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On the Diffraction by Biperiodic Anisotropic Structures

Gunther Schmidt[1]

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[1] Weierstrass Institute of Applied Analysis and Stochastics
E-Mail: schmidt@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper studies the scattering of electromagnetic waves by a nonmagnetic biperiodic structure. The structure consists of anisotropic optical materials and separates two regions with constant dielectric coefficients. The time harmonic Maxwell equations are transformed to an equivalent strongly elliptic variational problem for the magnetic field in a bounded biperiodic cell with nonlocal boundary conditions. This guarantees the existence of quasiperiodic magnetic fields in H^1 and electric fields in $H(\text{curl})$ solving Maxwell's equations. The uniqueness is proved for all frequencies excluding possibly a discrete set. The analytic dependence of these solutions on frequency and incident angles is studied.

1 Introduction

We consider the diffraction of a electromagnetic plane wave incident on a general biperiodic structure in $\mathbb{R}^3$. By biperiodic or doubly periodic we mean that the structure is periodic in two not necessarily orthogonal directions. This structure separates two homogeneous regions. The medium is assumed to be nonmagnetic, its optical property is characterized completely by the dielectric coefficient of the materials. Below and above the structure these coefficients are fixed constants. The structure itself is formed by inhomogeneous and anisotropic materials, we assume that the optical property of the structure is described by a dielectric tensor with essentially bounded and doubly periodic components.

The diffraction problem is to find, for a given incident wave of arbitrary polarization, the scattered far field pattern, i.e. the diffraction efficiencies and the states of polarization of all propagating diffracted waves. This problem is of considerable interest in several diverse areas of modern optics, where such doubly periodic structures are called crossed anisotropic gratings. The numerical modelling of those devices has recently received considerable attention within the engineering community ([13], [12], [14]).

Mathematical problems for isotropic biperiodic gratings have been considered in several papers. Dobson and Friedman [8] studied existence and uniqueness of solutions for biperiodic structures, consisting of two homogeneous materials separated by a piecewise C^2 interface by means of integral equations. Abboud introduced in [1] a variational approach for quite general biperiodic structures with variable magnetic permeability. He obtained a saddle point problem satisfying Fredholm's alternative. In Dobson [7], Bao [3], Bao and Dobson

[4] the existence and uniqueness of solutions of variational equations for the magnetic field are studied. The results are used to justify the stability of finite element methods for biperiodic diffraction problems.

In this note we extend the approach of [9] for diffractive structures which are constant in one direction to the analysis of crossed anisotropic gratings. We formulate a variational problem for the magnetic field which is equivalent to the underlying Maxwell equations. It has properties similar to those of the simpler twodimensional cases, where Maxwell's equations can be reduced to Helmholtz equations on $\mathbb{R}^2$. Using the concept of strong ellipticity we state general existence and uniqueness results under assumptions that are satisfied for any relevant practical application. We show that the variational problem is solvable for all frequencies and directions of the incident wave. The solutions are unique except a discrete sequence of frequencies accumulating at infinity. If the structure contains absorbing materials and the dielectric tensor is piecewise analytic, then the diffraction problem is uniquely solvable for all frequencies.

Specified to the isotropic case the variational formulation is close to that studied in [7] and [3], but our approach allows to treat more general situations. So we admit also negative real parts of the dielectric coefficients, which is quite common for metallic materials. Further we study the case of so called Rayleigh frequencies, which are associated with the appearance of new diffracted modes if the frequency increases.

The outline of the paper is as follows. In Section 2 we formulate Maxwell's equations and radiation conditions for the biperiodic diffraction problem. The variational formulation for the magnetic field in a bounded periodic cell is introduced in Section 3. We prove its strong ellipticity in H^1 and the equivalence to the partial differential problem, which justifies the discretization of the biperiodic diffraction problem with standard finite elements. In Section 4 we prove existence and uniqueness results for the problem and consider the dependence of solutions on frequency and incident angles of the incoming wave as well as the dependence on perturbations of the dielectric coefficients.

2 Diffraction problem

In the following we suppose that the whole space is filled with material of everywhere constant magnetic permeability $\mu > 0$ having a measurable and essentially bounded permittivity function ε, which is doubly periodic and homogeneous above and below the grating structure. More precisely, in Cartesian coordinates $(x_1, x_2, x_3) = (\boldsymbol{x}, x_3) = \mathrm{x} \in \mathbb{R}^3$ we assume that the permittivity function satisfies $\varepsilon(\boldsymbol{x} + B\boldsymbol{m}, x_3) = \varepsilon(\boldsymbol{x}, x_3)$ for all $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $\boldsymbol{x} \in \mathbb{R}^2$ with a nonsingular real 2×2 matrix B. There exists a constant

$b > 0$ such that for some $\delta > 0$

$$\varepsilon(\boldsymbol{x}, x_3) = \varepsilon_\pm \in \mathbb{C} \quad \text{for } \pm x_3 \geq b - \delta$$

with $\varepsilon_+ > 0$ (the grating is illuminated from above) and $0 \leq \arg \varepsilon_- < \pi$. Otherwise, for $|x_3| < b$ the permittivity function is given by a nonsingular 3×3 matrix $\varepsilon(\mathrm{x})$ with doubly periodic, complexvalued L^∞–components. We assume that almost everywhere

$$|\varepsilon(\mathrm{x})\, \boldsymbol{\xi} \cdot \overline{\boldsymbol{\xi}}| \geq c > 0 \,, \quad 0 \leq \arg(\varepsilon(\mathrm{x})\, \boldsymbol{\xi} \cdot \overline{\boldsymbol{\xi}}) \leq \phi < \pi \,, \quad \forall\, \boldsymbol{\xi} \in \mathbb{C}^3 \,, |\boldsymbol{\xi}| = 1 \,. \tag{2.1}$$

In the following $\varepsilon(\mathrm{x})$ denotes for $\pm x_3 \geq b$ the diagonal matrix $\varepsilon_\pm \mathbf{I}$.

The grating is illuminated by a plane wave

$$\mathbf{E}^i = \mathbf{p}\, e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t} \,, \quad \mathbf{H}^i = \mathbf{q}\, e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}$$

with the wave vector $\mathbf{k} = (\alpha_1, \alpha_2, -\beta) = \omega\sqrt{\varepsilon_+ \mu}\,(\sin\Phi_1 \cos\Phi_2, \sin\Phi_1 \sin\Phi_2, -\cos\Phi_1)$, given by the angles of incidence Φ_1, Φ_2 with $0 \leq \Phi_1 < \pi/2$, $0 \leq \Phi_2 < 2\pi$. In the following we denote $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. The coefficient vectors $\mathbf{p}, \mathbf{q}$ and the wave vector $\mathbf{k}$ satisfy

$$\mathbf{p} = \frac{\mathbf{q} \times \mathbf{k}}{\omega \varepsilon_+} \,, \ \mathbf{p} \cdot \mathbf{q} = 0 \,, \ \mathbf{k} \cdot \mathbf{k} = \omega^2 \mu \varepsilon_+ \,.$$

Dropping the factor $e^{-i\omega t}$, the electromagnetic field $(\mathbf{E}, \mathbf{H})$ solves the time–harmonic Maxwell equations

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \quad \text{and} \quad \nabla \times \mathbf{H} = -i\omega\varepsilon\mathbf{E} \,, \tag{2.2}$$

which hide also the equations on the divergence

$$\nabla \cdot (\varepsilon\mathbf{E}) = 0 \quad \text{and} \quad \nabla \cdot (\mu\mathbf{H}) = 0 \,. \tag{2.3}$$

We are interested in weak solutions of (2.2) belonging to $H(\mathrm{curl})$, i.e. possess locally finite energy

$$\mathbf{E} \,, \ \mathbf{H} \,, \ \nabla \times \mathbf{E} \,, \ \nabla \times \mathbf{H} \in L^2_{loc}(\mathbb{R}^3)^3 \,. \tag{2.4}$$

By the translation $\boldsymbol{x} \mapsto \boldsymbol{x} + B\boldsymbol{m}$ the diffraction problem is altered only in so far as the phase of the incident wave is modified by $e^{i(\boldsymbol{\alpha}, B\boldsymbol{m})}$. This motivates to look for quasiperiodic solutions, i.e. solutions $\mathbf{E}$ and $\mathbf{H}$ such that the vector fields

$$\vec{u}(\mathrm{x}) = e^{-i(\boldsymbol{\alpha}, \boldsymbol{x})}\, \mathbf{H}(\mathrm{x}) \,, \quad \vec{v}(\mathrm{x}) = e^{-i(\boldsymbol{\alpha}, \boldsymbol{x})}\, \mathbf{E}(\mathrm{x}) \tag{2.5}$$

are doubly periodic, $\vec{u}(\boldsymbol{x} + B\boldsymbol{m}, x_3) = \vec{u}(\boldsymbol{x}, x_3)$ for all $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$. Then $\mathbf{E}$ and $\mathbf{H}$ are defined by their values on $\overline{G} \times \mathbb{R}$ with $G = B((0,1)^2)$.

