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ROTHE'S METHOD FOR EQUATIONS MODELLING TRANSPORT OF DOPANTS IN SEMICONDUCTORS

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ABSTRACT. This paper is devoted to the investigation of some nonlinear reaction–diffusion system modelling the transport of dopants in semiconductors and arising in semiconductor technology. Besides of results on existence and qualitative properties of the solution to the problem itself we are interested in the investigation of corresponding discrete–time problems. Using Rothe's method in a fully implicite and a semi–implicite version, respectively, we get analogous results on existence and qualitative behaviour of solutions to the discrete–time equations. Moreover, convergence in some strong sense will be proved. Essential tools are estimates of the energy functional, L^∞–estimates obtained by De Giorgi's method, $L^q(S, W^{1,p})$–estimates for the continuous problem as well as a discrete version of Gronwall's lemma.

CONTENTS

1. Introduction

In this paper we continue the mathematical analysis of a reaction–diffusion system modelling the transport of dopants in semiconductors, see [3]. There the physical background of the equations under consideration is described and results concerning the existence, uniqueness as well as the asymptotic behaviour of solutions are obtained. Here we are mainly interested in the investigation of corresponding discrete–time problems. Furthermore, different to [3] we use more general boundary conditions and more general assumptions on the coefficients of volume and boundary reactions.

The investigation of the reaction–diffusion system as well as its discrete–time versions starts with estimates of the free energy. Here the main tool is some relation between the free energy and the dissipation rate (see Lemma 2.1). Because of the stronger assumptions on the coefficients of the boundary reactions the proof of this relation in [3] has been nearly trivial. Now we are forced to apply ideas of [7], additionally taking into account the fact that at least one activity nonlinearly depends on the concentration. Thus we obtain the fundamental result that along any solution to the reaction–diffusion system and its discrete–time versions, respectively, the free energy decays monotonously and exponentially to its equilibrium value as time tends to infinity. Using this result first global a–priori estimates are derived.

For getting global upper bounds, we intend to apply De Giorgi's method. Since the regularity results obtained by the energetic estimates are not sufficient to do so, firstly we have to improve these results. This will be done by multiplying the differential equations with suitable chosen test functions where we had to overcome the difficulty that in the boundary conditions some second order terms are present. Another delicacy which we have to manage in handling the discrete–time problems is that the test functions must be modified in such a way that a discrete version of Gronwall's Lemma (cf. [12]) is applicable.

The paper is organized as follows. In Section 2 we investigate the reaction–diffusion system itself. The basic assumptions are summarized in Subsection 2.1. Results on energetic estimates as in [3] are stated in Subsection 2.2. Here only the relation between the free energy and the dissipation rate is proved in detail. Subsection 2.3 summarizes results on existence and uniqueness.

Section 3 is devoted to discrete–time problems. We use Rothe's method (in other words, BDF of first degree) in two versions. The first one is a fully implicite scheme while the second one is a semi–implicite scheme in the sense that the diffusion coefficients (as far as they depend on the concentrations) are taken in the old time step. For both versions a–priori estimates are obtained in Subsection 3.2 and Subsection 3.3, existence results in Subsection 3.4. To prove existence we use some regularization technique as well as results on operators of variational type ([13]).

Our main result namely the convergence theorem is stated in Section 4. Besides of some additional regularity of the solution to the continuous problem (see [8]) once more the discrete version of Gronwall's Lemma is applied to prove strong convergence for both discrete problems.

Finally, let us note that one of our basic assumptions requires the existence of a thermodynamic equilibrium with nonzero concentrations uniquely determined in some sense. A more detailed discussion of this asssumption is given in Appendix A.

2. The reaction-diffusion system

2.1. Notation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\Gamma := \partial\Omega$ and $u = (u_0, u_1, u_2, u_3)\colon \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^4_+$ the vector of concentrations. We consider the system of differential equations

$$
\begin{aligned}
\frac{\partial u_0}{\partial t} &= -\ \mathrm{div}\ j_0 + R_1(u),\\
\frac{\partial u_1}{\partial t} &= -\ \mathrm{div}\ j_1 - R_1(u),\\
\frac{\partial u_2}{\partial t} &= -\ \mathrm{div}\ j_2 - R_2(u),\\
\frac{\partial u_3}{\partial t} &= -\ \mathrm{div}\ j_3 - R_1(u) - R_2(u),
\end{aligned}
\tag{2.1}
$$

$$
j_0 = -D_0\, u_0 \nabla \ln g(u_0),\ \ j_i = -D_i\, u_i \nabla \ln u_i,\ \ i = 1,2,3,
$$

$$
R_1(u) = \tilde{k}_1(u_1 u_3 - k_1\, g(u_0)),\ \ R_2(u) = \tilde{k}_2(u_2 u_3 - k_2)
$$

in $\mathbb{R}_+ \times \Omega$, complemented by the boundary conditions

$$
\begin{aligned}
j_{0,\nu} &= R_3(u) + R_4(u),\\
j_{1,\nu} &= R_5(u),\\
j_{2,\nu} &= R_6(u),\\
j_{3,\nu} &= R_7(u) - R_4(u),
\end{aligned}
\tag{2.2}
$$

$$
R_3(u) = \tilde{k}_3(g(u_0) - k_3),\ \ R_4(u) = \tilde{k}_4(g(u_0) - k_4 u_3),
$$

$$
R_5(u) = \tilde{k}_5(u_1 - k_5),\ \ R_6(u) = \tilde{k}_6(u_2 - k_6),\ \ R_7(u) = \tilde{k}_7(u_3 - k_7)
$$

on $\mathbb{R}_+ \times \Gamma$ as well as the initial condition

$$
u(0,\cdot) = U \tag{2.3}
$$

on Ω.

Let us put together the assumptions concerning the data in the equations formulated above, which will be used during the following sections:

$$
\begin{aligned}
&D_j \in L^\infty_+(\Omega), D_j \geq d_j > 0,\ j = 0,\dots,3,\\
&k_1,\dots,k_7 = \text{const} > 0,\\
&\tilde{k}_1, \tilde{k}_2 \in L^\infty_+(\Omega),\\
&\tilde{k}_j \in L^\infty_+(\Gamma),\ j = 3,\dots,7,\\
&U \in L^\infty_+(\Omega, \mathbb{R}^4) \cap W^{1,p}(\Omega, \mathbb{R}^4) \text{ for some } p > 2;
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&g \in C^2(\mathbb{R}_+),\\
&\varphi(y) := \frac{g'(y)y}{g(y)},\ \psi(y) := \frac{g(y)}{y},\ \sigma(y) := \frac{g''(y)y}{\sqrt{g(y)}},\ y > 0,\\
&\text{are such that}\\
&\psi(y) \geq \tau_1,\ |\psi(y_1) - \psi(y_2)| \leq \tau_2|y_1 - y_2|,\ y, y_1, y_2 > 0,\\
&\tau_3 \leq \varphi(y) \leq \tau_4,\ y > 0,\\
&|\sigma(y)| \leq \tau_5,\ y > 0;\ \tau_i = \text{const} > 0,\ i = 1, \dots, 5.
\end{aligned}
\tag{2.5}
$$

In order to formulate our last assumption we introduce the quantities

$$
\begin{aligned}
&\bar{k}_i := \tfrac{1}{\text{mes}\,\Omega} \int_\Omega \tilde{k}_i \mathrm{d}x,\ i = 1, 2,\ \bar{k}_i := \tfrac{1}{\text{mes}\,\Gamma} \int_\Gamma \tilde{k}_i \mathrm{d}\Gamma,\ i = 3,\dots,7,\\
&\bar{U} := \tfrac{1}{\text{mes}\,\Omega} \int_\Omega U \mathrm{d}x.
\end{aligned}
\tag{2.6}
$$

and denote by $\mathcal{S}$ the stoichiometric subspace of $\mathbb{R}^4$ belonging to the volume and surface reactions given in (2.1) and (2.2)

$$
\mathcal{S} = \text{span}\ \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7\}
$$

where $\rho_1 = \bar{k}_1(-1, 1, 0, 1)$, $\rho_2 = \bar{k}_2(0, 0, 1, 1)$, $\rho_3 = \bar{k}_3(1, 0, 0, 0)$, $\rho_4 = \bar{k}_4(1, 0, 0, -1)$, $\rho_5 = \bar{k}_5(0, 1, 0, 0)$, $\rho_6 = \bar{k}_6(0, 0, 1, 0)$, $\rho_7 = \bar{k}_7(0, 0, 0, 1)$. Finally, let

$$
\mathcal{R} := \left\{u \in \mathbb{R}^4_+ \colon R_i(u) = 0 \text{ a.e.}, i = 1, \dots, 7,\ u - \bar{U} \in \mathcal{S}\right\}.
$$

Now we assume that

$$
\text{there exists a } u^* \in \mathbb{R}^4,\ u^* > 0 \text{ such that } \mathcal{R} = \{u^*\}. \tag{2.7}
$$

Remark 2.1.

i) (2.5) implies that φ, ψ, $\sigma \in C([0, +\infty))$ and the inequalities in (2.5) are satisfied for $y = 0$, too. We have $g(0) = 0$, $\psi(0) = g'(0) > 0$, $\varphi(0) = 1$, $\sigma(0) = 0$.

ii) Further properties of the functions g, φ, ψ are summarized in [3].

iii) Because of $g \in C^2$, φ is locally Lipschitz continuous.

iv) For example, the function

$$g(y) := a\,y\left(y - y_0 + \sqrt{(y-y_0)^2 + y_1^2}\right),\ y \ge 0; \quad a, y_1 > 0,\ y_0 \in \mathbb{R}$$

satisfies (2.5). Its physical meaning is explained in [3].

v) Nessecary and sufficient conditions for (2.7), expressed in terms of $k_i, \bar{k}_i$, $i = 1, \dots, 7$ and $\bar{U}$ are given in Appendix A.

We use the notation $X := H^1(\Omega, \mathbb{R}^4)$, $Y := L^2(\Omega, \mathbb{R}^4)$, $Z := L^2(\Gamma, \mathbb{R}^4)$. Additionally let

$$V := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X) \colon u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^4(\Omega, \mathbb{R}^4))\right\},$$
$$W := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X) \colon u' \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X^*)\right\}.$$

We define $\mathcal{A} \colon X \times X \times X \longrightarrow X^*$, $A \colon X \longrightarrow X^*$ for $u, v, w, z \in X$ by

$$\begin{aligned}\langle \mathcal{A}(w, v, u), z\rangle := \int_\Omega \Big\{ & D_0 \varphi(w_0) \nabla u_0 \nabla z_0 + \sum_{i=1}^3 D_i \nabla u_i \nabla z_i \\ & + R_1(v)(z_1 + z_3 - z_0) + R_2(v)(z_2 + z_3)\Big\}\, \mathrm{d}x \\ + \int_\Gamma \Big\{ & R_3(v)\, z_0 + R_4(v)(z_0 - z_3) + \sum_{i=1}^3 R_{i+4}(v) z_i\Big\}\, \mathrm{d}\Gamma, \\ A(u) := & \mathcal{A}(u, u, u).\end{aligned} \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing of X and X^*. The problem we shall be concerned with consists in finding a solution to

$$u'(t) + A(u(t)) = 0 \quad \text{for a.e.}\, t \in \mathbb{R}_+,\ u(0) = U,\ u \in W \cap V,\ u \ge 0. \tag{P}$$

Here u' denotes the derivative of u with respect to time in the sense of X^*-valued distributions and $u \ge 0$ means that all components $u_i \ge 0$. For any $T \in \mathbb{R}_+$ we denote by S the finite time interval $[0, T]$ and

$$V_S := \left\{u \in L^2(S, X) \colon u \in L^\infty\left(S, L^4(\Omega, \mathbb{R}^4)\right)\right\},$$
$$W_S := \left\{u \in L^2(S, X) \colon u' \in L^2(S, X^*)\right\}.$$

In the canonical way we extend the definition of the operator A to functions from V_S. For any finite time interval S the reaction-diffusion system leads to the problem

$$u' + A(u) = 0,\ u(0) = U,\ u \in W_S \cap V_S,\ u \ge 0. \tag{$\mathrm{P_S}$}$$

Now we introduce several symbols and collect some basic results which we shall use in our considerations. Let be $u \in \mathbb{R}^4$, $\delta \in \mathbb{R}$. By $u + \delta$, $\sqrt{u}$, $\ln u$, $|u|$, u^+ and u^- we denote the vector whose i-th component is $u_i + \delta$, $\sqrt{u_i}$, $\ln u_i$, $|u_i|$, $\sup(u_i, 0)$ and $\sup(-u_i, 0)$, respectively; $u \ge c$ means $u_i \ge c$ for $i = 0, \dots, 3$. If there is no danger of misunderstanding we shall write shortly L^p instead of $L^p(\Omega, \mathbb{R}^k)$, $k \in \mathbb{N}$, and H^1 instead of $H^1(\Omega)$. We apply the Sobolev imbedding theorems as well as the

following form of the Gagliardo-Nirenberg inequality (cf. [9]):
Let $\Omega \subset \mathbb{R}^2$, $u \in H^1(\Omega)$, then

$$\|u\|_{L^r} \le c_0 \|u\|_{L^q}^{\theta} \|u\|_{H^1}^{1-\theta}, \text{ where } 1 \le q < r,\ \theta = \frac{q}{r}. \tag{2.9}$$

Additionally, for estimates of traces we use the inequality:
Let $\Omega \subset \mathbb{R}^2$, $u \in H^1(\Omega)$, then

$$\|u\|_{L^p(\partial\Omega)}^p \le c\, \|u\|_{L^{2(p-1)}}^{p-1} \|u\|_{H^1}. \tag{2.10}$$

A direct consequence of the Gagliardo-Nirenberg inequality is the following interpolation result:
Let $\Omega \subset \mathbb{R}^2$, $u \in L^\infty(\mathbb{R}_+, L^q(\Omega)) \cap L^2(\mathbb{R}_+, H^1(\Omega))$, $q \ge 1$, then $u \in L^p(\mathbb{R}_+, L^r(\Omega))$ with $p = 2/(1-\theta)$, $r = q/\theta$, $\theta \in (0,1)$ and

$$\|u\|_{L^p(\mathbb{R}_+,L^r(\Omega))}^p \le c_0^p \|u\|_{L^\infty(\mathbb{R}_+,L^q(\Omega))}^{p\theta} \|u\|_{L^2(\mathbb{R}_+,H^1(\Omega))}^2. \tag{2.11}$$

2.2. Estimates by the energy functional.

Let $\{\kappa_j\}_{j=1,\dots,l}$ be an orthogonal basis of $\mathcal{S}^\perp$ in $\mathbb{R}^4$ (if $\mathcal{S} = \mathbb{R}^4$ we set $l = 0$). We define the functions $i_j \colon \mathbb{R}^4 \to \mathbb{R}$ and the functionals $I_j \colon Y \to \mathbb{R}$ by

$$i_j(u) := (u, \kappa_j)_{\mathbb{R}^4},\ I_j(u) := \int_\Omega i_j(u(x))\,\mathrm{d}x,\ j = 1, \dots, l. \tag{2.12}$$

Obviously the functionals I_j are convex and continuous. For $u \in W_{[0,t]}$ it holds

$$I_j(u(t)) - I_j(u(0)) = \int_0^t \left\langle u'(s), \kappa_j \right\rangle \mathrm{d}s.$$

If u is a solution to (P) we thus obtain $\forall t \in \mathbb{R}_+$

$$\begin{aligned} I_j(u(t)) - I_j(U) &= \int_0^t \Big\{ \int_\Omega \sum_{k=1}^4 \kappa_{jk} \sum_{i=1}^2 \rho_{ik} R_i(u)\mathrm{d}x + \int_\Gamma \sum_{k=1}^4 \kappa_{jk} \sum_{i=3}^7 \rho_{ik} R_i(u)\mathrm{d}\Gamma \Big\} \mathrm{d}s \\ &= \int_0^t \Big\{ \int_\Omega \sum_{i=1}^2 (\rho_i, \kappa_j)_{\mathbb{R}^4} R_i(u)\mathrm{d}x + \int_\Gamma \sum_{i=3}^7 (\rho_i, \kappa_j)_{\mathbb{R}^4} R_i(u)\mathrm{d}\Gamma \Big\} \mathrm{d}s \\ &= 0, \qquad j = 1, \dots, l. \end{aligned} \tag{2.13}$$

Therefore I_j, $j = 1, \dots, l$ are invariants of problem (P). Let

$$I_j^0 := I_j(U), \qquad j = 1, \dots, l.$$

By assumption (2.7) there exists a $u^* \in \mathbb{R}^4$ such that $u^* > 0$, $R_k(u^*) = 0$ a.e., $k = 1, \dots, 7$, $i_j(u^*) = i_j(\bar{U})$, $j = 1, \dots, l$. By means of u^* we introduce the density of the free energy $f \colon \mathbb{R}^4 \to [0, +\infty]$,

$$f(u) := \begin{cases} e_g(u_0, u_0^*) + \sum_{i=1}^3 e(u_i, u_i^*) & \text{if } u \ge 0, \\ +\infty & \text{otherwise,} \end{cases}$$

as well as the free energy $F \colon Y \to [0, +\infty]$,

$$F(u) := \int_\Omega f(u(x))\mathrm{d}x, \tag{2.14}$$

where

$$e(y,y^*) := \int_{y^*}^{y} \ln \frac{\eta}{y^*} \mathrm{d}\eta, \ e_g(y,y^*) := \int_{y^*}^{y} \ln \frac{g(\eta)}{g(y^*)} \mathrm{d}\eta, \ y \geq 0, \ y^* > 0..$$

