Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Outwards pointing hysteresis operators and asymptotic behaviour of evolution equations

Olaf Klein[1,*] , Pavel Krejčí[2,**,***]

submitted: 20th June 2002

[1] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
E-mail: klein@wias-berlin.de

[2] Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25
11567 Praha 1
Czech Republic
E-mail: krejci@math.cas.cz

No. 748
Berlin 2002

2000 *Mathematics Subject Classification.* 34C55, 35B40, 74N30, 47J40, 35K60, 74K05.

Key words and phrases. hysteresis operators, Prandtl-Ishlinskii operator, asymptotic behaviour, visco-elasto-plasticity.

* Supported by DFG contract SP 212/10-3.

** A large part of this work has been done during the visit of the second author at WIAS Berlin in October-November 2001.

*** Supported by the GAČR under Grant No. 201/02/1058

.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The paper deals with the long-time behaviour of evolution systems described by ODEs and PDEs with hysteresis operators. The analysis is based on two concepts. The first one is the outward pointing property of the involved hysteresis operators which implies uniform a priori bounds for solutions, the second one is related to the hysteresis modelling itself and consists in introducing a class of thermodynamically consistent generalized Prandtl-Ishlinskii operators as a model for a nonlinear elastoplastic material law. A stability result for solutions in one-dimensional visco-elasto-plasticity is derived as an illustration of the theory.

1 Introduction

The investigation of the asymptotic stability of PDEs involving hysteresis operators is complicated by the fact that the state space is usually very complex and standard methods fail. The stability results are therefore either weaker than the corresponding ones in the case without hysteresis, see e. g. [9], or, if the operators have a special clockwise convexity property of hysteresis loops (cf. Figure 1 below), one can make use of a second order energy inequality like in [7, 10]. This is for instance the case of the so-called *Prandtl-Ishlinskii operator* which is often used as a model for elastoplastic material laws. The clockwise convexity property, however, is unstable with respect to small perturbations of the operator, and difficulties occur if Prandtl-Ishlinskii elastoplasticity is combined with an arbitrarily small nonlinearly elastic component represented by a functional strain-stress relation.

An alternative approach has been proposed in [5], where asymptotic stability results have been derived under the so-called "pointing outwards" assumption. We introduce here a new class of operators, the so-called generalized Prandtl-Ishlinskii operators, as a model for non-linear elastoplasticity which combines a Prandtl-Ishlinskii operator with nonlinear superposition operators, and find sufficient conditions for the validity of the pointing outwards property.

The extension of Prandtl-Ishlinskii operators cannot be done in an arbitrary way. The nonlinear perturbations may violate the thermodynamic consistency as an important admissibility criterion of the model in the sense that the associated internal energy in some cases turns out to be unbounded from below. We devote a large part of the paper to a discussion on this subject and identify important situations in which a lower bound for the internal energy can be found.

These concepts are then used for the investigation of a one dimensional visco-elasto-plasticity model, where the elastoplastic material law is given by an operator which is pointing outwards and thermodynamically consistent.

The paper is organized as follows. In Section 2 we give a formal definition of the pointing outwards property and show how it can be used for deriving a priori bounds for solutions to ordinary differential equations. Section 3 is devoted to a detailed analysis of generalized Prandtl-Ishlinskii operators which are shown, along with other classical hysteresis operators, to be pointing outwards under suitable hypotheses. Thermodynamic consistency criteria for these operators are established in Section 4. In Section 5 we apply the above results to estimating the long-time behaviour of solutions to a partial differential equation describing the wave propagation in visco-elasto-plastic media.

2 Outwards pointing maps

We deal with mappings $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denotes the space of continuous functions $u : \mathbb{R}_+ \to \mathbb{R}$, $\mathbb{R}_+ = [0, \infty[$, endowed with the family of seminorms

$$|u|_{[0,t]} \;=\; \max\{|u(s)|\,;\, 0 \le s \le t\}\,. \tag{2.1}$$

It can be considered as a Fréchet space with the metric

$$d(u,v) \;=\; \sup_{t\ge 0} \frac{|u-v|_{[0,t]}}{1+|u-v|_{[0,t]}}\,. \tag{2.2}$$

We start with the following definition.

