Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

On the uniqueness of solutions for nonlinear elliptic–parabolic equations

Herbert Gajewski[1], Igor V. Skrypnik[2]

submitted: 15 July 2002

[1] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: gajewski@wias-berlin.de

[2] Institute for Applied Mathematics and Mechanics
Rosa Luxemburg Str. 74
83114 Donetsk
Ukraine
E-Mail: skrypnik@iamm.ac.donetsk.ua

No. 754
Berlin 2002

2000 *Mathematics Subject Classification.* 35B45, 35K15, 35K20, 35K65.

Key words and phrases. Nonlinear parabolic equations, bounded solutions, uniqueness, non-standard assumptions, degenerate type.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We prove a priori estimates in $L^2(0,T;W^{1,2}(\Omega))$ and $L^\infty(Q_T)$, existence and uniqueness of solutions to Cauchy–Dirichlet problems for elliptic–parabolic systems

$$\frac{\partial\sigma(u)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho(u) b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\} + a\big(t,x,v,u\big) = 0,$$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big[k(x)\frac{\partial v}{\partial x_i}\Big] + \sigma(u) = f(t,x)\ ,\ (t,x)\in Q_T = (0,T)\times\Omega\ ,$$

where $\rho(u) = \frac{\partial\sigma(u)}{\partial u}$. Systems of such form arise as mathematical models of various applied problems, for instance, electron transport processes in semiconductors. Our basic assumption is that $\log\rho(u)$ is concave. Such assumption is natural in view of drift–diffusion models, where σ has to be specified as a probality distribution function like a Fermi integral and u resp. v have to be interpreted as chemical resp. electrostatic potential.

1 Introduction

We prove a priori estimates, existence and uniqueness of weak solutions to initial boundary value problems of the form

$$\frac{\partial\sigma(u)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho(u) b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\} + a\big(t,x,v,u\big) = 0, \quad (t,x)\in Q_T, \tag{1}$$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big[\kappa(x)\frac{\partial v}{\partial x_i}\Big] + \sigma(u) = f(t,x), \quad (t,x)\in Q_T, \tag{2}$$

$$u(t,x) = g_1(t,x), \quad (t,x)\in\Gamma = (0,T)\times\partial\Omega, \tag{3}$$

$$v(t,x) = g_2(t,x), \quad (t,x)\in\Gamma = (0,T)\times\partial\Omega, \tag{4}$$

$$u(0,x) = h(x), \quad x\in\Omega, \tag{5}$$

where $\sigma(u) = \int_0^u \rho(s)ds,\ \Omega$ is a bounded open set in $\mathbb{R}^n$ and $Q_T = (0,T)\times\Omega,\ T>0$.

Systems of the form (1), (2) arise as mathematical models of various applied problems, for instance reaction–drift–diffusion processes of electrically charged species, phase transition processes and transport processes in porous media. The investigation of nonlinear reaction–drift–diffusion systems has received much attention in recent years [1].

The equation (1) is degenerate because the function $\rho(u)$ can tend to zero. Cauchy–Dirichlet problems for degenerate parabolic equations have been studied by many authors (see for example [2], [3], [10]). But the structure of the equation (1) is different from that one considered in these papers. Boundary value problems for the equation of the structure (1) were studied by the authors in the stationary case in [8] and in the nonstationary case in [9].

The initial–boundary value problem for systems of the form (1) – (2) was studied in [5] under essentially stronger assumptions as in the presented paper. In [5] the solvability was proved for the special case $b_i(t,x,\xi) = \xi_i$, uniqueness was shown under the regularity assumption $v(x,t) \in L^\infty(0,T,W^{1,p}(\Omega))$, $p > n$.

We consider problem (1) – (5) under standard conditions for the functions $b_i(t,x,\xi)$ and some conditions for the function $a(t,x,v,u)$ to be formulated in Section 2. Our main specific assumption reads:

ρ) *$\rho \in (\mathbb{R}^1 \to \mathbb{R}^1)$ with $\rho(u) > 0$, $u \in \mathbb{R}^1$, is continuous and has a piecewise continuous derivative ρ' such that $\frac{\rho'(u)}{\rho(u)}$ is nonincreasing on $\mathbb{R}^1$.*

For the semiconductor theory [5] relevant examples for functions ρ satisfying condition ρ) are given by $\sigma = \mathcal{F}_{\gamma+1}$, $\rho = \sigma' = \mathcal{F}_\gamma$, where $\mathcal{F}_\gamma$ denotes the Fermi integral

$$\mathcal{F}_\gamma(u) = \frac{1}{\Gamma(\gamma+1)} \int_0^\infty \frac{s^\gamma \, ds}{1+\exp(s-u)} \qquad \gamma > -1 \, . \tag{6}$$

Another example comes from phase separation problems [7], where the Fermi function

$$\sigma(u) = \frac{1}{1+exp(-u)}\,, \qquad \rho(u) = \sigma'(u) = \frac{1}{(1+e^u)(1+e^{-u})}$$

plays a role corresponding to $\mathcal{F}_{\gamma+1}$.

We formulate our assumptions and main results in Section 2. First a priori estimates for solutions u, v are given in Section 3. In that Section we prove also regularity properties of the function v, important for further considerations. An L^∞ estimate of u is given in Section 4. Section 5 is devoted to the existence proof for solutions of problem (1) – (5). Our main result, uniqueness of solutions, is proved in Section 6.

Note that our considerations can be carried over to the case of Neumann boundary conditions instead of the Dirichlet conditions (3), (4).

We are planning in forthcoming papers to apply our approach to more general reaction–drift–diffusion systems, including more than one species and temperature.

2 Formulation of assumptions and main results

Let Ω be a bounded open set in $\mathbb{R}^n$ and $Q_T = (0,T)\times\Omega,\ T>0$. We shall assume that $n>2$. For $n\le 2$ it is necessary to make simple changes in our conditions that are connected with Sobolev's embedding theorem.

We assume following regularity condition on the boundary $\partial\Omega$ of the set Ω:

∂) there exist positive numbers χ, R_0, such that for an arbitrary point $x\in\partial\Omega$ the inequality $\operatorname{meas}\{B(x,R)\setminus\Omega\}\ge \chi R^n$ holds, where $0<R\le R_0$ and $B(x,R)$ is a ball of radius R with center x.

Let the coefficients $b_i,\ a,\ \kappa$ from (1), (2) satisfy following assumptions:

i) $a(t,x,v,u),\ b_i(t,x,\xi),\ i=1,\dots,n,$ *are measurable functions with respect to* t,x *for every* $u,v\in\mathbb{R}^1,\ \xi\in\mathbb{R}^n$ *and continuous with respect to* $u,v\in\mathbb{R}^1$, $\xi\in\mathbb{R}^n$, *for almost every* $(t,x)\in Q_T$; $b_i(t,x,0)=0$; $\kappa(x)$ *is measurable function of* x;

ii) *there exist positive constants* ν_1,ν_2 *such that for arbitrary* $\xi',\xi''\in\mathbb{R}^n$, $(t,x)\in Q_T$, *following inequalities hold*

ii)$_1$ $\sum_{i=1}^n\big[b_i(t,x,\xi')-b_i(t,x,\xi'')\big]\big(\xi_i'-\xi_i''\big)\ge\nu_1|\xi'-\xi'|^2$,

ii)$_2$ $|b_i(t,x,\xi)|\le\nu_2(|\xi|+1),\quad i=1,\dots,n,$

ii)$_3$ $\nu_1\le\kappa(x)\le\nu_2$;

iii) *there exists a nonnegative function* $\alpha\in L^{p_1}(Q_T)$, $p_1>\frac{n+2}{2}$, *such that for arbitrary* $(t,x)\in Q_T,\ v,u,u',u''\in\mathbb{R}^1$ *following inequalities hold*

$\big[a(t,x,v,u')-a(t,x,v,u'')\big](u'-u'')\ge\nu_1|u'-u''|^2$,

$|a(t,x,v,u)|\le\nu_2(|v|+|u|)+\alpha(t,x)$.

We note some simple consequences from condition ρ). Let

$$\alpha_\pm=\lim_{u\to\pm\infty}\rho(u). \tag{7}$$

Then for nonconstant functions ρ at least one of the numbers $\alpha_-,\ \alpha_+$ is zero [8]. Studying the behavior of the solution to (1) – (5) we have to distinguish the cases of zero or non–zero value of $\alpha_\pm$. In order to include both cases, we assume

$$\alpha_-=0,\quad \alpha_+\ne 0. \tag{8}$$

The considerations for the case $\alpha_-=\alpha_+=0$ are analogous. We remark only that the assumptions for the function $a(t,x,v,u)$ are connected with the behavior of the function ρ and that the condition *iii)* corresponds to the case (8).

We consider problem (1) – (5) with data such that

$$f \in C\big([0,T]; L^{p_2}(\Omega)\big), \quad \frac{\partial f}{\partial t} \in L^2\big(0,T; \big[\overset{\circ}{W}{}^{1,2}(\Omega)\big]^*\big), \quad p_2 > \frac{n}{2}, \tag{9}$$

$$\begin{gathered} g_i \in L^\infty\big(Q_T\big) \,\cap\, L^\infty\big(0,T; W^{1,2}(\Omega)\big) \,\cap\, L^1\big(0,T; W^{1,\infty}(\Omega)\big), \\ \frac{\partial g_1}{\partial t} \in L^1\big(0,T; L^\infty(\Omega)\big), \quad \frac{\partial g_2}{\partial t} \in L^2\big(0,T; L^2(\Omega)\big), \end{gathered} \tag{10}$$

$$h \in L^\infty(\Omega). \tag{11}$$

Definition 1 *A pair of functions (u,v), $u,v \in L^2\big(0,T; W^{1.2}(\Omega)\big)$ is called solution of problem (1) – (5) if following conditions are satisfied:*

i) $\sigma(u) \in C\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; L^{\frac{2n}{n+2}}(\Omega)\big)$,

$$\int_{Q_T}\!\!\int \rho(u)\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big)dx\,dt < \infty, \tag{12}$$

the time derivative of $\sigma(u)$ in the sense of distributions satisfies

$$\frac{\partial \sigma(u)}{\partial t} \in L^2\big(0,T; \big[\overset{\circ}{W}{}^{1,2}(\Omega)\big]^*\big) \tag{13}$$

and the integral identities

$$\int_0^\tau \Big\{ < \frac{\partial \sigma(u)}{\partial t}, \varphi > + \int_\Omega \Big[\sum_{i=1}^n \rho(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} + a(t,x,v,u)\varphi\Big]dx\Big\}dt = 0, \tag{14}$$

$$\int_\Omega \Big\{\kappa(x)\sum_{i=1}^n \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \sigma(u)\psi - f(t,x)\psi\Big\}dx = 0 \tag{15}$$

hold for arbitrary functions $\varphi \in C^\infty\big(\overline{Q}_T\big)$ vanishing near $\Gamma, \psi \in C_0^\infty(\Omega)$ and almost every $\tau \in (0,T)$;

ii)
$$u - g_1 \in L^2\big(0,T; \overset{\circ}{W}{}^{1,2}(\Omega)\big),\ v - g_2 \in L^2\big(0,T; \overset{\circ}{W}{}^{1,2}(\Omega)\big)\ ; \tag{16}$$

iii) for functions φ, as in (14) and satisfying additionally $\varphi(\tau,x) = 0$ for $x \in \Omega$ the equality

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \varphi > dt \ + \int_0^\tau \int_\Omega [\sigma(u) - \sigma(h)]\, \frac{\partial \varphi}{\partial t}\, dx\, dt = 0 \tag{17}$$

holds for $\tau \in (0,T)$.

In order to justify this definition it is sufficient to show that

$$\sigma(u) \in L^1(Q_T), \quad \rho(u) \in L^1(Q_T). \tag{18}$$

The first inclusion in (18) follows immediately from the assumption $\sigma(u) \in L^2(0,T; L^{\frac{2n}{n+2}}(\Omega))$. The second one follows from the inequality

$$\rho(u) \leq \frac{\rho(1)}{\sigma(1)}\,\sigma(u) \quad \text{for} \quad u \geq 1\,, \tag{19}$$

which is a consequence of condition $\rho)$ and

$$\frac{d}{du}\Big(\frac{\sigma(u)}{\rho(u)}\Big) = 1 - \frac{\rho'(u)}{\rho^2(u)}\int_0^u \rho(s)\; ds \geq 1 - \frac{1}{\rho(u)}\int_0^u \frac{\rho'(s)}{\rho(s)}\rho(s)\; ds = \frac{\rho(0)}{\rho(u)} > 0.$$

Remark 1 *Let (u,v) be a solution of problem (1)-(5). Since the set of functions from $C^\infty(\overline{Q}_T)$ vanishing near Γ is dense in $L^2\big(0,T;\overset{\circ}{W}{}^{1,2}(\Omega,\rho(u))\big)$, the integral identity (14) holds for all $\varphi \in L^2\big(0,T;\overset{\circ}{W}{}^{1,2}(\Omega)\big)$ such that*

$$\int_{Q_T}\!\!\int \rho(u)\Big|\frac{\partial\varphi}{\partial x}\Big|^2\, dx\; dt < \infty.$$

Analogously the identity (15) holds for arbitrary functions $\psi \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

Besides of (1), (2) we consider the regularized system

$$\frac{\partial\sigma(u)}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big\{\rho_\delta(u) b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\} + a\big(t,x,v,u\big) = 0, \tag{20}$$

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\Big[\kappa(x)\frac{\partial v}{\partial x_i}\Big] + \sigma(u) = f(t,x) \tag{21}$$

with

$$\rho_\delta(u) = \max\Big\{\rho(u), \rho\Big(-\frac{1}{\delta}\Big)\Big\} \quad \text{for} \quad \delta \in (0,1], \quad \rho_0(u) = \rho(u). \tag{22}$$

We understand solutions of the auxiliary problem (20), (21), (3) – (5) in the sense of Definition 1 after replacing $\rho(u)$ in (12) and (14) by $\rho_\delta(u)$.

