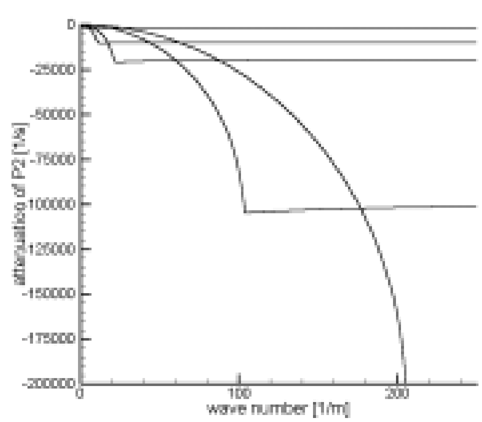

Figure 7: *Attenuation of P2-waves for permeabilities* π: 10^6 *(the smallest attenuation),* $5*10^6, 10^7, 5*10^7, 10^8$ *(the largest attenuation)* $[\frac{kg}{m^3 s}]$

The above described properties of initial value problems have an important influence on the construction of asymptotic solutions in the range of low frequencies. For instance, they lead to an entirely different structure of surface waves than this for high frequencies [7]. Asymptotic analysis of this problem has been performed by I. Edelman [6]. We shall discuss some aspects of this problem in the next section of this work.

2.3.3 Transversal modes of propagation

Let us introduce the following quantities

$$\mathbf{V}_{\perp}^{F} := \mathbf{V}^{F} - \left(\mathbf{V}^{F} \cdot \mathbf{n}\right) \mathbf{V}^{F}, \quad \mathbf{V}_{\perp}^{S} := \mathbf{V}^{S} - \left(\mathbf{V}^{S} \cdot \mathbf{n}\right) \mathbf{V}^{S}. \tag{25}$$

Then from (19) for arbitrary components of the above vectors $V_{\perp}^{F} := \mathbf{V}_{\perp}^{F} \cdot \mathbf{t}$, $V_{\perp}^{S} := \mathbf{V}_{\perp}^{S} \cdot \mathbf{t}$, with $\mathbf{t}$ being any unit vector perpendicular to $\mathbf{n}$ we obtain

$$\begin{pmatrix} \omega^2 - \frac{\mu^S}{\rho_0^S} k^2 + i\frac{\pi\omega}{\rho_0^S} & -i\frac{\pi\omega}{\rho_0^S} \\ -i\frac{\pi\omega}{\rho_0^F} & \omega^2 + i\frac{\pi\omega}{\rho_0^F} \end{pmatrix} \begin{pmatrix} V_{\perp}^{S} \\ V_{\perp}^{F} \end{pmatrix} = 0. \tag{26}$$

This is again an eigenvalue problem which yields the dispersion relation

$$\omega^3 + i\pi \left(\frac{1}{\rho_0^S} + \frac{1}{\rho_0^F}\right) \omega^2 - c_S^2 k^2 \omega - i c_S^2 k^2 \frac{\pi}{\rho_0^F} = 0. \tag{27}$$

As before we calculate limit speeds for high and low frequencies. We obtain

$$\lim_{\omega \to 0} \frac{\omega}{Re\, k} = \sqrt{\frac{\mu^S}{\rho_0^S + \rho_0^F}} =: c_{oS}, \quad \lim_{\omega \to \infty} \frac{\omega}{Re\, k} = \sqrt{\frac{\mu^S}{\rho_0^S}} \equiv c_S. \tag{28}$$

These are relations commonly used in geophysical applications.

We illustrate the solutions of the relation (26) in Figures 8 and 9 for the data

$$c_S = 1500 \frac{m}{s}, \quad \rho_0^S = 2500 \frac{kg}{m^3}, \quad \rho_0^F = 250 \frac{kg}{m^3}. \tag{29}$$

We obtain for the phase speed a behaviour quite similar to this of P1-waves. After the initial growth the phase speed goes to the limit value c_S for $k \to \infty$.

The behaviour of the attenuation is also similar to this of P1-waves. This is shown in Figure 9.

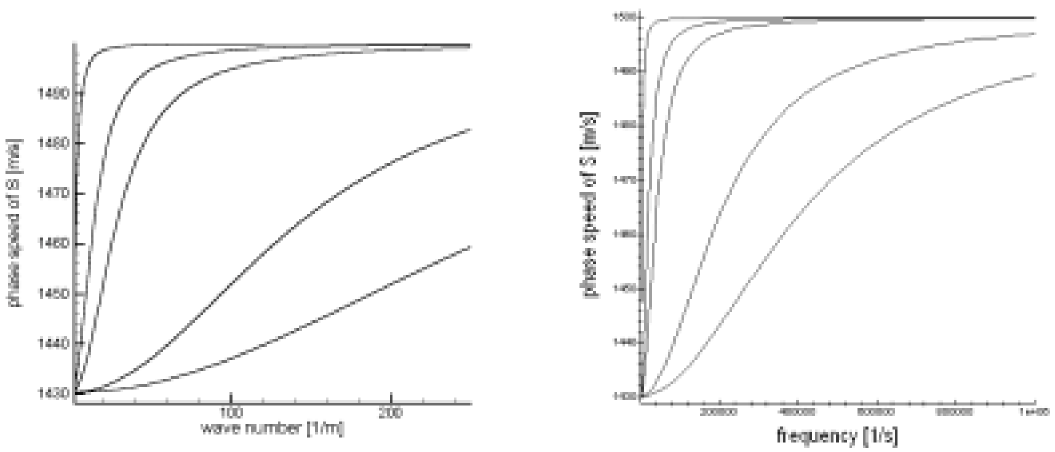

Figure 8: *Phase speed of S–waves for the permeabilities* π :

$10^6, 5*10^6, 10^7, 5*10^7, 10^8$ $[\frac{kg}{m^3 s}]$.

The left diagram corresponds to the initial value problem while the right diagram to the boundary vibrations. The upper curve corresponds to the lowest permeability.

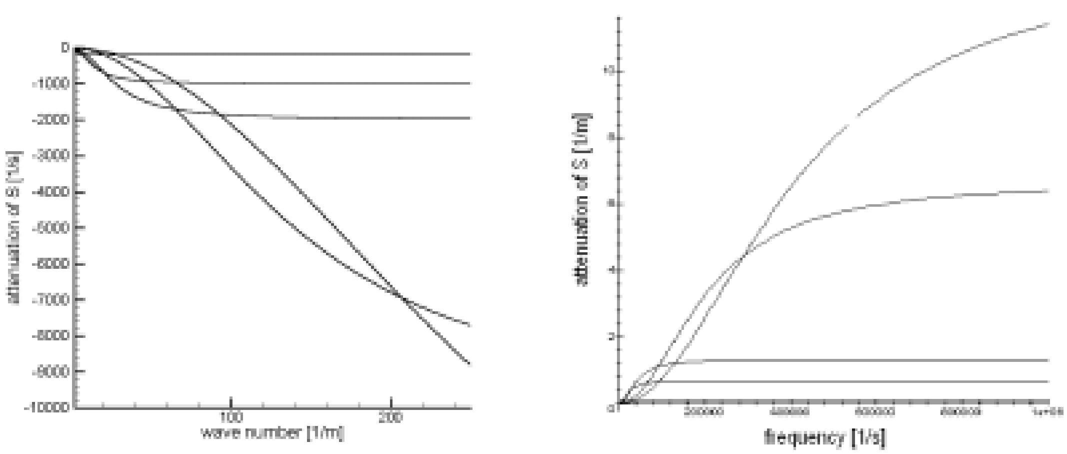

Figure 9: *Attenuation of S–waves for the permeabilities* π:

$10^6, 5*10^6, 10^7, 5*10^7, 10^8$ $[\frac{kg}{m^3 s}]$.