The functional F is proper, convex and lower semicontinuous (cf. [1]). For $u \in W_{[t_1,t_2]}$, $u \geq \delta > 0$, $\lambda \in \mathbb{R}$ it holds

$$e^{\lambda t_2} F(u(t_2)) - e^{\lambda t_1} F(u(t_1)) = \int_{t_1}^{t_2} e^{\lambda s} \Big\{ \lambda F(u(s)) + \big\langle u'(s), \nabla f(u(s)) \big\rangle \Big\} \, \mathrm{d}s. \tag{2.15}$$

If u is a solution to (P) with $u \geq \delta > 0$ then (2.15) implies

$$F(u(t_2)) - F(u(t_1)) = -\int_{t_1}^{t_2} D(u(s)) \, ds$$

where D denotes the dissipation rate:

$$D := D_{\text{diff}} + D_{\text{reac}} \colon \{u \in X \colon u \geq 0)\} \longrightarrow [0, +\infty],$$

$$D_{\text{diff}}(u) := 4 \int_\Omega \Big\{ D_0 \, \varphi(u_0)^2 \left|\nabla \sqrt{u_0}\right|^2 + \sum_{i=1}^{3} D_i \left|\nabla \sqrt{u_i}\right|^2 \Big\} \mathrm{d}x,$$

$$\begin{aligned} D_{\text{reac}}(u) := & \int_\Omega \Big\{ \tilde{k}_1 (u_1 \, u_3 - k_1 \, g(u_0)) \ln \frac{u_1 \, u_3}{k_1 \, g(u_0)} + \tilde{k}_2 (u_2 \, u_3 - k_2) \ln \frac{u_2 \, u_3}{k_2} \Big\} \mathrm{d}x \\ & + \int_\Gamma \Big\{ \tilde{k}_3 (g(u_0) - k_3) \ln \frac{g(u_0)}{k_3} + \tilde{k}_4 (g(u_0) - k_4 \, u_3) \ln \frac{g(u_0)}{k_4 u_3} \\ & \qquad + \sum_{i=1}^{3} \tilde{k}_{i+4} (u_i - k_{i+4}) \ln \frac{u_i}{k_{i+4}} \Big\} \mathrm{d}\Gamma. \end{aligned} \tag{2.16}$$

Theorem 2.1. *There exists a constant $c > 0$ such that*

$$\begin{aligned} &i) && F(u(t_2)) \leq F(u(t_1)) \quad \textit{for} \quad t_2 \geq t_1 \geq 0, \\ &ii) && \sup_{t \in \mathbb{R}_+} F(u(t)) \leq F(U), \\ &iii) && \|u\|_{L^\infty(\mathbb{R}_+, L^1(\Omega, \mathbb{R}^4))} \leq c, \\ &iv) && \|D(u)\|_{L^1(\mathbb{R}_+)} \leq c, \\ &v) && \left\|\nabla \sqrt{u_i}\right\|_{L^2(\mathbb{R}_+, L^2(\Omega, \mathbb{R}^2))} \leq c, \ i = 0, \ldots, 3 \end{aligned}$$

for any solution u to (P).

For the proof we refer to [3], Theorem 3.1. The additional boundary reaction R_3 does not produce new difficulties.

Lemma 2.1. *For every $R > 0$ there exists a $c_R > 0$ such that*

$$F(u) \leq c_R \Big(D(u) + \sum_{j=1}^{l} \big(I_j(u) - I_j^0 \big)^2 \Big)$$

for $u \in M_R := \{u \in X \colon \sqrt{u} \in X, \ F(u) \leq R\}$.

Proof. i) In the following and later c denotes (possibly different) positive constants the values of which depend only on the data. Let $u \in M_R$, $w_i := \sqrt{\frac{u_i}{u_i^*}} - 1$. Then (see [3])

$$c\,\|w\|_Y^2 \leq F(u) \leq c\left(\|w\|_Y^2 + \|w\|_{L^3}^3\right). \tag{2.17}$$

Furthermore we get

$$D(u) \geq c\,\tilde{D}(u),$$

$$\tilde{D}(u) = \int_\Omega \left\{ \sum_{i=0}^{3} \left|\nabla\sqrt{\frac{u_i}{u_i^*}}\right|^2 + \tilde{k}_1\left(\sqrt{\frac{u_1u_3}{u_1^*u_3^*}} - \sqrt{\frac{g(u_0)}{g(u_0^*)}}\right)^2 + \tilde{k}_2\left(\sqrt{\frac{u_2u_3}{u_2^*u_3^*}} - 1\right)^2 \right\}\mathrm{d}x$$

$$+ \int_\Gamma \left\{ \tilde{k}_3\left(\sqrt{\frac{g(u_0)}{g(u_0^*)}} - 1\right)^2 + \tilde{k}_4\left(\sqrt{\frac{g(u_0)}{g(u_0^*)}} - \sqrt{\frac{u_3}{u_3^*}}\right)^2 + \sum_{i=1}^{3}\tilde{k}_{i+4}\left(\sqrt{\frac{u_i}{u_i^*}} - 1\right)^2 \right\}\mathrm{d}\Gamma.$$

Let

$$\gamma(\xi) := \sqrt{\frac{g(u_0^*(1+\xi)^2)}{g(u_0^*)}},\ \xi \geq -1.$$

Then we have $\gamma(0) = 1$, $\gamma'(0) = \varphi(u_0^*)$ and

$$\gamma''(\xi) = \frac{u_0^*}{\sqrt{g(u_0^*)}}\left(2\,\sigma(u_0^*(1+\xi)^2) + \varphi(u_0^*(1+\xi)^2)\,\sigma_1(u_0^*(1+\xi)^2)\right)$$

where σ is defined in (2.5) and σ_1 is given by

$$\sigma_1(y) := \frac{g(y) - g'(y)\,y}{y\sqrt{g(y)}}.$$

Since $|\sigma(y)| \leq \tau_5$ and $|\sigma_1(y)| \leq |\sigma(\theta y)| \leq \tau_5,\ 0 < \theta < 1$, we find

$$\gamma(\xi) = 1 + \varphi(u_0^*)\,\xi + \bar{\gamma}(\xi)\,\xi^2,\ |\bar{\gamma}(\xi)| \leq c \quad \forall\,\xi \geq -1.$$

Thus we obtain

$$\tilde{D}(u) + \sum_{j=1}^{l}\left(I_j(u) - I_j^0\right)^2 = \sum_{i=0}^{3}\|\nabla w_i\|_{L^2}^2 + Q(w),$$

$$Q(w) = Q_1(w) + Q_2(w), \tag{2.18}$$

$$Q_1(w) = 4\sum_{j=1}^{l}\left(I_j^*(w)\right)^2 + \int_\Omega \left\{\tilde{k}_1(w_1 + w_3 - \varphi(u_0^*)\,w_0)^2 + \tilde{k}_2(w_2 + w_3)^2\right\}\mathrm{d}x$$

$$+ \int_\Gamma \left\{\tilde{k}_3\varphi(u_0^*)^2 w_0^2 + \tilde{k}_4(\varphi(u_0^*)w_0 - w_3)^2 + \sum_{i=1}^{3}\tilde{k}_{i+4}w_i^2\right\}\mathrm{d}\Gamma$$

where

$$I_j^*(w) := \int_\Omega (\kappa_j, \mathrm{diag}(u_0^*, u_1^*, u_2^*, u_3^*)w)_{\mathbb{R}^4}\mathrm{d}x$$

and

$$|Q_2(w)| \leq c\left(\|w\|_{L^3(\Omega)}^3 + \|w\|_{L^4(\Omega)}^4 + \|w\|_{L^3(\Gamma)}^3 + \|w\|_{L^4(\Gamma)}^4\right) \leq c\left(\|w\|_{H^1}^3 + \|w\|_{H^1}^4\right).$$

Let be $w = \text{const} \in \mathbb{R}^4$, $w \geq -1$. If $Q(w) = 0$ then $w = 0$ by (2.7). If $w = \text{const} \in \mathbb{R}^4$ and $Q_1(w) = 0$ then $w = 0$, too. Indeed, from $Q_1(w) = 0$ it follows that $M_1 w \in \mathcal{S}$, $M_2 w \in \mathcal{S}^\perp$, where $M_1 := \operatorname{diag}(u_0^*, u_1^*, u_2^*, u_3^*)$ and $M_2 := \operatorname{diag}(\varphi(u_0^*), 1, 1, 1)$. Therefore $(M_1 M_2 w, w)_{\mathbb{R}^4} = 0$, and because $M_1 M_2$ is positive definite, we conclude that $w = 0$.

Instead of the assertion of the lemma we shall prove the sharper inequality

$$F(u) \leq \tilde{c}_R \Big(\tilde{D}(u) + \sum_{j=1}^{l} \left(I_j(u) - I_j^0 \right)^2 \Big) \quad \forall u \in M_R,\ R > 0. \tag{2.19}$$

ii) Suppose that (2.19) is false. Then there exist $R > 0$ and sequences $c_n \in \mathbb{R}$, $u_n \in M_R$ such that $c_n \to \infty$ and

$$R \geq F(u_n) = c_n \Big(\tilde{D}(u_n) + \sum_{j=1}^{l} \left(I_j(u_n) - I_j^0 \right)^2 \Big) > 0.$$

Set $\lambda_n := \sqrt{F(u_n)}$ and $w_{ni} := \sqrt{\dfrac{u_{ni}}{u_i^*}} - 1$. Then (cf. (2.18))

$$R \geq {\lambda_n}^2 = c_n \Big(\sum_{i=0}^{3} \|\nabla w_{ni}\|_{L^2}^2 + Q(w_n) \Big) > 0. \tag{2.20}$$

This implies $\nabla w_{ni} \to 0$ in L^2 and since $\|w_n\|_Y^2 \leq c\,{\lambda_n}^2 \leq c\,R$ (cf. the left hand side of (2.17)) we may assume that w_n converges in H^1 to a constant vector $\bar{w} \in \mathbb{R}^4$. Furthermore,

$$0 \leq Q(\bar{w}) \leq \liminf_{n \to \infty} Q(w_n) = 0$$

such that $\bar{w} = 0$. Consequentely, $\lambda_n \to 0$ (cf. the right hand side of (2.17)).

Now, set $v_n := \dfrac{w_n}{\lambda_n}$. Dividing (2.20) by ${\lambda_n}^2$ we get

$$\frac{1}{c_n} = \sum_{i=0}^{3} \|\nabla v_{ni}\|_{L^2}^2 + Q_1(v_n) + \frac{1}{{\lambda_n}^2} Q_2(\lambda_n v_n). \tag{2.21}$$

This implies $\nabla v_{ni} \to 0$ in L^2 and since $\|v_n\|_Y^2 \leq c$ (cf. (2.17)) we may assume that v_n converges in H^1 to a constant vector $\bar{v} \in \mathbb{R}^4$. Since

$$\frac{1}{{\lambda_n}^2} \left| Q_2(\lambda_n v_n) \right| \leq c \left(\lambda_n \|v_n\|_{H^1}^3 + {\lambda_n}^2 \|v_n\|_{H^1}^4 \right) \to 0$$

from (2.21) it follows that $Q_1(\bar{v}) = 0$, $\bar{v} = 0$. On the other hand, because of (2.17) it holds

$$1 \leq c \left(\|v_n\|_Y^2 + \lambda_n \|v_n\|_{L^3}^3 \right) \to 0$$

which yields the contradiction. □

Theorem 2.2. *There exist positive constants* c, λ *such that*

$$\begin{array}{ll} i) & F(u(t)) \leq e^{-\lambda t} F(U) \quad \forall t \geq 0 \\ ii) & \|u(t) - u^*\|_{L^1(\Omega,\mathbb{R}^4)} \leq c\, e^{-\lambda t/2} \quad \forall t \geq 0, \\ iii) & \|u - u^*\|_{L^1(\mathbb{R}_+, L^1(\Omega,\mathbb{R}^4))},\ \|u - u^*\|_{L^2(\mathbb{R}_+, Y)} \leq c \end{array}$$

for any solution u *to (P).*

The proof of Theorem 2.2 is the same as in [3], now using Lemma 2.1 and (2.13).

2.3. Further regularity results, existence and uniqueness.

Theorem 2.3. *There exists a constant $c > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}_+, L^\infty(\Omega,\mathbb{R}^4))} \leq c$$

for any solution u to (P).

The proof of this result is devided into two steps. The first one is the iteration result

Lemma 2.2. *Let $j \in \mathbb{N}$, $j \geq 5$. Then there exists a constant $c > 0$ such that*

$$\left\| |u - u^*|^{\frac{j}{8}} \right\|_{L^2(\mathbb{R}_+, X)}, \ \left\| |u - u^*|^{\frac{j}{8}} \right\|_{L^\infty(\mathbb{R}_+, Y)} \leq c$$

for any solution u to (P).

The proof of Lemma 2.2 uses simultaneously different powers of the components of $u - u^*$ as test functions. This becomes nessecary because of the quadratic expressions in the boundary terms coming from the boundary reaction R_4. For the proof see [3, Theorem 4.1]. A similar proof, somewhat adapted to a discrete version of Gronwall's Lemma, will be done for discrete–time problems corresponding to (P) in Section 3. Lemma 2.2 ensures the regularity of $u - u^*$ which is used as starting point for the De Georgi method. Let $k \geq \max\{1, \|U\|_{L^\infty(\Omega,\mathbb{R}^4)}, \|u^*\|_{\mathbb{R}^4}\}$. If m_{jk} denotes the Lebesgue measure of the set $\{x \in \Omega : u_j > k\}$ and

$$\phi(k) := \left(\int_0^\infty \sum_{i=0}^{3} m_{ik}^{6/5} \mathrm{d}s \right)^{5/22},$$

one obtains from

$$\left\|(u-k)^+\right\|^2_{L^\infty(\mathbb{R}_+, Y)} + \left\|(u-k)^+\right\|^2_{L^2(\mathbb{R}_+, X)} \leq c\,\phi(k)^{11/5}$$

the measure estimate

$$(h-k)\phi(h) \leq c\,(\phi(k))^{11/10},$$

which guarantees by [6, Lemma 5] the boundedness of u. For the exact estimates see [3, Theorem 4.2] or the proof of the corresponding L^∞–estimate for the discrete–time version (see Theorem 3.4 in Section 3).

Theorem 2.4. *There exists a $p_0 > 2$ such that for every $q \in [1, \infty)$ and every $p \in [2, p_0]$ solutions u to (P_S) have the regularity property*

$$u \in L^q\left(S, W^{1,p}(\Omega, \mathbb{R}^4)\right) \cap W^{1,q}\left(S, W^{1,p'}(\Omega, \mathbb{R}^4)^*\right).$$

Proof. For the proof see [3, Theorem 6.1]. The L^∞–estimates for u (Theorem 2.3) and the assumption $U \in W^{1,p}(\Omega, \mathbb{R}^4)$ enable us to apply regularity results for parabolic equations of [8] with somewhat modified boundary conditions. □

Theorem 2.5. *There exists a solution to (P).*

Proof. We only give the main ideas of the proof, for a more detailed one see [3]. It sufficies to prove existence on finite time intervals. Let $T \in \mathbb{R}_+$ be arbitrarily fixed, $M > 1$ and

$$\rho(u) := \Big(\max\Big\{1, \sum_{i=0}^{3} |u_i|^2/M^2\Big\}\Big)^{-1}.$$

We define the regularized operators $\mathcal{A}_M : X \times X \times X \longrightarrow X^*$, $A_M : X \longrightarrow X^*$ by

$$\begin{aligned}
&\langle \mathcal{A}_M(w,v,u), z\rangle \\
&\quad := \int_\Omega \Big\{D_0\, \varphi(w_0^+)\nabla u_0 \nabla z_0 + \sum_{i=1}^{3} D_i \nabla u_i \nabla z_i \\
&\qquad\qquad + \rho(v)\big\{R_1(v^+)(z_1+z_3-z_0) + R_2(v^+)(z_2+z_3)\big\}\Big\}\,\mathrm{d}x \\
&\qquad + \int_\Gamma \rho(v)\Big\{R_3(v^+)z_0 + R_4(v^+)(z_0-z_3) + \sum_{i=1}^{3} R_{i+4}(v^+)z_i\Big\}\mathrm{d}\Gamma, \\
&A_M(u) := \mathcal{A}_M(u,u,u)
\end{aligned}$$

and consider the problem

$$u'(t) + A_M(u(t)) = 0 \quad \text{for a.e. } t \in S,\ u(0) = U,\ u \in W_S. \qquad (\mathrm{P_M})$$

If u is a solution to $(\mathrm{P_M})$ then $u \geq 0$ and u is bounded above by an L^∞–estimate not depending on M. To prove the existence of a solution to $(\mathrm{P_M})$ we use freezing techniques and fixed point arguments. For arbitrarily fixed $w \in W_S$ there exists exactly one solution to the problem

$$u'(t) + \mathcal{A}_M(u(t), w(t), u(t)) = 0 \quad \text{for a.e. } t \in S,\ u(0) = U,\ u \in W_S. \qquad (\mathrm{P_w})$$

This follows from standard results on evolution equations (see [2]). The mapping $W_S :\longrightarrow W_S$ assigning w the solution u to $(\mathrm{P_w})$ is completely continuous and maps W_S into a (bounded) ball in W_S. Thus by Schauder's Fixed Point Theorem there is a solution to $(\mathrm{P_M})$. Because of the a-priori estimates this is also a solution to (P) if we choose M sufficiently large. □

Theorem 2.6. *There is a unique solution to problem (P).*

For the proof we refer to [3]. Here the main idea consists in using a $L^q(S, L^p(\Omega))$–estimate for ∇u_0 to manage the problem that the diffusion coefficient for u_0 depends on u_0.