In what follows we understand as known parameters all numbers from the conditions *ii)*, *iii)*, norms of functions f, g_1, g_2, h, α in respective spaces and numbers that depend only on $n, \chi, R_0, \Omega, \rho$.

Theorem 1 *Let the conditions i) – iii), ρ), (9) – (11) be satisfied. Then there exists a constant M_1 depending only on known parameters and independent of $\delta \in [0,1]$ such that each solution u, v of problem (20), (21), (3) – (5) satisfies*

$$\operatorname*{ess\;sup}_{t\,\in\,(0,T)} \int_\Omega \Big\{\Lambda(u,(t,x)) + \Big|\frac{\partial v(t,x)}{\partial x}\Big|^2\Big\}\, dx + \int_{Q_T}\!\!\int \rho_\delta(u)\Big|\frac{\partial(u-v)}{\partial x}\Big|^2\, dt\, dx \leq M_1, \tag{23}$$

where

$$\Lambda(u) = \int_0^u s\,\rho(s)\,ds. \tag{24}$$

For proving regularity properties of the function v we need following growth condition

$$\rho_1^{-1}(u^\gamma + 1) \le \rho(u) \le \rho_1(u^\gamma + 1)\,, \quad u > 0\,, \quad 0 \le \gamma < \frac{2}{n-2} \tag{25}$$

with some positive constant ρ_1. (25) implies $\sigma(u) \le \rho_1\big(\frac{u^{\gamma+1}}{\gamma+1} + u\big)$ for $u > 0$ with $\gamma + 1 < \frac{n}{n-2}$. Remark that such type condition arised in [5] for $n > 2$ together with the stronger restriction $\gamma + 1 < \frac{2}{n-2}$.

Theorem 2 *Let the assumptions of Theorem 1 and condition (25) be satisfied. Then there exists a constant M_2, depending only on known parameters and independent of $\delta \in [0,1]$, such that each solution of problem (20), (21), (3) – (5) satisfies*

$$\int_{Q_T}\int \rho_\delta(u)\Big\{\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big\}\,dx\,dt \le M_2. \tag{26}$$

Theorem 3 *Let the assumptions of Theorem 2 be satisfied. Then the estimates*

$$||v||_{L^\infty(Q_T)} \le M_3, \quad |v(t,x') - v(t,x'')| \le H|x' - x''|^\eta \tag{27}$$

hold for arbitrary $t \in [0,T]$, $x', x'' \in \Omega$ with $\eta \in (0,1)$ and constants M_3, H, η depending only on known parameters and independent of δ.

In order to prove a priori estimates for u we need additional conditions with repect to ρ and a. In view of our uniqueness result we assume stronger conditions for a than needed if proving a priori estimates only:

a) $\frac{a(t,x,v,u)}{\rho(u)}$ is nondecreasing with respect to $u \in \mathbb{R}^1$, for arbitrary $(t,x) \in Q_T$, $v \in \mathbb{R}^1$;

ρ') there exists a positive constant ρ_2 such that $\rho'(u) \le \rho_2 \cdot \rho(u)$ holds for $u < 0$.

Theorem 4 *Let the conditions i) – iii), ρ), ρ'), a), (9) – (11), (25) be satisfied. Then there exists a constant M_4, depending only on known parameters and independent of $\delta \in [0, \frac{1}{M_4}]$, such that each solution u, v of problem (20), (21), (3) – (5) satisfies*

$$ess\ \sup\{|u(t,x)| : (t,x) \in Q_T\} \le M_4. \tag{28}$$

Theorem 5 *Let the conditions i) – iii), ρ), ρ'), a), (9) – (11), (25) be satisfied. Then the initial–boundary value problem (1) – (5) has at least one solution in the sense of Definition 1.*

Theorem 6 *Let the conditions i) – iii), ρ), ρ'), a), (9) – (11), (25) be satisfied and assume additionally that the functions $b_i(t,x,\xi)$, $\rho'(u)$, $a(t,x,v,u)$ are locally Lipschitzian with respect to ξ, u, v respectively. Then the initial–boundary value problem (1) – (5) has a unique solution u, v in the sense of the Definition 1.*

Proofs of theorems 1, 2, 3 are given in Section 3, proofs of theorems 4, 5, 6 are given in Sections 4, 5, 6 respectively.

3 Regularity of the function v

We start this section proving firstly the priori estimate (23). Next we shall prove boundedness and Hölder continuity of the function v.

Proof of Theorem 1. Denote by $v_0(x)$ the solution of problem (21), (4) for $t=0$ with $u(0,x)$ defined by (5) and let $(u(t,x),\, v(t,x))$ be the solution of problem (20), (21), (3) – (5). We extend functions $u(t,x)$, $v(t,x)$ by setting $u(t,x)=h(x)$, $v(t,x) = v_0(x)$ for $t<0$, $x\in\Omega$. In an analogous way we extend the functions $f(t,x)$, $g_2(t,x)$. Denote

$$\widetilde{u}(t,x) = u(t,x) - g_1(t,x), \quad \widetilde{v}(t,x) = v(t,x) - g_2(t,x).$$

Testing (15) with $\psi(x) = \widetilde{v}(t+s,x) - \widetilde{v}(t,x)$, we obtain for $\tau\in(0,T)$, $s\in(0,T-\tau)$

$$\int_{-s}^{\tau}\int_{\Omega}\Big\{\kappa(x)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big[v(t+s),x)+v(t,x)\big]\frac{\partial}{\partial x_i}\big[\widetilde{v}(t+s),x)-\widetilde{v}(t,x)\big]+$$

$$\big[\sigma(u(t+s,x))+\sigma(u(t,x))-f(t+s,x)-f(t,x)\big]\big[\widetilde{v}(t+s),x)-\widetilde{v}(t,x)\big]\Big\}\,dxdt=0.$$

Hence we get by simple calculations

$$\begin{aligned}
&\int_{\tau}^{\tau+s}\int_{\Omega}\kappa(x)\Big|\frac{\partial v}{\partial x}\Big|^2 dx\,dt - s\int_{\Omega}\kappa(x)\Big|\frac{\partial v_0}{\partial x}\Big|^2 dx\,dt-\\
&\int_{-s}^{\tau}\int_{\Omega}\kappa(x)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big[v(t+s),x)+v(t,x)\big]\frac{\partial}{\partial x_i}\big[g_2(t+s),x)-g_2(t,x)\big]\,dx\,dt+\\
&\int_{0}^{\tau}\int_{\Omega}\big[\sigma(u(t-s,x))+\sigma(u(t+s,x))-f(t-s,x)+f(t+s,x)\big]\widetilde{v}(t,x)\,dx\,dt+\\
&\int_{\tau}^{\tau+s}\int_{\Omega}\big[\sigma(u(t-s,x))+\sigma(u(t,x))-f(t-s,x)-f(t,x)\big]\widetilde{v}(t,x)\,dx\,dt-\\
&\int_{-s}^{0}\int_{\Omega}\big[\sigma(u(t,x))-f(t,x)+\sigma(u(t+s,x))-f(t+s,x)\big]\widetilde{v}(t,x)\,dxdt=0.
\end{aligned}\tag{29}$$

Dividing this equality by s and passing to the limit $s \to 0$, we obtain for almost every $\tau \in (0,T)$

$$\int_\Omega \kappa(x)\Big|\frac{\partial v(\tau,x)}{\partial x}\Big|^2\,dx \;-\; \int_\Omega \kappa(x)\Big|\frac{\partial v_0(x)}{\partial x}\Big|^2\,dx\; -$$

$$2\int_0^\tau \Big\{ \int_\Omega \kappa(x)\sum_{i=1}^n \frac{\partial v}{\partial x_i}\frac{\partial}{\partial x_i}\frac{\partial}{\partial t} g_2\,dx + < \frac{\partial \sigma(u)}{\partial t} - \frac{\partial f}{\partial t}, \widetilde{v} > \Big\}\,dt\; +$$

$$2\int_\Omega \Big\{ \big[\sigma(u(\tau,x)) - f(\tau,x)\big]\widetilde{v}(\tau,x)\,dx - \big[\sigma(h) - f(0,x)\big]\big[v_0(x) - g_2(0,x)\big]\Big\}dx = 0. \tag{30}$$

Using (15) with $\psi(x) = \widetilde{v}(\tau,x)$, we can rewrite the fifth term in (30) as

$$\int_\Omega \big[\sigma(u(\tau,x)) - f(\tau,x)\big]\widetilde{v}(\tau,x)\,dx = -\int_\Omega \kappa(x)\sum_{i=1}^n \frac{\partial v}{\partial x_i}\frac{\partial \widetilde{v}(\tau,x)\,dx}{\partial x_i}\,dx\,. \tag{31}$$

Remarking that it is simple to estimate the norm of $v_0(x)$ in $W^{1,2}(\Omega)$ and using the conditions (9), (10) and Cauchy's inequality we infer from (30), (31)

$$\int_\Omega \kappa(x)\Big|\frac{\partial v(\tau,x)}{\partial x}\Big|^2\,dx \;+ \int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \widetilde{v} > \,dt \le c_1\Big\{1 + \int_0^\tau\int_\Omega \Big|\frac{\partial v(t,x)}{\partial x}\Big|^2\,dx\,dt\Big\}. \tag{32}$$

Here and in what follows c_i denote constants depending only on known parameters. The conditions (10), (12) and Remark 1 allow us to substitute $\varphi = \widetilde{u} - \widetilde{v}$ in the identity

$$\int_0^\tau \Big\{ < \frac{\partial \sigma(u)}{\partial t}, \varphi > + \int_\Omega [\sum_{i=1}^n \rho_\delta(u)\, b_i\Big(t,x,\frac{\partial (u-v)}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} + a(t,x,v,u)\varphi]dx\Big\}dt = 0. \tag{33}$$

By (32) this gives

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, u - g_1 > dt \;+ \frac{1}{2}\int_\Omega \kappa(x)\Big|\frac{\partial v(\tau,x)}{\partial x}\Big|^2\,dx\; +$$

$$+ \int_0^\tau\int_\Omega \Big\{ \sum_{i=1}^n \rho_\delta(u)\, b_i\Big(t,x,\frac{\partial (u-v)}{\partial x}\Big)\frac{\partial (u-v)}{\partial x_i} \;+$$

$$[a(t,x,v,u) - a(t,x,v,v)]\,(u-v)\Big\}\,dx\,dt \;\le \tag{34}$$

$$\le \int_0^\tau\int_\Omega \Big\{ \sum_{i=1}^n \rho_\delta(u)\, b_i\Big(t,x,\frac{\partial (u-v)}{\partial x}\Big)\frac{\partial (g_1-g_2)}{\partial x_i} \;+ a(t,x,v,u)[g_1 - g_2]$$

$$- a(t,x,v,v)(u-v)\Big\}\,dx\,dt + c_1\Big\{1 + \int_0^\tau\int_\Omega \Big|\frac{\partial v(t,x)}{\partial x}\Big|^2\,dx\,dt\Big\}.$$

We write the first integral from (34) in the form

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, u - g_1 > dt = \int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, [u]_{-m}^m - g_1 > dt + \\ + \int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, u - [u]_{-m}^m > dt \tag{35}$$

with $m \geq ||g_1||_{L^\infty(Q_T)}$, $[u]_{-m}^m = \max\{\min[u, m], -m\}$. Then we can evaluate the first and the second integral of the right hand side of (35) by using Lemmas 2, 1 respectively [9]. So we obtain

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, u - g_1 > dt = \int_0^\tau \Big\{ \int_0^{u(x,\tau)} s\rho(s)\, ds - \int_0^{h(x)} s\rho(s)\, ds \Big\} dx + \\ + \int_0^\tau \int_\Omega \big[\sigma(u) - \sigma(h)\big] \frac{\partial g_1}{\partial t}\, dx\, dt - \int_\Omega \big[\sigma(u(\tau, x)) - \sigma(h(x))\big]\, g_1(\tau, x)\, dx. \tag{36}$$

Immediately from the definition of $\Lambda(u)$ we deduce

$$\sigma(u) < \varepsilon \Lambda(u) + c_\varepsilon \quad \text{for} \quad u \geq 0 \tag{37}$$

with arbitrary positive number ε and a constant c_ε depending only on ε and the function ρ. Using the conditions *ii)*, (10), (11) and the inequalities (19), (37), we obtain with arbitrary positive number ε and some function $\mu(t) \in L^1(0, T)$:

$$\Big| \int_0^\tau \int_\Omega \rho_\delta(u)\, b_i\Big(t, x, \frac{\partial(u - v)}{\partial x}\Big) \frac{\partial(g_1 - g_2)}{\partial x_i}\, dx\, dt \Big| \\ \leq \varepsilon \int_0^\tau \int_\Omega \rho_\delta(u) \Big| \frac{\partial(u - v)}{\partial x} \Big|^2 dx\, dt + \frac{c_2}{\varepsilon} \int_0^\tau \int_\Omega \Lambda(u)\mu(t)\, dx\, dt, \\ \int_0^\tau \int_\Omega \sigma(u) \frac{\partial g_1}{\partial t}\, dx\, dt \leq c_2 \Big\{ 1 + \int_0^\tau \int_\Omega \Lambda(u)\mu(t)\, dx\, dt \Big\}, \\ \int_\Omega \sigma(u(\tau, x)) g_1(\tau, x)\, dx \leq c_2 \Big\{ \varepsilon \int_\Omega \Lambda(u(\tau, x))\, dx + c_\varepsilon \Big\}. \tag{38}$$

We estimate terms in (34) involving the function a in standard way by using (10) and the condition *iii)*. Now from (34), (36), (38) and evident estimates for another terms in (36), we obtain

$$\int_\Omega \Lambda(u(\tau, x))\, dx + \int_\Omega \Big| \frac{\partial v(\tau, x)}{\partial x} \Big|^2 dx + \int_0^\tau \int_\Omega \rho(u) \Big| \frac{\partial(u - v)}{\partial x} \Big|^2 dx\, dt \leq \\ \leq c_3 \Big\{ 1 + \int_0^\tau \int_\Omega \big[1 + \mu(t)\big] \Big[\Lambda(u) + \Big| \frac{\partial v}{\partial x} \Big|^2 \Big] dx\, dt \Big\}. \tag{39}$$

Now the last inequality and Gronwall's lemma complete the proof of Theorem 1. □

In order to prove Theorem 2 we need auxiliary estimates.