Because the domain is unbounded in the x_3–direction, a radiation condition must be imposed. The physics requires that the diffracted fields remain bounded as $|x_3| \to \infty$, which leads to the so called outgoing wave condition for diffraction gratings (see [7], [4]):

$$\left.\begin{aligned} \mathbf{H}(\mathrm{x}) - \mathbf{q}\, e^{i(\boldsymbol{\alpha},\boldsymbol{x})-i\beta x_3} &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} H^+_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})+i\beta^+_{\boldsymbol{m}} x_3} \,, \\ \mathbf{E}(\mathrm{x}) - \mathbf{p}\, e^{i(\boldsymbol{\alpha},\boldsymbol{x})-i\beta x_3} &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} E^+_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})+i\beta^+_{\boldsymbol{m}} x_3} \,, \end{aligned}\right\} \quad x_3 \ge b\,,$$

$$\left.\begin{aligned} \mathbf{H}(\mathrm{x}) &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} H^-_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})-i\beta^-_{\boldsymbol{m}} x_3} \,, \\ \mathbf{E}(\mathrm{x}) &= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} E^-_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})-i\beta^-_{\boldsymbol{m}} x_3} \,, \end{aligned}\right\} \quad x_3 \le b\,, \tag{2.6}$$

with certain constant vectors $H^\pm_{\boldsymbol{m}}$ and $E^\pm_{\boldsymbol{m}}$, the so called Rayleigh coefficients. Here we denote $\boldsymbol{\alpha}_{\boldsymbol{m}} = \boldsymbol{\alpha} + 2\pi A\boldsymbol{m}$, $\boldsymbol{m} \in \mathbb{Z}^2$, with $A = (B^*)^{-1}$ and

$$\beta^\pm_{\boldsymbol{m}} = \beta^\pm_{\boldsymbol{m}}(\boldsymbol{\alpha}) = \sqrt{\omega^2\mu\varepsilon_\pm - |\boldsymbol{\alpha}_{\boldsymbol{m}}|^2}\,, \tag{2.7}$$

where the branch of the square root is chosen such that $\beta^\pm_{\boldsymbol{m}} > 0$ if $\omega^2\mu\varepsilon_\pm > |\boldsymbol{\alpha}_{\boldsymbol{m}}|^2$ and its branch-cut is $(-\infty, 0)$. Thus $\mathrm{Re}\,\beta^\pm_{\boldsymbol{m}} \ge 0$, $\mathrm{Im}\,\beta^\pm_{\boldsymbol{m}} \ge 0$. Note that $\beta^\pm_{\boldsymbol{m}}$ is real for at most finitely many $\boldsymbol{m}$, corresponding to propagating modes of the electromagnetic field

$$\begin{aligned} &\sum_{\boldsymbol{m}\in\mathbb{P}^+} H^+_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})+i\beta^+_{\boldsymbol{m}} x_3}\,, \quad \sum_{\boldsymbol{m}\in\mathbb{P}^+} E^+_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})+i\beta^+_{\boldsymbol{m}} x_3}\,, \qquad x_3 \to \infty\,, \\ &\sum_{\boldsymbol{m}\in\mathbb{P}^-} H^-_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})-i\beta^-_{\boldsymbol{m}} x_3}\,, \quad \sum_{\boldsymbol{m}\in\mathbb{P}^-} E^-_{\boldsymbol{m}}\, e^{i(\boldsymbol{\alpha}_{\boldsymbol{m}},\boldsymbol{x})-i\beta^-_{\boldsymbol{m}} x_3}\,, \qquad x_3 \to -\infty\,, \end{aligned}$$

where $\mathbb{P}^\pm = \{\boldsymbol{m} \in \mathbb{Z}^2 : |\boldsymbol{\alpha}_{\boldsymbol{m}}|^2 < \omega^2\mu\varepsilon_\pm\}$. The remaining plane waves in the sums (2.6) are exponentially decayed as $|x_3| \to \infty$, they carry no energy away from the inhomogeneous structure.

The Rayleigh coefficients are the main characteristics of diffraction gratings. They indicate the efficiencies, phase shifts and polarization of the propagating modes. The efficiency is defined as the ratio of the energies of the corresponding mode and of the incident wave and can be calculated from the formulas

$$e^+_{\boldsymbol{m}} = \frac{\beta^+_{\boldsymbol{m}} |H^+_{\boldsymbol{m}}|^2}{\beta\,|\mathbf{q}|^2} \quad \text{and} \quad e^-_{\boldsymbol{m}} = \frac{\varepsilon_+ \beta^-_{\boldsymbol{m}} |H^-_{\boldsymbol{m}}|^2}{\varepsilon_-\,\beta\,|\mathbf{q}|^2} \quad \text{for } \boldsymbol{m} \in \mathbb{P}^\pm\,.$$

3 Variational formulation

Finite energy solutions of Maxwell interface problems are in general not H^1–regular, see [6] and the literature cited therein. Therefore the natural variational spaces for those problems consist of vector fields from $H(\mathrm{curl})$ satisfying $\nabla\cdot(\varepsilon\mathbf{E}), \nabla\cdot(\mu\mathbf{H}) \in L^2_{loc}(\mathbb{R}^3)$, and only variational forms in the energy space give back a solution of the original Maxwell system. But constant μ implies H^1–regularity of the magnetic field $\mathbf{H}$. This follows from the fact, that any vector field $\vec{u} \in L^2(\Omega)^3$ with $\nabla \times \vec{u} \in L^2(\Omega)^3$, $\nabla \cdot \vec{u} \in L^2(\Omega)$ satisfies $\vec{u} \in H^1_{loc}(\Omega)^3$ for any bounded domain $\Omega \in \mathbb{R}^3$ (see [11]). Therefore we look similar to [7] for a H^1–variational equation for the doubly periodic vector field $\vec{u}(\mathrm{x}) = e^{-i(\boldsymbol{\alpha},\boldsymbol{x})}\,\mathbf{H}(\mathrm{x})$.

Define the vector valued differential operators

$$\nabla_{\boldsymbol{\alpha}} = (\partial_{1,\alpha_1}, \partial_{2,\alpha_2}, \partial_3) := \nabla + i\,(\boldsymbol{\alpha}, 0)\,,$$

which satisfies the usual curl and divergence identities. Then Maxwell's equations (2.2) transform in view of $\nabla_{\boldsymbol{\alpha}} \times \vec{u} = e^{-i(\boldsymbol{\alpha},\boldsymbol{x})} \nabla \times (\vec{u}\, e^{i(\boldsymbol{\alpha},\boldsymbol{x})})$ to the second order equation for doubly periodic vector fields

$$\nabla_{\boldsymbol{\alpha}} \times \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) - \mu\,\omega^2 \vec{u} = 0\,, \tag{3.1}$$

and one has to look for solutions with $\nabla_{\boldsymbol{\alpha}} \times \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \in L^2_{loc}(\mathbb{R}^3)^3$. The outgoing wave conditions (2.6) are transformed to nonlocal boundary conditions in a standard way by specifying the Dirichlet–to–Neumann map. Define the pseudodifferential operators $T^{\pm}_{\boldsymbol{\alpha}}$ acting on doubly periodic vector functions on $\mathbb{R}^2$ by the formula

$$T^{\pm}_{\boldsymbol{\alpha}} u(\boldsymbol{x}) = -i \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \beta^{\pm}_{\boldsymbol{m}} \hat{u}_{\boldsymbol{m}} e^{2\pi i(\boldsymbol{x}, A\boldsymbol{m})} \tag{3.2}$$

with the Fourier coefficients

$$\hat{u}_{\boldsymbol{m}} = |\det A| \int_G u(\boldsymbol{x})\, e^{-2\pi i(\boldsymbol{x}, A\boldsymbol{m})} d\boldsymbol{x}\ .$$

If $\mathbf{H}$ satisfies (2.6), then obviously

$$\partial_3 u_i(\boldsymbol{x}, b) = -T^{+}_{\boldsymbol{\alpha}} u_i(\boldsymbol{x}, b) - 2i\beta\, q_i\, e^{-i\beta b}\ , \quad \partial_3 u_i(\boldsymbol{x}, -b) = T^{-}_{\boldsymbol{\alpha}} u_i(\boldsymbol{x}, -b) \tag{3.3}$$

for each component of $\vec{u} = (u_1, u_2, u_3)$. The operators $T^{\pm}_{\boldsymbol{\alpha}}$ map the Sobolev space $H^s_p(G)$ of doubly periodic functions on G boundedly into $H^{s-1}_p(G)$, $s \in \mathbb{R}$. Here $H^s_p(G)$ denotes the closure of smooth doubly periodic functions with respect to the norm

$$\Big(|\hat{u}_0|^2 + \sum_{\boldsymbol{m} \in \mathbb{Z}^2 \setminus \{0\}} |\boldsymbol{m}|^{2s} |\hat{u}_{\boldsymbol{m}}|^2\Big)^{1/2}.$$

Further we introduce the domain $\Omega = G \times (-b, b)$ and denote by $\Gamma^{\pm}$ its upper and lower boundary, respectively. The doubly periodic Sobolev space $H^s_p(\Omega)$, $s \in \mathbb{R}$, is defined as the restriction to Ω of all functions in $H^s_{loc}(\mathbb{R}^3)$ which are doubly periodic in $\boldsymbol{x}$. Note that for $\vec{u} \in H^1_p(\Omega)$ the boundary values $\vec{u}|_{\Gamma^{\pm}} \in H^{1/2}_p(\Gamma^{\pm})^3$. Next we define the periodic pseudodifferential operators $\mathbf{R}^{\pm}_{\boldsymbol{\alpha}} := (\partial_{1,\alpha_1}, \partial_{2,\alpha_2}, \mp T^{\pm}_{\boldsymbol{\alpha}})$ acting on functions from $H^{1/2}_p(\Gamma^{\pm})$. For sufficiently smooth vector fields $\vec{u}$ obviously

$$\begin{aligned} &\vec{n} \times ((\nabla_{\boldsymbol{\alpha}} \times \vec{u})|_{\Gamma^{\pm}} - \mathbf{R}^{\pm}_{\boldsymbol{\alpha}} \times \vec{u}|_{\Gamma^{\pm}}) = (\partial_n u_1|_{\Gamma^{\pm}} - T^{\pm}_{\boldsymbol{\alpha}} u_1|_{\Gamma^{\pm}}, \partial_n u_2|_{\Gamma^{\pm}} - T^{\pm}_{\boldsymbol{\alpha}} u_2|_{\Gamma^{\pm}}, 0)\,, \\ &(\nabla_{\boldsymbol{\alpha}} \cdot \vec{u})|_{\Gamma^{\pm}} - \mathbf{R}^{\pm}_{\boldsymbol{\alpha}} \cdot \vec{u}|_{\Gamma^{\pm}} = \pm(\partial_n u_3|_{\Gamma^{\pm}} - T^{\pm}_{\boldsymbol{\alpha}} u_3|_{\Gamma^{\pm}})\,, \end{aligned} \tag{3.4}$$

where ∂_n is the normal derivative with respect to the exterior normal $\vec{n} = (0, 0, \pm 1)$ on $\Gamma^{\pm}$.