3. Discrete–time problems

3.1. Notation.

Our aim is to approximate problem (P) by discrete–time equations. We shall use the following notation, spaces and operators. We assume that we are given sequences of subdivisions $\{Z_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}_+$,

$Z_n = \left\{t_n^0, t_n^1, \ldots, t_n^k, \ldots\right\}$, $t_n^0 = 0$, $t_n^k \in \mathbb{R}_+$, $t_n^{k-1} < t_n^k$, $k \in \mathbb{N}$, $t_n^k \to +\infty$ as $k \to \infty$.
Let

$$h_n^k := t_n^k - t_n^{k-1},\ S_n^k := (t_n^{k-1}, t_n^k],\ H_n^k := \frac{h_n^k}{h_n^{k-1}},\ G_n^k := \frac{h_n^{k-1}}{h_n^k},$$
$$\bar{h}_n := \sup_{k\in\mathbb{N}} h_n^k,\ \bar{H}_n := \sup_{k\in\mathbb{N}} H_n^k.$$

Definition 3.1. Let h, $H > 0$ be given. A subdivision Z_n of $\mathbb{R}_+$ is called regular (with respect to h, $H > 0$) if $\bar{h}_n \le h$ and $\bar{H}_n \le H$.

Definition 3.2. For a given regular subdivision Z_n of $\mathbb{R}_+$ we define the mapping $\vartheta_n \colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\vartheta_n(t) := \begin{cases} t_n^{l-2} + G_n^l(t - t_n^{l-1}) & \text{on } S_n^l,\ l \ge 2, \\ -t_n^1 + t & \text{on } S_n^1. \end{cases}$$

Furthermore let

$$\tau_n^a u(t) := \begin{cases} u(\vartheta_n(t)) & \text{if } \vartheta_n(t) \in \mathbb{R}_+, \\ a & \text{if } \vartheta_n(t) \notin \mathbb{R}_+ \end{cases}$$

for $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$, $a \in X$.

In the remainder of the paper we fix some h, H and consider only subdivisions of $\mathbb{R}_+$ being regular in the sense of Definition 3.1.

For a given subdivision of $\mathbb{R}_+$ and a given Banach space E we introduce the space of piecewise constant functions

$$C_n(\mathbb{R}_+, E) := \left\{u : \mathbb{R}_+ \longrightarrow E\ :\ u(t) = u^k \quad \forall t \in S_n^k,\ u^k \in E,\ k \in \mathbb{N}\right\}.$$

Obviously it holds $C_n(\mathbb{R}_+, E) \subset L^r_{\mathrm{loc}}(\mathbb{R}_+, E)$ for $1 \le r \le \infty$. Analogously, for a finite subdivision of a finite interval S the space of piecewise constant functions is denoted by $C_n(S, E)$; then $C_n(S, E) \subset L^r(S, E)$ for $1 \le r \le \infty$.

Let U be the initial value of problem (P), we define the operators

$$\Delta_n : C_n(\mathbb{R}_+, X) \longrightarrow C_n(\mathbb{R}_+, X^*),\ K_n : C_n(\mathbb{R}_+, X) \longrightarrow C(\mathbb{R}_+, X)$$

by

$$\begin{aligned} (\Delta_n v)^k &:= \frac{1}{h_n^k}(v^k - v^{k-1}), \\ (K_n v)(t) &:= \frac{1}{h_n^k}\{(t_n^k - t)v^{k-1} + (t - t_n^{k-1})v^k\} \quad \forall t \in S_n^k, \\ v^0 &:= U. \end{aligned} \tag{3.1}$$

Obviously, $(K_n v)' = \Delta_n v$. For the sake of simplicity we denote τ_n^a for $a = U$ by τ_n. If $v \in C_n(\mathbb{R}_+, X)$ then $(\tau_n v)^k = v^{k-1}$, $k \ge 2$, $(\tau_n v)^1 = U$.

We consider simultanously a fully implicite (FI) and a semi–implicite (SI) discrete–time problem corresponding to subdivision Z_n. We define

$$\gamma_n : C_n(\mathbb{R}_+, X) \longrightarrow C_n(\mathbb{R}_+, X),\ \gamma_{n0} : C_n(\mathbb{R}_+, X) \longrightarrow C_n(\mathbb{R}_+, H^1(\Omega)),$$

$$A_n\colon C_n(\mathbb{R}_+, X) \longrightarrow C_n(\mathbb{R}_+, X^*)$$

by

$$\gamma_n u := \begin{cases} u & \text{for (FI)} \\ \tau_n u & \text{for (SI),} \end{cases}$$

$$\gamma_{n0}(u) := (\gamma_n u)_0 \,,$$
$$(A_n(u))(t) := \mathcal{A}((\gamma_n u)(t), u(t), u(t)) \quad \forall t \in \mathbb{R}_+.$$

where $\mathcal{A}$ is given in (2.8) and investigate the problem

$$\Delta_n u + A_n(u) = 0, \quad u \in C_n(\mathbb{R}_+, X),\ u \geq 0. \tag{P$_\mathrm{n}$}$$

Now we collect some results which are essential tools in our further considerations:

Lemma 3.1.

i) Let Z_n be a regular subdivision of $\mathbb{R}_+$ and $u \in L^2_{loc}(\mathbb{R}_+, X)$, $a \in X$. Then $\tau_n^a u \in L^2_{loc}(\mathbb{R}_+, X)$ and

$$\|\tau_n^a u\|_{L^2(S,X)} \leq \bar{H}_n^{\frac{1}{2}} \|u\|_{L^2(S,X)} + \bar{h}_n^{\frac{1}{2}} \|a\|_X$$

for any finite interval $S \subset \mathbb{R}_+$.

ii) Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of regular subdivisions of $\mathbb{R}_+$ with $\bar{h}_n \to 0$ for $n \to \infty$. Then

$$\|\tau_n^a u - u\|_{L^2(S,X)} \to 0 \qquad \text{as } n \to \infty$$

for any $u \in L^2_{loc}(\mathbb{R}_+, X)$, $a \in X$ and any finite interval $S \subset \mathbb{R}_+$.

Proof. For $a = 0$ the proof of both assertions is similar to that of [2, Lemma IV.1.5]. Now, let be $a \neq 0$ and S be any finite interval. Then

$$\|\tau_n^a u\|_{L^2(S,X)} \leq \left\|\tau_n^0 u\right\|_{L^2(S,X)} + \left\|\tau_n^a u - \tau_n^0 u\right\|_{L^2(S,X)} \leq \bar{H}_n^{\frac{1}{2}} \|u\|_{L^2(S,X)} + \bar{h}_n^{\frac{1}{2}} \|a\|_X$$

and

$$\begin{aligned}\|\tau_n^a u - u\|_{L^2(S,X)} &\leq \left\|\tau_n^0 u - u\right\|_{L^2(S,X)} + \left\|\tau_n^a u - \tau_n^0 u\right\|_{L^2(S,X)} \\ &\leq \left\|\tau_n^0 u - u\right\|_{L^2(S,X)} + \bar{h}_n^{\frac{1}{2}} \|a\|_X \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad \square\end{aligned}$$

Lemma 3.2. *Let S be any finite interval in $\mathbb{R}_+$, $\{Z_n\}_{n\in\mathbb{N}}$ a sequence of finite subdivisions of S with $\bar{h}_n \to 0$ for $n \to \infty$. Then for every $u \in W_S$ such that $u(0) = U$ there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ with the properties*

i) $w_n \in C_n(S, X)$, $K_n w_n(0) = U$,

ii) $\lim\limits_{n\to\infty} \left\{\|w_n - u\|_{L^2(S,X)} + \|K_n w_n - u\|_{C(S,Y)} + \|\Delta_n w_n - u'\|_{L^2(S,X^*)}\right\} = 0.$

Lemma 3.2 is an evident consequence of Lemma 1.2 in [4]. Another basic result which is used for the discrete-time problems is the following discrete analogue of Gronwall's Lemma:

Lemma 3.3. *Let $\{a_l\}$, $\{b_l\}$, $l = 0, \dots, k$ be real valued sequences, let $\{b_l\}$ be non-negative and $c \geq 0$ a constant. If*

$$a_l \leq c + \sum_{i=0}^{l-1} b_i a_i, \qquad l = 0, \dots, k,$$

then

$$a_l \leq c + c \sum_{i=0}^{l-1} b_i \exp \sum_{m=i+1}^{l-1} b_m, \qquad l = 0, \dots, k.$$

This result follows easily from [12, Lemma 2].

3.2. Estimates by the energy functional.

We modify the definition (2.16) of the dissipation rate as follows:

$$D(v,u) := D_{\text{diff}}(v,u) + D_{\text{reac}}(u), \; u \in \{u \in X \colon u \geq 0)\}, \; v \in H^1(\Omega), \; v \geq 0,$$

$$D_{\text{diff}}(v,u) := 4 \int_\Omega \Big\{ D_0 \, \varphi(v) \varphi(u_0) \left| \nabla \sqrt{u_0} \right|^2 + \sum_{i=1}^3 D_i \left| \nabla \sqrt{u_i} \right|^2 \Big\} \mathrm{d}x.$$

If u is a solution to $(\mathrm{P_n})$ we define

$$D_n(u)(s) := D(\gamma_{n0} u(s), u(s)).$$

Theorem 3.1. *There exist constants $c > 0$ such that*

$$\begin{array}{ll} i) & F(K_n u(t_2)) \leq F(K_n u(t_1)) \quad \text{for} \quad t_2 \geq t_1 \geq 0, \\ ii) & \sup\limits_{t \in \mathbb{R}_+} F(K_n u(t)) \leq F(U), \\ iii) & \|D_n(u)\|_{L^1(\mathbb{R}_+)} \leq c, \\ iv) & \|\nabla \sqrt{u_i}\|_{L^2(\mathbb{R}_+, L^2(\Omega, \mathbb{R}^2))} \leq c, \; i = 0, \dots, 3 \end{array}$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. Let u be a solution to $(\mathrm{P_n})$, assume that $0 \leq t_1 < t_2$, $0 < \delta \leq 1$ and $D_{n\delta}(u)(s) := D((\gamma_{n0} u)(s), u(s) + \delta)$. We use the differential formula (2.15) for $w = K_n u + \delta$, $\lambda = 0$. Taking into account that $(K_n u)' = \Delta_n u$,

$$\langle (\Delta_n u)(t), \nabla f((K_n u)(t) + \delta) \rangle \leq \langle (\Delta_n u)(t), \nabla f(u(t) + \delta) \rangle \quad \forall \, t \in [0, t_2] \quad (3.2)$$

and replacing $(\Delta_n u)(s)$ by $-(A_n(u))(s)$ we find

$$\begin{aligned} F((K_n u)(t_2) + \delta) &- F((K_n u)(t_1) + \delta) + \int_{t_1}^{t_2} D_{n\delta}(u)(s) \mathrm{d}s \\ &\leq \int_0^{t_2} \Big\{ c\delta \Big(1 + |\ln \delta| + \sum_{i=0}^3 \Big(\|u_i + 1\|_{L^2}^2 + \|u_i + 1\|_{L^2(\Gamma)}^2 \Big) \Big) \Big\} \mathrm{d}s \\ &\leq \int_0^{t_2} c\delta \Big(1 + |\ln \delta| + \sum_{i=0}^3 \|u_i + 1\|_{H^1}^2 \Big) \mathrm{d}s \\ &\leq t_2 c\delta (1 + |\ln \delta|) + c\delta \sum_{i=0}^3 \|u_i + 1\|_{L^2([0,t_2], H^1)}^2 . \end{aligned}$$

Since $u \in L^2_{\text{loc}}(\mathbb{R}^+, X)$, the norms on the right hand side are finite. Letting $\delta \downarrow 0$ we get by Fatou's Lemma

$$F((K_n u)(t_2)) + \int_{t_1}^{t_2} D_n(u(s))\,\mathrm{d}s \le F((K_n u)(t_1)).$$

This proves i). By setting $t_1 = 0$, $t_2 = t$ for $t \in \mathbb{R}_+$ we get ii) and iii). By the definition of $D_n(u)$, (2.4) and (2.5) the estimate iv) follows. $\square$

Corollary 3.1. *There exist constants $c > 0$ such that*

$$\begin{array}{ll} i) & F(u^k) \le F(u^l) \quad \textit{for} \quad k \ge l \ge 0, \\ ii) & \sup_{k\in\mathbb{N}} F(u^k) \le F(U), \\ iii) & \|u\|_{L^\infty(\mathbb{R}_+, L^1(\Omega,\mathbb{R}^4))} \le c, \\ iv) & \|u \ln u\|_{L^\infty(\mathbb{R}_+, L^1(\Omega,\mathbb{R}^4))} \le c \end{array}$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. Setting $t_1 = t_n^l$, $t_2 = t_n^k$ from Theorem 3.1 i) it follows i). From Theorem 3.1 ii) with $t = t_n^k$ we obtain ii). Because of

$$\|u \ln u\|_{L^1(\Omega)}\,, \ \|u\|_{L^1(\Omega)} \le F(u) + c$$

we get iii) and iv). $\square$

Theorem 3.2. *There exist constants c, $\lambda > 0$ such that*

$$\begin{array}{ll} i) & F(K_n u(t)) \le c\, e^{-\lambda t} F(U) \quad \forall t \ge 0, \\ ii) & F(u(t)) \le c\, e^{-\lambda t} F(U) \quad \forall t \ge 0, \\ iii) & \|u(t) - u^*\|_{L^1(\Omega,\mathbb{R}^4)} \le c\, e^{-\lambda t/2} \quad \forall t \ge 0 \end{array}$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. i) Let $t > 0$. Then $t \in S_n^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$. For $w = K_n u + \delta \in W_{[0,t]}$, $w \ge \delta > 0$ we use formula (2.15) and the inequality (3.2), λ will be specified during the proof. We replace $(\Delta_n u)(s)$ by $-(A_n(u))(s)$ and use the following two estimates

$$\begin{aligned} &\langle \Delta_n u, f(u+\delta)\rangle \le D_{n\delta}(u) + c\delta\left[1 + |\ln\delta| + \|u+1\|_X^2\right], \\ &F((K_n u)(t_2) + \delta) - F((K_n u)(t_1) + \delta) \\ &\le c\delta\left[(t_2 - t_1)(1 + |\ln\delta|) + \|u+1\|^2_{L^2([t_1,t_2],X)}\right]. \end{aligned}$$

which had been of importance also in the proof of Theorem 3.1. For $t_1 = t_n^{l-1}$, $t_2 = t \in S_n^l$ this implies

$$F((K_n u)(t) + \delta) \le F(u^{l-1} + \delta) + c\delta\left[h(1 + |\ln\delta|) + \|u+1\|^2_{L^2([0,t],X)}\right].$$

We define

$$S^*(\delta, t) := c\delta \int_0^t e^{\lambda t}\left[(1+h)(1 + |\ln\delta| + \|u+1\|^2_{L^2([0,t],X)} + \|u+1\|_X^2\right] \mathrm{d}s.$$