The left diagram corresponds to the initial value problem while the right diagram to the boundary vibrations.

The upper curve corresponds to the lowest permeability.

2.3.4 Group velocities

Propagation of waves with dispersion leads to the effect of propagation of packages of waves of frequencies from a certain interval in the form of an envelope of harmonic waves whose speed is different from a phase speed of any of the waves belonging to such a package. The speed of a package is called the *group velocity* and it is locally defined as the derivative of the frequency with respect to the wave number. It is well known (e.g. [15]) that measurements of speeds may give either a phase speed or a group speed depending on the way of excitation. Therefore it is essential to know the difference between both speeds for an arbitrary frequency.

In the case considered in this work the group velocity for the boundary value problem (a source vibrating with a given frequency ω) follows by the differentiation of the relation (22) with respect to ω

$$c_{gr} = \left(\frac{dk}{d\omega} \right)^{-1}. \tag{30}$$

The group velocity for the initial value problem (chopping) follows from the dispersion relation by differentiation with respect to k. One obtains easily

$$c_{gr} := \frac{d\omega}{dk} = 2\frac{\omega}{k} \frac{c_{P1}^2 G_2 + c_{P2}^2 G_1}{\left(\frac{\omega^2}{k^2} + c_{P1}^2 \right) G_2 + \left(\frac{\omega^2}{k^2} + c_{P2}^2 \right) G_1}, \tag{31}$$

where

$$\begin{aligned} G_1 \quad &: \quad = \frac{\omega^2}{k^2} - c_{P1}^2 + i\frac{\pi}{\rho_0^S}\frac{\omega}{k}, \\ G_1 \quad &: \quad = \frac{\omega^2}{k^2} - c_{P2}^2 + i\frac{\pi}{\rho_0^F}\frac{\omega}{k}, \end{aligned} \tag{32}$$

and the frequency ω is the function of the wave number k determined by the dispersion relation (21). In the present notation this relation has the form

$$G_1 G_2 + \frac{\pi^2}{\rho_0^S \rho_0^F k^2} \frac{\omega^2}{k^2} = 0. \tag{33}$$

In Figure 10 we show in juxtaposition the behaviour of group velocities for the boundary value problem (left), and for the initial value problem (right). We have used the data (24), and made the calculation for a single but representative value of the permeability coefficient $\pi = 10^7 \frac{kg}{m^3 s}$.

The behaviour of the group velocity for the boundary value problem is, as expected, smooth and it goes to the limit of c_{P1}, and c_{P2} respectively as $\omega \to \infty$. It exceeds a little the limit values for medium frequencies because the dispersion curves are in this range convex functions of the frequency.

The situation changes dramatically for the group velocity of P2-waves initiated by the initial impact in the range of wave numbers higher than a critical value (app. $20.556\frac{1}{m}$ in the numerical example). After the plateau of the zero velocity (right diagram of Figure

10) the group velocity decays from the *infinite value* to the c_{P2} limit. This inifinite critical value is related to the fact that the corresponding dispersion relation (Figure 3) possesses a vertical tangent in this point. This is a behaviour which is characteristic for *parabolic problems*. Our model is fully hyperbolic but its limit behaviour for P2-waves reminds this of the Darcy's parabolic model of diffusion in which one neglects inertial effects in the fluid component. In this sense the Darcy's model may be considered to be a low frequency approximation of the hyperbolic two-component model.

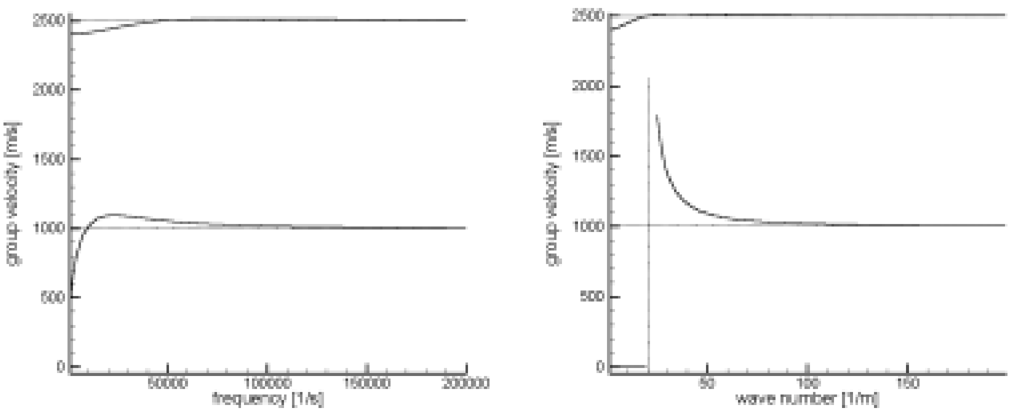

Figure 10: *Group velocities of P1-, and P2-waves for* $\pi = 10^7 \frac{kg}{m^3 s}$.
The left figure corresponds to the boundary value problem while the right one to the initial value problem. The critical value of the wave number: $k = 20.556 \frac{1}{m}$.

The above described properties of P2-waves for initial impact have important consequences for the existence of surface waves. We discuss this problem in the next section.

2.3.5 Conclusions for bulk waves in two-component media

Results presented in the above section show that the simplest possible model of saturated poroelastic materials yields qualitatively the same properties of wave motion as the more sophisticated Biot's model. However in contrast to the latter the model used in this work does not contradict any principal rules of modern continuum thermodynamics. In addition the notions such as tortuosity, anisotropic permeability, etc. which may be essential in some practical applications, are not needed in the construction of all important bulk modes of propagation in spite of claims in the literature on the Biot's model.

As the analysis of monochromatic waves shows the asymptotic behaviour for high frequencies checks with the expectation following from the analysis of singularities of fields. This is independent of the fact if one controls the propagation by harmonic excitations on the boundary (a given real frequency ω) or if one controls an initial condition in which a wave of a particular length (a real wave number k) is excited.