Lemma 3.1 *Any field $\vec{u} \in H_p^1(\Omega)^3$ with $\nabla_{\boldsymbol{\alpha}} \times \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \in L^2(\Omega)$ satisfies the nonlocal boundary conditions* (3.3) *if and only if in $H_p^{-1/2}(\Gamma^\pm)$*

$$\left.\begin{aligned} &\vec{n} \times \mathbf{R}_{\boldsymbol{\alpha}}^+ \times \vec{u} = \vec{n} \times \nabla_{\boldsymbol{\alpha}} \times \vec{u} + 2i\beta e^{-i\beta b}\, \vec{n} \times \vec{n} \times \mathbf{q} \\ &\mathbf{R}_{\boldsymbol{\alpha}}^+ \cdot \vec{u} = \nabla_{\boldsymbol{\alpha}} \cdot \vec{u} + 2i\beta e^{-i\beta b}\, \vec{n} \cdot \mathbf{q} \end{aligned}\right\} \text{ on } \Gamma^+ , \qquad \left.\begin{aligned} &\vec{n} \times \mathbf{R}_{\boldsymbol{\alpha}}^- \times \vec{u} = \vec{n} \times \nabla_{\boldsymbol{\alpha}} \times \vec{u} \\ &\mathbf{R}_{\boldsymbol{\alpha}}^- \cdot \vec{u} = \nabla_{\boldsymbol{\alpha}} \cdot \vec{u} \end{aligned}\right\} \text{ on } \Gamma^- . \tag{3.5}$$

Proof. Denoting by $\langle \cdot, \cdot \rangle_{\Gamma^\pm}$ the duality pairing between $H_p^{-1/2}(\Gamma^\pm)$ and $H_p^{1/2}(\Gamma^\pm)$ Green's formula

$$\int_\Omega (\nabla_{\boldsymbol{\alpha}} \times \vec{v}) \cdot \overline{\vec{\varphi}} - \int_\Omega \vec{v} \cdot (\overline{\nabla_{\boldsymbol{\alpha}} \times \vec{\varphi}}) = \langle \vec{n} \times \vec{v}, \vec{\varphi} \rangle_{\Gamma^+} + \langle \vec{n} \times \vec{v}, \vec{\varphi} \rangle_{\Gamma^-} , \; \vec{\varphi} \in H_p^1(\Omega)^3 , \tag{3.6}$$

is valid for $\vec{v} = \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u})$ (cf. [11]), hence the tangential trace $\vec{n} \times \vec{v}$ is well defined in $H^{-1/2}(\Gamma^\pm)$. Since ε is constant in a neighborhood of $\Gamma^\pm$ the assertion follows immediately from (3.3) and (3.4). ∎

Now we fix some complex number ρ and introduce the sesquilinear form

$$\begin{aligned} \mathcal{B}(\vec{u}, \vec{\varphi}) \;:=\; & \int_\Omega \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \cdot (\overline{\nabla_{\boldsymbol{\alpha}} \times \vec{\varphi}}) + \rho \int_\Omega (\nabla_{\boldsymbol{\alpha}} \cdot \vec{u})\, (\overline{\nabla_{\boldsymbol{\alpha}} \cdot \vec{\varphi}}) - \omega^2 \mu \int_\Omega \vec{u} \cdot \overline{\vec{\varphi}} \\ & + \frac{1}{\varepsilon_+} \Big(\langle \vec{n} \times (\mathbf{R}_{\boldsymbol{\alpha}}^+ \times \vec{u}), \vec{\varphi} \rangle_{\Gamma^+} - \langle \mathbf{R}_{\boldsymbol{\alpha}}^+ \cdot \vec{u}, \vec{n} \cdot \vec{\varphi} \rangle_{\Gamma^+} \Big) \\ & + \frac{1}{\varepsilon_-} \Big(\langle \vec{n} \times (\mathbf{R}_{\boldsymbol{\alpha}}^- \times \vec{u}), \vec{\varphi} \rangle_{\Gamma^-} - \langle \mathbf{R}_{\boldsymbol{\alpha}}^- \cdot \vec{u}, \vec{n} \cdot \vec{\varphi} \rangle_{\Gamma^-} \Big) . \end{aligned} \tag{3.7}$$

Let $(\mathbf{E},\mathbf{H})$ a quasiperiodic solution of Maxwell's equations (2.2) satisfying the outgoing wave condition (2.6). Using the notations (2.5) we get

$$\varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) = -i\omega \vec{v} , \quad \nabla_{\boldsymbol{\alpha}} \times \vec{v} = i\omega\mu \vec{u} , \quad \nabla_{\boldsymbol{\alpha}} \cdot \vec{u} = 0 .$$

Then Green's formula (3.6) leads together with (3.5) to

$$\begin{aligned} \mathcal{B}(\vec{u}, \vec{\varphi}) = & -i\omega \int_\Omega \vec{v} \cdot (\overline{\nabla_{\boldsymbol{\alpha}} \times \vec{\varphi}}) - \omega^2 \mu \int_\Omega \vec{u} \cdot \overline{\vec{\varphi}} - i\omega \langle \vec{n} \times \vec{v}, \vec{\varphi} \rangle_{\Gamma^+} \\ & + \frac{2i\beta e^{-i\beta b}}{\varepsilon_+} \int_{\Gamma^+} \Big(\vec{n} \times (\vec{n} \times \mathbf{q}) \cdot \overline{\vec{\varphi}} - \vec{n} \cdot \mathbf{q} (\vec{n} \cdot \overline{\vec{\varphi}}) \Big) - i\omega \langle \vec{n} \times \vec{v}, \vec{\varphi} \rangle_{\Gamma^-} \\ = & -i\omega \int_\Omega (\nabla_{\boldsymbol{\alpha}} \times \vec{v} - i\omega\mu \vec{u}) \cdot \overline{\vec{\varphi}} - \frac{2i\beta e^{-i\beta b}}{\varepsilon_+} \int_{\Gamma^+} (-\vec{n} \times \vec{n} \times \mathbf{q} + (\vec{n} \cdot \mathbf{q})\, \vec{n}) \cdot \overline{\vec{\varphi}} \end{aligned}$$

for any $\vec{\varphi} \in H_p^1(\Omega)^3$. From $-\vec{n} \times \vec{n} \times \mathbf{q} + (\vec{n} \cdot \mathbf{q})\, \vec{n} = \mathbf{q}$ the field $\vec{u}(\mathbf{x}) = e^{-i(\boldsymbol{\alpha},\boldsymbol{x})}\mathbf{H}(\mathbf{x})$ solves therefore the variational equation

$$\mathcal{B}(\vec{u}, \vec{\varphi}) = -\frac{2i\beta e^{-i\beta b}}{\varepsilon_+} \int_{\Gamma^+} \mathbf{q} \cdot \overline{\vec{\varphi}} , \qquad \forall \vec{\varphi} \in H_p^1(\Omega)^3 . \tag{3.8}$$

In the following we show that any solution of (3.8) with appropriate chosen ρ in $\mathcal{B}$ provides a quasiperiodic solution of the diffraction problem (2.2), (2.6).

Theorem 3.1 *If ε satisfies* (2.1), *then the number ρ with* $\operatorname{Im}\rho < 0$ *can be chosen such that the form $\mathcal{B}(\cdot,\cdot)$ is strongly elliptic in $H_p^1(\Omega)^3$, i.e. for all $\vec{v} \in H_p^1(\Omega)^3$*

$$\operatorname{Re}\left(\theta\,\mathcal{B}(\vec{v},\vec{v})\right) \geq c\,\|\vec{v}\|^2_{H^1(\Omega)} + q(\vec{v},\vec{v})$$

for some $\theta \in \mathbb{C}$, a constant $c > 0$, and a compact form $q(\cdot,\cdot)$.

Proof. Denote

$$\mathcal{B}_1(\vec{u},\vec{\varphi}) = \mathcal{B}(\vec{u},\vec{\varphi}) + \omega^2\mu\int_\Omega \vec{u}\cdot\overline{\vec{\varphi}}\,,$$

which for $\vec{u} = \vec{\varphi} = \vec{v}$ takes the form

$$\begin{aligned}\mathcal{B}_1(\vec{v},\vec{v}) = &\int_\Omega \left(\varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}}\times\vec{v})\cdot(\overline{\nabla_{\boldsymbol{\alpha}}\times\vec{v}}) + \rho\,|\nabla_{\boldsymbol{\alpha}}\cdot\vec{v}|^2\right)\\ &+\frac{1}{\varepsilon_+}\left(\langle\mathbf{T}^+_{\boldsymbol{\alpha}}\vec{v},\vec{v}\rangle_{\Gamma+} - 2\operatorname{Re}\langle\partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma+}\right)\\ &+\frac{1}{\varepsilon_-}\left(\langle\mathbf{T}^-_{\boldsymbol{\alpha}}\vec{v},\vec{v}\rangle_{\Gamma-} + 2\operatorname{Re}\langle\partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma-}\right)\,.\end{aligned}$$

The vector $\mathbf{T}^\pm_{\boldsymbol{\alpha}}\vec{v}$ denotes of course the action of $T^\pm_{\boldsymbol{\alpha}}$ on the components of $\vec{v}$ and we use that the adjoint $\partial^*_{j,\alpha_j} = (\partial_j + i\alpha_j)^* = -\partial_{j,\alpha_j} : H_p^{1/2}(G) \to H_p^{-1/2}(G)$, $j = 1, 2$.

First we choose θ with $0 < \arg\theta < \pi/2$ such that $\operatorname{Re}\left(\theta\,\varepsilon^{-1}\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}\right) \geq c\;|\boldsymbol{\xi}|^2$ in Ω and $\operatorname{Re}\left(-i\theta\beta^\pm_{\boldsymbol{m}}/\varepsilon_\pm\right) \geq 0$ with equality only if $\beta^\pm_{\boldsymbol{m}} = 0$.

If $\operatorname{Im}\varepsilon_- = 0$, then $\arg\left(-i\beta^\pm_{\boldsymbol{m}}/\varepsilon_\pm\right) \in \{-\pi/2, 0\}$, and one can choose $\theta = e^{i\phi/2}$ with $\phi = \max(\pi/2, \operatorname{ess\,sup}(\arg\varepsilon\,\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}))$.