Because of $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$ we have for all $t \in \mathbb{R}_+$, $S^*(\delta, t) \to 0$ for $\delta \to 0$. From

$$\begin{aligned}
& e^{\lambda t} F((K_n u)(t) + \delta) - F(U + \delta) \\
& \leq \int_0^t e^{\lambda s} \{\lambda F(\tau_n u + \delta) - D_{n\delta}(u)\} \, \mathrm{d}s + S^*(\delta, t) \\
& \leq \sum_{l=1}^k h_n^l \left\{\lambda e^{\lambda t_n^l} F(u^{l-1} + \delta) - e^{\lambda t_n^{l-1}} D_{n\delta}(u^l)\right\} + S^*(\delta, t) \\
& \quad + (t - t_n^k) \left\{\lambda e^{\lambda t_n^{k+1}} F(u^k + \delta) - e^{\lambda t_n^k} D_{n\delta}(u^{k+1})\right\} \\
& \leq h\lambda e^{\lambda h} F(U + \delta) - (t - t_n^k) e^{\lambda t_n^k} D_{n\delta}(u^{k+1}) + S^*(\delta, t) \\
& \quad + \sum_{l=1}^k e^{\lambda t_n^{l-1}} \left[\lambda h_n^{l+1} e^{2h\lambda} F(u^l + \delta) - h_n^l D_{n\delta}(u^l)\right]
\end{aligned}$$

we obtain by letting $\delta \to 0$

$$\begin{aligned}
& e^{\lambda t} F((K_n u)(t)) - (1 + h\lambda e^{\lambda h}) F(U) \\
& \qquad \leq \sum_{l=1}^k e^{\lambda t_n^{l-1}} \left[\lambda h_n^{l+1} e^{2h\lambda} F(u^l) - h_n^l D_n(u^l)\right] - (t - t_n^k) e^{\lambda t_n^k} D_n(u^{k+1}) \\
& \qquad \leq \sum_{l=1}^k e^{\lambda t_n^{l-1}} \left[\lambda h_n^{l+1} e^{2h\lambda} F(u^l) - \hat{c} h_n^l \tilde{D}(u^l)\right]
\end{aligned}$$

where $\tilde{D}$ is defined in (2.18). Taking into account that invariants of problem (P) (cf. (2.13)) are also invariants of problem ($\mathrm{P_n}$) and using (2.19) we choose $\lambda > 0$ such that $\lambda H e^{2\lambda h} < \hat{c}/\tilde{c}_R$ which implies that the terms on the right hand side are nonpositive. Therefore it follows i):

$$F((K_n u)(t)) \leq e^{-\lambda t}(1 + h\lambda e^{\lambda h}) F(U) \leq c e^{-\lambda t} F(U) \quad \forall t \in \mathbb{R}_+.$$

ii) Let $t \in S_n^k$. Then

$$F(u(t)) = F(u^k) = F((K_n u)(t_n^k)) \leq c\, e^{-\lambda t_n^k} F(U) \leq c\, e^{-\lambda t} F(U)$$

which proves ii).

iii) By the inequality

$$|y - y^*| \leq \left|\sqrt{y} - \sqrt{y^*}\right|^2 + 2\sqrt{y^*}\left|\sqrt{y} - \sqrt{y^*}\right|, \tag{3.3}$$

(2.17) and by ii) we obtain assertion iii):

$$\begin{aligned}
\|u(t) - u^*\|_{L^1(\Omega, \mathbb{R}^4)} & \leq \left\|\sqrt{u} - \sqrt{u^*}\right\|_Y^2 + c\left\|\sqrt{u} - \sqrt{u^*}\right\|_Y \\
& \leq c\left(F(U) e^{-\lambda t} + \sqrt{F(U)} e^{-\lambda t/2}\right) \\
& \leq c\, e^{-\lambda t/2}. \quad \square
\end{aligned}$$

Corollary 3.2. *There exist constants c, $c(p) > 0$ such that*

$$
\begin{array}{ll}
i) & \|u-u^*\|_{L^1(\mathbb{R}_+,L^1(\Omega,\mathbb{R}^4))}\,,\ \|u-u^*\|_{L^2(\mathbb{R}_+,L^2(\Omega,\mathbb{R}^4))} \le c,\\
ii) & \int_0^\infty \left\|\sqrt{u}-\sqrt{u^*}\right\|_Y^p\,ds \le c(p) \text{ where } 0<p<\infty,\\
iii) & \left\|\sqrt{u}-\sqrt{u^*}\right\|_{L^2(\mathbb{R}_+,X)}\,,\ \left\|\sqrt{u}-\sqrt{u^*}\right\|_{L^\infty(\mathbb{R}_+,Y)} \le c,\\
iv) & \|u-u^*\|_{L^{\frac{3}{2}}(\mathbb{R}_+,L^3(\Omega,\mathbb{R}^4))} \le c
\end{array}
$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. Integrating the inequality in Theorem 3.2 iii) over $\mathbb{R}_+$ we obtain $u-u^* \in L^1(\mathbb{R}_+, L^1(\Omega,\mathbb{R}^4))$. Let $w_i := \sqrt{u_i}-\sqrt{u_i^*}$. Because of (2.17), Theorem 3.2 ii) and Theorem 3.1 iv) we conclude that $w_i \in L^2(\mathbb{R}_+, H^1)$. Together with Theorem 3.1 ii) this yields iii). By interpolation we get $w_i \in L^4(\mathbb{R}_+, L^4)$, and therefore by (3.3)

$$
\begin{aligned}
\|u_i-u_i^*\|^2_{L^2(\mathbb{R}_+,L^2(\Omega))} &\le \int_0^\infty \int_\Omega \left\{|w_i|^4 + 4u_i^*\,|w_i|^2 + 4\sqrt{u_i^*}\,|w_i|^3\right\} dx\,ds\\
&\le c\left(\|w_i\|^4_{L^4(\mathbb{R}_+,L^4)} + \|w_i\|^2_{L^2(\mathbb{R}_+,L^2)}\right) \le c, \qquad i=0,\dots,3,
\end{aligned}
$$

which proves i). Assertion ii) is a direct consequence of (2.17) and Theorem 3.2 ii). For $i=0,\dots,3$ we estimate by (3.3) and the Gagliardo–Nirenberg inequality (2.9)

$$
\begin{aligned}
\|u_i-u_i^*\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}_+,L^3(\Omega))} &\le c\int_0^\infty \left(\int_\Omega \left(|w_i|^2+|w_i|\right)^3 dx\right)^{\frac{1}{2}} ds\\
&\le c\int_0^\infty \left(\|w_i\|^3_{L^6} + \|w_i\|^{\frac{3}{2}}_{L^3}\right) ds\\
&\le c\int_0^\infty \|w_i\|_{L^2}\left(\|w_i\|^2_{H^1} + \|w_i\|^{\frac{1}{2}}_{H^1}\right) ds\\
&\le c\,\|w_i\|_{L^\infty(\mathbb{R}_+,L^2)}\,\|w_i\|^2_{L^2(\mathbb{R}_+,H^1)} + c\int_0^\infty \|w_i\|_{L^2}\,ds.
\end{aligned}
$$

Thus iv) follows by ii) and iii). □

3.3. Further a-priori estimates.

We want to apply De Giorgi's method in order to show that solutions to $(\mathrm{P_n})$ are globally bounded. But the regularity results coming from the energy estimates are not good enough for starting the measure estimates. Therefore, at first we improve the regularity results. Here as well as in following sections we use the notation (cf.(2.4), (2.5))

$$
d := \min(\tau_3 d_0, d_1, d_2, d_3)\,. \tag{3.4}
$$

Lemma 3.4. *There exists a constant $K > 0$ depending only on the data with the following property: Let be $j \in \mathbb{N}$, $j \ge 5$. Then there exists a constant $c > 0$ such that*

$$
\left\|\left[(u-K)^+\right]^{\frac{j}{8}}\right\|_{L^\infty(\mathbb{R}_+,Y)}\,,\ \left\|\left[(u-K)^+\right]^{\frac{j}{8}}\right\|_{L^2(\mathbb{R}_+,X)} \le c
$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. The proof works by induction. The assertion for $j = 5$ is proved by using regularity results based on the energetic estimates (see Subsection 3.2). Here also the constant K will be fixed (cf. (3.5), (3.9)). In the following steps ($j > 5$) we apply additionally the estimates of Lemma 3.4 with $\tilde{j} < j$. For $j \in \mathbb{N}$, $j \geq 5$ let $m := j - 1$, $r := m/4$. For the sake of simplicity we shall handle the cases $j = 5$ ($r = 1$) and $j > 5$ ($r > 1$) simultaneously, as far as it is possible.

First, let

$$K \geq k^* := \max\left\{1, \|U\|_{L^\infty(\Omega,\mathbb{R}^4)}, \|u^*\|_{\mathbb{R}^4}\right\} \tag{3.5}$$

and $z := (u - K)^+$. We take formally [1] the test function

$$\left(z_0^r, z_1^r, z_2^r, z_3^{r-3/4}\right)$$

and use the following estimates

$$\begin{aligned} &g(u_0) \leq c(u_0^2 + u_0), \\ &|u_i| \leq z_i + K, && (3.6) \\ &(u - K)^+ \leq \tfrac{1}{\ln K}|u \ln u|. && (3.7) \end{aligned}$$

Because of

$$\int_0^{t_n^l} \left\langle (\Delta_n u)_i, z_i^r \right\rangle \mathrm{d}s \geq \tfrac{1}{r+1} \left\| z_i^l \right\|_{L^{r+1}}^{r+1}, \; i = 0, \ldots, 3, \tag{3.8}$$

we obtain for all $t_n^l \in Z_n$

$$\begin{aligned} &\sum_{i=0}^{2} \left[\tfrac{1}{r+1} \left\| z_i^l \right\|_{L^{r+1}}^{r+1} + d \int_0^{t_n^l} \tfrac{m}{(r+1)^2} \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 \mathrm{d}s \right] \\ &\qquad + \tfrac{4}{m+1} \left\| z_3^l \right\|_{L^{\frac{m+1}{4}}}^{\frac{m+1}{4}} + d \int_0^{t_n^l} \tfrac{16(m-3)}{(m+1)^2} \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^2 \mathrm{d}s \\ &\leq c \int_0^{t_n^l} \Bigg\{ \int_\Gamma \left\{ z_3 z_0^r + (z_0^2 + z_0) z_3^{\frac{m-3}{4}} + c(K) \left\{ (1 + z_0) z_3^{\frac{m-3}{4}} + z_0^r + z_1^r + z_2^r \right\} \right\} \mathrm{d}\Gamma \\ &\quad + \sum_{i=0}^{2} \left\| z_i^{\frac{r+1}{2}} \right\|_{L^2}^2 + \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2}^2 \\ &\quad + \int_\Omega \Big\{ z_1 z_3 z_0^r + z_0^2 z_1^r + z_2^r + z_0^2 z_3^{\frac{m-3}{4}} \\ &\qquad + c(K) \left\{ (1 + z_1 + z_3) z_0^r + (1 + z_0) z_1^r + (1 + z_0) z_3^{\frac{m-3}{4}} \right\} \Big\} \mathrm{d}x \Bigg\} \mathrm{d}s \end{aligned}$$

---

[1] More precisely the following estimates are obtained by test functions of this kind where z_i are replaced by $z_i^N := \min\{z_i, N\}$, $i = 0, 1, 2, 3$, $N > 0$. It is possible to get estimates which are independent of N. Letting $N \longrightarrow \infty$ we shall obtain the inequalities derived below.

$$
\begin{aligned}
&\leq \int_0^{t_n^l} \sum_{i=0}^{2} \Bigg\{ \int_\Gamma \left(c\left[z_i^2 z_3^{\frac{m-3}{4}} + z_i^r z_3 \right] + c(K) \left[z_i z_3^{\frac{m-3}{4}} + z_i^r + z_3^{\frac{m-3}{4}} \right] \right) \mathrm{d}\Gamma \\
&\quad + \int_\Omega \Bigg(c\left[z_i^{r+2} + z_i^{r+1} z_3 + z_i^2 z_3^{\frac{m-3}{4}} + z_3^{\frac{m+1}{4}} \right] \\
&\qquad + c(K) \left[z_i^{r+1} + z_3 z_i^r + z_i^r + z_i z_3^{\frac{m-3}{4}} + z_3^{\frac{m-3}{4}} \right] \Bigg) \mathrm{d}x \Bigg\} \mathrm{d}s. \\
&\leq \int_0^{t_n^l} \sum_{i=0}^{2} \Bigg\{ \int_\Gamma \left(c\left[z_i^2 z_3^{\frac{m-3}{4}} + z_i^r z_3 \right] + c(K) \left[z_i^{r+1} + z_i^r + z_3^{\frac{m+1}{4}} + z_3^{\frac{m-3}{4}} \right] \right) \mathrm{d}\Gamma \\
&\quad + \int_\Omega \left(c\left[z_i^{r+2} + z_i^{r+1} z_3 + z_3^{\frac{m+5}{4}} \right] + c(K) \left[z_i^r + z_3^{\frac{m-3}{4}} \right] \right) \mathrm{d}x \Bigg\} \mathrm{d}s.
\end{aligned}
$$

To estimate these different terms under the time integral on principle we follow this strategy: For boundary integrals at first the trace inequality (2.10) is applied. For all terms the Gagliardo-Nirenberg inequality (2.9) and the Young Inequality are used to obtain summands $\left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2$, $i = 0, 1, 2$ and $\left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^2$ with small coefficients and other expressions. To manage the other terms which arrise by this procedure we take advantage of $L^\infty(\mathbb{R}_+, Y) \cap L^2(\mathbb{R}_+, X)$-estimates for $z^{\tilde{j}/8}$, $5 \leq \tilde{j} \leq m$. If r is small some special estimates have to be provided. Then results obtained by the energetic estimates supply the desired estimates. Here are the exact evaluations:

1) $\int_\Gamma z_i^2\, z_3^{\frac{m-3}{4}}\, \mathrm{d}\Gamma,\ i = 0, 1, 2:$

$$
\begin{aligned}
\int_\Gamma z_i^2\, z_3^{\frac{m-3}{4}}\, \mathrm{d}\Gamma &\leq \left\| z_i^2 \right\|_{L^{\frac{m+1}{4}}(\Gamma)} \left\| z_3^{\frac{m-3}{4}} \right\|_{L^{\frac{m+1}{m-3}}(\Gamma)} = \left\| z_i^{\frac{r+1}{2}} \right\|_{L^{\frac{m+1}{r+1}}(\Gamma)}^{\frac{4}{r+1}} \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2(\Gamma)}^{\frac{2(m-3)}{m+1}} \\
&\leq c \left\| z_i^{\frac{r+1}{2}} \right\|_{L^{2(\beta-1)}}^{\frac{4(\beta-1)}{(r+1)\beta}} \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^{\frac{4}{(r+1)\beta}} \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2(\Gamma)}^{\frac{2(m-3)}{m+1}} \\
&\leq c \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^{\frac{4}{(r+1)\beta}(1+(\beta-1)(1-\theta))} \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2}^{\frac{m-3}{m+1}} \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^{\frac{m-3}{m+1}},
\end{aligned}
$$

where $\beta := \frac{m+1}{r+1}$, $\theta := \frac{r}{(\beta-1)(r+1)}$. Because the exponent of the H^1-norm of $z_i^{\frac{r+1}{2}}$ is smaller than two, we obtain from the Young inequality with $p' := \frac{(m+1)(r+1)}{4r^2-r-1}$

$$
\begin{aligned}
\int_\Gamma z_i^2\, z_3^{\frac{m-3}{4}}\, \mathrm{d}\Gamma &\leq \varepsilon \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2}^{\frac{(m-3)\,p'}{m+1}} \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^{\frac{(m-3)p'}{m+1}} \\
&\leq \varepsilon \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| z_3^{\frac{m+1}{8}} \right\|_{L^{\frac{2m}{m+1}}}^{\frac{(m-3)\,p' m}{(m+1)^2}} \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^{\frac{(m+2)(m-3)p'}{(m+1)^2}} \\
&\leq \varepsilon \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| z_3 \right\|_{L^r}^{\frac{(m-3)\,r\,p'}{2(m+1)}} \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^{\frac{(m+2)(m-3)p'}{(m+1)^2}}.
\end{aligned}
$$

The exponent of the H^1-norm of $z_3^{\frac{m+1}{8}}$ is also smaller than two. We use Young's inequality again. With $q' := (m+1)(4r^2-r-1)(8r^3-6r^2+2)^{-1}$, $z_3 \in L^\infty(\mathbb{R}_+, L^r)$

and the inequality $(m-3)(m+1)^{-1}\,p'\,q' \geq 2$ we conclude

$$\begin{aligned}
\int_\Gamma z_i^2\, z_3^{\frac{m-3}{4}}\,\mathrm{d}\Gamma &\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\,\|z_3\|_{L^r}^{\frac{(m-3)\,r\,p'\,q'}{2(m+1)}} \\
&\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\,\|z_3\|_{L^r}^{r} \\
&\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\left\|z_3^{\frac{r}{2}}\right\|_{L^2}^2 .
\end{aligned}$$

2) $\int_\Gamma z_3 z_i^r \mathrm{d}\Gamma,\ i=0,1,2:$

$$\begin{aligned}
\int_\Gamma z_3 z_i^r\,\mathrm{d}\Gamma &\leq \left\|z_3^{\frac{m+1}{8}}\right\|_{L^2(\Gamma)}^{\frac{8}{m+1}} \left\|z_i^{\frac{r+1}{2}}\right\|_{L^{\frac{m+1}{m-3}\frac{2r}{r+1}}(\Gamma)}^{\frac{2r}{r+1}} \\
&\leq c\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^{\frac{2r}{r+1}} \left\|z_3^{\frac{m+1}{8}}\right\|_{L^2}^{\frac{4}{m+1}} \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^{\frac{4}{m+1}} \\
&\leq \varepsilon\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|z_3^{\frac{m+1}{8}}\right\|_{L^2}^{\frac{4(r+1)}{m+1}} \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^{\frac{4(r+1)}{m+1}} \\
&\leq \varepsilon\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|z_3^{\frac{m+1}{8}}\right\|_{L^{\frac{2m}{m+1}}}^{\frac{4(r+1)m}{(m+1)^2}} \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^{\frac{4(r+1)}{m+1}\left(1+\frac{1}{m+1}\right)} \\
&\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\left\|z_3^{\frac{m+1}{8}}\right\|_{L^{\frac{2m}{m+1}}}^{\frac{4(r+1)m}{2mr-m-3}} \\
&\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\,\|z_3\|_{L^r}^{\frac{2r(m+1)(r+1)}{2mr-m-3}} \\
&\leq \varepsilon\left(\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2\right) + c\,\|z_3\|_{L^r}^{r} .
\end{aligned}$$

3) $\int_\Gamma z_i^{r+1}\mathrm{d}\Gamma,\ i=0,1,2:$

$$\begin{aligned}
c(K)\int_\Gamma z_i^{r+1}\mathrm{d}\Gamma &\leq \varepsilon\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c(K)\left\|z_i^{\frac{r+1}{2}}\right\|_{L^2}^2 \\
&\leq \varepsilon\left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + \int_\Omega \left(z_i^{r+2} + c(K)\,z_0^r\right)\mathrm{d}x.
\end{aligned}$$

Now we apply the estimates of step 7) and 10) derived below.