However the situation changes if we consider the low frequency limit. This limit is smooth independently of the external control for the classical two modes of propagation – P1-waves and S-waves. Both these waves have finite phase speeds for $\omega \to 0$ and these are a bit smaller than the speeds of propagation of the corresponding fronts. This is not the case for the P2-mode. This mode behaves like a wave for harmonic excitations on the boundary. The phase speed of this wave goes to zero as $\omega \to 0$. In the vicinity of the zero frequency it has approximately a parabolic character. The behaviour changes entirely in the case of initial conditions. In the vicinity of the zero point of the wave number k (infinitely long waves) the P2-mode has the zero phase velocity and it is solely damped. After a plateau of the zero velocity whose length depends on the value of the permeability coefficient π this mode behaves again as a wave and in the limit of high frequencies (short waves) this behaviour is the same as this of the P2-waves excited by harmonic vibrations.

Such a behaviour has a very important practical bearing. First of all the lack of positive results for the P2-waves in *in situ* measurements may be related not only to the high attenuation of P2-waves but also to the nonexistence of these waves for low frequency initial excitations. It is also very important in the analysis of surface waves in the range of low frequencies commonly used in geophysical applications. We will return to this question in the next section of this work.

Let us mention finally that the attenuation properties of all modes are caused by the relative motion of components reflected by the permeability coefficient π. As the examples presented above clearly show these properties check well with the expectations.

3 Surface waves

3.1 Surface waves in single component media

3.1.1 Introduction

In contrast to bulk waves surface waves propagate along surfaces and their penetration in the direction perpendicular to the surface decays so fast that amplitudes of disturbances can be assumed to be zero in the depth of a few wave lengths. Consequently, their geometrical dispersion is determined in a two dimensional space rather than the three dimensional space of bulk waves. Hence for the point source the amplitude of surface waves decays as r^2 and not as r^3 for the bulk waves, where r is the distance from the source. This property is the main reason for destructive actions of surface waves in earthquakes and, simultaneously, it is the reason for their importance in nondestructive testing of soils. The latter means that one can investigate properties of soils to the depth of app. twice the wave length without the necessity to make boreholes and to investigate laboratory samples distorted by boring and transport.

In the following subsections we present some properties of surface waves for single component heterogeneous elastic media as well as surface waves in two-component porous materials.

Single component models apply to processes in porous materials in which a relative motion of components is not essential. Such are many quasistatic geophysical processes as well as far-field properties of waves in which an influence of P2-waves does not appear anymore. An extensive presentation of wave properties under such conditions can be found in books on seismology of the earth (e.g. [1], [2], [14]).

In the case of surface waves in two-component materials we consider separately two types of boundary conditions for a far-field approximation of a harmonic boundary source. Finally we review some properties of surface waves in two-component porous materials initiated by an impact.

3.1.2 Rayleigh waves in a homogeneous elastic materials

We begin with a brief reminder of the classical Rayleigh problem. It is a two-dimensional dynamical solution of the boundary value problem for a semi-infinite elastic body described by the equations

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \operatorname{div} \mathbf{T}, \quad \mathbf{T} = \lambda \left(\operatorname{tr} \mathbf{e} \right) \mathbf{1} + 2\mu \mathbf{e}, \quad \mathbf{e} := \operatorname{sym} \operatorname{grad} \mathbf{u}, \tag{34}$$

where ρ is a constant mass density, and λ, μ denote Lamé constants.

It is known that the decomposition of the displacement vector $\mathbf{u}$ into a potential and solenoidal parts: $\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T$ yields the equivalent set of equations

$$\begin{aligned} \frac{\partial^2 \mathbf{u}_L}{\partial t^2} &= c_L^2 \triangle \mathbf{u}_L, \quad \operatorname{rot} \mathbf{u}_L = 0, \quad c_L := \sqrt{\frac{\lambda + 2\mu}{\rho}}, \\ \frac{\partial^2 \mathbf{u}_T}{\partial t^2} &= c_T^2 \triangle \mathbf{u}_T, \quad \operatorname{div} \mathbf{u}_T = 0, \quad c_T := \sqrt{\frac{\mu}{\rho}}. \end{aligned} \tag{35}$$

Each of these equations describes a bulk wave in the infinite medium – the first one a longitudinal wave (called a P-wave in geophysics), and the second one a transversal wave (called S-wave in geophysics).

We seek the solution of the following boundary value problem:

$$\left. \mathbf{T}\mathbf{n} \right|_{z=0} = 0, \quad \mathbf{n} \equiv -\mathbf{e}_3, \quad \left. \mathbf{u} \right|_{z \to \infty} = 0. \tag{36}$$

The direction $\mathbf{n}$ coincides with the negative direction of the z-axis.

We make the following ansatz

$$\begin{aligned} \mathbf{u}_L &= A_L e^{-\gamma z} e^{i(kx - \omega t)} \mathbf{e}_1 + B_L e^{-\gamma z} e^{i(kx - \omega t)} \mathbf{e}_3, \\ \mathbf{u}_T &= A_T e^{-\beta z} e^{i(kx - \omega t)} \mathbf{e}_1 + B_T e^{-\beta z} e^{i(kx - \omega t)} \mathbf{e}_3, \end{aligned} \tag{37}$$

where $A_L, B_L, A_T, B_T, \gamma, \beta, k, \omega$ are constants.

Substitution of these relations in conditions $(35)_{2,4}$ yields the compatibility conditions

$$\frac{\gamma^2}{k^2} = 1 - \frac{c_R^2}{c_L^2}, \quad \frac{\beta^2}{k^2} = 1 - \frac{c_R^2}{c_T^2}, \quad c_R := \frac{\omega}{k}. \tag{38}$$

Simultaneously the substitution in equations $(35)_{1,3}$ leads to the following form of the solution

$$\begin{aligned} B_L &= i\frac{\gamma}{k}A_L \quad \Rightarrow \quad \mathbf{u}_L = \left(\mathbf{e}_1 + i\frac{\gamma}{k}\mathbf{e}_3\right) A_L e^{-\gamma z} e^{i(kx-\omega t)}, \\ B_T &= i\frac{k}{\beta}A_T \quad \Rightarrow \quad \mathbf{u}_T = \left(\mathbf{e}_1 + i\frac{k}{\beta}\mathbf{e}_3\right) A_T e^{-\beta z} e^{i(kx-\omega t)}. \end{aligned} \tag{39}$$

It remains to exploit the boundary conditions $(36)_1$. The second condition is identically satisfied provided the constants γ, β are chosen to be positive. According to $(34)_2$ the boundary conditions for stresses can be written in the following form

$$\begin{aligned} \left(c_L^2 - 2c_T^2\right) \frac{\partial \mathbf{u} \cdot \mathbf{e}_1}{\partial x} + c_L^2 \frac{\partial \mathbf{u} \cdot \mathbf{e}_3}{\partial z}\bigg|_{z=0} &= 0, \\ \frac{\partial \mathbf{u} \cdot \mathbf{e}_1}{\partial z} + \frac{\partial \mathbf{u} \cdot \mathbf{e}_3}{\partial x}\bigg|_{z=0} &= 0. \end{aligned} \tag{40}$$

Finally the substitution of relations (39) in the above conditions leads to the homogeneous set of equations for the constants A_L, A_T. Consequently the determinant of this set should be zero, and this gives rise to the following equation

$$\mathcal{P}_R := \left(2 - \frac{c_R^2}{c_T^2}\right)^2 - 4\sqrt{1 - \frac{c_R^2}{c_T^2}}\sqrt{1 - \frac{c_R^2}{c_L^2}} = 0. \tag{41}$$

This equation determines the phase speed $c_R = \frac{\omega}{k}$ of the wave described by the solution (39). It is clear that this solution is independent of the choice of the frequency ω. Hence these waves are nondispersive. Their amplitudes decay with the depth z in an exponential way. For this reason they are called *surface waves*. They have been discovered by Rayleigh. It can be shown that the equation (41) possesses a single real positive solution $c_R < c_T$.