Lemma 1 *Assume that the conditions of Theorem 1 are satisfied and following inequality*

$$\underset{t\in(0,T)}{ess\ \sup} \int_\Omega \sigma^q(u(t,x))\,dx \le K_1 \tag{40}$$

is fulfilled with some numbers $q \in \left(\frac{2n}{n+2}\,,\,\frac{n}{2}\right)$, K_1, *depending only on known parameters. Then the estimate*

$$\underset{t\in(0,T)}{ess\ \sup} \left\{ \int_\Omega |v(t,x)|^{\frac{pn}{n-2}}\,dx \;+ \int_\Omega |v(t,x)|^{p-2} \left|\frac{\partial v(t,x)}{\partial x}\right|^2 dx \right\} \le K_2 \tag{41}$$

holds with a number $p>2$ *defined by the equality*

$$p\frac{n}{n-2} = (p-1)\frac{q}{q-1} \tag{42}$$

and with a constant K_2 *depending only on known parameters.*

Proof. Denote

$$m_0 = ||g_1||_{L^\infty(Q_T)} + ||g_2||_{L^\infty(Q_T)} + ||h||_{L^\infty(\Omega)} + 1 \tag{43}$$

and use following notations for $k\in\mathbb{R}^1$ and arbitrary function w defined on Q_T

$$w_k(t,x) = \big[w(t,x)\big]_k = \min\{w(t,x),k\},$$

$$w_+(t,x) = \big[w(t,x)\big]_+ = \max\{w(t,x),0\}.$$

We test the integral identity

$$\sum_{i-1}^{n} \int_\Omega \kappa(x)\frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i}\,dx \;+ \int_\Omega \big[\sigma(u)-f\big]\psi\,dx \;= 0 \tag{44}$$

with $\psi = sign\ v\cdot\big[|v|_k - m_0\big]^{p-1}$ with $k>m_0$. Using the conditions *ii)*, (9), (40), and Hölder's inequality we obtain

$$\int_\Omega \big[|v|_k - m_0\big]_+^{p-2}\left|\frac{\partial v_k}{\partial x}\right|^2 dx \;\le c_4\Big\{\int_\Omega \big[|v|_k-m_0\big]_+^{(p-1)\frac{q}{q-1}}\,dx\Big\}^{\frac{q-1}{q}}. \tag{45}$$

From this inequality and the embedding theorem we have

$$\Big\{\int_\Omega \big[|v|_k - m_0\big]_+^{\frac{pn}{n-2}}\,dx\Big\}^{\frac{n-2}{n}} \le c_5\Big\{\int_\Omega \big[|v|_k-m_0\big]_+^{(p-1)\frac{q}{q-1}}\,dx\Big\}^{\frac{q-1}{q}}. \tag{46}$$

Taking into account the restriction on q and the choice of p we deduce (41) from (45), (46), (23) and the proof is completed. □

Proof of Theorem 2. We assume firstly that $\frac{2+\gamma}{1+\gamma} < \frac{n}{2}$. It is simple to check ([8], inequality (8)) that the conditions ρ) and (8) imply

$$|\sigma(u)| \le c_6 \quad \text{for} \quad u<0\,. \tag{47}$$

From this and (25) we find

$$|\sigma(u)|^{q_0} \le c_7\big[\Lambda(u)+1\big] \quad \text{with} \quad q_0 = \frac{2+\gamma}{1+\gamma}\,. \tag{48}$$

Using (48), (23) and Lemma 1, we obtain (41) with p_0 defined by the equality

$$p_0 \frac{n}{n-2} = (p_0-1)(2+\gamma).$$

This p_0 satisfies the inequality $p_0 - 2 > \frac{n}{n-2} > \gamma$. Consequently, (41), (25) imply

$$\int\!\!\int_{\{|u|\le 2|v|\}} \rho_\delta(u)\Big|\frac{\partial v}{\partial x}\Big|^2\, dx\; dt \;\le c_8. \tag{49}$$

Here $\{|u|\le 2|v|\} = \{(t,x)\in Q_T : |u(t,x)| \le 2|v(t,x)|\}$ and analogous notations we shall use further.

We want to establish a estimate analogous to (49) with respect to set $\{|u| > 2|v|\}$. Taking into account that $\rho_\delta(u) \le 1+\rho(0)$ for $u<0$, we can restrict ourselves to the set $\{u > 2|v|\}$. We substitute the test function

$$\psi = (|v|_k - g_2)\{[[u-|v|_k]_+]_k + |v|_k + m_0\}^{\widetilde{\gamma}} sign\; v$$

with $k > m_0$, $\widetilde{\gamma} > 0$, in (44). After standard calculations we obtain

$$I_1 \equiv \int\!\!\int_{\{|v|<k\}} \big\{[[u-|v|]_+]_k + |v| + m_0\big\}^{\widetilde{\gamma}}\Big|\frac{\partial v}{\partial x}\Big|^2\, dx\; dt \le c_9(I_2+I_3)\;, \tag{50}$$

where

$$I_2 = \int\!\!\int_{\{|v|_k<u\},} (|v|_k+m_0)\big\{[u-|v|_k]_k + |v|_k + m_0\big\}^{\widetilde{\gamma}-1}\cdot\Big|\frac{\partial u}{\partial x}\Big|\Big|\frac{\partial v}{\partial x}\Big|\, dx\; dt$$

$$I_3 = \int\!\!\int_{Q_T} \big\{(u_+ +1)^{\gamma-1} + |f(t,x)|\big\}(|v|_k+1)\big\{[u_+]_{2k} + |v|_k + 1\big\}^{\widetilde{\gamma}}\, dx\; dt.$$

The integral I_2 will be estimated in different ways for $\widetilde{\gamma}\le 1$ and for $\widetilde{\gamma} > 1$. For $\widetilde{\gamma}\le 1$ we have

$$\begin{aligned} I_2 \le & \int\!\!\int_{\{|v|_k<u\}} (|v|_k+m_0)^{\widetilde{\gamma}}\Big[\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big]\, dx\; dt \;\le \\ \le 3 & \int\!\!\int_{\{u>0\}} \Big\{(u+m_0)^{\widetilde{\gamma}}\Big|\frac{\partial(u-v)}{\partial x}\Big|^2 + (|v|+m_0)^{\widetilde{\gamma}}\Big|\frac{\partial v}{\partial x}\Big|^2\Big\}\, dx\; dt \;\le c_{10}. \end{aligned} \tag{51}$$

Here we used (41) and the inequality

$$\underset{t\in(0,T)}{ess\;\sup}\int_\Omega u_+^{2+\gamma}(t,x)\, dx + \int\!\!\int_{\{u>0\}} (1+u)^\gamma\Big|\frac{\partial(u-v)}{\partial x}\Big|^2\, dx\; dt \;\le c_{11}\;, \tag{52}$$

that follows from (23), (25).

For $\widetilde{\gamma} > 1$ we estimate I_2 by using the evident inequality

$$\big[[[u - |v|_k]_+]_k\big] + |v|_k + m_0 \le 2|v|_k + m_0$$

on the set $\{|v| \ge k\}$. Then we have

$$I_2 \le \varepsilon I_1 + c_{12} \iint_{\{u>0\}} \Big\{(u+m_0)^{\widetilde{\gamma}}\Big|\frac{\partial(u-v)}{\partial x}\Big|^2 + \frac{1}{\varepsilon^{\widetilde{\gamma}-1}}(|v|+m_0)^{\widetilde{\gamma}}\Big|\frac{\partial v}{\partial x}\Big|^2\Big\}\, dx\, dt\,, \tag{53}$$

where the last integral can be estimated analogously to (51).

Using Hölder's inequality and the embedding theorem we obtain for $\delta \ge 0$

$$\begin{aligned}
&\int_{\Omega_T}\!\!\int \big|[u_+]_k - g_{1,+}\big|^{(2+\gamma)\frac{2}{n}+2+\delta}\, dx\, dt \le \\
&\le \int_0^T \Big\{\int_\Omega \big|[u_+]_k - g_{1,+}\big|^{2+\gamma}\, dx\Big\}^{\frac{2}{n}} \Big\{\int_\Omega \Big(\big|[u_+]_k - g_{1,+}\big|^{1+\frac{\delta}{2}}\Big)^{\frac{2n}{n-2}}\, dx\Big\}^{\frac{n-2}{n}}\, dt \\
&\le c_{13}\Big\{\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \big|[u_+]_k - g_{1,+}\big|^{2+\gamma}\, dx\Big\}^{\frac{2}{n}} \cdot \\
&\qquad \cdot \int_{\Omega_T}\!\!\int \big|[u_+]_k - g_{1,+}\big|^{\delta}\Big|\frac{\partial}{\partial x}\big([u_+]_k - g_{1,+}\big)\Big|^2\, dx\, dt.
\end{aligned} \tag{54}$$

Choosing $\delta = 0$, the inequalities (23), (52), and condition (10) imply

$$\int_{Q_T}\!\!\int u_+^{(2+\gamma)\frac{2}{n}+2}\, dx\, dt \le c_{14}\,. \tag{55}$$

We estimate I_3 by Young's unequality and condition (9) and obtain

$$I_3 \le c_{15}\Big\{1 + \int_{Q_T}\!\!\int u_+^{\gamma+\widetilde{\gamma}+2}\, dx\, dt + \int_{Q_T}\!\!\int |v|^{\gamma+\widetilde{\gamma}+2}\, dx\, dt\Big\}\,. \tag{56}$$

The integral with v can be estimated by a constant in virtue of the inequality (41) in the case that $\widetilde{\gamma} \in [0, \gamma]$. If γ is such that

$$2\gamma + 2 \le (2+\gamma)\frac{2}{n} + 2,$$

the integral with u_+ and $\widetilde{\gamma} = \gamma$ in (56) can be also estimated by a constant because of the inequality (55). In the opposite case we choose $\widetilde{\gamma}$ satisfying the condition

$$\gamma + \widetilde{\gamma} + 2 \le (2+\gamma)\frac{2}{n} + 2\,.$$

For example we can take $\widetilde{\gamma} = \widetilde{\gamma}_1 = \frac{2}{n}$. For such choice of $\widetilde{\gamma}$ we get from (50), (52), (53), (56) $I_1 \le c_{16}$, which implies

$$\iint_{\{u>|v|\}} (u - |v|)^{\widetilde{\gamma}}\Big|\frac{\partial v}{\partial x}\Big|^2\, dx\, dt \le c_{17}$$

and consequently

$$\int\int_{\{u>2|v|\}} \big[u(t,x)\big]^{\widetilde{\gamma}} \Big|\frac{\partial v}{\partial x}\Big|^2 \,dx\,dt \;\le c_{18}\;. \tag{57}$$

From (23), (49), (57) we obtain

$$\int_{Q_T}\int |u|^{\widetilde{\gamma}}\Big|\frac{\partial v}{\partial x}\Big|^2 \,dx\,dt \;\le c_{19}, \quad \int_{Q_T}\int |u|^{\widetilde{\gamma}}\Big|\frac{\partial u}{\partial x}\Big|^2 \,dx\,dt \;\le c_{19} \tag{58}$$

and this ends the proof of Theorem 2 in the case that $\frac{2+\gamma}{1+\gamma} < \frac{n}{2}$, $\widetilde{\gamma} = \gamma$.

If $\widetilde{\gamma} = \widetilde{\gamma}_1 < \gamma$, we can iterate our discussions with respect to $\widetilde{\gamma}$. Using (58), we obtain from (54)

$$\int_{Q_T}\int u_+^{(2+\gamma)\frac{n}{2}+2+\widetilde{\gamma}_1} \,dx\,dt \;\le c_{20}\;,$$

that allows us to choose $\widetilde{\gamma}_2 = \min\left\{\gamma, \frac{4}{n}\right\}$. Repeating this argument, if necessary, we can choose $\widetilde{\gamma}_3 = \gamma$ and we proved the Theorem if $\frac{2+\gamma}{1+\gamma} < \frac{n}{2}$.