4) $\int_\Gamma z_i^r \mathrm{d}\Gamma,\ i=0,1,2:$

a) $r=1$:

$$\begin{aligned}
\int_0^t\int_\Gamma z_i\mathrm{d}\Gamma\mathrm{d}s &\leq \int_0^t \|u_i - u_i^*\|_{L^1(\Gamma)}\,\mathrm{d}s \\
&\leq c\int_0^t\left(\left\|\sqrt{u_i}-\sqrt{u_i^*}\right\|_{L^2(\Gamma)}^2 + \left\|\sqrt{u_i}-\sqrt{u_i^*}\right\|_{L^2(\Gamma)}\right)\mathrm{d}s \\
&\leq c\int_0^t\left(\left\|\sqrt{u_i}-\sqrt{u_i^*}\right\|_{H^1}^2 + \left\|\sqrt{u_i}-\sqrt{u_i^*}\right\|_{L^2}^{\frac{2}{3}}\right)\mathrm{d}s \leq c
\end{aligned}$$

because of Corollary 3.2 ii) and iii).

b) $r > 1$:
By assumption $z_i^{\frac{r}{2}} \in L^2(\mathbb{R}_+, H^1)$. Thus it follows $z_i^{\frac{r}{2}} \in L^2(\mathbb{R}_+, L^2(\Gamma))$.

5) $\int_\Gamma z_3^{\frac{m+1}{4}} \mathrm{d}\Gamma$:

$$c(K) \int_\Gamma z_3^{\frac{m+1}{4}} \mathrm{d}\Gamma \leq \varepsilon \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^2 + c(K) \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2}^2$$
$$\leq \varepsilon \left\| z_3^{\frac{m+1}{8}} \right\|_{H^1}^2 + \int_\Omega \left(z_3^{\frac{m+5}{4}} + c(K)\, z_3^{\frac{m-3}{4}} \right) \mathrm{d}x.$$

Now we apply the estimates of step 9) and 11).

6) $\int_\Gamma z_3^{\frac{m-3}{4}} \mathrm{d}\Gamma$:
a) $r \leq \frac{7}{4}$:
Let $q := \frac{m-3}{4}$ then by Corollary 3.2 ii) and iii)

$$\int_0^t \int_\Gamma z_3^q \,\mathrm{d}\Gamma \mathrm{d}s \leq c \int_0^t \left(\left\| \sqrt{u_3} - \sqrt{u_3^*} \right\|_{L^2(\Gamma)}^{2q} + \left\| \sqrt{u_3} - \sqrt{u_3^*} \right\|_{L^2(\Gamma)}^{q} \right) \mathrm{d}s$$
$$\leq c \int_0^t \left(\left\| \sqrt{u_i} - \sqrt{u_i^*} \right\|_{H^1}^2 + \left\| \sqrt{u_i} - \sqrt{u_i^*} \right\|_{L^2}^{\frac{2q}{2-q}} + \left\| \sqrt{u_i} - \sqrt{u_i^*} \right\|_{L^2}^{\frac{2q}{4-q}} \right) \mathrm{d}s$$
$$\leq c.$$

b) $r > \frac{7}{4}$:
By assumption $z_3^{\frac{m-3}{8}} \in L^2(\mathbb{R}_+, H^1)$ which implies $z_3^{\frac{m-3}{8}} \in L^2(\mathbb{R}_+, L^2(\Gamma))$.

7) $\int_\Omega z_i^{r+2} \mathrm{d}x,\ i = 0, 1, 2$:
a) $r = 1$:
Using inequality (3.7), Corollary 3.1 iv) and Corollary 3.2 iv) we estimate

$$\int_\Omega z_i^3 \mathrm{d}x \leq \varepsilon \left\| z_i \right\|_{H^1}^2 + c \left\| z_i \right\|_{L^2}^4 \leq \varepsilon \left\| z_i \right\|_{H^1}^2 + c \left\| z_i \right\|_{L^1}^{\frac{1}{2}} \left\| z_i \right\|_{L^3}^{\frac{3}{2}} \left\| z_i \right\|_{L^2}^2$$
$$\leq \varepsilon \left\| z_i \right\|_{H^1}^2 + \left[\frac{c}{\ln K} \left\| u_i \ln u_i \right\|_{L^\infty(\mathbb{R}_+, L^1)} \right]^{\frac{1}{2}} \left\| z_i \right\|_{L^3}^{\frac{3}{2}} \left\| z_i \right\|_{L^2}^2 .$$

b) $r > 1$:

$$\int_\Omega z_i^{r+2} \mathrm{d}x \leq \left\| z_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r+1}} \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^{\frac{4}{r+1}} \leq \varepsilon \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| z_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r-1}}$$
$$\leq \varepsilon \left\| z_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| z_i \right\|_{L^r}^{r} .$$

8) $\int_\Omega z_i^{r+1} z_3 \, dx, \ i = 0,1,2:$

a) $r = 1$:

By (3.7), Corollary 3.1 iv) and Corollary 3.2 iv) we obtain

$$\begin{aligned}\int_\Omega z_i^2 z_3 dx &\le \|z_i\|_{L^4}^2 \|z_3\|_{L^2} \le \|z_i\|_{L^2} \|z_i\|_{H^1} \|z_3\|_{L^2} \\ &\le \varepsilon \|z_i\|_{H^1}^2 + c\|z_3\|_{L^2}^2 \|z_i\|_{L^2}^2 \le \varepsilon \|z_i\|_{H^1}^2 + c\|z_3\|_{L^1}^{\frac{1}{2}} \|z_3\|_{L^3}^{\frac{3}{2}} \|z_i\|_{L^2}^2 \\ &\le \varepsilon \|z_i\|_{H^1}^2 + \left[\frac{c}{\ln K} \|u_3 \ln u_3\|_{L^\infty(\mathbb{R}_+, L^1)}\right]^{\frac{1}{2}} \|z_3\|_{L^3}^{\frac{3}{2}} \|z_i\|_{L^2}^2 .\end{aligned}$$

b) $r > 1$:

Because of $z_3 \in L^\infty(\mathbb{R}_+, L^r)$

$$\begin{aligned}\int_\Omega z_i^{r+1} z_3 \, dx &\le \left\|z_i^{r+1}\right\|_{L^{\frac{r}{r-1}}} \|z_3\|_{L^r} \le c\left\|z_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r-1}}}^2 \le c\left\|z_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^{\frac{2(r-1)}{r+1}} \left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^{\frac{4}{r+1}} \\ &\le \varepsilon \left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|z_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^2 \le \varepsilon \left\|z_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\|z_i\|_{L^r}^r .\end{aligned}$$

9) $\int_\Omega z_3^{\frac{m+5}{4}} \, dx:$

a) $r = 1:$

$$\begin{aligned}\int_\Omega z_3^{\frac{9}{4}} dx &\le \left\|z_3^{\frac{5}{8}}\right\|_{L^{\frac{8}{5}}}^{\frac{8}{5}} \left\|z_3^{\frac{5}{8}}\right\|_{H^1}^2 = \|z_3\|_{L^1} \left\|z_3^{\frac{5}{8}}\right\|_{H^1}^2 \\ &\le \frac{1}{\ln K} \|u_3 \ln u_3\|_{L^\infty(\mathbb{R}_+, L^1)} \left\|z_3^{\frac{5}{8}}\right\|_{H^1}^2 .\end{aligned}$$

b) $r > 1:$

$$\begin{aligned}\int_\Omega z_3^{\frac{m+5}{4}} \, dx &\le \left\|z_3^{\frac{m+1}{8}}\right\|_{L^{\frac{2m}{m+1}}}^{\frac{2m}{m+1}} \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^{\frac{10}{m+1}} \le \varepsilon \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2 + c\left\|z_3^{\frac{m+1}{8}}\right\|_{L^{\frac{2m}{m+1}}}^{\frac{2r}{r-1}} \\ &\le \varepsilon \left\|z_3^{\frac{m+1}{8}}\right\|_{H^1}^2 + c\|z_3\|_{L^r}^r .\end{aligned}$$

10) $\int_\Omega z_i^r dx, \ i = 0,1,2:$

For $r = 1$ we get by Corollary 3.2 i) $z_i \in L^1(\mathbb{R}_+, L^1)$, if $r > 1$ by assumption $z_i \in L^r(\mathbb{R}_+, L^r)$.

11) $\int_\Omega z_3^{\frac{m-3}{4}} \, dx:$

a) $r < \frac{7}{4}:$

Let $q := \frac{m-3}{4}$, we use Corollary 3.2 ii)

$$\begin{aligned}\int_0^t \int_\Omega z_3^q dx ds &\le \int_0^t \|u_3 - u_3^*\|_{L^q}^q \, ds \\ &\le c\int_0^t \left(\left\|(\sqrt{u_3} - \sqrt{u_3^*}\right\|_{L^{2q}}^{2q} + \left\|(\sqrt{u_3} - \sqrt{u_3^*}\right\|_{L^q}^q\right) ds \\ &\le c\int_0^t \left(\left\|(\sqrt{u_3} - \sqrt{u_3^*}\right\|_{L^2}^{2q} + \left\|(\sqrt{u_3} - \sqrt{u_3^*}\right\|_{L^2}^q\right) ds \le c(q).\end{aligned}$$

b) $r \geq \frac{7}{4}$:
By assumption $z_3 \in L^{\frac{m-3}{4}}\left(\mathbb{R}_+, L^{\frac{m-3}{4}}\right)$.

Using the above estimates **1)** to **11)** we continue as follows:
a) $r = 1$:
Using Corollary 3.1 iv) we find $\forall l \in \mathbb{N}$

$$\sum_{i=0}^{2}\left(\left\|z_i^l\right\|_{L^2}^2 + \int_0^{t_n^l}\|z_i\|_{H^1}^2\,\mathrm{d}s\right) + \left\|z_3^l\right\|_{L^{5/4}}^{5/4} + \int_0^{t_n^l}\left\|z_3^{5/8}\right\|_{H^1}^2\mathrm{d}s$$
$$\leq \int_0^{t_n^l}\left\{\frac{1}{4}\left(\sum_{i=0}^{2}\left\|z_i^l\right\|_{H^1}^2 + \left\|z_3^{5/8}\right\|_{H^1}^2\right) + \frac{\hat{c}}{\ln K}\|u_3\ln u_3\|_{L^\infty(\mathbb{R}_+,L^1)}\left\|z_3^{5/8}\right\|_{H^1}^2\right.$$
$$+\sum_{i=0}^{2}\left[\frac{\hat{c}}{\ln K}\|u_i\ln u_i\|_{L^\infty(\mathbb{R}_+,L^1)}\right]^{1/2}\|z_i\|_{L^3}^{3/2}\|z_i\|_{L^2}^2$$
$$\left.+\left[\frac{\hat{c}}{\ln K}\|u_3\ln u_3\|_{L^\infty(\mathbb{R}_+,L^1)}\right]^{1/2}\|z_3\|_{L^3}^{3/2}\sum_{i=0}^{2}\|z_i\|_{L^2}^2\right\}\mathrm{d}s + c_1(K).$$

Now we fix $K \geq k^*$ such that

$$\begin{aligned}\frac{\hat{c}}{\ln K}\|u_3\ln u_3\|_{L^\infty(\mathbb{R}_+,L^1)} &\leq \frac{1}{4},\\ \left[\frac{\hat{c}}{\ln K}\|u_i\ln u_i\|_{L^\infty(\mathbb{R}_+,L^1)}\right]^{1/2}\|u_i - u_i^*\|_{L^{3/2}(\mathbb{R}_+,L^3)}^{3/2} &\leq \frac{1}{4}, \quad i = 0,1,2,3.\end{aligned} \tag{3.9}$$

Thus the conditions for the choice of K (see (3.5), too) depend only on the data (cf. Corollary 3.1 iv) and Corollary 3.2 iv)). Let

$$a_l := \sum_{i=0}^{2}\left(\left\|z_i^l\right\|_{L^2}^2 + \int_0^{t_n^l}\|z_i\|_{H^1}^2\,\mathrm{d}s\right) + \left\|z_3^l\right\|_{L^{5/4}}^{5/4} + \int_0^{t_n^l}\left\|z_3^{5/8}\right\|_{H^1}^2\mathrm{d}s$$

then we obtain for this fixed K the estimate

$$a_l \leq c_1 + c\sum_{k=1}^{l-1}a_k\,b_k \qquad \forall l \in \mathbb{N},$$

where

$$b_k := h_n^j\sum_{i=0}^{3}\left\|u_i^k - u_i^*\right\|_{L^3}^{3/2}.$$

According to Lemma 3.3 we conclude that $\forall l \in \mathbb{N}$

$$
\begin{aligned}
a_l &\le c_1 + c_1\, c \sum_{k=1}^{l-1} b_k \exp\left\{ c \sum_{m=k+1}^{l-1} b_m \right\} \\
&\le c_1 + c_1\, c \sum_{k=1}^{l-1} b_k \exp\left\{ c \, \|u - u^*\|^{3/2}_{L^{3/2}(\mathbb{R}_+, L^3(\Omega,\mathbb{R}^4))} \right\} \\
&\le c_1 + c_1\, c \, \|u - u^*\|^{3/2}_{L^{3/2}(\mathbb{R}_+, L^3(\Omega,\mathbb{R}^4))} \exp\left\{ c \, \|u - u^*\|^{3/2}_{L^{3/2}(\mathbb{R}_+, L^3(\Omega,\mathbb{R}^4))} \right\} \le c.
\end{aligned}
$$

By the definition of a_l it follows that

$$
\|z_i\|_{L^\infty(\mathbb{R}_+,L^2)}, \left\| z_3^{5/8} \right\|_{L^\infty(\mathbb{R}_+,L^2)}, \|z_i\|_{L^2(\mathbb{R}_+,H^1)}, \left\| z_3^{5/8} \right\|_{L^2(\mathbb{R}_+,H^1)} \le c, \; i = 0,1,2.
$$

b) $r > 1$:

We obtain from the above estimates with K fixed in the step for $r = 1$

$$
\begin{aligned}
&\sum_{i=0}^{2} \left[\left\| z_i^l \right\|^{r+1}_{L^{r+1}} + \int_0^{t_n^l} \left\| z_i^{\frac{r+1}{2}} \right\|^2_{H^1} \mathrm{d}s \right] + \left\| z_3^l \right\|^{\frac{m+1}{4}}_{L^{\frac{m+1}{4}}} + \int_0^{t_n^l} \left\| z_3^{\frac{m+1}{8}} \right\|^2_{H^1} \mathrm{d}s \\
&\le c + \tfrac{1}{2} \int_0^{t_n^l} \left[\sum_{i=0}^{2} \left\| z_i^{\frac{r+1}{2}} \right\|^2_{H^1} + \left\| z_3^{\frac{m+1}{8}} \right\|^2_{H^1} \right] \mathrm{d}s \qquad \forall l \in \mathbb{N}.
\end{aligned}
$$

Thus

$$
\left\| z_i^{\frac{r+1}{2}} \right\|_{L^\infty(\mathbb{R}_+,L^2)}, \left\| z_i^{\frac{r+1}{2}} \right\|_{L^2(\mathbb{R}_+,H^1)}, \left\| z_3^{\frac{m+1}{8}} \right\|_{L^\infty(\mathbb{R}_+,L^2)}, \left\| z_3^{\frac{m+1}{8}} \right\|_{L^2(\mathbb{R}_+,H^1)} \le c.
$$