The above solution is not the only surface wave solution of the classical elasticity. In the next subsection we show another one discovered by Love.

3.1.3 Waves in a layer of an ideal fluid and Love waves

In order to appreciate the influence of heterogeneities on the propagation of surface waves we investigate first a simple example of a layer of *ideal fluid* $-\infty < x < \infty$, $0 \leq z \leq H$. The upper surface $z = H$ is free of loading and the lower surface $z = 0$ is in contact with a *rigid body*. The problem is described by the equations of mass and momentum conservation

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0, \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p, \quad p = p_0 + \kappa \left(\rho - \rho_0\right), \tag{42}$$

where ρ_0, p_0 are reference constant values of the mass density and pressure, respectively, and κ denotes a constant compressibility coefficient of the fluid.

Simple manipulations lead to the following wave equation for the pressure p

$$\frac{\partial^2 p}{\partial t^2} = \kappa \triangle p, \quad (x, z) \in (-\infty, \infty) \times (0, H), \tag{43}$$

and $\triangle$ is the Laplace operator. The solution of this equation must satisfy the following boundary conditions

$$p\left(x, z = H, t\right) = 0, \quad v_z\left(x, z = 0, t\right) = 0. \tag{44}$$

We seek the solution in the form of a monochromatic wave of the frequency ω

$$p = \left(Ae^{irkz} + Be^{-irkz}\right) e^{i(kx - \omega t)}. \tag{45}$$

Then the second boundary condition can be replaced by the following one

$$\frac{\partial p}{\partial z}\left(x, z = 0, t\right) = 0. \tag{46}$$

Substitution of (45) in the equation (43) yields the compatibility relation

$$r^2 = \frac{c_{ph}^2}{c^2} - 1, \quad c_{ph} := \frac{\omega}{k}, \quad c := \sqrt{\kappa}. \tag{47}$$

Simultaneously the evaluation of boundary conditions with the ansatz (45) yields the set of homogeneous algebraic relations for the constants A and B

$$\begin{aligned} Ae^{irkH} + Be^{-irkH} &= 0, \\ A - B &= 0. \end{aligned} \tag{48}$$

Consequently the determinant of this set must be equal to zero and we obtain

$$\cos\left(rkH\right) = 0. \tag{49}$$

In order to obtain nontrivial solutions we have to require that r is real. This means, however, that the phase velocities c_{ph} are bigger than the speed of propagation c appearing in the wave equation for the pressure (43)($\kappa \equiv c^2$). If we require that waves of the form

(45) do exist then this seems to violate the basic property of the hyperbolic problem. This result follows from the assumption that the foundation of the fluid is a *rigid body* in which all disturbances propagate with an *infinite speed.* As we see further a modification of the boundary condition $(44)_2$ for the case of contact with an elastic body which we make for the so-called Love waves eliminates this paradox.

Solution of the equation (49) yields immediately the following relation between the phase speed and the frequency

$$c_{ph} = \frac{c}{\sqrt{1 - \frac{\omega_{cr}^2}{\omega^2}}}, \quad \omega_{cr} := \left(n + \frac{1}{2}\right)\pi \frac{c}{H}, \quad n = 1, 2, \ldots \tag{50}$$

This relation is illustrated in Figure 11.

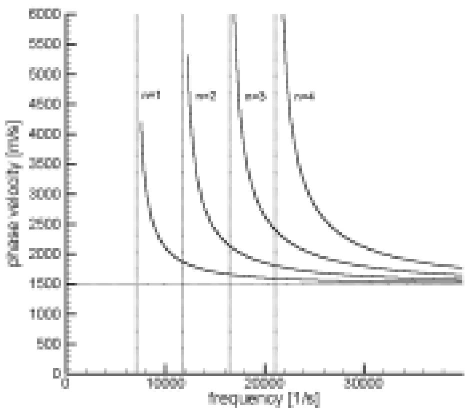

Figure 11: *Phase velocity for a layer of ideal fluid. Numerical data:* $c = 1500\frac{m}{s}, H = 1m.$

Modes: $n = 1, 2, 3, 4$ *are shown in the Figure.*

The paradox of infinite phase speeds does not appear anymore in the case of surface waves which propagate in an elastic layer over an elastic half-space. Transversal waves in such a system have been described in 1911 by Love. We proceed to present briefly these results. They form the simplest illustration of the problem of surface waves in heterogeneous materials.

We consider the propagation of a wave whose amplitude has solely an $\mathbf{e}_2$-component $u_2 \equiv \mathbf{u} \cdot \mathbf{e}_2$ (perpendicular to the (x, z)-plane). The body consists of a layer of thickness H in the z-direction in which the mass density is ρ' and the speed of shear waves is c'_T. This layer is connected to the elastic half-space $z \leq 0$ whose mass density is ρ and the speed of shear waves c_T. We seek the solution of wave equations

$$\begin{aligned} \frac{\partial^2 u'_2}{\partial t^2} &= c'^2_T \triangle u'_2, \quad 0 < z < H, \\ \frac{\partial^2 u_2}{\partial t^2} &= c^2_T \triangle u_2, \quad z < 0, \end{aligned} \tag{51}$$

in the form

$$
\begin{aligned}
u_2' &= \left(A'e^{iks'z} + B'e^{-iks'z}\right) e^{i(kx-\omega t)}, \\
u_2 &= Be^{ksz}e^{i(kx-\omega t)},
\end{aligned}
\tag{52}
$$

i.e. in a form of a monochromatic wave which propagates in the direction of the x-axis with the frequency ω, wave number k in this direction, and with the phase speed $c := \frac{\omega}{k}$. The wave should decay in the z-direction, i.e. s must be positive. We check now if the ansatz (52) can fulfil equations (51), and the following boundary conditions

1) shear stress on the plane $z = H$ is equal to zero, i.e.

$$
\frac{\partial u_2'}{\partial z}(x, z = H, t) = 0, \tag{53}
$$

2) shear stress and the displacement must be continuous on the interface $z = 0$

$$
\begin{aligned}
\rho' c_T' \frac{\partial u_2'}{\partial z}(x, z = 0, t) &= \rho c_T \frac{\partial u_2}{\partial z}(x, z = 0, t), \\
u_2'(x, z = 0, t) &= u_2(x, z = 0, t).
\end{aligned}
\tag{54}
$$