If $\arg\varepsilon_- = \pi - \tau$, $\tau \in (0,\pi)$, then $\arg\left(-i\beta^-_{\boldsymbol{m}}/\varepsilon_-\right) \in (\tau/2 - \pi, \tau - \pi)$. Then with $\theta = e^{i\phi/2}$, $\phi = \max(\pi - \tau/2, \operatorname{ess\,sup}(\arg\varepsilon\,\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}))$, obviously

$$|\arg\left(\theta\,\varepsilon^{-1}\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}\right)| \leq \phi/2\,,\ \arg\frac{\theta\beta^-_{\boldsymbol{m}}}{i\,\varepsilon_-} \in \left(\frac{\tau}{4} - \frac{\pi}{2}, \tau - \frac{\pi}{2}\right)\,,\ \arg\frac{\theta\beta^+_{\boldsymbol{m}}}{i\,\varepsilon_+} \in \left(-\frac{\tau}{4}, \frac{\phi}{2}\right)\,.$$

Thus, with $C = \operatorname{ess\,inf}\left(\operatorname{Re}\left(\theta\,\varepsilon^{-1}\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}\right)\right)$, $|\boldsymbol{\xi}| = 1$, and $\rho = \theta^{-1}C$ we obtain

$$\begin{aligned}\operatorname{Re}\left(\theta\,\mathcal{B}_1(\vec{v},\vec{v})\right) \geq\ &C\int_\Omega \left(|\nabla_{\boldsymbol{\alpha}}\times\vec{v}|^2 + |\nabla_{\boldsymbol{\alpha}}\cdot\vec{v}|^2\right)\\ &+\operatorname{Re}\frac{\theta}{\varepsilon_+}\left(\langle\mathbf{T}^+_{\boldsymbol{\alpha}}\vec{v},\vec{v}\rangle_{\Gamma+} - 2\operatorname{Re}\langle\partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma+}\right)\\ &+\operatorname{Re}\frac{\theta}{\varepsilon_-}\left(\langle\mathbf{T}^-_{\boldsymbol{\alpha}}\vec{v},\vec{v}\rangle_{\Gamma-} + 2\operatorname{Re}\langle\partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma-}\right)\,.\end{aligned}$$

Integration by parts and the periodicity of $\vec{v}$ give

$$\int_\Omega (|\nabla_{\boldsymbol{\alpha}} \times \vec{v}|^2 + |\nabla_{\boldsymbol{\alpha}} \cdot \vec{v}|^2) = \int_\Omega |\nabla_{\boldsymbol{\alpha}} \vec{v}|^2$$
$$+2\mathrm{Re}\,\Big(\langle \partial_{1,\alpha_1} v_1 + \partial_{2,\alpha_2} v_2, v_3\rangle_{\Gamma+} - \langle \partial_{1,\alpha_1} v_1 + \partial_{2,\alpha_2} v_2, v_3\rangle_{\Gamma-}\Big),$$

hence we arrive at the inequality

$$\begin{aligned} \mathrm{Re}\,(\theta\,\mathcal{B}_1(\vec{v},\vec{v})) \geq C \int_\Omega |\nabla_{\boldsymbol{\alpha}} \vec{v}|^2 \\ +\mathrm{Re}\,\frac{\theta}{\varepsilon_+}\,\langle \mathbf{T}^+_{\boldsymbol{\alpha}} \vec{v}, \vec{v}\rangle_{\Gamma+} - 2\,(\mathrm{Re}\,\frac{\theta}{\varepsilon_+} - C)\mathrm{Re}\,\langle \partial_{1,\alpha_1} v_1 + \partial_{2,\alpha_2} v_2, v_3\rangle_{\Gamma+} \\ +\mathrm{Re}\,\frac{\theta}{\varepsilon_-}\,\langle \mathbf{T}^-_{\boldsymbol{\alpha}} \vec{v}, \vec{v}\rangle_{\Gamma+} + 2\,(\mathrm{Re}\,\frac{\theta}{\varepsilon_-} - C)\mathrm{Re}\,\langle \partial_{1,\alpha_1} v_1 + \partial_{2,\alpha_2} v_2, v_3\rangle_{\Gamma-}\,. \end{aligned} \tag{3.9}$$

Using the Fourier series of $\vec{v}|_{\Gamma\pm} \in H^{1/2}_p(\Gamma^\pm)^3$ the boundary terms can be written as

$$\begin{aligned} \mathrm{Re}\,\frac{\theta}{\varepsilon_\pm}\,\langle \mathbf{T}^\pm_{\boldsymbol{\alpha}} \vec{v}, \vec{v}\rangle_{\Gamma\pm} \mp 2\,(\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)\mathrm{Re}\,\langle \partial_{1,\alpha_1} v_1 + \partial_{2,\alpha_2} v_2, v_3\rangle_{\Gamma\pm} \\ = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \mathbf{C}^\pm_{\boldsymbol{m}} \vec{v}_{\boldsymbol{m}}(\pm b) \cdot \overline{\vec{v}_{\boldsymbol{m}}(\pm b)} \end{aligned} \tag{3.10}$$

with the matrices

$$\mathbf{C}^\pm_{\boldsymbol{m}} = \begin{pmatrix} -\mathrm{Re}\,\dfrac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} & 0 & \pm i(\mathrm{Re}\,\dfrac{\theta}{\varepsilon_\pm} - C)(\boldsymbol{\alpha}_{\boldsymbol{m}})_1 \\ 0 & -\mathrm{Re}\,\dfrac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} & \pm i(\mathrm{Re}\,\dfrac{\theta}{\varepsilon_\pm} - C)(\boldsymbol{\alpha}_{\boldsymbol{m}})_2 \\ \mp i(\mathrm{Re}\,\dfrac{\theta}{\varepsilon_\pm} - C)(\boldsymbol{\alpha}_{\boldsymbol{m}})_1 & \mp i(\mathrm{Re}\,\dfrac{\theta}{\varepsilon_\pm} - C)(\boldsymbol{\alpha}_{\boldsymbol{m}})_2 & -\mathrm{Re}\,\dfrac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} \end{pmatrix}$$

$(\boldsymbol{\alpha}_{\boldsymbol{m}})_j$ denoting the components of $\boldsymbol{\alpha}_{\boldsymbol{m}} \in \mathbb{R}^2$. The eigenvalues of $\mathbf{C}^\pm_{\boldsymbol{m}}$ are

$$-\mathrm{Re}\,\frac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm}\,, \quad -\mathrm{Re}\,\frac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} \pm (\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)|\boldsymbol{\alpha}_{\boldsymbol{m}}|\,, \tag{3.11}$$

hence, because of $\mathrm{Re}\,(-i\theta\beta^\pm_{\boldsymbol{m}}/\varepsilon_\pm) \geq 0$ and $C \leq \mathrm{Re}\,(\theta/\varepsilon_\pm)$,

$$\begin{aligned} \Big(-\mathrm{Re}\,\frac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} - (\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)|\boldsymbol{\alpha}_{\boldsymbol{m}}|\Big)|\vec{v}_{\boldsymbol{m}}(\pm b)|^2 \leq \mathbf{C}^\pm_{\boldsymbol{m}} \vec{v}_{\boldsymbol{m}}(\pm b) \cdot \overline{\vec{v}_{\boldsymbol{m}}(\pm b)} \\ \leq \Big(-\mathrm{Re}\,\frac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} + (\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)|\boldsymbol{\alpha}_{\boldsymbol{m}}|\Big)|\vec{v}_{\boldsymbol{m}}(\pm b)|^2\,. \end{aligned} \tag{3.12}$$

The constant of the lower bound can be nonpositive only for a finite number of indices $\boldsymbol{m} \in \mathbb{Z}^2$. This follows immediately from

$$\begin{aligned} -\mathrm{Re}\,\frac{i\,\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} - (\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)|\boldsymbol{\alpha}_{\boldsymbol{m}}| = C|\boldsymbol{\alpha}_{\boldsymbol{m}}| + \mathrm{Im}\,\frac{\theta\,\beta^\pm_{\boldsymbol{m}}}{\varepsilon_\pm} - \mathrm{Re}\,\frac{\theta}{\varepsilon_\pm}|\boldsymbol{\alpha}_{\boldsymbol{m}}| \\ = C|\boldsymbol{\alpha}_{\boldsymbol{m}}| + \mathrm{Re}\,\frac{\theta}{\varepsilon_\pm}\Big(\mathrm{Im}\,\beta^\pm_{\boldsymbol{m}} - |\boldsymbol{\alpha}_{\boldsymbol{m}}|\Big) + \mathrm{Im}\,\frac{\theta}{\varepsilon_\pm}\,\mathrm{Re}\,\beta^\pm_{\boldsymbol{m}} \end{aligned} \tag{3.13}$$

and $\mathrm{Re}\,\beta_{\boldsymbol{m}}^{\pm} \to 0$, $\mathrm{Im}\,\beta_{\boldsymbol{m}}^{\pm} - |\boldsymbol{\alpha}_{\boldsymbol{m}}| \to 0$ as $|\boldsymbol{m}| \to \infty$ in view of the definition (2.7). Moreover, (3.13) and (3.10) imply that for fixed ω and $\boldsymbol{\alpha}$ there exist a constant $c_1 > 0$ and finite–dimensional operators $K^{\pm}$ such that for all $\vec{v} \in H_p^{1/2}(\Gamma^{\pm})^3$

$$\mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}}\,\langle \mathbf{T}_{\boldsymbol{\alpha}}^{\pm}\vec{v},\vec{v}\rangle_{\Gamma^{\pm}} \mp 2\,(\mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}} - C)\mathrm{Re}\,\langle \partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma^{\pm}} \\ \geq c_1\|\vec{v}\|^2_{H_p^{1/2}(\Gamma^{\pm})} - \langle K^{\pm}\vec{v},\vec{v}\rangle_{\Gamma^{\pm}}\,.$$

Thus

$$\mathrm{Re}\,(\theta\mathcal{B}(\vec{v},\vec{v})) \geq C\int\limits_{\Omega} |\nabla_{\boldsymbol{\alpha}}\vec{v}|^2 + c_1\Big(\|\vec{v}\|^2_{H_p^{1/2}(\Gamma^+)} + \|\vec{v}\|^2_{H_p^{1/2}(\Gamma^-)}\Big) \\ -\langle K^+\vec{v},\vec{v}\rangle_{\Gamma^+} - \langle K^-\vec{v},\vec{v}\rangle_{\Gamma^-} - \mathrm{Re}\,\theta\omega^2\mu\|\vec{v}\|^2_{L^2(\Omega)}\,.$$

From

$$\|\nabla_{\boldsymbol{\alpha}}\psi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} \geq (\alpha^2+1)^{-1}(\|\nabla\psi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)})$$

easily follows that

$$\Big(\|\nabla_{\boldsymbol{\alpha}}\vec{v}\|^2_{L^2(\Omega)} + \|\vec{v}\|^2_{H_p^{1/2}(\Gamma^+)} + \|\vec{v}\|^2_{H_p^{1/2}(\Gamma^-)}\Big)^{1/2}$$

is an equivalent norm on $H_p^1(\Omega)^3$, hence the proof is complete. ∎

Corollary 3.1 *The form $\mathcal{B}$ generates a bounded linear operator $H_p^1(\Omega)^3 \longrightarrow (H_p^1(\Omega)^3)'$ which is Fredholm with index zero.*

Remark 3.1 Obviously Theorem 3.1 is valid for any $\rho \in \mathbb{C}$ with $\mathrm{Re}\,(\theta\rho) \geq C$. But for the following we require that $\mathrm{Im}\,\rho < 0$. Then one can prove, following Costabel and Dauge [5], that the divergence of solutions of (3.8) vanishes for all frequencies.