Therefore $\left\| z_i^{\frac{m+1}{8}} \right\|_{L^\infty(\mathbb{R}_+,L^2)} \le c$, $i = 0,1,2$. To show a $L^2(\mathbb{R}_+, H^1)$–estimate for $z_i^{\frac{m+1}{8}}$ we take the test function $z_i^{\frac{m-3}{4}}$ for $i = 0,1$ or 2 and use (3.6):

$$
\begin{aligned}
&\|z_i(t)\|^{\frac{m+1}{4}}_{L^{\frac{m+1}{4}}} + \int_0^t \left\| z_i^{\frac{m+1}{8}} \right\|^2_{H^1} \mathrm{d}s \\
&\quad \le c \int_0^t \left\{ \sum_{k=0}^{2} \int_\Omega \left\{ u_3 u_k + u_k^2 + u_k + 1 \right\} z_i^{\frac{m-3}{4}} \mathrm{d}x + \int_\Gamma (u_3 + 1) z_i^{\frac{m-3}{4}} \mathrm{d}\Gamma \right\} \mathrm{d}s \\
&\quad \le c \int_0^t \left\{ \sum_{k=0}^{2} \int_\Omega \left\{ z_3 z_k + z_k^2 + z_3 + z_k + 1 \right\} z_i^{\frac{m-3}{4}} \mathrm{d}x \right. \\
&\qquad\qquad \left. + \int_\Gamma (z_3 + 1)\, z_i^{\frac{m-3}{4}} \mathrm{d}\Gamma \right\} \mathrm{d}s \\
&\quad \le \int_0^t \left\{ c \Big[\sum_{k=0}^{3} \left(\|z_k(t)\|^{\frac{m+5}{4}}_{L^{\frac{m+5}{4}}} + \|z_k(t)\|^{\frac{m+1}{4}}_{L^{\frac{m+1}{4}}} \right) + \int_\Omega z_i^{\frac{m-3}{4}} \mathrm{d}x \right. \\
&\qquad \left. + \left\| z_3(t)^{\frac{m+1}{8}} \right\|^2_{L^2(\Gamma)} + \int_\Gamma z_i^{\frac{m-3}{4}} \mathrm{d}\Gamma \Big] + \tfrac{1}{2} \left\| z_i(t)^{\frac{m+1}{8}} \right\|^2_{H^1} \right\} \mathrm{d}s \qquad \forall t \in \mathbb{R}_+.
\end{aligned}
$$

From $z_3^{\frac{m+1}{8}}$, $z_i^{\frac{r+1}{2}} \in L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, H^1)$, $i = 0,1,2$, and results obtained during the energy estimates we thus conclude that $\left\| z_i^{\frac{m+1}{8}} \right\|_{L^2(\mathbb{R}_+,H^1)} \le c$, $i = 0,1,2$. $\square$

Theorem 3.3. *For given $r \in \mathbb{N}$, $r \geq 2$ there exist a constant $c > 0$ such that*

$$\left\| |u-u^*|^{\frac{r+1}{2}} \right\|_{L^\infty(\mathbb{R}_+,Y)}, \quad \left\| |u-u^*|^{\frac{r+1}{2}} \right\|_{L^2(\mathbb{R}_+,X)} < c$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. Let u be a solution to $(\mathrm{P_n})$, $w := u - u^*$, $v := |w|$. By $|(u_i - K)^+| \leq |u_i - u_i^*|$ and Lemma 3.4 we find that for $p \in \mathbb{R}$, $p \geq 1$, there exists a constant $c(p) > 0$ such that

$$\left\| v^{\frac{p}{2}} \right\|_{L^\infty(\mathbb{R}_+,Y)} \leq \left\| \left[(u-K)^+\right]^{\frac{p}{2}} \right\|_{L^\infty(\mathbb{R}_+,Y)} + c \leq c(p)$$

with K defined in (3.5), (3.9). By Corollary 3.2 i) we have $v \in L^1(\mathbb{R}_+, L^1(\Omega, \mathbb{R}^4))$. We show: For every $r \in \mathbb{N}$, $r \geq 2$ from $v^{\frac{r}{2}} \in L^2(\mathbb{R}_+, Y)$ it follows $v^{\frac{r+1}{2}} \in L^2(\mathbb{R}_+, X)$. Let $r \in \mathbb{N}$ be given. Using the test function $v^r \mathrm{sgn}(u - u^*)$ and the following representation of the reaction and boundary terms

$$\begin{aligned}
R_1(u) &= \tilde{k}_1 \left(w_1 w_3 + u_1^* w_3 + u_3^* w_1 - k_1 \left(g(u_0) - g(u_0^*)\right)\right), \\
R_2(u) &= \tilde{k}_2 \left(w_2 w_3 + u_2^* w_3 + u_3^* w_2\right), \\
R_4(u) &= \tilde{k}_4 \left(g(u_0) - g(u_0^*) - k_4 w_3\right), \\
|g(u_0) - g(u_0^*)| &\leq c\left(|u_0 - u_0^*|^2 + |u_0 - u_0^*|\right)
\end{aligned} \tag{3.10}$$

we obtain for all $t_n^k \in Z_n$

$$\begin{aligned}
&\sum_{i=0}^{3} \left(\left\| v_i(t_n^k) \right\|_{L^{r+1}}^{r+1} + \int_0^{t_n^k} \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 \mathrm{d}s \right) \\
&\leq c + c \sum_{i=0}^{3} \int_0^{t_n^k} \left\{ \|v_i\|_{L^{r+2}}^{r+2} + \|v_i\|_{L^{r+1}}^{r+1} + \|v_i\|_{L^{r+2}(\Gamma)}^{r+2} + \|v_i\|_{L^{r+1}(\Gamma)}^{r+1} \right\} \mathrm{d}s \\
&\leq c + \int_0^{t_n^k} \sum_{i=0}^{3} \left\{ \tfrac{1}{2} \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| v_i^{\frac{r}{2}} \right\|_{L^2}^2 \right\} \mathrm{d}s
\end{aligned}$$

which really proves the theorem but we want to explain in more detail how the last line in this estimate follows from the last but one:

$$\begin{aligned}
\int_\Gamma v_i^{r+2} \mathrm{d}\Gamma &= \left\| v_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2(r+2)}{r+1}}(\Gamma)}^{\frac{2(r+2)}{r+1}} \leq c \left\| v_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2(r+3)}{r+1}}}^{\frac{r+3}{r+1}} \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1} \\
&\leq \varepsilon \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| v_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2(r+3)}{r+1}}}^{\frac{2(r+3)}{r+1}} \leq \varepsilon \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| v_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2(r+3)}{r+1}}}^{\frac{2(r+3)}{r+4}} \\
&\leq \varepsilon \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| v_i^{\frac{r+1}{2}} \right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r+4}} \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^{\frac{6}{r+4}} \leq \tilde{\varepsilon} \left\| v_i^{\frac{r+1}{2}} \right\|_{H^1}^2 + c \left\| v_i^{\frac{r}{2}} \right\|_{L^2}^2,
\end{aligned}$$

$$\int_\Gamma v_i^{r+1}\mathrm{d}\Gamma \le c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^2}\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1} \le c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^{\frac{r}{r+1}}\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^{\frac{r+2}{r+1}}$$

$$\le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^2 \le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r}{2}}\right\|_{L^2}^2,$$

$$\int_\Omega v_i^{r+2}\mathrm{d}x \le c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r+1}}\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^{\frac{4}{r+1}} \le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r-1}}$$

$$\le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(r+1)}{r-1}} \le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r}{2}}\right\|_{L^2}^2,$$

$$\int_\Omega v_i^{r+1}\mathrm{d}x \le c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^{\frac{2r}{r+1}}\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^{\frac{2}{r+1}} \le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r+1}{2}}\right\|_{L^{\frac{2r}{r+1}}}^2$$

$$\le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(r+1)}{r}} \le \varepsilon\left\|v_i^{\frac{r+1}{2}}\right\|_{H^1}^2 + c\left\|v_i^{\frac{r}{2}}\right\|_{L^2}^2. \quad \square$$

Theorem 3.4. *There exists a constant $c > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R}_+,L^\infty(\Omega,\mathbb{R}^4))} \le c$$

for any regular subdivision Z_n of $\mathbb{R}_+$ and any solution u to (P_n).

Proof. i) Let u be a solution to $(\mathrm{P_n})$, $v := |u - u^*|$. Because of Theorem 3.3, imbedding and trace theorems and (2.11) it is easy to check that there exists an element $b \in L^4\left(\mathbb{R}_+,(W^{1,5/4}(\Omega))^*\right)$ such that

$$\langle b,\bar v\rangle := \int_0^\infty \sum_{j=0}^3 \left[\int_\Omega (v_j^2+v_j)\bar v\,\mathrm{d}x + \int_\Gamma (v_j^2+v_j)\bar v\,\mathrm{d}\Gamma\right]\mathrm{d}s\; \forall\, \bar v \in L^{4/3}\left(\mathbb{R}_+, W^{1,5/4}(\Omega)\right).$$

ii) Let $k \ge K$ with K defined in (3.5), (3.9). We test by $z := (u-k)^+$ and denote by m_{jk} the Lebesgue measure of the set $\{x \in \Omega : u_j > k\}$. Because of (3.8) and (3.10) we find $\forall\, t_n^l \in Z_n$

$$\begin{aligned}\sum_{i=0}^3 \left[\left\|z_i(t_n^l)\right\|_{L^2}^2 + 2\,d\int_0^{t_n^l}\|z_i\|_{H^1}^2\,\mathrm{d}s\right]&\\ &\le c\int_0^{t_n^l}\sum_{i=0}^3 \|b\|_{(W^{1,5/4})^*}\|z_i\|_{W^{1,5/4}}\,\mathrm{d}s\\ &\le c\int_0^{t_n^l}\sum_{i=0}^3 \|b\|_{(W^{1,5/4})^*}\|z_i\|_{H^1}\,m_{ik}^{3/10}\mathrm{d}s\\ &\le \int_0^{t_n^l}\sum_{i=0}^3 \left(d\,\|z_i\|_{H^1}^2 + c\,\|b\|_{(W^{1,5/4})^*}^2\,m_{ik}^{3/5}\right)\mathrm{d}s.\end{aligned}$$

Thus we obtain the inequality

$$
\begin{aligned}
\sum_{i=0}^{3} \left[\|z_i(t)\|_{L^2}^2 + d\int_0^t \|z_i\|_{H^1}^2 \,\mathrm{d}s \right] & \\
&\leq c\sum_{i=0}^{3} \|b\|^2_{L^4\left(\mathbb{R}_+,(W^{1,5/4})^*\right)} \|m_{ik}\|^{3/5}_{L^{6/5}(\mathbb{R}_+)} \qquad (3.11)\\
&\leq c\sum_{i=0}^{3} \|m_{ik}\|^{3/5}_{L^{6/5}(\mathbb{R}_+)} \quad \forall t\in\mathbb{R}_+.
\end{aligned}
$$

iii) Let $\phi(k) := \left(\int_0^\infty \sum_{i=0}^3 m_{ik}^{6/5}\mathrm{d}s\right)^{5/22}$. Then we obtain for $h>k$ by the Gagliardo-Nirenberg inequality (2.9) and (3.11) the estimate

$$
\begin{aligned}
(h-k)\phi(h) &= \left(\int_0^\infty \sum_{i=0}^{3} \left[(h-k)^{11/3} m_{ih}\right]^{6/5} \mathrm{d}s\right)^{5/22} \leq \left(\int_0^\infty \sum_{i=0}^{3} \|z_i\|^{22/5}_{L^{11/3}} \,\mathrm{d}s\right)^{5/22}\\
&\leq \left(c\int_0^\infty \sum_{i=0}^{3} \|z_i\|^{12/5}_{L^2} \|z_i\|^2_{H^1}\,\mathrm{d}s\right)^{5/22}\\
&\leq c\left(\sum_{i=0}^{3} \|z_i\|^{12/5}_{L^\infty(\mathbb{R}_+,L^2)} \|z_i\|^2_{L^2(\mathbb{R}_+,H^1)}\right)^{5/22}\\
&\leq c\left(\sum_{i=0}^{3} \|m_{ik}\|^{3/5}_{L^{6/5}(\mathbb{R}_+)}\right)^{(5/22)\cdot(11/5)} = c\left(\sum_{i=0}^{3}\int_0^\infty m_{ik}^{6/5}\mathrm{d}s\right)^{(5/22)\cdot(11/10)}\\
&\leq c\phi(k)^{11/10}.
\end{aligned}
$$

By [6, Lemma 5] we conclude that there is a $\tilde{k}$ such that $\phi(k)=0$ for all $k\geq\tilde{k}$. Therefore by the definition of ϕ it follows $u\in L^\infty\left(\mathbb{R}_+, L^\infty(\Omega,\mathbb{R}^4)\right)$. □

Remark 3.1. Theorem 3.4 implies that $\|u|_\Gamma\|_{L^\infty(\mathbb{R}_+,L^\infty(\Gamma,\mathbb{R}^4))}\leq c$.

3.4. Existence.

Let Z_n be any regular subdivision of $\mathbb{R}_+$. To show the solvability of ($\mathrm{P_n}$) it suffices to prove existence on any finite interval $S := \left[0, t_n^k\right]$, $k\in\mathbb{N}$. That means, we have to solve the problem

$$u^l + h_n^l \mathcal{A}\left((\gamma_n u)^l, u^l, u^l\right) = u^{l-1},\ u^0 = U,\ u^l\in X,\ u^l\geq 0,\ l=1,\dots,k.$$

We consider the corresponding regularized problem

$$u^l + h_n^l \mathcal{A}_M\left((\gamma_n u)^l, u^l, u^l\right) = u^{l-1},\ u^0 = U,\ u^l\in X,\ l=1,\dots,k. \qquad (\mathrm{P_{Mn}})$$

Lemma 3.5. *If u is a solution to (P_{Mn}) then $u\geq 0$ and*

$$\|u\|_{L^\infty(S,L^\infty(\Omega,\mathbb{R}^4))},\ \|u|_\Gamma\|_{L^\infty(S,L^\infty(\Gamma,\mathbb{R}^4))}\leq c(S)$$

where c is independent of M.

Proof. i) By the testfunction $-u^- \in L^2(S,X)$ it follows from $(\mathrm{P_{Mn}})$ that for $l = 1,\dots,k$

$$\sum_{m=1}^{l} \left(\left\|(u^m)^-\right\|_{L^2}^2 + \int_\Omega (u^m)^- u^{m-1}\mathrm{d}x + h_n^m\, d \left\|\nabla (u^m)^-\right\|_{L^2}^2 \right) \le 0.$$

For $l=1$ we obtain from $u^0 \ge 0$ that $u^1 \ge 0$ which implies (for $l=2$) that $u^2 \ge 0$ and so on.
ii) Applying the procedure of the proof of Theorem 3.1 to $(\mathrm{P_{Mn}})$ we get

$$\|u\|_{L^\infty(S,L^1(\Omega,\mathbb{R}^4))} \le c,\ \|\nabla\sqrt{u_i}\|_{L^2(S,L^2(\Omega,\mathbb{R}^2))} \le c,\ i=0,\dots,3,$$

independently of M. Therefore, because S is a finite time interval we obtain

$$\|u-u^*\|_{L^1(S,L^1(\Omega,\mathbb{R}^4))},\ \left\|\sqrt{u}\right\|_{L^2(S,X)},\ \left\|\sqrt{u}\right\|_{L^4(S,L^4(\Omega,\mathbb{R}^4))},\ \|u-u^*\|_{L^2(S,Y)} \le c(S),$$

$$\int_S \left\|\sqrt{u}-\sqrt{u^*}\right\|_{L^2}^p \mathrm{d}s \le c(p,S) \quad \text{for } 0<p<\infty,$$

$$\left\|\sqrt{u}-\sqrt{u^*}\right\|_{L^2(S,X)},\ \left\|\sqrt{u}-\sqrt{u^*}\right\|_{L^\infty(S,Y)} \le c(S).$$

These estimates correspond to the assertions of Corollary 3.2 formulated on the finite time interval S, they guarantee that Theorem 3.3 and Theorem 3.4 are true also for solutions to $(\mathrm{P_{Mn}})$. □

Lemma 3.6. *There exist solutions to* (P_{Mn}).

Proof. To demonstrate the existence of solutions to $(\mathrm{P_{Mn}})$, we show: For given $u^{l-1} \in X$, $h_n^l \in \mathbb{R}_+$ there exist solutions $u^l \in X$ of

$$u^l + h_n^l \mathcal{A}_M\left((\gamma_n u)^l, u^l, u^l\right) = u^{l-1} \tag{3.12}$$

such that solutions to $(\mathrm{P_{Mn}})$ can be obtained step by step. For fixed u^{l-1}, h_n^l let $B: X \to X^*$ be defined by

$$B(u^l) := u^l + h_n^l \mathcal{A}_M\left((\gamma_n u)^l, u^l, u^l\right).$$

Then B is an operator of variational type (cf. [13], we use for (FI) the decomposition $B(v) = B(v,v)$ with

$$\begin{aligned}\langle B(v,w),z\rangle := \int_\Omega \Big\{ \sum_{i=0}^{3} w_i z_i + h_n^l \Big[D_0\varphi(v_0)\nabla w_0 \nabla z_0 + \sum_{i=1}^{3} D_i \nabla w_i \nabla z_i \\ + \rho(v)\left(R_1(v)(z_1+z_3-z_0) + R_2(v)(z_2+z_3)\right)\Big]\Big\}\mathrm{d}x \\ + h_n^l \int_\Gamma \rho(v)\Big(R_3(v)z_0 + R_4(v)(z_0-z_3) + \sum_{i=1}^{3} R_{i+4}(v) z_i\Big)\mathrm{d}\Gamma\end{aligned}$$

for all $v,w,z \in X$. For (SI) the argument of φ has to be replaced by the given $(u^{l-1})_0$.) By [13, Corollaire 2.1] the problem (3.12) has at least one solution. □

Thus the L^∞–estimates for the solutions to (P_n) and $(\mathrm{P_{Mn}})$ lead to the following existence result:

Theorem 3.5. *For every regular subdivision Z_n of $\mathbb{R}_+$ there exists at least one solution to (P_n).*

4. Convergence theorem

Here we state and prove our main result.