Substitution of the ansatz (52) in equations (51) yields

$$
s'^2 = \frac{c^2}{c_T'^2} - 1, \quad s^2 = 1 - \frac{c^2}{c_T^2}, \quad c \equiv \frac{\omega}{k}. \tag{55}
$$

Boundary conditions (54) lead to a homogeneous set of three algebraic relations for the constants A', B', B. Consequently its determinant must be zero and this condition yields

$$
\omega = \frac{c}{Hs'}\left[\arctan\left(\frac{\rho c_T^2 s}{\rho' c_T'^2 s'}\right) + n\pi\right], \quad n = 1, 2, 3, \ldots, \tag{56}
$$

and both s, and s' must be real, i.e.

$$
c_T' \leq c \leq c_T. \tag{57}
$$

This is the condition for the *existence* of Love waves. Hence the Love waves can propagate solely in layers which are **softer than the foundation**. In addition there exist infinitely many modes of propagation whose existence is limited from below by a corresponding critical frequency. All these modes are dispersive because the phase speeds depend on the frequency given by the inverse relation to (56).

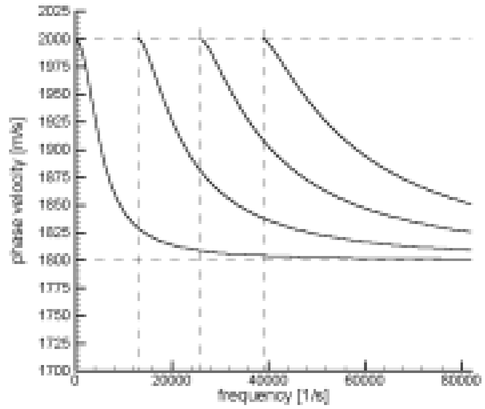

Figure 12: *Fundamental mode and three higher modes ($n = 1, 2, 3$) of the Love wave*

In Figure 12 we show an example of the solution of relation (56) for the following data:

$$c_T = 2000\frac{m}{s}, \quad c_T' = 1800\frac{m}{s}, \quad \frac{\rho'}{\rho} = 0.8, \quad H = 1m. \tag{58}$$

3.1.4 Surface waves in elastic heterogeneous materials

Surface waves observed in geotechnics propagate always over a heterogeneous soil and in such a case not only Love waves but also Rayleigh waves possess multiple modes of propagation. They are described by equations following from the momentum conservation law in which one has to substitute Hooke's law with coefficients dependent on the depth z. Instead of equations (34) we have then

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\triangle\mathbf{u} + (\lambda+\mu)\,\text{grad div}\,\mathbf{u} + \mathbf{e}_3\frac{d\lambda}{dz}\,\text{div}\,\mathbf{u} + \frac{d\mu}{dz}\left(\mathbf{e}_3 \times rot\mathbf{u} + 2\frac{\partial\mathbf{u}}{\partial z}\right), \tag{59}$$

where ρ, λ, μ are function of z and $\mathbf{e}_3$ is the unit vector perpendicular to the boundary (in the direction of z-axis). A solution for harmonic waves is sought in the form

$$\mathbf{u} = \left(u_1\left(z,k,\omega\right)\mathbf{e}_1 + u_3\left(z,k,\omega\right)\mathbf{e}_3\right)e^{i(kx-\omega t)}. \tag{60}$$

Substitution of this ansatz in (59) yields the following set of ordinary differential equations

$$\frac{d\mathbf{f}}{dz} = \mathbf{A}\left(z\right)\mathbf{f}, \quad \mathbf{f} \in\Re^4, \quad \mathbf{A} \in\Re^4 \times \Re^4, \tag{61}$$

where the vector $\mathbf{f}$ and the matrix $\mathbf{A}$ are defined as follows

$$\mathbf{f} := \left(u_1, u_3, f_3, f_4\right)^T, \quad f_3 := \mu\left(\frac{du_1}{dz} + ku_3\right), \quad f_4 := \left(\lambda + 2\mu\right)\frac{du_3}{dz} - k\lambda u_1, \tag{62}$$

$$\mathbf{A} := \begin{pmatrix} 0 & -k & \mu^{-1} & 0 \\ k\lambda\left(\lambda+2\mu\right)^{-1} & 0 & 0 & \left(\lambda+2\mu\right)^{-1} \\ k^2\zeta-\omega^2\rho & 0 & 0 & -k\lambda\left(\lambda+2\mu\right)^{-1} \\ 0 & -\omega^2\rho & k & 0 \end{pmatrix}, \quad \zeta := 4\mu\frac{\lambda+\mu}{\lambda+2\mu}. \tag{63}$$

With this notation the shear stress, τ_{xz}, and the stress component normal to the boundary, σ_z, can be written in the form

$$\tau_{xz} = f_3 e^{i(kx-\omega t)}, \quad \sigma_z = f_4 e^{i(kx-\omega t)}. \tag{64}$$

The set of equations (61) defines a linear differential eigenvalue problem with eigenfunctions $\mathbf{f}$. Boundary conditions associated to this problem follow from the requirement that the stress components (64) vanish for $z=0$, and the eigenvector $\mathbf{f}$ vanishes as $z\to\infty$.

Nontrivial solutions of this eigenvalue problem for a given frequency ω exist only for some values of the wave number, k, say k_j, $j=1,\ldots,M$ which are called eigenvalues of the problem. The relation between the frequency, and eigenvalues is known solely in implicit form

$$\mathcal{D}_R\left(\lambda\left(z\right),\mu\left(z\right),\rho\left(z\right),k_j,\omega\right)=0, \tag{65}$$

called Rayleigh dispersion relation. Solutions of this relation are complex which means that Rayleigh waves in heterogeneous materials are attenuated in contrast to Rayleigh waves in homogeneous materials. In addition they depend on the frequency which means that Rayleigh waves in heterogeneous materials are dispersive.

Several numerical techniques have been developed to solve the above eigenvalue problem. The most popular and successful is most likely the method of discretizing the problem in z-direction (multilayer system) introduced by Eduardo Kausel. We shall not present here any details of those techniques refering to the work of C. Lai [9] for their presentation with corresponding references.

3.2 Surface waves in two-component poroelastic materials

The theory of surface waves in two-component systems differs qualitatively from such a theory for one-component continua. Such waves are produced in linear models by a combination of bulk waves. In the case of a one-component continuum there are two bulk modes of propagation which yield the single Rayleigh wave. For two-component systems we have three bulk modes: P1-waves, P2-waves and S-waves which produce two surface modes in the case of impermeable boundary. For the permeable boundary, i.e. for the case of an additional system – a fluid in the exterior, there may exists three surface modes, etc.

In this section we consider surface waves in two-component homogeneous poroelastic materials with the impermeable boundary. However we indicate as well some properties related to the permeable boundary condition. This condition has been proposed by Deresiewicz and Skalak in the 60ies. However a recent work on problems in which this condition is incorporated indicate some flaws which has not been successfully removed as yet.