If $\frac{2+\gamma}{1+\gamma} = \frac{n}{2}$ we can use Lemma 1 with $q' < q$ instead of q. We can choose such q' that the corresponding p' satisfies $p' - 2 > \gamma$ and then we keep all discussions of the previous proof. If $\frac{2+\gamma}{1+\gamma} > \frac{n}{2}$, then the boundedness of solutions of the equation (21) under the conditions (9), (10), (40) and the assumption formulated above is well known [10]. In this case we can keep the previous discussions with corresponding simplification. The proof of Theorem 2 is completed. □

Lemma 2 *Assume that the conditions of Theorem 2 are satisfied and*

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \sigma^q(u_+(t,x))\,dx \;+\; \int\int_{\{u>1\}} \rho_\delta^2(u)\sigma^{q-2}(u)\Big|\frac{\partial u}{\partial x}\Big|^2 \,dx\,dt \le K_3 \tag{59}$$

holds with numbers $q \in \left[\frac{2+\gamma}{1+\gamma}, \frac{n}{2}\right)$, K_3, depending only on known parameters. Then there exist positive constants β, K_4 depending only on known parameters such that

$$\int\int_{\{u>1\}} \rho_\delta^2(u)\sigma^{q-2+\beta}(u)\Big|\frac{\partial v}{\partial x}\Big|^2 \,dx\,dt \le K_4\;. \tag{60}$$

Proof. By Theorem 2 follows that (59) holds for $q = q_0 = \frac{2+\gamma}{1+\gamma}$. We shall prove (60) for this value of q. The proof of the lemma for $\frac{2+\gamma}{1+\gamma} < q < \frac{n}{2}$ is the same as for $q = \frac{2+\gamma}{1+\gamma}$.

From Lemma 1 with $q = \frac{2+\gamma}{1+\gamma}$ we obtain analogously to (49)

$$\int_{\{|u|<2|v|\}} \rho_\delta^2(u)\sigma^{q_0-2+\beta_1}(u)\Big|\frac{\partial v}{\partial x}\Big|^2 \,dx\,dt \le c_{21}\;,\; \beta_1 = \frac{2-(n-2)\gamma}{(1+\gamma)(n-2)}. \tag{61}$$

For the proof of (60) it is sufficient to check that the integral I_1 in (50) can be estimated by a constant for $\widetilde{\gamma} = \gamma + (1+\gamma)\beta_2$ with positive β_2 depending only on

γ, n. This estimation of I_1 runs analogously to the corresponding estimation in the proof of Theorem 2. Hence we make only some remarks.

We change the inequality (51) for $\widetilde{\gamma} \leq 1$, $\widetilde{\gamma} \leq \gamma + \frac{1}{2}(p_0 - 2 - \gamma)$, $p_0 = \frac{q_0(n-2)}{n-2q_0} > 2 + \frac{2}{n-2}$, in the following way

$$I_2 \leq 3 \iint\limits_{\{u>0\}} \Big\{ (u+m_0)^{\gamma} \Big| \frac{\partial(u-v)}{\partial x} \Big|^2 + (|v|+m_0)^{p_0-2} \Big| \frac{\partial v}{\partial x} \Big|^2 \Big\} \, dx \, dt \leq c_{22} \quad (62)$$

after using Theorem 2 and Lemma 1. Analogously we change (53) for $\widetilde{\gamma} > 1$.

In order to estimate I_3 we remark that (54) and Theorem 2 imply

$$\int_{Q_T}\!\!\int u_+^{(2+\gamma)(1+\frac{2}{n})} \, dx \, dt \ \leq c_{23}. \quad (63)$$

From (56), (63), (41), we see that the integral I_3 can be estimated by a constant, provided

$$\gamma + \widetilde{\gamma} + 2 \leq (2+\gamma)\Big(1 + \frac{2}{n}\Big), \ \gamma + \widetilde{\gamma} + 2 \leq \frac{p_0 n}{n-2}.$$

But both of these restrictions can be satisfied with $\widetilde{\gamma} = \gamma + (1+\gamma)\beta_3$ and some positive β_3 depending only on n, γ. Therefore we can choose positive β_2 such that the integral I_1 with $\widetilde{\gamma} = \gamma + (1+\gamma)\beta_2$ is estimated by a constant depending only on known parameters. From this estimate and (61) we obtain the inequality (60). □

Lemma 3 *Assume that the conditions of Theorem 2 are satisfied. Then there exist numbers $\overline{q}, K_5$, depending only on known parameters, such that $\overline{q} > \frac{n}{2}$ and*

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_{\Omega} \sigma^{\overline{q}}(u_+(t,x)) \, dx \ + \iint\limits_{\{u>1\}} \rho_\delta^2(u) \sigma^{\overline{q}-2}(u) \Big| \frac{\partial u}{\partial x} \Big|^2 \, dx \, dt \leq K_5 \, . \quad (64)$$

Proof. We substitute the function

$$\varphi = \big[\sigma(u_k) - \sigma(m_0)\big]_+^2 \{1 + \big[\sigma(u_k) - \sigma(m_0)\big]^3\}^r \ , \ r \in (-\frac{2}{3}, \infty), \quad (65)$$

in the integral identity

$$\int_0^\tau \Big\{ < \frac{\partial \sigma(u)}{\partial t}, \varphi > + \int_\Omega \Big[\sum_{i=1}^n \rho_\delta(u) b_i \Big(t, x, \frac{\partial(u-v)}{\partial x}\Big) \frac{\partial \varphi}{\partial x_i} + a(t,x,v,u)\varphi \Big] dx \Big\} dt = 0. \quad (66)$$

Then, using Lemma 1 from [9], we can evaluate the first summand of (66) to obtain

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \varphi > dt = \int_\Omega \Lambda^{(r)}(u(\tau,x)) \, dx \ , \quad (67)$$

where

$$\begin{aligned} \Lambda^{(r)}(u) &= \int_0^u \rho(s) \big[\sigma(s_k) - \sigma(m_0)\big]_+^2 \Big\{ \frac{1}{2} + [\sigma(s_k) - \sigma(m_0)]^3 \Big\}^r \, ds \\ &\geq \frac{1}{3(r+1)} \Big\{ \frac{1}{2} + \big[\sigma(u_k) - \sigma(m_0)\big]^3 \Big\}^{r+1} \quad \text{for} \quad u > m_0. \end{aligned} \quad (68)$$

Here $s_k = \min[s,k]$ and the value of u_k is analogous.

We write the derivative of φ in the form

$$\frac{\partial \varphi}{\partial x_i} = \Big[\widetilde{\Phi}^{(r)}(u_k)\,\frac{\partial(u-v)}{\partial x_i} + \widetilde{\Phi}^{(r)}(u_k)\,\frac{\partial v}{\partial x_i}\Big]\chi(m_0 < u < k) \tag{69}$$

where $\chi(m_0 < u < k)$ is the characteristic function of the set $\{m_0 < u < k\}$ and the function $\widetilde{\Phi}^r(u)$ satisfies for $r > -\frac{2}{3}$ the estimate

$$c_{24}k(r)\Phi^{(r)}(u)\rho(u) \le \widetilde{\Phi}^{(r)}(u) \le c_{25}(r+1)\Phi^{(r)}(u)\rho(u) \tag{70}$$

with $k(r) = \min(1, 2+3r)$,

$$\Phi^{(r)}(u) = \big[\sigma(u) - \sigma(m_0)\big]_+ \Big\{\frac{1}{2} + \big[\sigma(u) - \sigma(m_0)\big]^3\Big\}^r. \tag{71}$$

Using (67) – (70) and conditions *ii), iii)* we obtain from (66) with the function φ defined by (65)

$$\begin{aligned}
&\int_\Omega \Big\{\frac{1}{2} + \big[\sigma(u_k(\tau,x)) - \sigma(m_0)\big]_+^3\Big\}^{r+1} dx \, + \\
&+ \int_0^\tau \int_\Omega \rho_\delta^2(u)\Phi^{(r)}(u_k)\chi(m_0 < u < k)\Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt \;\le \\
&\le c_{26}\Big\{\Big[\frac{r+1}{\kappa(r)}\Big]^2 \int_0^\tau \int_\Omega \rho_\delta^2(u)\Phi^{(r)}(u_k)\chi(m_0 < u < k)\Big|\frac{\partial v}{\partial x}\Big|^2 dx\, dt \, + \\
&+ \frac{r+1}{\kappa(r)} \int_0^\tau \int_\Omega (1+|u|+|v|)\big[\sigma(u_k) - \sigma(m_0)\big]\Phi^{(r)}(u_k)\, dx\, dt\Big\}.
\end{aligned} \tag{72}$$

Let us assume now that for some $q \in \big[\frac{2+\gamma}{1+\gamma}, \frac{n}{2}\big)$ the inequality (59) is fulfilled. Then we obtain from Lemma 2 that the first integral of the right hand side of (72) can be estimated by a constant independent on k for $r = \frac{1}{3}[q - 3 + \beta]$.

We shall check now that the second integral of the right hand site of (72) for $r = \frac{1}{3}[q - 3 + \beta']$ and some positive β' depending only on γ can be also estimated by a constant independent on k. Analogously to inequalities (54), (55) we obtain from (59)

$$\int_{Q_T}\int u_+^{q(1+\gamma)(1+\frac{2}{n})}\, dx\, dt \;\le c_{27}. \tag{73}$$

From (59) and Lemma 1 we have

$$\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega \big|v(t,x)\big|^{\frac{qn}{n-2q}}\, dx \le c_{28}. \tag{74}$$

(73), (74) imply the needed estimate for the last integral in (72) provided

$$\beta' \le \frac{1}{1+\gamma}\Big\{q(1+\gamma)\Big(1+\frac{2}{n}\Big) + \gamma\Big\} - q, \quad \beta' \le \frac{1}{1+\gamma}\Big\{\frac{qn}{n-2q} + \gamma\Big\} - q.$$

For that purpose it is sufficient to choose $\beta' = \frac{\gamma}{1+\gamma}$.

We proved that for $\overline{\beta} = \min(\beta, \beta')$ the left hand side of (72) is estimated by constant depending only on known parameters if $r = \frac{1}{3}\big(q - 3 + \overline{\beta}\big)$. This estimate implies that the inequality (59) is fulfilled with $q + \overline{\beta}$ instead of q. We can guarantee also by small change of $\overline{\beta}$ that the number $\frac{1}{\overline{\beta}}\big[\frac{n}{2} - \frac{2+\gamma}{1+\gamma}\big]$ is not integer, and denote by N its integer part. Recalling that the estimate (59) is fulfilled with $q = q_0 = \frac{2+\gamma}{1+\gamma}$ and choosing the sequence $q_i = q_0 + i\overline{\beta}$. We obtain after $N+1$ iterations our previous discussions that the inequality (59) is fulfilled with $q = q_{N+1} > \frac{n}{2}$. Consequently the inequality (64) is satisfied with $\overline{q} = q_{N+1}$ and this ends the proof of Lemma 3. □

Proof of Theorem 3. The result of Theorem 3 follows immediately from the estimates (47), (64), the conditions *ii)*, (9), (10) and the assumption on the set Ω. It is necessary to apply only well known results on regularity of solutions of elliptic equations to equation (21) (see, for example, [10]). □

4 Boundedness of the function u

We assume in this section that the conditions of Theorem 4 are satisfied. We shall prove estimates for u separately for the sets $\{u > 0\}$ and $\{u < 0\}$. These estimates will be given in Lemmas 4, 6.

Lemma 4 *Let the conditions of Theorem 4 be satisfied. Then there exists a constant M_5 depending only on known parameters such that*

$$ess\ \sup\{u(t,x) : (t,x) \in Q_T\} \leq M_5. \tag{75}$$

Proof. We shall use the inequality (72). We start estimating the first integral of the right hand side of (72).

Let $\{\varphi_j^2(x)\}$, $j = 1, \dots, J$, be a partition of unity such that

$$\sum_{j=1}^{J} \varphi_j^2(x) = 1, \quad \left|\frac{\partial \varphi_j}{\partial x}\right| \leq \frac{K_0}{R} \quad \text{for} \quad x \in \Omega,$$
$$\varphi_j(x) \in C^\infty(\mathbb{R}^n),\ supp\ \varphi_j \subset B(x_j, R),\ J \leq \frac{K_0}{R^n},\ R < 1, \tag{76}$$

where $B(x_j, R)$ is a ball of radius R with centre $x_j \in \Omega$, K_0 is a number depending only on n. The number R will be choosen later on.