Theorem 4.1. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of regular subdivisions of $\mathbb{R}_+$ with $\bar{h}_n \to 0$ for $n \to \infty$. Let, for $n \in \mathbb{N}$, the function u_n be a solution to (P_n) and let u be the solution to (P). If $S := [0,T]$ is any finite time interval, then*

i) $u_n \longrightarrow u$ *in* $L^2(S,X)$ *and* $u_n \longrightarrow u$ *in* $L^\infty(S,Y)$ *as* $n \longrightarrow \infty$.

ii) $K_n u_n \longrightarrow u$ *in* W_S *and* $K_n u_n \longrightarrow u$ *in* $C(S,Y)$ *as* $n \longrightarrow \infty$.

Proof. Because T is not nessesarily a nodal point of the subdivisions Z_n we apply the following construction: For every $n \in \mathbb{N}$ let k_n denote that index k such that $t_n^{k-1} \le T < t_n^k$, let

$$\tilde{T} := \max_{n\in\mathbb{N}} t_n^{k_n},\ \tilde{S} := [0,\tilde{T}],\ \tilde{Z}_n := \left\{t_n^j : t_n^j \le \tilde{T}\right\} \cup \{\tilde{T}\}.$$

1. From the definition of Δ_n and K_n it follows easily

$$\int_0^t \langle \Delta_n v - \Delta_n w, v - w - K_n v + K_n w\rangle \mathrm{d}s \ge 0 \qquad\qquad (4.1)$$
$$\forall t \in [0, t_n^{k_n}],\ \forall v, w \in C_n([0,t_n^{k_n}], X).$$

2. By Lemma 3.2 we find for the solution u to (P) a sequence $\{w_n\}_{n\in\mathbb{N}} \subset C_n(\tilde{S}, X)$, $K_n w_n(0) = U$ with

$$\lim_{n\to\infty}\left\{\|w_n - u\|_{L^2(\tilde{S},X)} + \left\|\tilde{K}_n w_n - u\right\|_{C(\tilde{S},Y)} + \left\|\tilde{\Delta}_n w_n - u'\right\|_{L^2(\tilde{S},X^*)}\right\} = 0$$

where $\tilde{K}_n$, $\tilde{\Delta}_n$ are the operators corresponding to the subdivision $\tilde{Z}_n$. For $s \in [0, t_n^{k_n}]$, $w_n \in C_n(\tilde{S}, X)$ we have

$$\tilde{\Delta}_n w_n(s) = \Delta_n w_n(s),\ \tilde{K}_n w_n(s) = K_n w_n(s).$$

Using (4.1) we obtain for every $t \in [0, t_n^{k_n}]$

$$\begin{aligned}
&\frac{1}{2}\|(K_n u_n - K_n w_n)(t)\|_Y^2 + \int_0^t \langle A_n(u_n) - A(u), u_n - u\rangle \mathrm{d}s \\
&\le \int_0^t \Big\{\langle \Delta_n u_n - \Delta_n w_n, u_n - w_n\rangle + \langle A_n(u_n) - A(u), u_n - w_n + w_n - u\rangle\Big\}\mathrm{d}s \\
&= \int_0^t \Big\{\langle u' - \Delta_n w_n, u_n - w_n\rangle + \langle A_n(u_n) - A(u), w_n - u\rangle\Big\}\mathrm{d}s \qquad (4.2) \\
&\le c\Big(\left\|u' - \tilde{\Delta}_n w_n\right\|_{L^2(\tilde{S},X^*)} + \|w_n - u\|_{L^2(\tilde{S},X)}\Big)
\end{aligned}$$

since $A_n(u_n)$ is uniformly bounded in $L^2\left(\tilde{S}, X^*\right)$, u_n and w_n are uniformly bounded in $L^2\left(\tilde{S}, X\right)$.

3. Because of the L^∞– estimates for the solutions u_n and u and the local Lipschitz continuity of φ we get with that p occuring in the $L^q(\tilde{S}, W^{1,p})$–estimates for u (cf. Theorem 2.4) and $q = 2p/(p-2)$

$$\begin{aligned}
&\int_0^{t_n^k} \langle A_n(u_n) - A(u), u_n - u\rangle \mathrm{d}s \\
&\quad \geq \int_0^{t_n^k} \Big\{ d\,\|u_n - u\|_X^2 - c\left(\|u_n - u\|_Y^2 + \|u_n - u\|_Z^2\right) \\
&\qquad\qquad - c\int_\Omega |\varphi(\gamma_{n0}u_n) - \varphi(u_0)||\nabla u_0||\nabla(u_{n0} - u_0)|\mathrm{d}x\Big\}\mathrm{d}s \\
&\quad \geq \int_0^{t_n^k} \Big\{ d\,\|u_n - u\|_X^2 - c\left(\|u_n - u\|_Y^2 + \|u_n - u\|_Z^2\right) \\
&\qquad\qquad - c\,\|\gamma_{n0}u_n - u_0\|_{L^q}\,\|\nabla u_0\|_{L^p}\,\|\nabla(u_{n0} - u_0)\|_{L^2}\Big\}\mathrm{d}s \\
&\quad \geq \int_0^{t_n^k} \left\{ \tfrac{3d}{4}\,\|u_n - u\|_X^2 - c\,\|u_n - u\|_Y^2 - c\,\|\gamma_{n0}u_n - u_0\|_{L^q}^2\,\|\nabla u_0\|_{L^p}^2 \right\}\mathrm{d}s.
\end{aligned}$$

4. Taking into account the L^∞-bounds for u and u_n the term $\|\gamma_{n0}u_n - u_0\|_{L^q}^2$ will be estimated by

$$\|\gamma_{n0}u_n - u_0\|_{L^q}^2 \leq c\Big\{ \|u_{n0} - (\tau_n u_n)_0\|_{L^2}^{\frac{4}{q}} + \|(\tau_n u_n)_0 - (\tau_n u)_0\|_{L^q}^2 + \|(\tau_n u)_0 - u_0\|_{L^2}^{\frac{4}{q}} \Big\}.$$

Since $A_n(u_n)$ is uniformly bounded in $L^2\left(\tilde{S}, X^*\right)$, u_n and $\tau_n u_n$ are uniformly bounded in $L^2\left(\tilde{S}, X\right)$ we conclude by $(\mathrm{P_n})$ that

$$\begin{aligned}
\|u_n - \tau_n u_n\|_{L^2([0,t_n^{k_n}],Y)}^2 &= \sum_{l=1}^{k_n} h_n^l \langle u_n^l - u_n^{l-1}, u_n^l - u_n^{l-1}\rangle \\
&\leq \bar{h}_n \int_0^{\tilde{T}} |\langle A_n(u_n), u_n - \tau_n u_n\rangle|\,\mathrm{d}s \leq c\,\bar{h}_n.
\end{aligned} \tag{4.3}$$

Furthermore, on S_n^k, $k = 1, \ldots, k_n$ we have

$$\begin{aligned}
\|(u_n - u)(t)\|_Y \leq \left\|u_n^k - w_n^k\right\|_Y + \left\|\tilde{K}_n w_n - u\right\|_{C(\tilde{S},Y)} \\
+ \sup_{r,s\in\tilde{S},\ |r-s|\leq\bar{h}_n} \|u(s) - u(r)\|_Y .
\end{aligned} \tag{4.4}$$

5. Using the inequalities of step 4, $L^r\left(\tilde{S}, L^p\right)$–estimates for ∇u_0, $r \in [1, \infty)$ and

the boundedness of $\{H_n^l\}_{l,n\in\mathbb{N}}$ we obtain for $k=1,\dots,k_n$

$$\begin{aligned}
&\sum_{l=1}^{k}\int_{t_n^{l-1}}^{t_n^l}\|\gamma_{n0}u_n-u_0\|_{L^q}^2\,\|\nabla u_0\|_{L^p}^2\,\mathrm{d}s\\
&\le c\Bigg\{\int_0^{t_n^k}\left(\|(u_{n0}-(\tau_n u_n)_0)(s)\|_{L^2}^{\frac{4}{q}}+\|(u_0-(\tau_n u)_0)(s)\|_{L^2}^{\frac{4}{q}}\right)\|\nabla u_0\|_{L^p}^2\,\mathrm{d}s\\
&\quad+\sum_{l=2}^{k}\int_{t_n^{l-1}}^{t_n^l}\left\|u_{n0}^{l-1}-(\tau_n u)_0(s)\right\|_{L^q}^2\|\nabla u_0\|_{L^p}^2\,\mathrm{d}s\Bigg\}\\
&\le c\Bigg\{\left(\bar h_n^{\frac{2}{q}}+\|u-\tau_n u\|_{L^2(\tilde S,Y)}^{\frac{4}{q}}\right)\|\nabla u_0\|_{L^p(\tilde S,L^p)}^2\\
&\quad+\sum_{l=2}^{k}H_n^l\int_{t_n^{l-2}}^{t_n^{l-1}}\left\|u_{n0}^{l-1}-u_0(\tilde t)\right\|_{L^2}^{\frac{4}{q}}\left\|u_{n0}^{l-1}-u_0(\tilde t)\right\|_{H^1}^{\frac{2(q-2)}{q}}\left\|\nabla u_0\left(\vartheta_n^{-1}(\tilde t)\right)\right\|_{L^p}^2\,\mathrm{d}\tilde t\Bigg\}\\
&\le c\left\{\bar h_n^{\frac{2}{q}}+\|u-\tau_n u\|_{L^2(\tilde S,Y)}^{\frac{4}{q}}\right\}\\
&\quad+\sum_{l=1}^{k-1}\int_{t_n^{l-1}}^{t_n^l}\left(\varepsilon\left\|u_{n0}^l-u_0(\tilde t)\right\|_{H^1}^2+c\,H_n^{l+1}\left\|u_{n0}^l-u_0(\tilde t)\right\|_{L^2}^2\left\|\nabla u_0\left(\vartheta_n^{-1}(\tilde t)\right)\right\|_{L^p}^q\right)\mathrm{d}\tilde t\\
&\le c\left\{\bar h_n^{\frac{2}{q}}+\|u-\tau_n u\|_{L^2(\tilde S,Y)}^{\frac{4}{q}}\right\}\\
&\quad+\sum_{l=1}^{k-1}\int_{t_n^{l-1}}^{t_n^l}\Bigg(\varepsilon\left\|u_{n0}^l-u_0(\tilde t)\right\|_{H^1}^2+c\,H_n^{l+1}\Bigg(\sup_{r,s\in\tilde S,\ |r-s|\le\bar h_n}\|u(s)-u(r)\|_Y^2\\
&\qquad\qquad+\left\|u_n^l-w_n^l\right\|_Y^2+\left\|\tilde K_n w_n-u\right\|_{C(\tilde S,Y)}^2\Bigg)\left\|\nabla u_0\left(\vartheta_n^{-1}(\tilde t)\right)\right\|_{L^p}^q\Bigg)\mathrm{d}\tilde t\\
&\le c\Bigg\{\sup_{r,s\in\tilde S,\ |r-s|\le\bar h_n}\|u(s)-u(r)\|_Y^2\\
&\qquad\qquad+\bar h_n^{\frac{2}{q}}+\|u-\tau_n u\|_{L^2(\tilde S,Y)}^{\frac{4}{q}}+\left\|\tilde K_n w_n-u\right\|_{C(\tilde S,Y)}^2\Bigg\}\\
&\quad+\sum_{l=1}^{k-1}\left(\int_{t_n^{l-1}}^{t_n^l}\varepsilon\left\|u_{n0}^l-u_0(\tilde t)\right\|_{H^1}^2+c\,H_n^{l+1}\left\|u_n^l-w_n^l\right\|_Y^2\left\|\nabla u_0\left(\vartheta_n^{-1}(\tilde t)\right)\right\|_{L^p}^q\right)\mathrm{d}\tilde t.
\end{aligned}$$

6. Combining estimate (4.2) and steps 3 and 5 we get for $k=1,\dots,k_n$

$$\begin{aligned}
\left\|u_n^k-w_n^k\right\|_Y^2+\int_0^{t_n^k}\|u_n-u\|_X^2\,\mathrm{d}s\le&\,\hat c\sum_{l=1}^{k-1}\left\|u_n^l-w_n^l\right\|_Y^2\int_{t_n^l}^{t_n^{l+1}}\|\nabla u_0(s)\|_{L^p}^q\,\mathrm{d}s\\
&+\hat c\sum_{l=1}^{k}h_n^l\left\|u_n^l-w_n^l\right\|_Y^2+\tilde c(n),
\end{aligned}$$

where

$$\tilde{c}(n) := c\Bigg\{ \|w_n - u\|_{L^2(\tilde{S},X)} + \left\|u - \tilde{K}_n w_n\right\|^2_{C(\tilde{S},Y)} + \left\|u' - \tilde{\Delta}_n w_n\right\|_{L^2(\tilde{S},X^*)}$$

$$+ \bar{h}_n^{\frac{2}{q}} + \|u - \tau_n u\|^{\frac{4}{q}}_{L^2(\tilde{S},Y)} + \sup_{r,s\in\tilde{S},\ |r-s|\le\bar{h}_n} \|u(s) - u(r)\|^2_Y \Bigg\}.$$

With

$$a_n^k := \left\|u_n^k - w_n^k\right\|^2_Y + \int_0^{t_n^k} \|u_n - u\|^2_X \,\mathrm{d}s$$

we obtain for $k = 1, \ldots, k_n$

$$a_n^k \le \hat{c}\sum_{l=1}^{k} a_n^l h_n^l + \hat{c}\sum_{l=1}^{k-1} a_n^l \|\nabla u_0\|^q_{L^q(S_n^{l+1},L^p)} + \tilde{c}(n).$$

For sufficiently large n we have $\hat{c}\bar{h}_n < \frac{1}{2}$ and therefore for $k = 1, \ldots, k_n$

$$a_n^k \le 2\hat{c}\sum_{l=1}^{k-1} \left(h_n^l + \|\nabla u_0\|^q_{L^q(S_n^{l+1},L^p)}\right) a_n^l + 2\tilde{c}(n).$$

By Lemma 3.3 we conclude that

$$\begin{aligned} a_n^k &\le 2\tilde{c}(n) + 4\tilde{c}(n)\hat{c}\left(\tilde{T} + \|\nabla u_0\|^q_{L^q(\tilde{S},L^p)}\right) \exp\left\{2\hat{c}\left(\tilde{T} + \|\nabla u_0\|^q_{L^q(\tilde{S},L^p)}\right)\right\} \\ &\le \tilde{c}(n)\ \mathrm{const}(\tilde{T}) \end{aligned}$$

for $k = 1, \ldots, k_n$ and sufficiently large n. From Lemma 3.2, Lemma 3.1 ii) and $u \in C(\tilde{S}, Y)$ we obtain that $\tilde{c}(n) \to 0$ for $n \to \infty$. By the definition of a_n^k and by $S \subset [0, t_n^{k_n}]$, $n \in \mathbb{N}$ it follows $u_n \to u$ in $L^2(S,X)$ and $u_n - w_n \to 0$ in $L^\infty(S,Y)$. Because of (4.4) we find $u_n \to u$ in $L^\infty(S,Y)$ which finally proves i).

7. Let

$$I_n := \left(\int_0^T \int_\Omega |\varphi(u_0) - \varphi(u_{n0})|^2 \,|\nabla u_0|^2 \,\mathrm{d}x\mathrm{d}s\right)^{1/2}.$$

Because of the L^∞–estimates for u and u_n we estimate for $\bar{u} := u_n - u$ and $v \in L^2(S,X)$

$$\begin{aligned} &\left|\int_0^T \langle A_n(u_n) - A(u), v\rangle \,\mathrm{d}s\right| \\ &\le c\int_0^T \left\{\sum_{i=0}^{3} \|\nabla \bar{u}_i\|_{L^2} \|\nabla v_i\|_{L^2} + \|\bar{u}\|_Y \|v\|_Y + \|\bar{u}\|_Z \|v\|_Z\right\}\mathrm{d}s + I_n \|v\|_{L^2(S,X)} \\ &\le c\|v\|_{L^2(S,X)} \left\{\|\bar{u}\|_{L^2(S,X)} + I_n\right\}. \end{aligned}$$

From $u_n \to u$ in $L^2(S,Y)$ and the properties of superposition operators it follows $I_n \to 0$ for $n \to \infty$. Therefore for all $\varepsilon > 0$ there is a $n_0(\varepsilon)$ such that $\forall\, n \ge n_0$

$$\|\bar{u}\|_{L^2(S,X)} + I_n \le \varepsilon, \quad \left|\int_0^T \langle A_n(u_n) - A(u), v\rangle \,\mathrm{d}s\right| \le c\varepsilon \|v\|_{L^2(S,X)}\ \forall\, v \in L^2(S,X).$$

Thus $A_n(u_n) \to A(u)$ in $L^2(S,X^*)$ which proves $\Delta_n u_n = (K_n u_n)' \to u'$ in $L^2(S,X^*)$.