3.2.1 Compatibility conditions and dispersion relation

As discussed in [25] we seek a solution of the set (1)–(5) in which we introduce the displacement vector $\mathbf{u}^S$ for the skeleton, and formally the displacement vector $\mathbf{u}^F$ for the fluid. The latter is introduced solely for the technical symmetry of considerations and it does not have any physical bearing. Then

$$\begin{aligned} \mathbf{u}^S &= \operatorname{grad}\varphi^S + \operatorname{rot}\psi^S, \quad \mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}, \quad \mathbf{e}^S = \operatorname{sym}\operatorname{grad}\mathbf{u}^S, \\ \mathbf{u}^F &= \operatorname{grad}\varphi^F + \operatorname{rot}\psi^F, \quad \mathbf{v}^F = \frac{\partial \mathbf{u}^F}{\partial t}. \end{aligned} \tag{66}$$

As the problem is assumed to be two-dimensional we make the following ansatz for solutions harmonic in the x-direction

$$\begin{aligned} \varphi^S &= A^S(z)\exp\left[i(kx-\omega t)\right], \quad \varphi^F = A^F(z)\exp\left[i(kx-\omega t)\right], \\ \psi_z^S &= B^S(z)\exp\left[i(kx-\omega t)\right], \quad \psi_z^F = B^F(z)\exp\left[i(kx-\omega t)\right], \\ \psi_x^S &= \psi_y^S = \psi_x^F = \psi_y^F = 0, \end{aligned} \tag{67}$$

and

$$\begin{aligned} \rho^S - \rho_0^S &= A_\rho^S(z)\exp\left[i(kx-\omega t)\right], \quad \rho^F - \rho_0^F = A_\rho^F(z)\exp\left[i(kx-\omega t)\right], \\ n - n_0 &= A^\Delta \exp\left[i(kx-\omega t)\right]. \end{aligned} \tag{68}$$

Substitution in field equations leads after straightforward calculations to the following compatibility conditions

$$\begin{aligned} B^F &= \frac{i\pi}{\rho_0^F\omega + i\pi}B^S, \quad A^\Delta = -\frac{n_0\omega\tau}{i+\omega\tau}\left(\frac{d^2}{dz^2}-k^2\right)\left(A^F - A^S\right), \\ A_\rho^S &= -\rho_0^S\left(\frac{d^2}{dz^2}-k^2\right)A^S, \quad A_\rho^F = -\rho_0^F\left(\frac{d^2}{dz^2}-k^2\right)A^F, \end{aligned} \tag{69}$$

as well as

$$\left[\kappa\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2\right]A^F + \left[\frac{n_0\beta\omega\tau}{\rho_0^F(i+\omega\tau)}\left(\frac{d^2}{dz^2}-k^2\right)+\frac{i\pi}{\rho_0^F}\omega\right]\left(A^F - A^S\right) = 0, \tag{70}$$

$$\left[\frac{\lambda^S+2\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2\right]A^S - \left[\frac{n_0\beta\omega\tau}{\rho_0^S(i+\omega\tau)}\left(\frac{d^2}{dz^2}-k^2\right)+\frac{i\pi}{\rho_0^S}\omega\right]\left(A^F - A^S\right) = 0, \tag{71}$$

$$\left[\frac{\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2\right]B^S+\frac{i\pi\rho_0^F}{\rho_0^S\left(\rho_0^F\omega+i\pi\right)}\omega^2 B^S=0. \tag{72}$$

It is convenient to introduce a dimensionless notation. In order to do so we define the following auxiliary quantities

$$\begin{aligned} c_s &:= \frac{c_S}{c_{P1}}<1, \quad c_f:=\frac{c_{P2}}{c_{P1}}, \quad \pi':=\frac{\pi\tau}{\rho_0^S}>0, \quad \beta':=\frac{n_0\beta}{\rho_0^S c_{P1}^2}>0, \\ r &:= \frac{\rho_0^F}{\rho_0^S}<1, \qquad z':=\frac{z}{c_{P1}\tau}, \quad k':=kc_{P1}\tau, \quad \omega':=\omega\tau, \end{aligned} \tag{73}$$

where the speeds c_{P1}, c_S, c_{P2} are defined by the relations (14). Further we omit the prime for typographical reasons. Substitution of (73) in equations (70), (71), (72) yields

$$\begin{aligned} \left[c_f^2\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2\right]A^F+\left[\frac{\beta\omega}{r\left(i+\omega\right)}\left(\frac{d^2}{dz^2}-k^2\right)+i\frac{\pi}{r}\omega\right]\left(A^F-A^S\right) &= 0, \\ \left[\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2\right]A^S-\left[\frac{\beta\omega}{i+\omega}\left(\frac{d^2}{dz^2}-k^2\right)+i\pi\omega\right]\left(A^F-A^S\right) &= 0, \\ \left[c_s^2\left(\frac{d^2}{dz^2}-k^2\right)+\omega^2+\frac{i\pi\omega}{\omega+i\frac{\pi}{r}}\right]B^S &= 0. \end{aligned} \tag{74}$$

This differential eigenvalue problem can be easily solved because the matrix of coefficients for homogeneous materials is independent of z. Consequently we seek solutions in the form

$$A^F=A_f^1e^{\gamma_1 z}+A_f^2e^{\gamma_2 z}, \quad A^S=A_s^1e^{\gamma_1 z}+A_s^2e^{\gamma_2 z}, \quad B^S=B_se^{\zeta z}, \tag{75}$$

where the exponents $\gamma_1, \gamma_2, \zeta$ must possess negative real parts. Substitution in (74) yields them in the form

$$\left(\frac{\zeta}{k}\right)^2=1-\frac{1}{c_s^2}\left(1+\frac{i\pi}{\omega+i\frac{\pi}{r}}\right)\left(\frac{\omega}{k}\right)^2, \tag{76}$$

and

$$\begin{aligned} &\left[c_f^2+\left(c_f^2+\frac{1}{r}\right)\frac{\beta\omega}{i+\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2-1\right]^2+\left[1+\left(1+\frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left(\frac{\omega}{k}\right)^4 \\ &+\left[1+c_f^2+\left(1+\frac{1}{r}\right)\frac{\beta\omega}{i+\omega}+\left(c_f^2+\frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2-1\right]\left(\frac{\omega}{k}\right)^2=0. \end{aligned} \tag{77}$$

Simultaneously we obtain the following relations for eigenvectors

$$\mathbf{R}^1=\left(B_s, A_s^1, A_f^1\right)^T, \quad \mathbf{R}^2=\left(B_s, A_s^2, A_f^2\right)^T, \tag{78}$$

where

$$A_f^1 = \delta_f A_s^1, \quad A_s^2 = \delta_s A_f^2, \tag{79}$$

$$\delta_f := \frac{1}{r} \frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}{\left(c_f^2 + \frac{1}{r}\frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi}{\omega r}\frac{\omega^2}{k^2}}, \tag{80}$$

$$\delta_s := \frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}{\left(1 + \frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}. \tag{81}$$

The above solution for the exponents still leaves three unknown constants B_s, A_f^2, A_s^1 which must be specified from boundary conditions. This is the subject of the next subsection. However for technical reasons we solve the problem under a simplifying assumption $\beta = 0$. We have already mentioned in the section on bulk waves that this simplification does not change qualitative properties of acoustic waves and the quantitative influence is small for practically relevant values of β. In addition we solve solely the limit problems in the range of high and low frequencies.