We test the integral identity (44) with the function

$$\psi = \sum_{j=1}^{J} \rho_\delta^2(u_k)\Phi^{(r)}(u_k)[v - v_j]\varphi_j^2(x),\ v_j(t) = v(x_j, t)\ . \tag{77}$$

Integration with respect to t yields

$$\int_{Q_\tau}\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\Big|\frac{\partial v}{\partial x}\Big|^2\,dx\,dt \;=J_1+J_2+J_3\;, \tag{78}$$

where $Q_\tau=\{(t,x):0<t<\tau, x\in\Omega\}$,

$$J_1=-\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)\Phi_1^{(r)}(u_k)[v-v_j]\varphi_j^2\frac{\partial u_k}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx\,dt,$$

$$J_2=-2\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)[v-v_j]\varphi_j\frac{\partial \varphi_j}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx\,dt, \tag{79}$$

$$J_3=-\sum_{j=1}^{J}\int_{Q_\tau}\int [\sigma(u)-f]\rho_\delta^2(u_k)\Phi^{(r)}(u_k)[v-v_j]\varphi_j^2(x)\,dx\,dt$$

and $\Phi_1^{(r)}(u_k)$ is defined by

$$\Phi_1^{(r)}(u_k)=\rho_\delta^2(u_k)\rho(u_k)\Big\{\frac{1}{2}+\big[\sigma(u_k)-\sigma(m_0)\big]^3\Big\}^{r-1}\cdot$$

$$\cdot\Big\{\frac{1}{2}+(3r+1)\big[\sigma(u_k)-\sigma(m_0)\big]^3\Big\}\chi(m_0<u<k)+2\,\rho_\delta(u_k)\rho'(u_k)\Phi^{(r)}(u_k)\chi(u<k). \tag{80}$$

Denote $u_0=\sigma^{-1}\big[\sigma(m_0)+\frac{1}{2}\big]$. Analogously to (19) we obtain

$$\begin{aligned}\rho'(u)\sigma(u)\le\rho^2(u)\quad&\text{for}\quad u>0\;,\\ \rho(u)\le 2\,\rho(u_0)\big[\sigma(u)-\sigma(m_0)\big]\quad&\text{for}\quad u>u_0.\end{aligned} \tag{81}$$

Hence we getfor $r\ge-\frac{1}{2},\ k>u_0$

$$\Phi_1^{(r)}(u_k)\le c_{29}(r+1)\Big\{\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\chi(u_0<u<k)+\chi(m_0<u\le u_0\Big\}. \tag{82}$$

We assume further that $r\ge-\frac{1}{2}$ and we choose the number R from (76) according to

$$R^\varepsilon=\frac{\varepsilon}{(r+1)^2},\quad \varepsilon<\frac{1}{4}\;, \tag{83}$$

where ε will be specified later on. Using (27), (76), (82), (83), we obtain

$$\begin{aligned}|J_1|\le\varepsilon\Big\{&\int_{Q_\tau}\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\Big|\frac{\partial v}{\partial x}\Big|^2\,dx\,dt+\\ &+c_{30}\frac{1}{(r+1)^2}\int_{Q_\tau}\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\chi(u_0<u<k)\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt+\\ &+c_{30}\frac{1}{r+1}\int_{Q_\tau}\int \Big[\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial v}{\partial x}\Big|^2\Big]\chi(u>m_0)\,dx\,dt\Big\}.\end{aligned} \tag{84}$$

By (76) and Cauchy's inequality we have

$$|J_2| \le \int_{Q_\tau}\!\!\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\Big\{\varepsilon\Big|\frac{\partial v}{\partial x}\Big|^2 + \frac{1}{\varepsilon}\frac{c_{31}}{R^{n+2}}\Big\}\,dx\,dt\,. \tag{85}$$

From (78), (79), (83) – (85) we infer

$$\begin{aligned}
&\int_{Q_\tau}\!\!\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\Big|\frac{\partial v}{\partial x_i}\Big|^2\,dx\,dt \;\le\\
&\le c_{32}\Big\{\frac{\varepsilon}{(r+1)^2}\int_{Q_\tau}\!\!\int \kappa(x)\rho_\delta^2(u_k)\Phi^{(r)}(u_k)\chi(u_0<u<k)\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt\,+\\
&+\frac{1}{r+1}\int_{Q_\tau}\!\!\int\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial v}{\partial x}\Big|^2\Big)\chi(u>m_0)\,dx\,dt\,+\\
&+\int_{Q_\tau}\!\!\int \rho_\delta^2(u_k)\Phi^{(r)}(u_k)\Big[\sigma(u)+|f|+\frac{1}{\varepsilon}\Big[\frac{(r+1)^2}{\varepsilon}\Big]^{n+2}\Big]\,dx\,dt\,.
\end{aligned} \tag{86}$$

Applying the last estimate to the first integral of the right hand side of (72) and choosing ε small enough, we get from (72), (26), (27), (19), (81)

$$\begin{aligned}
&\operatorname*{ess\ sup}_{\tau\in(0,T)}\int_\Omega\Big\{\frac{1}{2}+\big[\sigma(u_k(\tau,x))-\sigma(m_0)\big]_+^{\frac{3}{}}\Big\}^{r+1}\,dx\,+\\
&+\int_{Q_T}\!\!\int \rho_\delta^2(u)\Phi^{(r)}(u_k)\chi(m_0<u<k)\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt\;\le\\
&\le c_{33}(r+1)^{\lambda_1}\Big\{\int_{Q_T}\!\!\int \Phi^{(r)}(u_k)\big[\sigma(u_k)-\sigma(m_0)\big]^2\big[\sigma(u)+|f|\big]\,dx\,dt\;+1\Big\}
\end{aligned} \tag{87}$$

with $\lambda_1 = 2(n+2)+2$.

We want to apply Moser iteration with respect to the integral

$$I_k(r) = \int_{Q_T}\!\!\int \Phi^{(r)}(u_k)\big[\sigma(u_k)-\sigma(m_0)\big]^2\big[\sigma(u)+|f|\big]\,dx\,dt\,. \tag{88}$$

To this end we use the embedding inequality

$$\begin{aligned}
&\int_0^T\Big\{\int_\Omega |v(t,x)|^{2(1+\frac{2p}{n})}\,dx\Big\}^{\frac{1}{p}}\,dt\;\le\\
&\le C(n,p)\Big\{\operatorname*{ess\ sup}_{t\in(0,T)}\int_\Omega v^2(t,x)\,dx\Big\}^{\frac{1}{p}+\frac{2}{n}-1}\int_{Q_T}\!\!\int\Big|\frac{\partial v}{\partial x}\Big|^2\,dx\,dt\,,
\end{aligned} \tag{89}$$

which is fulfilled for $1\le p<\frac{n}{n-2}$ with a constant $C(n,p)$ depending only on n,p and with an arbitrary function $v\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;\overset{\circ}{W}{}^{1,2}(\Omega))$. From condition (9) and inequality (64) we have $\sigma(u)+|f|\in L^\infty(0,T,L^{p'}(\Omega))$ for some $p'>\frac{n}{2}$.

Applying Hölder's inequality to (88) we obtain

$$I_k(r) \le c_{34} \int_0^T \Big\{ \int_\Omega \Big[\Phi^{(r)}(u_k)\big[\sigma(u_k)-\sigma(m_0)\big]^2\Big]^p \, dx\Big\}^{\frac{1}{p}} \, dt \ \le$$

$$\le c_{35}\Big\{ \operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \Big[\Phi_2^{(r)}(u_k)\Big]^2 \, dx\Big\}^{\frac{1}{p}+\frac{2}{n}-1} \int_{Q_T}\!\!\int \Big|\frac{\partial \Phi_2^{(r)}(u_k)}{\partial x}\Big|^2 \, dx\, dt \ , \tag{90}$$

where

$$\Phi_2^{(r)}(u_k) = \Big[\Phi^{(r)}(u_k)\big[\sigma(u_k)-\sigma(m_0)\big]^2\Big]^{\frac{p}{2(1+\frac{2p}{n})}}. \tag{91}$$

Simple calculations give

$$\Big[\Phi_2^{(r)}(u_k)\Big]^2 \le \Big\{\frac{1}{2} + \big[\sigma(u_k)-\sigma(u_0)\big]_+^{3}\Big\}^{(r+1)\theta}, \ \theta = \frac{p}{1+\frac{2p}{n}} < 1 \ ,$$

$$\Big|\frac{\partial \Phi_2^{(r)}(u_k)}{\partial x}\Big|^2 \le c_{36}(r+1)^2\Big\{\rho_\delta^2(u_k)\Phi^{(\theta r+\theta-1)}(u_k)+1\Big\}\chi(m_0<u<k). \tag{92}$$

For $r \ge -\frac{1}{2}$ we get from (90), (92) and (87)

$$\widetilde{I}_k(r) \le c_{37}(r+1)^{\lambda_2}\big\{\widetilde{I}_k(\theta r+\theta-1)\big\}^{\frac{1}{\theta}} \ , \ \lambda_2 = 2+\frac{\lambda_1}{\theta}, \ \widetilde{I}_k(r) = I_k(r)+1 \ . \tag{93}$$

We choose $r_j = \frac{1}{2}\theta^{-j}-1, \ j=0,1,\dots,$ and obtain from (93)

$$\big\{\widetilde{I}_k(r_j)\big\}^{\theta^{-j}} \le c_{37}^{\theta^{-j}}\,\theta^{-j\lambda_2\theta^{-j}}\big\{\widetilde{I}_k(r_{j-1})\big\}^{\theta^{j-1}}.$$

Iterating this estimate yields for arbitrary j

$$\big\{\widetilde{I}_k(r_j)\big\}^{\theta^{-j}} \le c_{38}\,\widetilde{I}_k(-\frac{1}{2}) \ . \tag{94}$$

Now (90), (92) and (64) imply

$$\widetilde{I}_k(-\frac{1}{2}) \le c_{39}\Big\{1+\Big[\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \big[\sigma(u_+(t,x))\big]^{\frac{3\theta}{2}} \, dx\Big]^{\frac{1}{p}+\frac{2}{n}-1}\cdot$$

$$\cdot \iint_{\{u>1\}} \rho_\delta^2(u)[\sigma(u)]^{\frac{3\theta}{2}-2}\Big|\frac{\partial u}{\partial x}\Big|^2 \, dx\, dt\Big\} \le c_{40} \ , \tag{95}$$

where the constant c_{40} is independent of k. Recall that we consider the case $n>2$, i. e., $\frac{3\theta}{2} < \frac{3}{2} \le \overline{q}$, where $\overline{q}$ is the number from Lemma 2. Hence the desired estimate (75) follows from (94), (95). □

We shall use the notations

$$w^{(k)}(t,x) = [w(t,x)]^{(k)} = \max\{w(t,x),k\}, \ w_-(t,x) = [w(t,x)]_- = \min\{w(t,x),0\} \tag{96}$$

for $k \in \mathbb{R}^1$ and arbitrary functions w defined on Q_T.

Lemma 5 *Let the conditions of Theorem 4 be satisfied. Then there exists a constant M_6 depending only on known parameters such that*

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \left|u^{(k)}(t,x)\right| dx + \int_{Q_T}\int \left|\frac{\partial u^{(k)}}{\partial x}\right|^2 dx\, dt \le M_6 \ , \ for\ k > -\frac{1}{\delta}. \quad (97)$$

Proof. We test the integral identity (33) with

$$\varphi = \frac{1}{\rho(u^{(k)})}\left[\sigma(u^{(k)}) - \sigma(-m_0)\right]_- \left|u^{(k)} + m_0\right]^r, \ k < -m_0 - 1\,,\ r \ge 0\,,$$

to obtain

$$\begin{aligned}
&\int_0^\tau < \frac{\partial\sigma(u)}{\partial t}, \frac{1}{\rho(u^{(k)})}\left[\sigma(u^{(k)}) - \sigma(-m_0)\right]_- \left|u^{(k)} + m_0\right|^r > dt + \\
&+ \sum_{i=1}^n \int_{Q_\tau}\int \rho_\delta(u) b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\frac{\partial(u-v)}{\partial x_i}\psi^{(r)}(u)\chi(k<u<-m_0)\, dx\, dt + \\
&+ \int_{Q_\tau}\int a(t,x,v,u)\frac{1}{\rho(u^{(k)})}\left[\sigma(u^{(k)}) - \sigma(-m_0)\right]_- \left|u^{(k)} + m_0\right|^r dx\, dt + \\
&+ \sum_{i=1}^n \int_{Q_\tau}\int \rho_\delta(u) b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\frac{\partial v}{\partial x_i}\psi^{(r)}(u)\chi(k<u<-m_0)\, dx\, dt = 0\,,
\end{aligned} \quad (98)$$

where

$$\psi^{(r)}(u) = \Big[1 + \frac{\rho'(u)}{\rho^2(u)}\int_u^{-m_0}\rho(s)\, ds\Big]|u+m_0|^r - r\frac{1}{\rho(u)}\big[\sigma(u) - \sigma(-m_0)\big]|u+m_0|^{r-1}. \quad (99)$$

We evaluate the first integral in (98) by Lemma 1 in [9] and find

$$\begin{aligned}
&\int_0^\tau < \frac{\partial\sigma(u)}{\partial t}, \frac{1}{\rho(u^{(k)})}\left[\sigma(u^{(k)}) - \sigma(-m_0)\right]_- \left|u^{(k)} + m_0\right|^r > dt = \\
&= \int_\Omega \Lambda_-^{(r)}(u(\tau,x))\, dx\ ,
\end{aligned} \quad (100)$$

where

$$\Lambda_-^{(r)}(u) = \int_0^u \rho(s)\frac{1}{\rho(s^{(k)})}\left[\sigma(s^{(k)}) - \sigma(-m_0)\right]_- \left|s^{(k)} + m_0\right|^r ds\ ,\ s^{(k)} = \max\{s,k\}.$$

By simple calculations we obtain for $u > -m_0$

$$\Lambda_-^{(r)}(u) \ge \frac{c_{41}}{r+1}\left|u^{(k)} + m_0\right|^{r+1} - c_{42}\,. \quad (101)$$

For $u < -m_0$T condition $\rho)$ implies

$$\frac{\rho'(u)}{\rho^2(u)}\int_u^{-m_0}\rho(s)\, ds \ge \frac{1}{\rho(u)}\int_u^{-m_0}\frac{\rho'(s)}{\rho(s)}\rho(s)\, ds = \frac{\rho(-m_0)}{\rho(u)} - 1 \quad (102)$$

and hence

$$\psi^{(r)}(u) \geq \frac{\rho(-m_0)}{\rho(u)} \big|u + m_0\big|^r. \tag{103}$$

Condition ρ') and inequality (47) yield a corresponding estimate from above

$$\psi^{(r)}(u) \leq c_{43} \frac{r+1}{\rho(u)} \big[\big|u + m_0\big|^r + r\big]. \tag{104}$$