Theorem 3.2 *If $\vec{u}$ solves (3.8), then $\nabla_{\boldsymbol{\alpha}} \cdot \vec{u} = 0$.*

Proof. Introduce the subspace $\widetilde{H}_p^2(\Omega) \subset H_p^2(\Omega)$ of functions ψ, which satisfy on $\Gamma^{\pm}$ the boundary conditions $\partial_{\boldsymbol{n}}\psi|_{\Gamma^{\pm}} = -(T_{\boldsymbol{\alpha}}^{\pm})^*(\psi|_{\Gamma^{\pm}})$, where $(T_{\boldsymbol{\alpha}}^{\pm})^* : H_p^{1/2}(\Gamma^{\pm}) \to H_p^{-1/2}(\Gamma^{\pm})$ is the adjoint to $T_{\boldsymbol{\alpha}}^{\pm}$. If $\vec{\varphi} = \nabla_{\boldsymbol{\alpha}}\psi$, $\psi \in \widetilde{H}_p^2(\Omega)$, then

$$\langle \vec{n}\times(\mathbf{R}_{\boldsymbol{\alpha}}^{\pm}\times\vec{u}),\vec{\varphi}\rangle_{\Gamma^{\pm}} - \langle \mathbf{R}_{\boldsymbol{\alpha}}^{\pm}\cdot\vec{u},\vec{n}\cdot\vec{\varphi}\rangle_{\Gamma^{\pm}} = \langle \partial_{1,\alpha_1}u_1 + \partial_{2,\alpha_2}u_2 \mp T_{\boldsymbol{\alpha}}^{\pm}u_3, (T_{\boldsymbol{\alpha}}^{\pm})^*\psi\rangle_{\Gamma^{\pm}} \\ +\langle T_{\boldsymbol{\alpha}}^{\pm}u_1 \pm \partial_{1,\alpha_1}u_3, \partial_{1,\alpha_1}\psi\rangle_{\Gamma^{\pm}} + \langle T_{\boldsymbol{\alpha}}^{\pm}u_2 \pm \partial_{2,\alpha_2}u_3, \partial_{2,\alpha_2}\psi\rangle_{\Gamma^{\pm}} \\ = -\langle \vec{n}\cdot\vec{u}, (\partial_{1,\alpha_1}^2 + \partial_{2,\alpha_2}^2 + ((T_{\boldsymbol{\alpha}}^{+})^*)^2)\psi\rangle_{\Gamma^{\pm}} = \mu\varepsilon_{\pm}\,\omega^2\langle \vec{n}\cdot\vec{u},\psi\rangle_{\Gamma^{\pm}}\,.$$

Here we use the relation

$$(\partial_{1,\alpha_1}^2 + \partial_{2,\alpha_2}^2 + ((T_{\boldsymbol{\alpha}}^{\pm})^*)^2)\psi = -\sum_{\boldsymbol{m}\in\mathbb{Z}^2}\Big(|\boldsymbol{\alpha}_{\boldsymbol{m}}|^2 + (\overline{\beta_{\boldsymbol{m}}^{\pm}})^2\Big)\hat{\psi}_{\boldsymbol{m}}e^{2\pi i(x,A\boldsymbol{m})}$$

and the definition (2.7). Since $\nabla_{\boldsymbol\alpha} \times \nabla_{\boldsymbol\alpha}\psi = 0$ and μ is constant after applying Green's formula

$$\int\limits_{\Omega} \left(\vec{u} \cdot \overline{\nabla_{\boldsymbol\alpha}\psi} + (\nabla_{\boldsymbol\alpha} \cdot \vec{u})\,\overline{\psi}\right) = \int\limits_{\Gamma^+} (\vec{n} \cdot \vec{u})\,\overline{\psi} + \int\limits_{\Gamma^-} (\vec{n} \cdot \vec{u})\,\overline{\psi} \tag{3.14}$$

we get therefore

$$\mathcal{B}(\vec{u}, \nabla_{\boldsymbol\alpha}\psi) = \int\limits_{\Omega} (\nabla_{\boldsymbol\alpha} \cdot \vec{u})\,\overline{(\overline{\rho}\,\Delta_{\boldsymbol\alpha} + \omega^2\mu)\,\psi}$$

with $\Delta_{\boldsymbol\alpha} = \nabla_{\boldsymbol\alpha} \cdot \nabla_{\boldsymbol\alpha}$. Because of $(\mathbf{q}, \mathbf{k}) = 0$ the right–hand side

$$\frac{-2i\beta e^{-i\beta b}}{\mu\varepsilon_+} \int\limits_{\Gamma^+} \mathbf{q}\,\overline{\nabla_{\boldsymbol\alpha}\psi} = \frac{-2i\beta e^{-i\beta b}}{\mu\varepsilon_+} \int\limits_{\Gamma^+} (-i\alpha_1 q_1 - i\alpha_2 q_2 + i\beta q_3)\overline{\psi} = 0\,.$$

Thus, if $\vec{u}$ solves (3.8), then

$$\int\limits_{\Omega} (\nabla_{\boldsymbol\alpha} \cdot \vec{u})\,\overline{(\overline{\rho}\,\Delta_{\boldsymbol\alpha} + \omega^2\mu)\psi} = 0$$

for all $\psi \in \widetilde{H}^2_p(\Omega)$. Now the assertion is a consequence of the following Lemma. ∎

Lemma 3.2 *The boundary value problem*

$$(\Delta_{\boldsymbol\alpha} + \frac{\omega^2\mu}{\overline{\rho}})\psi = f\,, \quad \partial_{\boldsymbol n}\psi = -(T^{\pm}_{\boldsymbol\alpha})^*\psi \quad \text{on } \Gamma^{\pm}\,, \tag{3.15}$$

has for any $f \in L^2(\Omega)$ and $\omega > 0$ a unique solution $\psi \in H^2_p(\Omega)$.

Proof. (3.15) admits the weak formulation: for all $\varphi \in H^1_p(\Omega)$

$$a(\psi, \varphi) := \int\limits_{\Omega} (\nabla_{\boldsymbol\alpha}\psi \cdot \overline{\nabla_{\boldsymbol\alpha}\varphi} - \frac{\mu\omega^2}{\overline{\rho}}\,\psi\,\overline{\varphi}) + \langle (T^+_{\boldsymbol\alpha})^*\psi, \varphi\rangle_{\Gamma^+} + \langle (T^-_{\boldsymbol\alpha})^*\psi, \varphi\rangle_{\Gamma^-} = -\int\limits_{\Omega} f\overline{\varphi}\,.$$

Analogously to the proof of Theorem 3.1 one concludes from

$$\mathrm{Re}\Big(a(\psi, \psi) + \frac{\omega^2\mu}{\overline{\rho}} \int\limits_{\Omega} |\psi|^2\Big) = \int\limits_{\Omega} |\nabla_{\boldsymbol\alpha}\psi|^2 + \sum_{\boldsymbol m \in \mathbb{Z}^2} \mathrm{Im}\,\beta^+_{\boldsymbol m}\,|\widehat{\psi}_{\boldsymbol m}(b)|^2 + \sum_{\boldsymbol m \in \mathbb{Z}^2} \mathrm{Im}\,\beta^-_{\boldsymbol m}\,|\widehat{\psi}_{\boldsymbol m}(-b)|^2$$

that $a(\cdot,\cdot)$ generates a Fredholm operator of index zero from $H^1_p(\Omega)$ into its dual $(H^1_p(\Omega))'$. Additionally, since $\mathrm{Im}\,\rho < 0$ we obtain

$$\begin{aligned} \mathrm{Im}\,a(\psi, \psi) &= \mathrm{Im}\,\frac{\omega^2\mu}{\rho} \int\limits_{\Omega} |\psi|^2 + \sum_{\boldsymbol m \in \mathbb{Z}^2} \mathrm{Re}\,\beta^+_{\boldsymbol m}\,|\widehat{\psi}_{\boldsymbol m}(b)|^2 + \sum_{\boldsymbol m \in \mathbb{Z}^2} \mathrm{Re}\,\beta^-_{\boldsymbol m}\,|\widehat{\psi}_{\boldsymbol m}(-b)|^2 \\ &\geq c\,\omega^2 \|\psi\|^2_{L^2(\Omega)}\,, \end{aligned}$$

hence the kernel of this operator is trivial for all $\omega > 0$. Consequently, (3.15) is solvable for all $f \in L^2(\Omega)$ and due to the elliptic regularity of the Laplacian in convex domains the solution $\psi \in H^1_p(\Omega)$ belongs to $\widetilde{H}^2_p(\Omega)$. ∎

Theorem 3.3 *Let $\vec{u}$ be a solution of the equation* (3.8)*. Then* $\mathbf{H}(\mathrm{x}) = \vec{u}(\mathrm{x})e^{i(\boldsymbol{\alpha},\boldsymbol{x})}$ *and* $\mathbf{E}(\mathrm{x}) = i\omega^{-1}\varepsilon^{-1}(\nabla \times (\vec{u}(\mathrm{x})e^{i(\boldsymbol{\alpha},\boldsymbol{x})}))$ *solve the Maxwell system* (2.2) *for* $|x_3| \le b$*. Outside* Ω *the electromagnetic field is given by* (2.6) *with the Rayleigh coefficients*

$$H^+_{\boldsymbol{m}} = e^{-i\beta^+_{\boldsymbol{m}} b}\,|\det A| \int_G (\vec{u}(\boldsymbol{x}, b) - \mathbf{q}\, e^{-i\beta b})e^{-2\pi i(\boldsymbol{x},A\boldsymbol{m})}d\boldsymbol{x}\,,$$

$$H^-_{\boldsymbol{m}} = e^{i\beta^-_{\boldsymbol{m}} b}\,|\det A| \int_G \vec{u}(\boldsymbol{x}, -b)e^{-2\pi i(\boldsymbol{x},A\boldsymbol{m})}d\boldsymbol{x}\,,$$

$$E^\pm_{\boldsymbol{m}} = -\frac{(\boldsymbol{\alpha}_{\boldsymbol{m}}, \pm\beta^\pm_{\boldsymbol{m}}) \times H^\pm_{\boldsymbol{m}}}{\omega\varepsilon_\pm}\,.$$