8. By (4.3) we find

$$\int_0^T \|K_n u_n - u_n\|_X^2 \,\mathrm{d}s \le \sum_{l=1}^{k_n} \left(h_n^l\right)^{-2} \int_{t_n^{l-1}}^{t_n^l} (t-t_n^l)^2 \left\|u_n^l - u_n^{l-1}\right\|_X^2 \mathrm{d}t$$
$$\le \frac{1}{3} \|u_n - \tau_n u_n\|^2_{L^2(\tilde{S},X)} \quad \longrightarrow 0 \text{ for } n \longrightarrow \infty$$

which together with Theorem 4.1 i) yields $K_n u_n \to u$ in $L^2(S,X)$. Combined with the result of step 7 we conclude that $K_n u_n \to u$ in W_S. Thus $K_n u_n \to u$ in $C(S,Y)$, too. $\square$

Appendix A. Equilibrium states

The assumption (2.7) is essential for deriving energetic estimates as well as results on the asymptotic behaviour of solutions to the initial boundary value problem and its time discrete versions. The properties of the set

$$\mathcal{R} := \left\{u \in \mathbb{R}_+^4\colon R_i(u) = 0 \text{ a.e.}, i = 1,\dots,7,\ u - \bar{U} \in \mathcal{S}\right\}$$

depend on the data k_i, $\bar{k}_i$, $i = 1,\dots,7$ and $\bar{U}$ such that necessary and sufficient conditions for the validity of (2.7) should be expressed in terms of this data. This could be done as in [5], [7], [10], [11]. Here we emphasize only some important cases; a complete survey is given in Table 1–4.

Remember, that we have assumed $k_i > 0$, $\bar{k}_i \ge 0$, $i = 1,\dots,7$ and $\bar{U} \ge 0$ (see (2.6), (2.4)).

1) $\bar{k}_i = 0$, $i = 1,\dots,7$ (neither volume nor boundary reactions, No. 1 in Table 1): Then $\mathcal{S} = \{0\}, \mathcal{S}^\perp = \mathbb{R}^4$ and $\mathcal{R} = \left\{\bar{U}\right\}$. Assumption (2.7) now means $\bar{U}_i > 0$, $i = 1,\dots,4$, and no conditions on the k_i's.

2) $\bar{k}_i > 0$, $i = 1,\dots,7$ (all volume and boundary reactions, No. 128 in Table 4): We obtain $\mathcal{S} = \mathbb{R}^4$, thus (2.7) does not contain any condition on $\bar{U}$. The relations

$$k_3 = k_4 k_7,\ k_2 = k_6 k_7,\ k_5 = k_1 k_4 \tag{A.1}$$

ensure that $\mathcal{R} \ne \emptyset$. From $R_i(u) = 0$, $i = 3,5,6,7$, and the strong monotonicity of g we get $u_0 = g^{-1}(k_3) > 0$, $u_i = k_{i+4} > 0$, $i = 1,2,3$. Because of (A.1) $R_i(u) = 0$, $i = 1,2,4$, too. Thus $\mathcal{R}$ consists of only one element, it is positive and (2.7) is equivalent to (A.1).

3) $\bar{k}_1$, $\bar{k}_2 > 0$, $\bar{k}_i = 0$, $i = 3,\dots,7$ (only volume reactions, No. 9 in Table 1): Because of $\mathcal{S} = \operatorname{span}\{(-1,1,0,1), (0,0,1,1)\}$ we can choose $\kappa_1 := (1,1,0,0)$, $\kappa_2 := (1,0,-1,1)$ to describe the invariants $I_j(u)$, $j = 1,2$, to problem (P). For $u \in \mathcal{R}$ we have

$$u_1 u_3 = k_1 g(u_0), \tag{A.2}$$
$$u_2 u_3 = k_2, \tag{A.3}$$
$$u_0 + u_1 = \bar{U}_0 + \bar{U}_1, \tag{A.4}$$
$$u_0 - u_2 + u_3 = \bar{U}_0 - \bar{U}_2 + \bar{U}_3. \tag{A.5}$$

By (A.3) $u_2,\ u_3 > 0$. By (A.4) $\bar{U}_0 + \bar{U}_1 > 0$ is a necessary condition for $u > 0$. Let $\bar{U}_0 + \bar{U}_1 > 0$ then at least one of u_0 or u_1 is positive.
a) $u_1 > 0$: Replacing $u_2,\ u_3$ in (A.5) by

$$u_2 = \frac{k_2}{u_3},\ u_3 = \frac{k_1 g(u_0)}{\bar{U}_0 + \bar{U}_1 - u_0}$$

(cf. (A.2), (A.4)) we get

$$u_0 - \frac{k_2(\bar{U}_0 + \bar{U}_1 - u_0)}{k_1 g(u_0)} + \frac{k_1 g(u_0)}{\bar{U}_0 + \bar{U}_1 - u_0} = \bar{U}_0 - \bar{U}_2 + \bar{U}_3. \tag{A.6}$$

No.	$\bar{k}_1$	$\bar{k}_2$	$\bar{k}_3$	$\bar{k}_4$	$\bar{k}_5$	$\bar{k}_6$	$\bar{k}_7$	Conditions on k_i	Conditions on $\bar{U}_i$
1									$\bar{U}_0,\ \bar{U}_1,\ \bar{U}_2,\ \bar{U}_3 > 0$
2	*								$\bar{U}_2,\ \bar{U}_0 + \bar{U}_1,\ \bar{U}_0 + \bar{U}_3 > 0$
3		*							$\bar{U}_0,\ \bar{U}_1 > 0$
4			*						$\bar{U}_1,\ \bar{U}_2,\ \bar{U}_3 > 0$
5				*					$\bar{U}_1,\ \bar{U}_2,\ \bar{U}_0 + \bar{U}_3 > 0$
6					*				$\bar{U}_0,\ \bar{U}_2,\ \bar{U}_3 > 0$
7						*			$\bar{U}_0,\ \bar{U}_1,\ \bar{U}_3 > 0$
8							*		$\bar{U}_0,\ \bar{U}_1,\ \bar{U}_2 > 0$
9	*	*							$\bar{U}_0 + \bar{U}_1 > 0$
10	*		*						$\bar{U}_2 > 0$
11	*			*					$\bar{U}_2,\ \bar{U}_0 + \bar{U}_3 > 0$
12	*				*				$\bar{U}_2,\ \bar{U}_0 + \bar{U}_3 > 0$
13	*					*			$\bar{U}_0 + \bar{U}_1,\ \bar{U}_0 + \bar{U}_3 > 0$
14	*						*		$\bar{U}_2,\ \bar{U}_0 + \bar{U}_1 > 0$
15		*	*						$\bar{U}_1 > 0$
16		*		*					$\bar{U}_1 > 0$
17		*			*				$\bar{U}_0 > 0$
18		*				*			$\bar{U}_0,\ \bar{U}_1 > 0$
19		*					*		$\bar{U}_0,\ \bar{U}_1 > 0$
20			*	*					$\bar{U}_1,\ \bar{U}_2 > 0$
21			*		*				$\bar{U}_2,\ \bar{U}_3 > 0$
22			*			*			$\bar{U}_1,\ \bar{U}_3 > 0$
23			*				*		$\bar{U}_1,\ \bar{U}_2 > 0$
24				*	*				$\bar{U}_2,\ \bar{U}_0 + \bar{U}_3 > 0$
25				*		*			$\bar{U}_1,\ \bar{U}_0 + \bar{U}_3 > 0$
26				*			*		$\bar{U}_1,\ \bar{U}_2 > 0$
27					*	*			$\bar{U}_0,\ \bar{U}_3 > 0$
28					*		*		$\bar{U}_0,\ \bar{U}_2 > 0$
29						*	*		$\bar{U}_0,\ \bar{U}_1 > 0$

Table 1. Necessary and sufficient conditions for the validity of assumption (2.7) if no, one or two reactions (indicated by $*$) are present.

No.	$\bar{k}_1$	$\bar{k}_2$	$\bar{k}_3$	$\bar{k}_4$	$\bar{k}_5$	$\bar{k}_6$	$\bar{k}_7$	Conditions on k_i	Conditions on $\bar{U}_i$
30	*	*	*						
31	*	*		*					
32	*	*			*				$\bar{U}_0 + \bar{U}_1 > 0$
33	*	*				*			
34	*	*					*		$\bar{U}_0 + \bar{U}_1 > 0$
35	*		*	*					$\bar{U}_2 > 0$
36	*		*		*				$\bar{U}_2 > 0$
37	*		*			*			
38	*		*				*		$\bar{U}_2 > 0$
39	*			*	*			$k_5 = k_1 k_4$	$\bar{U}_2,\ \bar{U}_0 + \bar{U}_3 > 0$
40	*			*		*			$\bar{U}_0 + \bar{U}_3 > 0$
41	*			*			*		$\bar{U}_2 > 0$
42	*				*	*			$\bar{U}_0 + \bar{U}_3 > 0$
43	*				*		*		$\bar{U}_2 > 0$
44	*					*	*		$\bar{U}_0 + \bar{U}_1 > 0$
45		*	*	*					$\bar{U}_1 > 0$
46		*	*		*				
47		*	*			*			$\bar{U}_1 > 0$
48		*	*				*		$\bar{U}_1 > 0$
49		*		*	*				
50		*		*		*			$\bar{U}_1 > 0$
51		*		*			*		$\bar{U}_1 > 0$
52		*			*	*			$\bar{U}_0 > 0$
53		*			*		*		$\bar{U}_0 > 0$
54		*				*	*	$k_2 = k_6 k_7$	$\bar{U}_0,\ \bar{U}_1 > 0$
55			*	*	*				$\bar{U}_2 > 0$
56			*	*		*			$\bar{U}_1 > 0$
57			*	*			*	$k_3 = k_4 k_7$	$\bar{U}_1,\ \bar{U}_2 > 0$
58			*		*	*			$\bar{U}_3 > 0$
59			*		*		*		$\bar{U}_2 > 0$
60			*			*	*		$\bar{U}_1 > 0$
61				*	*	*			$\bar{U}_0 + \bar{U}_3 > 0$
62				*	*		*		$\bar{U}_2 > 0$
63				*		*	*		$\bar{U}_1 > 0$
64					*	*	*		$\bar{U}_0 > 0$

Table 2. Necessary and sufficient conditions for the validity of assumption (2.7) if three reactions (indicated by $*$) are present.

No.	$\bar{k}_1$	$\bar{k}_2$	$\bar{k}_3$	$\bar{k}_4$	$\bar{k}_5$	$\bar{k}_6$	$\bar{k}_7$	Conditions on k_i	Conditions on $\bar{U}_i$
65	$*$	$*$	$*$	$*$					
66	$*$	$*$	$*$		$*$				
67	$*$	$*$	$*$			$*$			
68	$*$	$*$	$*$				$*$		
69	$*$	$*$		$*$	$*$			$k_5 = k_1 k_4$	
70	$*$	$*$		$*$		$*$			
71	$*$	$*$		$*$			$*$		
72	$*$	$*$			$*$	$*$			
73	$*$	$*$			$*$		$*$		
74	$*$	$*$				$*$	$*$	$k_2 = k_6 k_7$	$\bar{U}_0 + \bar{U}_1 > 0$
75	$*$		$*$	$*$	$*$			$k_5 = k_1 k_4$	$\bar{U}_2 > 0$
76	$*$		$*$	$*$		$*$			
77	$*$		$*$	$*$			$*$	$k_3 = k_4 k_7$	$\bar{U}_2 > 0$
78	$*$		$*$		$*$	$*$			
79	$*$		$*$		$*$		$*$	$k_1 k_3 = k_5 k_7$	$\bar{U}_2 > 0$
80	$*$		$*$			$*$	$*$		
81	$*$			$*$	$*$	$*$		$k_5 = k_1 k_4$	$\bar{U}_0 + \bar{U}_3 > 0$
82	$*$			$*$	$*$		$*$	$k_5 = k_1 k_4$	$\bar{U}_2 > 0$
83	$*$			$*$		$*$	$*$		
84	$*$				$*$	$*$	$*$		
85		$*$	$*$	$*$	$*$				
86		$*$	$*$	$*$		$*$		$k_2 k_4 = k_3 k_6$	$\bar{U}_1 > 0$
87		$*$	$*$	$*$			$*$	$k_3 = k_4 k_7$	$\bar{U}_1 > 0$
88		$*$	$*$		$*$	$*$			
89		$*$	$*$		$*$		$*$		
90		$*$	$*$			$*$	$*$	$k_2 = k_6 k_7$	$\bar{U}_1 > 0$
91		$*$		$*$	$*$	$*$			
92		$*$		$*$	$*$		$*$		
93		$*$		$*$		$*$	$*$	$k_2 = k_6 k_7$	$\bar{U}_1 > 0$
94		$*$			$*$	$*$	$*$	$k_2 = k_6 k_7$	$\bar{U}_0 > 0$
95			$*$	$*$	$*$	$*$			
96			$*$	$*$	$*$		$*$	$k_3 = k_4 k_7$	$\bar{U}_2 > 0$
97			$*$	$*$		$*$	$*$	$k_3 = k_4 k_7$	$\bar{U}_1 > 0$
98			$*$		$*$	$*$	$*$		
99				$*$	$*$	$*$	$*$		

Table 3. Necessary and sufficient conditions for the validity of assumption (2.7) if four reactions (indicated by $*$) are present.

No.	$\bar{k}_1$	$\bar{k}_2$	$\bar{k}_3$	$\bar{k}_4$	$\bar{k}_5$	$\bar{k}_6$	$\bar{k}_7$	Conditions on k_i	Conditions on $\bar{U}_i$
100	*	*	*	*	*			$k_5 = k_1 k_4$	
101	*	*	*	*		*		$k_2 k_4 = k_3 k_6$	
102	*	*	*	*			*	$k_3 = k_4 k_7$	
103	*	*	*		*	*		$k_2 k_5 = k_1 k_3 k_6$	
104	*	*	*		*		*	$k_1 k_3 = k_5 k_7$	
105	*	*	*			*	*	$k_2 = k_6 k_7$	
106	*	*		*	*	*		$k_5 = k_1 k_4$	
107	*	*		*	*		*	$k_5 = k_1 k_4$	
108	*	*		*		*	*	$k_2 = k_6 k_7$	
109	*	*			*	*	*	$k_2 = k_6 k_7$	
110	*		*	*	*	*		$k_5 = k_1 k_4$	
111	*		*	*	*		*	$k_3 = k_4 k_7, k_1 k_3 = k_5 k_7$	$\bar{U}_2 > 0$
112	*		*	*		*	*	$k_3 = k_4 k_7$	
113	*		*		*	*	*	$k_1 k_3 = k_5 k_7$	
114	*			*	*	*	*	$k_5 = k_1 k_4$	
115		*	*	*	*	*		$k_2 k_4 = k_3 k_6$	
116		*	*	*	*		*	$k_3 = k_4 k_7$	
117		*	*	*		*	*	$k_2 = k_6 k_7, k_3 = k_4 k_7$	$\bar{U}_1 > 0$
118		*	*		*	*	*	$k_2 = k_6 k_7$	
119		*		*	*	*	*	$k_2 = k_6 k_7$	
120			*	*	*	*	*	$k_3 = k_4 k_7$	
121	*	*	*	*	*	*		$k_5 = k_1 k_4, k_2 k_4 = k_3 k_6$	
122	*	*	*	*	*		*	$k_3 = k_4 k_7, k_5 = k_1 k_4$	
123	*	*	*	*		*	*	$k_3 = k_4 k_7, k_2 = k_6 k_7$	
124	*	*	*		*	*	*	$k_2 = k_6 k_7, k_1 k_3 = k_5 k_7$	
125	*	*		*	*	*	*	$k_2 = k_6 k_7, k_5 = k_1 k_4$	
126	*		*	*	*	*	*	$k_3 = k_4 k_7, k_1 k_3 = k_5 k_7$	
127		*	*	*	*	*	*	$k_3 = k_4 k_7, k_2 = k_6 k_7$	
128	*	*	*	*	*	*	*	$k_3 = k_4 k_7, k_2 = k_6 k_7, k_5 = k_1 k_4$	

Table 4. Necessary and sufficient conditions for the validity of assumption (2.7) if five, six or all reactions (indicated by $*$) are present.

Because $f\colon (0,r) \to (-\infty, +\infty)$ defined by

$$f(y) := y - c\frac{r-y}{g(y)} + d\frac{g(y)}{r-y}, \quad r > 0$$

is bijective, (A.6) has for every right hand side $\bar{U}_0 - \bar{U}_2 + \bar{U}_3$ exactly one solution u_0, and $u_0 > 0$. Then u_i, $i = 1, 2, 3$, are uniquely determined and positive.
b) $u_0 > 0$: Then $g(u_0) > 0$ and by (A.2) $u_1 > 0$, too. Now we can conclude like in case a).
Thus (2.7) is equivalent to $\bar{U}_0 + \bar{U}_1 > 0$.

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