Further, condition a), Theorem 3 and (47) imply the following estimate for the term involving a in (98):

$$\begin{aligned} &\frac{1}{\rho(u)} a(t,x,v,u) \frac{\rho(u)}{\rho(u^{(k)})} \big[\sigma(u^{(k)}) - \sigma(-m_0)\big]_- \geq \\ &\geq \frac{a(t,x,v,0)}{\rho(0)} \cdot \frac{\rho(u)}{\rho(u^{(k)})} \big[\sigma(u^{(k)}) - \sigma(-m_0)\big]_- \geq -c_{44}. \end{aligned} \tag{105}$$

Using the inequalities (101), (103) – (105) we get from (98)

$$\begin{aligned} &\frac{1}{r+1} \int_\Omega \big|\big[u^{(k)} + m_0\big]_-\big|^{r+1} \, dx + \int_{Q_\tau}\!\!\int \big|u + m_0\big|^r \Big|\frac{\partial u}{\partial x}\Big|^2 \cdot \chi(k < u < -m_0) \, dx \, dt \leq \\ &\leq c_{45} \int_{Q_\tau}\!\!\int \Big\{(r+1)^2 \big|u + m_0\big|^r \Big|\frac{\partial v}{\partial x}\Big|^2 \chi(k < u < -m_0) \\ &+ r(r+1)\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big)\chi(k < u < -m_0) + \big|\big[u^{(k)} + m_0\big]_-\big|^r\Big\} \, dx \, dt. \end{aligned} \tag{106}$$

Finally, inequality (97) follows immediately from (106) with $r = 0$ and Theorem 1. $\square$

Lemma 6 *Let the conditions of Theorem 4 be satisfied. Then*

$$ess \ \inf\{u(t,x) : (t,x) \in Q_T\} \geq -M_7 \tag{107}$$

holds for $\delta \in \big[0, \frac{1}{M_7}\big]$ *with a positive constant* M_7 *depending only on known parameters.*

Proof. We shall use inequality (106). To this end we start estimating the first integral of the right hand side of (106). We assume further that $k > -\frac{1}{\delta}$.

Let $\{\varphi_j^2(x)\}, \ j = 1, \dots J$, be a partition of unity satisfying (76) with a number R to be fixed later on. We test the integral identity (44) with

$$\psi = \sum_{j=1}^{J} [v - v_j] \big|\big[u^{(k)} + m_0\big]_-\big|^r \varphi_j^2(x), \ r \geq 2, v_j(t) = v(t, x_j). \tag{108}$$

After integration wit respect to t we get

$$\int_{Q_\tau}\int \kappa(x)\big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial v}{\partial x}\right|^2 dx\ dt\ = J_1^- + J_2^- + J_3^- \tag{109}$$

where

$$J_1^- = r\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)|[u+m_0]_-|^{r-1}\cdot[v-v_j]\varphi_j^2\frac{\partial u^{(k)}}{\partial x_i}\frac{\partial v}{\partial x_i}\ dx\ dt,$$

$$J_2^- = -2\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)[v-v_j]|[u+m_0]_-|^r\varphi_j\frac{\partial \varphi_j}{\partial x_i}\frac{\partial v}{\partial x_i}\ dx\ dt, \tag{110}$$

$$J_3^- = -\sum_{j=1}^{n}\int_{Q_\tau}\int[\sigma(u)-f][v-v_j]|[u^{(k)}+m_0]_-|^r\varphi_j^2\ dx\ dt.$$

Repeating arguments used for estimating J_1 in Lemma 4 and choosing the R from (83), we get

$$\begin{aligned}|J_1^-| \le \varepsilon\Big\{\int_{Q_\tau}\int \kappa(x)\big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial v}{\partial x}\right|^2 dx\ dt\ + \\ + c_{46}\frac{1}{(r+1)^2}\int_{Q_\tau}\int \kappa(x)\big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial u^{(k)}}{\partial x}\right|^2 dx\ dt\ + c_{46}\Big\}.\end{aligned} \tag{111}$$

We apply Cauchy's inequality to J_2^-, J_3^-, use (111) and obtain from (109)

$$\begin{aligned}&\int_{Q_\tau}\int \kappa(x)\big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial v}{\partial x}\right|^2 dx\ dt\ \le \\ &\le c_{47}\Big\{\frac{\varepsilon}{(r+1)^2}\int_{Q_\tau}\int \kappa(x)\big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial u^{(k)}}{\partial x}\right|^2 dx\ dt\ + \\ &+\Big[1+\frac{1}{\varepsilon}\Big(\frac{(r+1)^2}{\varepsilon}\Big)^{n+2}\Big]\int_{Q_\tau}\int \big|\big[u^{(k)}+m_0\big]_-\big|^r\ dx\ dt\Big\}.\end{aligned} \tag{112}$$

Now (23), (97), (106) and the last estimate taken with sufficiently small ε imply

$$\begin{aligned}&\underset{\tau\in(0,T)}{ess\ \sup}\int_\Omega \big|\big[u^{(k)}(\tau,x)+m_0\big]_-\big|^{r+1}dx + \int_{Q_T}\int \big|\big[u^{(k)}+m_0\big]_-\big|^r\left|\frac{\partial u^{(k)}}{\partial x}\right|^2 dx\,dt \\ &\le c_{48}(r+1)^{\lambda_3}\Big\{\int_{Q_T}\int \big|\big[u^{(k)}+m_0\big]_-\big|^r\ dx\ dt\ +1\Big\}\ ,\ \lambda_3 = 2(n+3)\,.\end{aligned} \tag{113}$$

From this and Gronwall's Lemma we infer

$$\int_{Q_T}\int \big|\big[u^{(k)}+m_0\big]_-\big|^r\ dx\ dt \le c(r) \tag{114}$$

for an arbitrary $r \geq 2$ and a constant $c(r)$ depending only on r and known parameters and independent of k. Using Moser's iteration process and inequality (114) we obtain

$$ess\ \sup\left\{\left|\left[u^{(k)}(t,x)+m_0\right]_-\right| : (t,x) \in Q_T\right\} \leq c_{49} \tag{115}$$

for $k > -\frac{1}{\delta}$ with a constant c_{49} depending only on known parameters. Inequality (115) means that the desired inequality (107) holds with $M_7 = m_0 + c_{49} + 1$, $0 \leq \delta \leq \frac{1}{M_7}$. □**Proof of Theorem 4.** The assertion of Theorem 4 follows immediately from Lemmas 4 and 6. □

5 Proof of existence of solutions

We modify the functions ρ and a in the following way

$$\rho^*(u) = \rho(\min[u, M_4]), \quad a^*(t,x,v,u) = a(t,x,v,\min[u,M_4])\,, \tag{116}$$

where M_4 is the constant from Theorem 4.

These new functions ρ^*, a^* satisfy the conditions ρ), ρ'), i), iii), (25), a) with the same parameters as the functions ρ, a. Now we consider for $\delta = \frac{1}{M_4}$ the initial boundary value problem for the system

$$\frac{\partial\sigma^*(u)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{\rho_\delta^*(u) b_i\left(t, x, \frac{\partial(u-v)}{\partial x}\right)\right\} + a^*(t,x,v,u) = 0\ , \tag{117}$$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\kappa(x)\frac{\partial v}{\partial x_i}\right] + \sigma^*(u) = f(t,x)\ , \tag{118}$$

completed by the conditions (3) – (5). By Theorem 4 arbitrary solutions (u, v) of problem (117), (118) satisfy the a priori estimate

$$ess\ \sup\{|u(t,x)| : (t,x) \in Q_T\} \leq M_4 \tag{119}$$

with the constant M_4 from Theorem 4.

From (116) and

$$\rho_\delta^*(u) = \max\{\rho^*(u), \rho^*(-M_4)\} = \max\{\rho(u), \rho(-M_4)\}$$

we see that a solution of problem (117), (118), (3) – (5) with $\delta = \frac{1}{M_4}$ is automatically a solution of problem (1) – (5).

We don't want to go into details of proving solvability of the problem (117), (118), (3) – (5) with $\delta = \frac{1}{M_4}$. That could be done via Euler's backward time discretization. Such approach was used in [2], [5]. We remark only that solvability of the arising elliptic problem can be proved by using degree theory for operators of class (S_+) [12]. □

6 Proof of Uniqueness

For proving the uniqueness of the solution for problem (1) – (5) we assume that there exist two solutions (u_1, v_1), (u_2, v_2) in the sense of the Definition 1 and show that $u_1 = u_2, v_1 = v_2$. By Theorems 2, 3, we have for $j = 1, 2$

$$||u_j||_{L^\infty(Q_T)} + ||v_j||_{L^\infty(Q_T)} + \left|\left|\frac{\partial u_j}{\partial x}\right|\right|^2_{L^2(Q_T)} + \left|\left|\frac{\partial v_j}{\partial x}\right|\right|_{L^2(Q_T)} \leq M \tag{120}$$

with some constant M depending only on known parameters.

The proof of Theorem 6 will be given in four steps corresponding to four different choices of test functions in the integral identities (14), (15).

First step. We test (14) for $u = u_1,\ v = v_1$ with

$$\varphi_1 = \frac{1}{\rho(u_1)}[\sigma(u_1) - \sigma(u_2)]$$

and for $u = u_2,\ v = v_2$ with $\varphi_2 = u_1 - u_2$. Taking the difference of the obtained equalities we find

$$\begin{aligned}
&\int_0^\tau \Big\{ < \frac{\partial \sigma(u_1)}{\partial t}, \frac{1}{\rho(u_1)}\big[\sigma(u_1) - \sigma(u_2)\big] > - < \frac{\partial \sigma(u_2)}{\partial t}, (u_1 - u_2) > \Big\}\, dt + \\
&+ \sum_{i=1}^n \int_{Q_\tau}\!\!\int \Big\{ b_i\Big(t, x, \frac{\partial(u_1 - v_1)}{\partial x}\Big)\Big[\Big(\rho(u_1) - \frac{\rho'(u_1)}{\rho(u_1)}\int_{u_2}^{u_1} \rho(s)\, ds\Big)\frac{\partial u_1}{\partial x_i} - \\
&- \rho(u_2)\frac{\partial u_2}{\partial x_i}\Big] - \rho(u_2) b_i\Big(t, x, \frac{\partial(u_2 - v_2)}{\partial x}\Big) \cdot \frac{\partial(u_1 - u_2)}{\partial x_i}\Big\}\, dx\, dt + \\
&+ \int_{Q_\tau}\!\!\int \Big\{ a(t, x, v_1, u_1)\frac{1}{\rho(u_1)}\big[\sigma(u_1) - \sigma(u_2)\big] - a(t, x, v_2, u_2)(u_1 - u_2)\Big\}\, dx\, dt = 0.
\end{aligned} \tag{121}$$

We shall evaluate the left hand side of (121) term by term. We start with the first integral applying Lemma 2 from [9] with respect to the function

$$F(z_1, z_2) = F_1(z_1, z_2) = \int_{\sigma^{-1}(z_2)}^{\sigma^{-1}(z_1)} \big[\sigma^{-1}(z_1) - s\big]\rho(s)\, ds.$$

We obtain by (120)

$$\begin{aligned}
&\int_0^\tau \Big\{ < \frac{\partial \sigma(u_1)}{\partial t}, \frac{1}{\rho(u_1)}\big[\sigma(u_1) - \sigma(u_2)\big] > - < \frac{\partial \sigma(u_2)}{\partial t}, u_1 - u_2 > \Big\}\, dt = \\
&= \int_\Omega F_1\big(\sigma(u_1(\tau, x)), \sigma(u_2(\tau, x))\big)\, dx = \\
&= \int_\Omega \Big\{ \int_{u_2(\tau,x)}^{u_1(\tau,x)} \big[u_1(\tau, x) - s\big]\rho(s)\, ds\Big\}\, dx \geq c_{50} \int_\Omega \big|u_1(\tau, x) - u_2(\tau, x)\big|^2\, dx.
\end{aligned} \tag{122}$$

We shall estimate the second integral in (121) by using the inequalities

$$\rho(u_1) - \frac{\rho'(u_1)}{\rho(u_1)} \int_{u_2}^{u_1} \rho(s)\, ds \ \geq \rho(u_1) - \int_{u_2}^{u_1} \frac{\rho'(s)}{\rho(s)} \rho(s)\, ds \ = \rho(u_2), \tag{123}$$

$$\Big| \rho(u_1) - \rho(u_2) - \frac{\rho'(u_1)}{\rho(u_1)} \int_{u_2}^{u_1} \rho(s)\, ds \Big| \leq c_{51} |u_1 - u_2|^2, \tag{124}$$

that follow from condition ρ) and the local Lipschitz condition for ρ' respectively.