Proof. Setting $\vec{v} = -(i\omega\varepsilon)^{-1}\nabla_{\boldsymbol{\alpha}} \times \vec{u}$ from Theorem 3.2

$$\mathcal{B}(\vec{u}, \vec{\varphi}) = -i\omega \int_\Omega (\nabla_{\boldsymbol{\alpha}} \times \vec{v} - i\omega\mu\vec{u}) \cdot \overline{\vec{\varphi}} = 0$$

for any $\vec{\varphi} \in H^1_p(\Omega)^3$ with $\vec{\varphi}|_{\Gamma\pm} = 0$. Thus $(\mathbf{H}, \mathbf{E})$ satisfies Maxwell's equations in Ω in weak sense and moreover $\nabla_{\boldsymbol{\alpha}} \times \varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \in L^2(\Omega)^3$. Then Green' formula (3.6) leads to

$$\begin{aligned}\mathcal{B}(\vec{u}, \vec{\varphi}) &= \frac{1}{\varepsilon_+}\Big(\langle \vec{n} \times (\mathbf{R}^+_{\boldsymbol{\alpha}} \times \vec{u}) - \vec{n} \times (\nabla_{\boldsymbol{\alpha}} \times \vec{u}), \vec{\varphi}\rangle_{\Gamma+} - \langle \mathbf{R}^+_{\boldsymbol{\alpha}} \cdot \vec{u}, \vec{n} \cdot \vec{\varphi}\rangle_{\Gamma+}\Big)\\ &+\frac{1}{\varepsilon_-}\Big(\langle \vec{n} \times (\mathbf{R}^-_{\boldsymbol{\alpha}} \times \vec{u}) - \vec{n} \times (\nabla_{\boldsymbol{\alpha}} \times \vec{u}), \vec{\varphi}\rangle_{\Gamma-} - \langle \mathbf{R}^-_{\boldsymbol{\alpha}} \cdot \vec{u}, \vec{n} \cdot \vec{\varphi}\rangle_{\Gamma-}\Big)\end{aligned}$$

for all $\varphi \in H^1_p(\Omega)^3$, which implies together with $(\nabla_{\boldsymbol{\alpha}} \cdot \vec{u})|_{\Gamma\pm} = 0$ and Lemma 3.1 that the boundary conditions (3.3) are satisfied. ∎

Remark 3.2 For isotropic gratings (ε is a scalar function) the sesquilinear form $\mathcal{B}$ differs from the form introduced in [7] only in the choice of the coefficient ρ, which is chosen as the function $\rho = 1/\varepsilon(\mathrm{x})$. In this paper the biperiodic diffraction problem is considered under the assumption that inside Ω the dielectric coefficient satisfies $\mathrm{Re}\,\varepsilon \ge \delta > 0$, $\mathrm{Im}\,\varepsilon \ge 0$, and that $\beta^\pm_{\boldsymbol{m}} \ne 0$ for all $\boldsymbol{m} \in \mathbb{Z}^2$. However, the well–posedness of the problem is stated by using the relation

$$\begin{aligned}&\int_\Omega \frac{1}{\varepsilon}\,(|\nabla_{\boldsymbol{\alpha}} \times \vec{v}|^2 + |\nabla_{\boldsymbol{\alpha}} \cdot \vec{v}|^2) = \int_\Omega \frac{1}{\varepsilon}\,|\nabla_{\boldsymbol{\alpha}}\vec{v}|^2\\ &+\frac{2}{\varepsilon_+}\mathrm{Re}\,\langle \partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma+} - \frac{2}{\varepsilon_-}\mathrm{Re}\,\langle \partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma-}\,,\end{aligned}$$

which is true only for constant ε. It can be easily seen that the proofs given here can also be applied to the case of variable $\rho = 1/\varepsilon$ in the form $\mathcal{B}$. They lead to the same results for the corresponding variational equation with the only exception, that possibly $\nabla_{\boldsymbol{\alpha}} \cdot \vec{u} \ne 0$ for real ε and a discrete sequence $\omega_j \to \infty$.

4 Existence and uniqueness results

Lemma 4.1 *If* $\mathcal{B}(\vec{u},\vec{\varphi})=0$ *for all* $\vec{\varphi}\in H_p^1(\Omega)^3$*, then* $|\vec{u}_{\boldsymbol{m}}(\pm b)|=0$ *if* $\operatorname{Re}\beta_{\boldsymbol{m}}^{\pm}>0$.

Proof. Suppose that $\vec{u}\neq 0$ solves the homogeneous equation. The domain integrals of $\operatorname{Im}\mathcal{B}(\vec{u},\vec{u})$ are nonpositive, we show that this is also true for the boundary terms. This is obvious for $\varepsilon_{\pm}>0$:

$$\operatorname{Im}\frac{1}{\varepsilon_{\pm}}\Big(\langle \mathbf{T}_{\boldsymbol{\alpha}}^{\pm}\vec{u},\vec{u}\rangle_{\Gamma^{\pm}} \mp 2\operatorname{Re}\langle \partial_{1,\alpha_1}u_1+\partial_{2,\alpha_2}u_2,u_3\rangle_{\Gamma^{\pm}}\Big) = -\frac{1}{\varepsilon_{\pm}}\sum_{\boldsymbol{m}\in\mathbb{Z}^2}\operatorname{Re}\beta_{\boldsymbol{m}}^{\pm}|\vec{u}_{\boldsymbol{m}}(\pm b)|^2,$$

hence $\operatorname{Im}\mathcal{B}(\vec{u},\vec{u})=0$ if and only if $\operatorname{Re}\beta_{\boldsymbol{m}}^{\pm}|\vec{u}_{\boldsymbol{m}}(\pm b)|^2=0$ for all $\boldsymbol{m}\in\mathbb{Z}^2$.

Let $\operatorname{Im}\varepsilon_->0$. Then the operator $T_{\boldsymbol{\alpha}}^-$ is invertible and in view of $\mathbf{R}_{\boldsymbol{\alpha}}^-\cdot\vec{u}|_{\Gamma^-}=0$ the components of $\vec{u}$ satisfy

$$u_3=-(T_{\boldsymbol{\alpha}}^-)^{-1}(\partial_{1,\alpha_1}u_1+\partial_{2,\alpha_2}u_2)\quad\text{on }\Gamma^-.$$

Therefore we derive the representation

$$\begin{aligned}&\frac{1}{\varepsilon_-}\Big(\langle \mathbf{T}_{\boldsymbol{\alpha}}^-\vec{u},\vec{u}\rangle_{\Gamma^-}+2\operatorname{Re}\langle\partial_{1,\alpha_1}u_1+\partial_{2,\alpha_2}u_2,u_3\rangle_{\Gamma^-}\Big)\\ &=\frac{1}{\varepsilon_-}\langle T_{\boldsymbol{\alpha}}^-u_1+(T_{\boldsymbol{\alpha}}^-)^{-1}(\partial_{1,\alpha_1}^2u_1+\partial_{1,\alpha_1}\partial_{2,\alpha_2}u_2),u_1\rangle_{\Gamma^-}\\ &\quad+\frac{1}{\varepsilon_-}\langle T_{\boldsymbol{\alpha}}^-u_2+(T_{\boldsymbol{\alpha}}^-)^{-1}(\partial_{1,\alpha_1}\partial_{2,\alpha_2}u_1+\partial_{2,\alpha_2}^2u_2),u_2\rangle_{\Gamma^-}=\sum_{\boldsymbol{m}\in\mathbb{Z}^2}(\mathbf{D}_{\boldsymbol{m}}U_{\boldsymbol{m}},U_{\boldsymbol{m}})\end{aligned}$$

with the 2×2 matrices

$$\mathbf{D}_{\boldsymbol{m}}=-\frac{i}{\varepsilon_-}\begin{pmatrix}\beta_{\boldsymbol{m}}^-+\dfrac{(\boldsymbol{\alpha_m})_1^2}{\beta_{\boldsymbol{m}}^-} & \dfrac{(\boldsymbol{\alpha_m})_1(\boldsymbol{\alpha_m})_2}{\beta_{\boldsymbol{m}}^-}\\ \dfrac{(\boldsymbol{\alpha_m})_1(\boldsymbol{\alpha_m})_2}{\beta_{\boldsymbol{m}}^-} & \beta_{\boldsymbol{m}}^-+\dfrac{(\boldsymbol{\alpha_m})_2^2}{\beta_{\boldsymbol{m}}^-}\end{pmatrix}\quad\text{and}\quad U_{\boldsymbol{m}}=\begin{pmatrix}\widehat{u}_{1\boldsymbol{m}}(-b)\\ \widehat{u}_{2\boldsymbol{m}}(-b)\end{pmatrix}.$$

The eigenvalues of the matrix $\operatorname{Im}\,\mathbf{D}_{\boldsymbol{m}}$

$$-\operatorname{Re}\frac{\beta_{\boldsymbol{m}}^-}{\varepsilon_-}\quad\text{and}\quad -\operatorname{Re}\Big(\frac{\beta_{\boldsymbol{m}}^-}{\varepsilon_-}+\frac{|\boldsymbol{\alpha_m}|^2}{\varepsilon_-\beta_{\boldsymbol{m}}^-}\Big)=-\operatorname{Re}\frac{\omega^2\mu}{\beta_{\boldsymbol{m}}^-}$$

are negative for all $\boldsymbol{m}\in\mathbb{Z}^2$. ∎

Corollary 4.1 *If* $\vec{u}$ *is a solution of the homogeneous equation, then* $|\nabla_{\boldsymbol{\alpha}}\times\vec{u}|=0$ *on* $\operatorname{supp}\operatorname{Im}\varepsilon$.

Theorem 4.1 *Let* $\Phi_0\in(0,\pi/2)$*. There exists a frequency* $\omega_0>0$ *such that the variational problem* (3.8) *admits a unique solution* $u\in H_p^1(\Omega)^3$ *for all incidence angles* Φ_1,Φ_2 *with* $\Phi_1\le\Phi_0$ *and any frequency* $0<\omega<\omega_0$.