In the case of *high frequency approximation* we immediately obtain from relations (76) and (77)

$$\frac{1}{\omega} \ll 1: \qquad \left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2,$$
$$\left(\frac{\gamma_1}{k}\right)^2 = 1 - \left(\frac{\omega}{k}\right)^2, \quad \left(\frac{\gamma_2}{k}\right)^2 = 1 - \frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2, \tag{82}$$

and

$$\delta_f = \delta_s = 0 \quad \Rightarrow \quad \mathbf{R}^1 = \left(B_s, A_s^1, 0\right)^T, \quad \mathbf{R}^2 = \left(B_s, 0, A_f^2\right)^T. \tag{83}$$

For the case of *low frequency approximation* the equation (77) becomes singular. It can be written in the following form

$$\omega \ll 1: \qquad c_f^2\omega^2\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]^2 + \omega\left[\omega + i\pi\left(1 + \frac{1}{r}\right)\right]\left(\frac{\omega}{k}\right)^4 +$$
$$+ \left[\omega\left(1 + c_f^2\right) + i\pi\left(c_f^2 + \frac{1}{r}\right)\right]\omega\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]\left(\frac{\omega}{k}\right)^2 = 0. \tag{84}$$

Making the following substitution

$$W := \omega \left[\left(\frac{\gamma}{k} \right)^2 - 1 \right] + \omega \left(\frac{\omega}{k} \right)^2 \frac{1 + c_f^2}{2c_f^2}, \tag{85}$$

we obtain the quadratic equation for W

$$c_f^2 W^2 + i\pi \left(c_f^2 + \frac{1}{r} \right) \left(\frac{\omega}{k} \right)^2 W - \tag{86}$$

$$- \left\{ i\pi \left[\frac{\left(c_f^2 + \frac{1}{r} \right) \left(1 + c_f^2 \right)}{2c_f^2} - \left(1 + \frac{1}{r} \right) \right] \left(\frac{\omega}{k} \right)^4 \right\} \omega + O\left(\omega^2 \right) = 0,$$

which for small ω can be solved by the regular perturbation method

$$W = W_0 + \omega W_1 + O\left(\omega^2 \right). \tag{87}$$

After easy calculations we obtain

$$W = \begin{cases} \left[\frac{1+c_f^2}{2c_f^2} - \frac{r+1}{rc_f^2+1} \right] \left(\frac{\omega}{k} \right)^2 \omega, \\ - \left[\frac{1+c_f^2}{2c_f^2} - \frac{r+1}{rc_f^2+1} \right] \left(\frac{\omega}{k} \right)^2 \omega - i\pi \frac{rc_f^2+1}{rc_f^2} \left(\frac{\omega}{k} \right)^2 \end{cases}. \tag{88}$$

Bearing the relation (85) in mind we arive at the following results for the exponents

$$\omega \ll 1: \qquad \left(\frac{\zeta}{k} \right)^2 = 1 - \frac{r+1}{c_s^2} \left(\frac{\omega}{k} \right)^2,$$

$$\left(\frac{\gamma_1}{k} \right)^2 = 1 - \frac{r+1}{rc_f^2+1} \left(\frac{\omega}{k} \right)^2, \tag{89}$$

$$\left(\frac{\gamma_2}{k} \right)^2 = 1 - \frac{rc_f^4+1}{c_f^2 \left(rc_f^2+1 \right)} \left(\frac{\omega}{k} \right)^2 - \frac{i\pi}{\omega} \frac{rc_f^2+1}{rc_f^2} \left(\frac{\omega}{k} \right)^2,$$

and for the coefficients of amplitudes

$$\delta_f = 1 - \frac{\omega r}{i\pi} \frac{1-c_f^2}{1+rc_f^2}, \quad \delta_s = -rc_f^2 \left(1 - \frac{\omega r}{i\pi} \frac{1-c_f^2}{1+rc_f^2} \right). \tag{90}$$

Obviously due to the singular character of the equation (84) the last contribution to $\frac{\gamma_2}{k}$ becomes singular for $\omega \to 0$.

3.2.2 Boundary value problems for surface waves

In order to determine surface waves in saturated poroelastic medium we need conditions for $z = 0$. We discuss in details the problem in which this boundary is impermeable, i.e. a poroelastic medium is in contact with vacuum. Boundary conditions have then the form

$$T_{13}\big|_{z=0} \equiv T_{13}^S\Big|_{z=0} = \mu^S \left(\frac{\partial u_1^S}{\partial z} + \frac{\partial u_3^S}{\partial x} \right)\Bigg|_{z=0} = 0, \tag{91}$$

$$T_{33}\big|_{z=0} \equiv \left(T_{33}^S - p^F\right)\Big|_{z=0} =$$

$$= c_{P1}^2 \rho_0^S \left(\frac{\partial u_1^S}{\partial x} + \frac{\partial u_3^S}{\partial z} \right) - 2c_S^2 \rho_0^S \frac{\partial u_1^S}{\partial x} - c_{P2}^2 \left(\rho^F - \rho_0^F \right)\Bigg|_{z=0} = 0, \tag{92}$$

$$\frac{\partial}{\partial t} \left(u_3^F - u_3^S \right)\Bigg|_{z=0} = 0, \tag{93}$$

where the first two conditions mean that the surface $z = 0$ is stress-free (far-field approximation), and the last condition means that there is no transport of fluid mass through this surface (impermeable boundary). u_1^S, u_3^S denote the components of the displacement $\mathbf{u}^S$ in the direction of x-axis and z-axis, respectively, while u_3^F is the z-component of the displacement $\mathbf{u}^F$.

In the case of permeable boundary the last condition would not hold. Instead the mass transport through the surface must be specified by a relation to a driving force. According to the proposition of Deresiewicz and Skalak such a driving force is proportional to the difference of pore pressures on both sides of the boundary. In the earlier paper on surface waves on such boundaries [7] we have used the following condition

$$\rho_0^F \frac{\partial}{\partial t} \left(u_3^F - u_3^S \right) - \alpha \left(p^F - n_0 p_{ext} \right)\Bigg|_{z=0} = 0, \tag{94}$$

where α denotes a surface permeability coefficient and p_{ext} is an external pressure. This condition relies on the assumption that the pore pressure p and the partial pressure p^F satisfy the relation $p^F \approx n_0 p$ at least in a small vicinity of the surface. In some cases it may be a good approximation and the results presented in [7] check qualitatively very well with observations. However the condition seems to be violated on boundaries of granular materials with a relatively small porosity. We leave this issue unclarified in this work and refer to a future research.