From *ii)*, (123), (124) and the local Lipschitz condition for the function b_i we get

$$\sum_{i=1}^{n} \int_{Q_\tau}\!\!\int \Big\{ b_i\Big(t, x, \frac{\partial(u_1 - v_1)}{\partial x}\Big) \Big[\Big(\rho(u_1) - \frac{\rho'(u_1)}{\rho(u_1)} \int_{u_2}^{u_1} \rho(s)\, ds \Big) \cdot$$

$$\cdot \Big(\frac{\partial(u_1 - v_1)}{\partial x_i} + \frac{\partial v_1}{\partial x_i} \Big) - \rho(u_2) \frac{\partial u_2}{\partial x_i} \Big] - \rho(u_2) b_i\Big(t, x, \frac{\partial(u_2 - v_2)}{\partial x}\Big) \frac{\partial(u_1 - u_2)}{\partial x_i} \Big\}\, dx\, dt$$

$$\geq \sum_{i=1}^{n} \int_{Q_\tau}\!\!\int \rho(u_2) \Big[b_i\Big(t, x, \frac{\partial(u_1 - v_1)}{\partial x}\Big) - b_i\Big(t, x, \frac{\partial(u_2 - v_2)}{\partial x}\Big) \Big] \cdot$$

$$\cdot \frac{\partial(u_1 - u_2 - v_1 + v_2)}{\partial x_i}\, dx\, dt \ - c_{52} \int_{Q_\tau}\!\!\int \Big\{ |u_1 - u_2|^2 \Big[\Big| \frac{\partial(u_1 - v_1)}{\partial x} \Big| + 1 \Big] \Big| \frac{\partial v_1}{\partial x} \Big| +$$

$$+ \Big| \frac{\partial(u_1 - v_1 - u_2 + v_2)}{\partial x} \Big| \Big| \frac{\partial(v_1 - v_2)}{\partial x} \Big| \Big\}\, dx\, dt \geq c_{53} \int_{Q_\tau}\!\!\int \Big| \frac{\partial(u_1 - u_2)}{\partial x} \Big|^2 dx\, dt -$$

$$- c_{54} \int_{Q_\tau}\!\!\int \Big\{ |u_1 - u_2|^2 \Big[1 + \Big| \frac{\partial(u_1 - v_1)}{\partial x} \Big| \Big] \Big| \frac{\partial v_1}{\partial x} \Big| + \Big| \frac{\partial(v_1 - v_2)}{\partial x} \Big|^2 \Big\}\, dx\, dt\, . \tag{125}$$

The last integral in (121) we estimate by using condition a), *iii)*, the local Lipschitz condition for a and the inequality

$$\Big| \frac{1}{\rho(u_2)} \big[\sigma(u_1) - \sigma(u_2)\big] - (u_1 - u_2) \Big| \leq c_{55} |u - u_2|^2,$$

that follows from local boundedness of ρ'. We obtain

$$\int_{Q_\tau}\!\!\int \Big\{ a(t, x, v_1, u_1) \frac{1}{\rho(u_1)} \big[\sigma(u_1) - \sigma(u_2)\big] - a(t, x, v_2, u_2)(u_1 - u_2) \Big\}\, dx\, dt \geq$$

$$\geq \int_{Q_\tau}\!\!\int \Big\{ a(t, x, v_1, u_2) \Big[\frac{1}{\rho(u_2)} \big(\sigma(u_1) - \sigma(u_2)\big) - (u_1 - u_2) \Big] +$$

$$+ \big[a(t, x, v_1, u_2) - a(t, x, v_2, u_2) \big] (u_1 - u_2) \Big\}\, dx\, dt \geq$$

$$\geq -c_{55} \int_{Q_\tau}\!\!\int \Big\{ [1 + \alpha(t, x)](u_1 - u_2)^2 + (v_1 - v_2)^2 \Big\}\, dx\, dt. \tag{126}$$

Now the (121) and (122), (125), (126) and Poincaré's inequality imply

$$\int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\,dx\ +\int_{Q_\tau}\int\Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\,dx\,dt\le$$
$$\le c_{56}\int_{Q_\tau}\int\Big\{\Big|\frac{\partial(v_1-v_2)}{\partial x}\Big|^2+ \tag{127}$$
$$+\Big[\Big(1+\Big|\frac{\partial(u_1-v_1)}{\partial x}\Big|\Big)\Big|\frac{\partial v_1}{\partial x}\Big|+1+\alpha(t,x)\Big]|u_1-u_2|^2\Big\}\,dx\,dt.$$

Second step. We test the integral identity (15) for $u=u_i,\ v=v_i,\ i=1,2$, with $\psi_1=v_1-v_2$. Taking the difference of the obtained equalities, applying condition $ii)_3$ and the inequalities of Cauchy and Poincaré, we get

$$\int_\Omega\Big|\frac{\partial(v_1-v_2)}{\partial x}\Big|^2\,dx\le c_{57}\int_\Omega|u_1-u_2|^2\,dx. \tag{128}$$

Third step. We test the integral identity (15) for $u=u_1,\ v=v_1$ with

$$\varphi_3=\frac{1}{\rho(u_1)}\big[\exp(N\sigma(u_1))-\exp(N\sigma(u_2))\big]_+ \tag{129}$$

and for $u=u_2,\ v=v_2$ with

$$\varphi_4=N[u_1-u_2]_+\exp(N\sigma(u_2))\ , \tag{130}$$

where N is a positive number depending only on known parameters and satisfying

$$N\rho^2(s)+2\rho'(s)\ge 1\quad\text{for}\quad |s|\le M \tag{131}$$

with the constant M from (120). Taking the difference of the obtained equalities we get

$$\int_0^\tau\Big\{<\frac{\partial\sigma_1}{\partial t},\frac{1}{\rho(u_1)}\big[\exp(N\sigma(u_1))-\exp(N\sigma(u_2))\big]_+>-$$
$$-<\frac{\partial\sigma_2}{\partial t},N[u_1-u_2]_+\exp(N\sigma(u_2))>\Big\}\,dt\ +I^{(1)}+I^{(2)}+I^{(3)}=0, \tag{132}$$

where

$$I^{(1)}=N\sum_{j=1}^n\int_{Q_\tau^+}\int b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\Big[\rho(u_1)\exp(N\sigma(u_1))\frac{\partial u_1}{\partial x_i}-$$
$$-\rho(u_2)\exp(N\sigma(u_2))\frac{\partial u_2}{\partial x_i}-\frac{\rho'(u_1)}{\rho(u_1)}\int_{u_2}^{u_1}\rho(s)\exp(N\sigma(s))\,ds\ \cdot \tag{133}$$
$$\cdot\Big(\frac{\partial(u_1-v_1)}{\partial x_i}+\frac{\partial v_1}{\partial x_i}\Big)\Big]\,dx\,dt,$$

$$I^{(2)} = -N \sum_{j=1}^{n} \int_{Q_\tau^+} \int \rho(u_2) b_i\Big(t, x, \frac{\partial(u_2 - v_2)}{\partial x}\Big) \cdot$$
$$\cdot \Big[\frac{\partial(u_1 - u_2)}{\partial x_i} + N\rho(u_2)(u_1 - u_2)\frac{\partial u_2}{\partial x_i}\Big] \exp(N\sigma(u_2))\, dx\, dt, \tag{134}$$

$$I^{(3)} = \int_{Q_\tau^+} \int \Big\{ \frac{a(t, x, v_1, u_1)}{\rho(u_1)} \big[\exp(N\sigma(u_1)) - \exp(N\sigma(u_2))\big] -$$
$$- Na(t, x, v_2, u_2)(u_1 - u_2) \exp(N\sigma(u_2)) \Big\}\, dx\, dt. \tag{135}$$

Here $Q_\tau^+ = \big\{(t, x) \in Q_\tau : u_1(t, x) > u_2(t, x)\big\}$.

We shall evaluate the terms of the left hand side of (132). To the first one we apply Lemma 2 from [9] with respect to the function

$$F(z_1, z_2) = F_2(z_1, z_2) = N\Big[\int_{\sigma^{-1}(z_2)}^{\sigma^{-1}(z_1)} \big(\sigma^{-1}(z_1) - s\big)\rho(s) \exp(N\sigma(s))\, ds\Big]_+.$$

Using (120) we obtain

$$\int_0^\tau \Big\{ < \frac{\partial \sigma_1}{\partial t}, \frac{1}{\rho(u_1)}\big[\exp(N\sigma(u_1)) - \exp(N\sigma(u_2))\big]_+ > -$$
$$- < \frac{\partial \sigma_2}{\partial t}, N[u_1 - u_2]_+ \exp(N\sigma(u_2)) \Big\}\, dt \;= \tag{136}$$
$$= \int_\Omega F_2\big(\sigma(u_1(\tau, x)), \sigma(u_2(\tau, x))\big)\, dx \;\geq c_{58} \int_\Omega \big[u_1(\tau, x) - u_2(\tau, x)\big]_+^2\, dx.$$

As to the second summand in (132) we use the inequality

$$-\frac{\rho'(u_1)}{\rho(u_1)} \int_{u_2}^{u_1} \rho(s) \exp(N\sigma(s))\, ds \;\geq -\int_{u_2}^{u_1} \rho'(s) \exp(N\sigma(s))\, ds \;=$$
$$= \rho(u_2) \exp(N\sigma(u_2)) - \rho(u_1) \exp(N\sigma(u_1)) + N \int_{u_2}^{u_1} \rho^2(s) \exp(N\sigma(s))\, ds \tag{137}$$

that follows from condition ρ). We obtain

$$I^{(1)} \geq N^2 \sum_{j=1}^{n} \int_{Q_\tau^+} \int b_i\Big(t, x, \frac{\partial(u_1 - v_1)}{\partial x}\Big) \frac{\partial(u_1 - v_1)}{\partial x_i} \cdot$$
$$\cdot \int_{u_2}^{u_1} \rho^2(s) \exp(N\sigma(s))\, ds\, dx\, dt \;+ I_1^{(1)} + I_2^{(1)} \tag{138}$$

where

$$I_1^{(1)} = N\sum_{j=1}^{n}\int_{Q_\tau^+}\int \rho(u_2)\exp(N\sigma(u_2))b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial(u_1-u_2)}{\partial x_i}\;dx\;dt,$$

$$\begin{aligned} I_2^{(1)} = N\sum_{j=1}^{n}\int_{Q_\tau^+}\int b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial v_1}{\partial x_i}\Big\{\rho(u_1)\exp(N\sigma(u_1))-\\ -\rho(u_2)\exp(N\sigma(u_2))-\frac{\rho'(u_1)}{\rho(u_1)}\int_{u_2}^{u_1}\rho(s)\exp(N\sigma(s)\;ds\Big\}\;dx\;dt. \end{aligned} \tag{139}$$

We transform the integral from (134) in the following way

$$\begin{aligned} I^{(2)} = -\,N^2\sum_{i=1}^{n}\int_{Q_\tau^+}\int \rho^2(u_2)b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial(u_1-v_1)}{\partial x_i}\cdot\\ \cdot(u_1-u_2)\exp(N\sigma(u_2))\;dx\;dt + I_1^{(2)} + I_2^{(2)} + I_3^{(2)}\;, \end{aligned} \tag{140}$$

where

$$\begin{aligned} I_1^{(2)} &= -N\sum_{i=1}^{n}\int_{Q_\tau^+}\int \rho(u_2)\exp(N\sigma(u_2))b_i\Big(t,x,\frac{\partial(u_2-v_2)}{\partial x}\Big)\frac{\partial(u_1-u_2)}{\partial x_i}\;dx\;dt\;,\\ I_2^{(2)} &= -N^2\sum_{i=1}^{n}\int_{Q_\tau^+}\int \rho^2(u_2)b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial v_1}{\partial x_i}(u_1-u_2)\exp(N\sigma(u_2))\;dx\;dt\;,\\ I_3^{(2)} &= -\,N^2\sum_{i=1}^{n}\int_{Q_\tau^+}\int \rho^2(u_2)(u_1-u_2)\exp(N\sigma(u_2))\Big[b_i\Big(t,x,\frac{\partial(u_2-v_2)}{\partial x}\Big)\frac{\partial u_2}{\partial x_i}-\\ &\quad -b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial u_1}{\partial x_i}\Big]\;dx\;dt. \end{aligned} \tag{141}$$

Now we shall estimate summands from $I^{(1)}+I^{(2)}$ that arise from (138) and (140). In view of the choice of N ((132) we have for $u_1 > u_2$

$$\begin{aligned} &\int_{u_2}^{u_1}\rho^2(s)\exp(N\sigma(s))\;ds - \rho^2(u_2)\exp(N\sigma(u_2))(u_1-u_2)\\ &=\int_{u_2}^{u_1}\int_{u_2}^{s}\big[2\rho(z)\rho'(z)+N\rho^3(z)\big]\exp(N\sigma(z))\;dz\;ds\;\geq c_{58}|u_1-u_2|^2. \end{aligned} \tag{142}$$

Hence we get

$$\begin{aligned} N^2\sum_{i=1}^{n}\int_{Q_\tau^+}\int b_i\Big(t,x,\frac{\partial(u_1-v_1)}{\partial x}\Big)\frac{\partial(u_1-v_1)}{\partial x_i}\Big\{\int_{u_2}^{u_1}\rho^2(s)\exp(N\sigma(s))\;ds-\\ -\rho^2(u_2)\exp(N\sigma(u_2))(u_1-u_2)\Big\}\;dx\;dt \geq c_{59}\int_{Q_\tau^+}\int |u_1-u_2|^2\Big|\frac{\partial(u_1-v_1)}{\partial x}\Big|^2\;dx\;dt. \end{aligned} \tag{143}$$

Using condition *ii)* we get

$$I_1^{(1)} + I_2^{(1)} \geq c_{60} \int_{Q_\tau^+}\int \left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 dx\, dt - c_{61}\int_{Q_\tau^+}\int \left|\frac{\partial(v_1-v_2)}{\partial x}\right|^2 dx\, dt. \quad (144)$$