Proof. Consider the constants of the lower bound in (3.12) for $\Phi_1 \leq \Phi_0$ and small ω. Note that $\Big||\boldsymbol{\alpha_m}| - 2\pi|A\boldsymbol{m}|\Big| \leq \omega\sqrt{\mu\varepsilon_+}\sin\Phi_0$ and let $\omega^2\mu|\varepsilon_\pm| < |\boldsymbol{\alpha_m}|^2$ for all $\boldsymbol{m} \in \mathbb{Z}^2\backslash\{0\}$. Then from

$$\Big| - i\beta_{\boldsymbol{m}}^\pm - |\boldsymbol{\alpha_m}|\Big| = |\boldsymbol{\alpha_m}|\Big|\Big(1 - \frac{\omega^2\mu\varepsilon_\pm}{|\boldsymbol{\alpha_m}|^2}\Big)^{1/2} - 1\Big)\Big| \leq \frac{\omega^2\mu\varepsilon_\pm}{|\boldsymbol{\alpha_m}|}$$

one easily concludes that there exists $\omega_0 > 0$ for which

$$-\mathrm{Re}\,\frac{i\,\theta\,\beta_{\boldsymbol{m}}^\pm}{\varepsilon_\pm} - (\mathrm{Re}\,\frac{\theta}{\varepsilon_\pm} - C)|\boldsymbol{\alpha_m}| = C|\boldsymbol{\alpha_m}| + \mathrm{Re}\,\frac{\theta}{\varepsilon_\pm}(-i\beta_{\boldsymbol{m}}^\pm - |\boldsymbol{\alpha_m}|) \geq c_1|\boldsymbol{m}|$$

with a constant c_1 not depending on $\Phi_1 \leq \Phi_0$ and $\omega < \omega_0$. Let $\mathcal{B}(\vec{u},\vec{u}) = 0$. Then (3.9), (3.10) lead to

$$\begin{aligned} \mathrm{Re}\,(\theta\,\mathcal{B}(\vec{u},\vec{u})) = 0 \geq C\int_\Omega |\nabla_{\boldsymbol{\alpha}}\vec{u}|^2 - \omega^2\mu\int_\Omega |\vec{u}|^2 + c_1\sum_{\boldsymbol{m}\neq 0}|\boldsymbol{m}||\vec{\hat{u}}_{\boldsymbol{m}}(b)|^2 \\ +c_1\sum_{\boldsymbol{m}\neq 0}|\boldsymbol{m}||\vec{\hat{u}}_{\boldsymbol{m}}(-b)|^2 + \mathbf{C}_0^+\vec{\hat{u}}_0(b)\cdot\overline{\vec{\hat{u}}_{\boldsymbol{m}}(b)} + \mathbf{C}_0^-\vec{\hat{u}}_0(-b)\cdot\overline{\vec{\hat{u}}_0(-b)}\,. \end{aligned} \tag{4.1}$$

In view of Lemma 4.1 we have $|\vec{\hat{u}}_0(b)| = 0$ and from (3.12)

$$|\mathbf{C}_0^-\vec{\hat{u}}_0(-b)\cdot\overline{\vec{\hat{u}}_0(-b)}| \leq c_2\,\omega\,|\vec{\hat{u}}_0(-b)|^2\,.$$

Therefore (4.1) provides the inequality

$$||\nabla\vec{u}||^2_{L^2(\Omega)} + ||\vec{u}||^2_{H^{1/2}(\Gamma^+)} + \sum_{\boldsymbol{m}\neq 0}|\boldsymbol{m}||\vec{\hat{u}}_{\boldsymbol{m}}(-b)|^2 \leq c(\omega||\vec{u}||^2_{H^1(\Omega)} + \omega^2||\vec{u}||^2_{L^2(\Omega)})$$

valid for all $\Phi_1 \leq \Phi_0$ and $\omega < \omega_0$. Since the square root of the expression on the left is an equivalent norm on $H^1_{\boldsymbol{p}}(\Omega)^3$ it follows that $\vec{u} = 0$. ∎

We now study the diffraction problem in the case of arbitrary frequencies $\omega > 0$. Introduce the set of exceptional values (the Rayleigh frequencies), where at least one of the numbers $\beta_{\boldsymbol{m}}^\pm$ vanishes:

$$\mathcal{R}(\varepsilon) = \Big\{(\omega,\Phi_1,\Phi_2) : \exists\,\boldsymbol{m}\in\mathbb{Z}^2 \text{ such that } |\boldsymbol{\alpha_m}|^2 = \omega^2\mu\varepsilon_\pm\Big\}\,.$$

The Rayleigh frequencies are associated with the appearance of new propagating modes when ω increases, and important physical phenomena arise in the vicinity of those frequencies. Here we generalize a result obtained in [9] for the classical TE and TM diffraction problems by periodic gratings.

Theorem 4.2

(i) For all but a sequence of countable frequencies ω_j, $\omega_j \to \infty$, the biperiodic diffraction problem (3.8) *has a unique solution $\vec{u} \in H^1_{\boldsymbol{p}}(\Omega)^3$.*

(ii) If (3.8) *is uniquely solvable for given $(\omega,\Phi_1,\Phi_2) \notin \mathcal{R}(\epsilon)$, then the solution $\vec{u}$ depends analytically on the frequency and the incident angles in a neighborhood of this point.*

Proof. We split the sesquilinear form $\mathcal{B}$ into a coercive and a compact part. The proof of Theorem 3.1 shows that $\theta\mathcal{B}_1$ is not coercive because of nonpositive eigenvalues of the matrices $\mathbf{C}^{\pm}_{\boldsymbol{m}}$. But from (3.11) the eigenvalues of

$$\mathbf{C}^{\pm}_{\boldsymbol{m}} + \Big(\mathrm{Re}\,\frac{i\,\theta\,\beta^{\pm}_{\boldsymbol{m}}}{\varepsilon_{\pm}} + \mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}}|\boldsymbol{\alpha}_{\boldsymbol{m}}|\Big)\,\mathbf{I}$$

are positive for all $\boldsymbol{m} \in \mathbb{Z}^2$ except the case $|\boldsymbol{\alpha}_{\boldsymbol{m}}| = 0$. So we set $a_{\boldsymbol{m}} = \sqrt{|\boldsymbol{\alpha}_{\boldsymbol{m}}|^2 + d}$ for some fixed $d > 0$, introduce the pseudodifferential operator

$$\mathbf{A}_{\boldsymbol{\alpha}}\vec{v}(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} a_{\boldsymbol{m}}\vec{v}_{\boldsymbol{m}} e^{2\pi i(\boldsymbol{x},A\boldsymbol{m})}$$

and consider the perturbed form

$$\mathcal{B}_2(\vec{u},\vec{\varphi}) = \mathcal{B}_1(\vec{u},\vec{\varphi}) + \frac{1}{\varepsilon_+}\langle(\mathbf{A}_{\boldsymbol{\alpha}} - \mathbf{T}^{+}_{\boldsymbol{\alpha}})\,\vec{u},\vec{\varphi}\rangle_{\Gamma+} + \frac{1}{\varepsilon_-}\langle(\mathbf{A}_{\boldsymbol{\alpha}} - \mathbf{T}^{-}_{\boldsymbol{\alpha}})\,\vec{u},\vec{\varphi}\rangle_{\Gamma-}\,.$$

Analogously to the proof of Theorem 3.1 we obtain with the parameters θ and C

$$\mathrm{Re}\,(\theta\,\mathcal{B}_2(\vec{v},\vec{v})) \ge C\int_{\Omega} |\nabla_{\boldsymbol{\alpha}}\vec{v}|^2 + J^+(\vec{v}) + J^-(\vec{v})$$

and the boundary terms

$$\begin{aligned} J^{\pm}(\vec{v}) &= \mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}}\langle\mathbf{A}_{\boldsymbol{\alpha}}\vec{v},\vec{v}\rangle_{\Gamma\pm} \mp 2(\mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}} - C)\mathrm{Re}\,\langle\partial_{1,\alpha_1}v_1 + \partial_{2,\alpha_2}v_2, v_3\rangle_{\Gamma\pm} \\ &\ge \sum_{\boldsymbol{m}\in\mathbb{Z}^2}\Big(\mathrm{Re}\,\frac{\theta}{\varepsilon_{\pm}}(a_{\boldsymbol{m}} - |\boldsymbol{\alpha}_{\boldsymbol{m}}|) + C|\boldsymbol{\alpha}_{\boldsymbol{m}}|\Big)|\vec{v}_{\boldsymbol{m}}(\pm b)|^2\,, \end{aligned}$$

which implies

$$\mathrm{Re}\,(\theta\,\mathcal{B}_2(\vec{v},\vec{v})) \ge C\|\nabla_{\boldsymbol{\alpha}}\vec{v}\|^2_{L^2(\Omega)} + c_1\Big(\|\vec{v}\|^2_{H^{1/2}_p(\Gamma^+)} + \|\vec{v}\|^2_{H^{1/2}_p(\Gamma^-)}\Big)\,.$$

The representation

$$\mathcal{B}(\vec{u},\vec{\varphi}) = \mathcal{B}_2(\vec{u},\vec{\varphi}) - \omega^2\mu\int_{\Omega}\vec{u}\cdot\overline{\vec{\varphi}} + \frac{1}{\varepsilon_+}\langle(\mathbf{T}^{+}_{\boldsymbol{\alpha}} - \mathbf{A}^{+}_{\boldsymbol{\alpha}})\,\vec{u},\vec{\varphi}\rangle_{\Gamma+} + \frac{1}{\varepsilon_-}\langle(\mathbf{T}^{-}_{\boldsymbol{\alpha}} - \mathbf{A}^{-}_{\boldsymbol{\alpha}})\,\vec{u},\vec{\varphi}\rangle_{\Gamma-}$$

shows, that for all valid Φ_1, Φ_2 the form $\mathcal{B}$ generates linear bounded mappings from $H^1_{\boldsymbol{p}}(\Omega)^3$ into $(H^1_{\boldsymbol{p}}(\Omega)^3)'$ which are compact perturbations of an invertible operator function, depending on $\omega > 0$. Moreover, this function and the perturbations depend analytically on ω if $(\omega,\Phi_1,\Phi_2) \notin \mathcal{R}(\varepsilon)$. Therefore one can repeat the arguments of the proof of Theorem 3.3 in [9], which are based on Gohberg's Theorem for analytic operator functions. It implies that the number of linearly independent solutions of (3.8) is constant for all $\omega \in \mathbb{R}^+\backslash\mathcal{R}(\epsilon)$ with the possible exception of certain isolated points in that domain. It was shown in [9] that the exceptional points do not have finite accumulation points. ∎

Remark 4.1 Note that the diffraction problem (3.8) is solvable for all $\omega > 0$ and incidence angles Φ_1, Φ_2. From Lemma 4.1 any solution $\vec{v}$ of the adjoint homogeneous equation $\mathcal{B}(\vec{\varphi}, \vec{v}) = 0$, $\vec{\varphi} \in H^1(\Omega)^3$, satisfies $|\vec{\hat{v}}_0(b)| = 0$, thus is orthogonal to the right hand side of equation (3.8).