Substitution of results of subsection 3.2.1. in the boundary conditions (91)-(93) yields the following equations for three unknown constants B_s, A_f^2, A_s^1

$$\mathbf{AX} = \mathbf{0}, \tag{95}$$

where

$$\mathbf{A} := \begin{pmatrix} \left(\frac{\zeta}{k}\right)^2 + 1 & 2i\frac{\gamma_2}{k}\delta_s & 2i\frac{\gamma_1}{k} \\ -2ic_s^2\frac{\zeta}{k} & \begin{array}{c}\left[\left(\frac{\gamma_2}{k}\right)^2 - 1 + 2c_s^2\right]\delta_s + \\ +rc_f^2\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right]\end{array} & \begin{array}{c}\left(\frac{\gamma_1}{k}\right)^2 - 1 + 2c_s^2 + \\ +rc_f^2\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right]\delta_f\end{array} \\ i\frac{r\omega}{r\omega + i\pi} & -\left(\delta_s - 1\right)\frac{\gamma_2}{k} & \left(\delta_f - 1\right)\frac{\gamma_1}{k} \end{pmatrix}, \tag{96}$$

$$\mathbf{X} := \left(B_s, \quad A_f^2, \quad A_s^1 \right)^T.$$

This homogeneous set yields the *dispersion relation:* $\det \mathbf{A} = 0$ determining the $\omega - k$ relation. We investigate separately solutions of this equations for high and low frequencies.

3.2.3 High frequency approximation

In the case of high frequencies $\frac{1}{\omega} \ll 1$ we have $\delta_s = \delta_f = 0$ and the dispersion relation follows in the form

$$\mathcal{P}_R\sqrt{1 - c_f^2\left(\frac{\omega}{k}\right)^2} + \frac{r}{c_s^4}\left(\frac{\omega}{k}\right)^4\sqrt{1 - \left(\frac{\omega}{k}\right)^2} = 0, \tag{97}$$

where

$$\mathcal{P}_R := \left(2 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2 - 4\sqrt{1 - \left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2}. \tag{98}$$

Hence for $r = 0$ the relation (97) reduces to $\mathcal{P}_R = 0$ which is the Rayleigh dispersion relation for single component continua. Otherwise we obtain the relation identical with this analysed by I. Edelman and K. Wilmanski [7] in the limit of short waves (i.e. $\frac{1}{k} \ll 1$). Consequently the conclusions for this case are the same as well. As shown in the paper [7] the equation (97) possesses two roots defining two surface waves: a true Stoneley wave which propagates almost without attenuation with the speed a bit smaller than c_f as well as a generalized Rayleigh wave which is leaky (i.e. it radiates the energy to the P2-wave) and propagates with the speed c_R: $c_f < c_R < c_s$.

3.2.4 Low frequency approximation

A limit of long waves has been recently analysed by I. Edelman [6]. She has shown that the existence of a critical value k_{cr} of the wave number k for initial value problems as described in Section 2 for bulk waves yields the nonexistence of Stoneley surface waves in the range $0 \leq k \leq k_{cr}$. This is not the case for boundary value problems in which we consider the limit of small frequencies ω rather than the limit of long waves. In this limit surface waves do exist and we proceed to investigate this problem in some details.

If we account for the relations (89) and (90) in the condition $\det \mathbf{A} = 0$ then we obtain the dispersion relation reflecting a dependence of $\frac{\omega}{k}$ on ω. The expansion with respect to $\sqrt{\omega}$ yields the identity in the zeroths order and the following relation for the higher order

$$\left(\frac{\omega}{k}\right)\left\{\left(2-\frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{r+1}{rc_f^2+1}\left(\frac{\omega}{k}\right)^2}\right\}+ +O\left(\sqrt{\omega}\right)=0. \tag{99}$$

Clearly we obtain two soultions:

1. Rayleigh wave whose speed is different from zero in the limit $\omega \to 0$ and whose attenuation is of the order $O\left(\sqrt{\omega}\right)$. The relation for the speed reminds the relation (41) with the speeds of bulk waves replaced by the low frequency limits. Namely we have

$$\frac{r+1}{c_s^2}=c_{P1}^2\frac{\rho_0^S+\rho_0^F}{\mu^S}\equiv\frac{c_{P1}^2}{c_{oS}^2},\quad \frac{r+1}{rc_f^2+1}=c_{P1}^2\frac{\rho_0^S+\rho_0^F}{\lambda^S+2\mu^S+\rho_0^F}\equiv\frac{c_{P1}^2}{c_{oP1}^2}, \tag{100}$$

   and these relations follow from the definitions (23), (28). Consequently

$$\left(2-\frac{c_{P1}^2}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\frac{c_{P1}^2}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{c_{P1}^2}{c_{oP1}^2}\left(\frac{\omega}{k}\right)^2}=0. \tag{101}$$

2. Stoneley wave has the speed of propagation of the order $O\left(\sqrt{\omega}\right)$. Hence it goes to zero in the same way as the speed of propagation of P2-wave.

4 Final remarks

The results for a two-component model of porous solid-fluid mixtures presented in this work should be compared with those obtained by means of the Biot's model and with experimental observations. We shall not go into details of such a comparison in this work. However it can be easily checked that there is a very good qualitative agreement of both models as far as propagation of acoustic waves is concerned and these check well with experimental observations. One should mention the following features following from the above considerations.

1. The analysis of acoustic waves based on the model proposed by K. Wilmanski does not contain the flaws of the Biot's model: the violation of the second law in the case

of the simple model (without contributions of higher gradients) and the violation of the material objectivity principle by the contribution of relative acceleration in the Biot's model.

2. There exist three bulk modes of propagation: P1-waves, S-waves and P2-waves.

3. Speeds of propagation of these waves check with experimental observations in both limits of high and low frequencies.

4. There exist two modes of surface waves for impermeable boundaries: Rayleigh wave and Stoneley wave and these check again with the observations.

In contrast to the high frequency and short wave limits which coincide, the low frequency limit is different from that obtained earlier for the long waves. Such a limit exists for $\omega \to 0$ while the long wave limit contains the zone $0 \leq k \leq k_{cr}$ forbidden for the propagation.

Let us mention finally that the motivation for a wave analysis with a real wave number k rather than with a real frequency ω, used in the papers [5], [6], [7] was primarily based on the observation that this corresponds better with the way in which acoustic waves are initiated in engineering applications. In contrast to the far field approximation of seismic waves which is usually based on the frequency analysis, engineering applications were primarily concerned with waves initiated by chopping or explosions which led to initial value problems. This is not the case any more. Numerous experiments and measurements are made nowadays by divices producing harmonic vibrations (e.g. [13]) and this leads to a boundary value problem in which the real frequency ω is the proper choice of the control variable.

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