Since, as a consequence of the local Lipschitz continuity of ρ',

$$\Big|\rho(u_1)\exp(N\sigma(u_1)) - \rho(u_2)\exp(N\sigma(u_2)) - \frac{\rho'(u_1)}{\rho(u_1)}\int_{u_2}^{u_1}\rho(s)\exp N\sigma(s)\, ds - $$
$$- N\rho^2(u_2)(u_1-u_2)\exp(N\sigma(u_2))\Big| \leq c_{62}|u_1-u_2|^2 , \quad (145)$$

condition *ii)* yields

$$\left|I_2^{(1)} + I_2^{(2)}\right| \geq c_{63}\int_{Q_\tau^+}\int \left(1 + \left|\frac{\partial(u_1-v_1)}{\partial x}\right|\right)\left|\frac{\partial v_1}{\partial x}\right||u_1-u_2|^2\, dx\, dt. \quad (146)$$

The next estimate follows from the local Lipschitz condition for b_i:

$$\left|I_3^{(2)}\right| \leq c_{64}\int_{Q_\tau^+}\int \Big\{\varepsilon|u_1-u_2|^2\left|\frac{\partial(u_1-v_1)}{\partial x}\right|^2 + \frac{1}{\varepsilon}\left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 +$$
$$+ \frac{1}{\varepsilon}\left|\frac{\partial(v_1-v_2)}{\partial x}\right|^2 + \left|\frac{\partial v_1}{\partial x}\right|^2|u_1-u_2|^2\Big\}\, dx\, dt\,. \quad (147)$$

Here $\varepsilon \in (0,1)$ is an arbitrary number. The term $I^{(3)}$ defined by (135) can be estimated analogously to (126) such that we get

$$I^{(3)} \geq -c_{65}\int_{Q_\tau^+}\int (1+\alpha(t,x))|u_1-u_2|^2\, dx\, dt. \quad (148)$$

Finally, we obtain from (132), (136), (138), (140), (143), (144), (146) – (148) for sufficiently small ε

$$\int_\Omega \left[u_1(\tau,x) - u_2(\tau,x)\right]_+^2 dx + \int_{Q_\tau^+}\int |u_1-u_2|^2\left|\frac{\partial(u_1-v_1)}{\partial x}\right|^2 dx\, dt \leq$$
$$\leq c_{66}\int_{Q_\tau^+}\int \Big\{\left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 + \left|\frac{\partial(v_1-v_2)}{\partial x}\right|^2 + \left|\frac{\partial v_1}{\partial x}\right|^2|u_1-u_2|^2 + \quad (149)$$
$$+ (1+\alpha(t,x))|u_1-u_2|^2\Big\}\, dx\, dt.$$

Changing the places of u_1 and u_2 in the last inequality we get immediately

$$\int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\, dx + \int_{Q_\tau}\int |u_1-u_2|^2\left(\left|\frac{\partial u_1}{\partial x}\right|^2 + \left|\frac{\partial u_2}{\partial x}\right|^2\right) dx\, dt \leq$$
$$\leq c_{67}\int_{Q_\tau}\int \Big\{\left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 + \left|\frac{\partial(v_1-v_2)}{\partial x}\right|^2 + \left(\left|\frac{\partial v_1}{\partial x}\right|^2 + \left|\frac{\partial v_2}{\partial x}\right|^2\right)|u_1-u_2|^2 +$$
$$+ (1+\alpha(t,x))|u_1-u_2|^2\Big\}\, dx\, dt.$$
$$(150)$$

Fourth step. Let $\{\varphi_j(x)\}, j = 1, \ldots, J$ be a partition satisfying the conditions (76) with a number R to be fixed chosen later on. We the integral identity (15) for $u = u_1, v = v_1$ with

$$\psi_2 = \sum_{j=1}^{J} [v_1 - v_{1,j}]\, \varphi_j^2 |u_1 - u_2|^2 \tag{151}$$

where $v_{1,j} = v_{1,j}(t) = v(t, x_j)$. We obtain after integration with respect to t

$$\int_{Q_\tau}\int \kappa(x)|u_1 - u_2|^2 \left|\frac{\partial v_1}{\partial x}\right|^2 dx\, dt \;=\; J^{(1)} + J^{(2)} + J^{(3)}\,, \tag{152}$$

where

$$\begin{aligned} J^{(1)} &= -2\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)\cdot\frac{\partial v_1}{\partial x_i}\cdot\frac{\partial(u_1 - u_2)}{\partial x_i}[v_1 - v_{1,j}](u_1 - u_2)\varphi_j^2\, dx\, dt\,, \\ J^{(2)} &= -2\sum_{j=1}^{J}\sum_{i=1}^{n}\int_{Q_\tau}\int \kappa(x)\frac{\partial v_1}{\partial x_i}\cdot\frac{\partial \varphi_j}{\partial x_i}[v_1 - v_{1,j}]\varphi_j|u_1 - u_2|^2\, dx\, dt\,, \\ J^{(3)} &= -\sum_{j=1}^{J}\int_{Q_\tau}\int [\sigma(u_j) - f][v_1 - v_{1,j}]\varphi_j^2|u_1 - u_2|^2\, dx\, dt\,. \end{aligned}$$

We estimate $J^{(1)}, J^{(2)}, J^{(3)}$ by Cauchy's inequality, Theorem 3 and (76) and obtain

$$\begin{aligned} |J^{(1)}| &\le \frac{1}{4}\int_{Q_\tau}\int \kappa(x)\left|\frac{\partial v_1}{\partial x}\right|^2|u_1 - u_2|^2\, dx\, dt + c_{68}R^\alpha\int_{Q_\tau}\int \left|\frac{\partial(u_1 - u_2)}{\partial x}\right|^2 dx\, dt \\ |J^{(2)}| &\le \frac{1}{4}\int_{Q_\tau}\int \kappa(x)\left|\frac{\partial v_1}{\partial x}\right|^2|u_1 - u_2|^2\, dx\, dt + \frac{c_{69}}{R^2}\int_{Q_\tau}\int |u_1 - u_2|^2\, dx\, dt \\ |J^{(3)}| &\le c_{70}\int_{Q_\tau}\int (1 + |f|)|u_1 - u_2|^2\, dx\, dt. \end{aligned} \tag{153}$$

The equality (152) and inequalities (153) imply immediately

$$\begin{aligned} &\int_{Q_\tau}\int |u_1 - u_2|^2\left|\frac{\partial v_1}{\partial x}\right|^2 dx\, dt \;\le c_{71}\int_{Q_\tau}\int \left\{R^\alpha\left|\frac{\partial(u_1 - u_2)}{\partial x}\right|^2 + \right. \\ &\left. + \left(\frac{1}{R^2} + |f|\right)|u_1 - u_2|^2\right\} dx\, dt. \end{aligned} \tag{154}$$

End of the proof of Theorem 6. Applying Cauchy's inequality to the term in (127) involving the derivative of $u_1 - v_1$ and choosing a suitable value of R, we obtain from (127), (128), (150), (154)

$$\begin{aligned} &\int_\Omega |u_1(\tau, x) - u_2(\tau, x)|^2\, dx \;+ \int_{Q_\tau}\int \left|\frac{\partial(u_1 - u_2)}{\partial x}\right|^2 dx\, dt \;\le \\ &\le c_{72}\int_{Q_\tau}\int (1 + |\alpha| + |f|)|u_1 - u_2|^2\, dx\, dt. \end{aligned} \tag{155}$$

We estimate the integral on the right hand site of (155) by Hölder's inequality and use the conditions on α, f to get

$$\begin{aligned} &\underset{\tau \in (0,\theta)}{ess\ \sup} \int_\Omega |u_1(\tau,x) - u_2(\tau,x)|^2\, dx \ + \int_{Q_\theta}\int \Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\, dx\, dt \ \le \\ &\le c_{73}\Big\{\int_{Q_\theta}\int |u_1-u_2|^{2p_1'}\, dx\, dt\ \Big\}^{\frac{1}{p_1'}} + c_{73}\int_0^\theta \Big\{\int_\Omega |u_1-u_2|^{2p_2'}\, dx\ \Big\}^{\frac{1}{p_2'}}\, dt \end{aligned} \tag{156}$$

for an arbitrary $\theta \in (0,T)$.

Estimating the first integral on the right hand site of (156) by Hölder's inequality, using the embedding $V^2(Q_T) \subset L^{2\frac{(n+2)}{2}}(Q_T)$ and setting $q_1 = n+2-p_1'n$, we find

$$\begin{aligned} &\Big\{\int_{Q_\theta}\int |u_1-u_2|^{2p_1'}\, dx\, dt\ \Big\}^{\frac{1}{p_1'}} \le \Big(\int_{Q_\theta}\int |u_1-u_2|^2\, dx\, dt\ \Big)^{\frac{q_1}{2p_1'}}\cdot \\ &\cdot\Big\{\int_{Q_\theta}\int |u_1-u_2|^{\frac{2(n+2}{n}}\, dx\, dt\Big\}^{\frac{1}{p_1'}-\frac{q_1}{2p_1'}} \le \varepsilon^{-\frac{2p_1'}{q_1}}\int_{Q_\theta}\int |u_1-u_2|^2\, dx\, dt\, + \\ &+ c_{74}\varepsilon^{\frac{2p_1'}{2p_1'-q_1}}\Big\{\ \underset{\tau \in (0,\theta)}{ess\ \sup} \int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\, dx\ + \int_{Q_\theta}\int \Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\, dx\, dt\Big\} \end{aligned} \tag{157}$$

with an arbitrary $\varepsilon \in (0,1)$ and a constant c_{74} depending only on n.

In analogous way we estimate the last integral in (156). We define γ by the equality

$$\frac{2-\gamma}{2p_2'-\gamma} = \frac{1}{2}\Big(\frac{1}{p_2'} + \frac{n-2}{n}\Big).$$

Then $q_2 = \frac{2p_2'-\gamma}{2-\gamma} < \frac{n}{n-2}$ and $2q_2 < 2\big(1+\frac{2q_2}{n}\big)$.

Estimating the last integral in (156) by Hölder's inequality and (88) we find

$$\begin{aligned} &\int_Q^\theta \Big\{\int_\Omega |u_1-u_2|^{2p_2'}\, dx\ \Big\}^{\frac{1}{p_2'}}\, dt \le c_{75}\Big(\int_{Q_\theta}\int |u_1-u_2|^2\, dx\, dt\Big)^{\frac{\gamma}{2p_2'}}\cdot \\ &\cdot\Big\{\int_Q^\theta \Big[\int_\Omega |u_1-u_2|^{2(1+\frac{2q_2)}{n}}\, dx\Big]^{\frac{1}{q_2}}\, dx\Big\}^{\left(1+\frac{\gamma}{2p_2'}\right)\frac{nq_2}{n+2q_2}} \le \\ &\le c_{76}\Big\{\varepsilon^{-\frac{2p_2'}{\gamma}}\int_{Q_\theta}\int |u_1-u_2|^2\, dx\, dt\, + \\ &+\varepsilon^{\frac{2p_2'}{2p_2'-\gamma}}\Big[\ \underset{\tau \in (0,\theta)}{ess\ \sup} \int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\, dx + \int_{Q_\theta}\int \Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\, dx\, dt\ \Big]\Big\} \end{aligned} \tag{158}$$

with $\varepsilon \in (0,1)$.

The inequalities (156) – (158) imply with suitable ε

$$\int_\Omega |u_1(\theta,x) - u_2(\theta,x)|^2 \leq c_{77} \int_{Q_\theta}\int |u_1 - u_2|^2 \, dx \, dt$$

for arbitrary $\theta \in (0,T)$. Finally, Gronwall's lemma yields $u_1 = u_2$ and the equality $v_1 = v_2$ follows now from (128). □

References

[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in: Function spaces, differential operators and nonlinear analysis (H.-J. Schmeisser, H. Triebel, eds.), B. G. Teubner Verlagsgesellschaft, Teubner-Texte Math., **133**, Stuttgart 1993, 9–126.

[2] H.W. Alt, S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., **183** (1983), 311–341.

[3] Ph. Benilan, P. Wittbold, *On mild and weak solutions of elliptic–parabolic systems*, Adv. Differ. Equ., vol. **1** (1996), 1053–1073.

[4] H. Gajewski, *On a variant of monotonicity and its application to differential equations*, Nonlinear Anal., TMA, vol. **22** (1994), 73–80.

[5] H. Gajewski, K. Gröger, *Reaction–diffusion processes of electrically charged species*, Math. Nachr., **177** (1996), 109–130.

[6] H. Gajewski, K. Zacharias, *Global behavior of a reaction–diffusion system modelling chemotaxis*, Math. Nachr., **195** (1998), 77–114.

[7] H. Gajewski, K. Zacharias, On a nononlocal phase separation model, Preprint 656 WIAS Berlin, 2001, Jnl. Math. Anal. Appl. (to appear)

[8] H. Gajewski, I.V. Skrypnik, *To the uniqueness problem for nonlinear elliptic equations*, Preprint No. 527, WIAS (1999), Nonlinear Analysis (2002).

[9] H. Gajewski, I.V. Skrypnik, *On the uniqueness of solutions for nonlinear parabolic equations*, Preprint No. 658, WIAS (2001), Discrete and Continuous Dynamical Systems (to appear) .

[10] O.A. Ladyzhenskaya, N.N. Uraltseva, *Linear and quasilinear elliptic equations*, Nauka, Moscow (1973) (Russian).

[11] F. Otto, *L^1–contraction and uniqueness for quasilinear elliptic–parabolic equations*, C.R. Acad. Sci. Paris, **318**, Serie 1 (1995), 1005–1010.

[12] I.V. Skrypnik, *Methods for analysis of nonlinear elliptic boundary value problems*, Transl. Math. Monographs, A.M.S., Providence, vol. **139** (1994).