The variational formulation allows also to treat unique solvability of the biperiodic diffraction problem for small perturbations ε_h of the dielectric tensor ε.

Theorem 4.3 *If* (3.8) *has for given parameters* ω, Φ_1, Φ_2 *a unique solution* $\vec{u}$ *and the perturbations* ε_h *of the dielectric tensor satisfy uniformly the conditions* (2.1) *and*

$$\|(\varepsilon_h - \varepsilon)\vec{v}\|_{L^2(\Omega)} \to 0 \text{ as } h \to 0 \,, \quad \forall\, \vec{v} \in L^2(\Omega)^3 \,, \tag{4.2}$$

then for all sufficiently small h the problems with the perturbed dielectric tensors are also uniquely solvable. Their solutions converge to $\vec{u}$ in the norm of $H^1(\Omega)$.

Proof. We use arguments from the theory of projection methods similar to [10]. Consider the sesquilinear form of the perturbed problem

$$\begin{aligned} \mathcal{B}_h(\vec{u}, \vec{\varphi}) \;:=& \int_\Omega \varepsilon_h^{-1} (\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \cdot (\overline{\nabla_{\boldsymbol{\alpha}} \times \vec{\varphi}}) + \rho \int_\Omega (\nabla_{\boldsymbol{\alpha}} \cdot \vec{u})\, (\overline{\nabla_{\boldsymbol{\alpha}} \cdot \vec{\varphi}}) - \omega^2 \mu \int_\Omega \vec{u} \cdot \overline{\vec{\varphi}} \\ &+ J_h^+(\vec{u}, \vec{\varphi}) + J_h^-(\vec{u}, \vec{\varphi})\,, \end{aligned} \tag{4.3}$$

where $J_h^\pm(\vec{u}, \vec{\varphi})$ denote the boundary terms corresponding to the perturbed dielectric coefficients $\varepsilon_\pm^h$ of the upper and lower medium and ρ is the same as in $\mathcal{B}(\cdot,\cdot)$. Since $\varepsilon_\pm^h \to \varepsilon_\pm$ it is easy to check that $\|(T_{\boldsymbol{\alpha}}^\pm)_h - T_{\boldsymbol{\alpha}}^\pm\|_{H_p^s(G)} \to 0$, where the pseudodifferential operators $(T_{\boldsymbol{\alpha}}^\pm)_h$ are defined analogously to (3.2) with $\varepsilon_\pm$ replaced by the numbers $\varepsilon_\pm^h$ in (2.7).

The forms $\mathcal{B}^h(\cdot,\cdot)$ generate a sequence of bounded operators, denoted again by $\mathcal{B}^h$ and acting from $H_p^1(\Omega)^3$ into $(H_p^1(\Omega)^3)'$. From (4.2) the sequence $\mathcal{B}^h$ converge strongly to the operator $\mathcal{B}$ generated by the original form (3.7). Suppose that there exists a sequence $\vec{u}_h \in H_p^1(\Omega)^3$, $\|\vec{u}_h\| = 1$, such that $\mathcal{B}^h \vec{u}_h \to 0$. A subsequence, again denoted by $\{\vec{u}_h\}$, converges weakly to some $\vec{u} \in H_p^1(\Omega)^3$, hence $\mathcal{B}^h \vec{u}_h$ weakly to $\mathcal{B}\vec{u}$ and therefore $\vec{u} = 0$.

On the other hand, the proof of Theorem 4.2 indicates the representation $\mathcal{B}^h = \mathcal{B}_2^h + \mathcal{T}^h$, where

$$\operatorname{Re} \theta(\mathcal{B}_2^h \vec{u}_h, \vec{u}_h) \ge c\, \|\vec{u}_h\|_{H_p^1(\Omega)} \tag{4.4}$$

with a constant c depending only on ess sup $\varepsilon_h \boldsymbol{\xi} \cdot \overline{\boldsymbol{\xi}}$, and compact $\mathcal{T}^h$ converging in operator norm to the operator defined by $(\mathcal{T}\vec{u}, \vec{\varphi}) = \mathcal{B}(\vec{u}, \vec{\varphi}) - \mathcal{B}_2(\vec{u}, \vec{\varphi})$. Hence $\mathcal{T}^h \vec{u}_h \to 0$ which together with (4.4) gives the contradiction to the assumption $\|\vec{u}_h\| = 1$. Thus for $h < h_0$ we have

$$\sup_{\vec{\varphi} \in H_p^1(\Omega)^3} |\mathcal{B}^h(\vec{u}, \vec{\varphi})| \ge c^{-1} \|\vec{u}\|_{H^1} \|\vec{\varphi}\|_{H^1}$$

for all $\vec{u} \in H_p^1(\Omega)^3$ with a constant c not depending on h. Denoting by $\vec{u}$ and $\vec{u}_h$ the unique solutions of the unperturbed and the perturbed problem, respectively, we obtain therefore

$$\|\vec{u}-\vec{u}_h\|_{H^1} \leq c \sup_{\|\vec{\varphi}\|_{H^1}=1} |\mathcal{B}^h(\vec{u}-\vec{u}_h,\vec{\varphi})| = c \sup_{\|\vec{\varphi}\|_{H^1}=1} |\mathcal{B}^h(\vec{u},\vec{\varphi}) - \mathcal{B}(\vec{u},\vec{\varphi})| \, .$$

which together with the equation

$$\mathcal{B}^h(\vec{u},\vec{\varphi}) - \mathcal{B}(\vec{u},\vec{\varphi}) = \int_\Omega (\varepsilon_h^{-1} - \varepsilon^{-1})(\nabla_{\boldsymbol{\alpha}} \times \vec{u}) \cdot (\overline{\nabla_{\boldsymbol{\alpha}} \times \vec{\varphi}})$$
$$+\frac{1}{\varepsilon_+^h}\langle((\mathbf{T}_{\boldsymbol{\alpha}}^+)_h - \mathbf{T}_{\boldsymbol{\alpha}}^+)\,\vec{u},\vec{\varphi}\rangle_{\Gamma+} + \frac{\varepsilon_+ - \varepsilon_+^h}{\varepsilon_+\,\varepsilon_+^h}\Big(\langle\vec{n}\times(\mathbf{R}_{\boldsymbol{\alpha}}^+\times\vec{u}),\vec{\varphi}\rangle_{\Gamma+} - \langle\mathbf{R}_{\boldsymbol{\alpha}}^+\cdot\vec{u},\vec{n}\cdot\vec{\varphi}\rangle_{\Gamma+}\Big)$$
$$+\frac{1}{\varepsilon_-^h}\langle((\mathbf{T}_{\boldsymbol{\alpha}}^-)_h - \mathbf{T}_{\boldsymbol{\alpha}}^-)\,\vec{u},\vec{\varphi}\rangle_{\Gamma+} + \frac{\varepsilon_- - \varepsilon_-^h}{\varepsilon_-\,\varepsilon_-^h}\Big(\langle\vec{n}\times(\mathbf{R}_{\boldsymbol{\alpha}}^-\times\vec{u}),\vec{\varphi}\rangle_{\Gamma-} - \langle\mathbf{R}_{\boldsymbol{\alpha}}^-\cdot\vec{u},\vec{n}\cdot\vec{\varphi}\rangle_{\Gamma-}\Big)$$

proves the convergence of $\vec{u}_h$ to $\vec{u}$. ∎

We conclude with a uniqueness result for structures containing absorbing materials. In order to apply unique continuation we assume that Ω can be divided into subdomains Ω_j with piecewise analytic boundaries $\partial\Omega_j$ such that the dielectric tensor is analytic on Ω_j. Note that at interfaces Λ between adjacent subdomains the tangential components of solutions to (3.8) are continuous

$$[\vec{n}\times\vec{u}]_\Lambda = 0 \quad \text{and} \quad [\vec{n}\times\varepsilon^{-1}(\nabla_{\boldsymbol{\alpha}}\times\vec{u})]_\Lambda = 0 \, . \tag{4.5}$$

Here $[\mathbf{n}\times\vec{\varphi}]_\Lambda$ denotes the jump across the interface equal to $\vec{n}\times(\vec{\varphi}_1-\vec{\varphi}_2)$, where $\vec{\varphi}_j$ are the restrictions of $\vec{\varphi}$ to the domains separated by Λ and $\vec{n}$ is the unit normal to the interface.

Theorem 4.4 *Suppose that the imaginary part of the dielectric tensor is positive in some subdomain $\Omega_1 \subset \Omega$, $\operatorname{Im}\,\varepsilon(\mathrm{x}) > 0$, $\mathrm{x}\in\Omega_1$. Then the variational problem* (3.8) *is uniquely solvable for all $\omega > 0$.*

Proof. Let $\vec{u}$ be a solution of the homogeneous equation. From Corollary 4.1 we obtain $\nabla\times\mathbf{H} = e^{i(\boldsymbol{\alpha},\boldsymbol{x})}\nabla_{\boldsymbol{\alpha}}\times\vec{u} = 0$ in Ω_1. Maxwell's equations yield therefore $\mathbf{H}=\mathbf{E}=0$ in Ω_1, hence $\vec{n}\times\mathbf{E} = \vec{n}\times\mathbf{H} = 0$ on $\partial\Omega_1$. Now (4.5) implies $\mathbf{H}=\mathbf{E}=0$ in adjacent subdomains by using a result of Abboud and Nedelec [2], which states that $\mathbf{H}$ and $\mathbf{E}$ vanish identically on Ω_j if $\vec{n}\times\mathbf{E}$ and $\vec{n}\times\mathbf{H}$ vanish on an analytic part of its boundary. ∎

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