Definition 2.1 *A mapping $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is said to be*

(i) locally Lipschitz*, if there exists a non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality*

$$|\mathcal{H}[u_1](t) - \mathcal{H}[u_2](t)| \;\le\; \psi\left(|u_1|_{[0,t]} + |u_2|_{[0,t]}\right) |u_1 - u_2|_{[0,t]} \tag{2.3}$$

holds for every $u_1, u_2 \in C(\mathbb{R}_+)$ and every $t \ge 0$,

(ii) causal*, if the implication*

$$|u_1 - u_2|_{[0,t]} = 0 \;\Rightarrow\; \mathcal{H}[u_1](t) = \mathcal{H}[u_2](t) \tag{2.4}$$

holds for every $u_1, u_2 \in C(\mathbb{R}_+)$ and every $t \ge 0$,

(iii) rate-independent, *if*

$$\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t)) \tag{2.5}$$

holds for every $v \in C(\mathbb{R}_+)$, *every* $t \geq 0$, *and every continuous non-decreasing function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\alpha(0) = 0$ *and* $\alpha(\infty) = \infty$,

(iv) *a* hysteresis operator, *if it is causal and rate-independent,*

(v) pointing outwards with bound h in the δ-neighbourhood of $[A, B]$ for initial values in $[a, b]$, *if for every* $t \geq 0$ *and every* $u \in C(\mathbb{R}_+)$ *such that*

$$u(0) \in [a, b], \qquad u(s) \in \,]A - \delta, B + \delta[\quad \forall\, s \in [0, t] \tag{2.6}$$

we have

$$(\mathcal{H}[u](t) - h)(u(t) - B)^+ \;\geq\; 0, \quad (\mathcal{H}[u](t) + h)(u(t) - A)^- \;\leq\; 0 \tag{2.7}$$

for some given values of $\delta > 0$, $h \geq 0$, $A \leq a \leq b \leq B$, *where* $z^+ = \max\{z, 0\}$ *and* $z^- = \max\{-z, 0\}$ *for* $z \in \mathbb{R}$ *denote the positive and negative part of* z, *respectively.*

Throughout the paper, we use the terms “increasing” and “decreasing” in the strict sense, i.e. a function which is constant on a part of its domain is neither increasing nor decreasing. The term “non-decreasing function” as in point (iii) of the definition describes a function which is nowhere decreasing on its domain.

Condition (2.7) needs some comment. In plain words, one might say that if the value of u at time t exceeds the limits $[A, B]$ and the values of u in the past remained in a δ-neighbourhood of $[A, B]$, the $\mathcal{H}[u](t)$ “points outwards” with respect to the relative position of $u(t)$ and the interval $[A, B]$, that is,

$$u(t) > B \;\Rightarrow\; \mathcal{H}[u](t) \geq h \geq 0, \quad u(t) < A \;\Rightarrow\; \mathcal{H}[u](t) \leq -h \leq 0. \tag{2.8}$$

The set of all mappings $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ satisfying the implication (2.6) $\Rightarrow$ (2.7) will be denoted by $PO(a, b, A, B, \delta, h)$.

For $a \leq b$ and $h \geq 0$ we further introduce the sets

$$PO_h^{\mathrm{weak}}(a, b) \;=\; \bigcup_{\delta > 0} \; \bigcup_{[A,B] \supset [a,b]} PO(a, b, A, B, \delta, h), \tag{2.9}$$

$$PO_h(a, b) \;=\; \bigcap_{\delta > 0} \; \bigcup_{[A,B] \supset [a,b]} PO(a, b, A, B, \delta, h), \tag{2.10}$$

$$PO_\infty(a, b) \;=\; \bigcap_{h \geq 0} PO_h(a, b). \tag{2.11}$$

We immediately see that for every $a \leq b$ and every $h' > h \geq 0$ we have

$$PO_h^{\mathrm{weak}}(a, b) \supset PO_h(a, b) \supset PO_{h'}(a, b) \supset PO_\infty(a, b). \tag{2.12}$$

In the easy case of a superposition operator $\mathcal{H}[u](t) = g(u(t))$ generated by a continuous function $g : \mathbb{R} \to \mathbb{R}$, the implication (2.6) $\Rightarrow$ (2.7) can be written equivalently in the form

$$g(u) \geq h \quad \text{for} \quad u \in [B, B+\delta[\,, \quad g(u) \leq -h \quad \text{for} \quad u \in]A-\delta, A]\,. \tag{2.13}$$

Conditions of the form (2.13) have been used e. g. in [11, 12] to derive uniform estimates for solutions to equations of viscoelasticity. A generalization of this idea to visco-elasto-plasticity with outwards pointing hysteresis operators will be shown below in Section 5.

We first illustrate here the meaning of the concept of pointing outwards mappings on the example of the differential equation

$$\dot{u}(t) + \mathcal{H}[u](t) = 0\,, \quad u(0) = u^0\,. \tag{2.14}$$

Proposition 2.2 *Let $\mathcal{H} \in PO_0^{\text{weak}}(a,b)$ for some $a \leq b$ be a locally Lipschitz causal operator in the sense of Definition 2.1. Then for every $u^0 \in [a,b]$, Problem (2.14) admits a unique classical globally bounded solution.*

Proof. The local existence of a unique classical solution u in an interval $[0,\tau]$ for $\tau > 0$ sufficiently small is obtained by the standard Banach contraction argument. We then extend u into a maximal solution defined in an interval $[0,T[$. To derive an a priori bound, we fix $\delta > 0$ and $[A,B] \supset [a,b]$ such that $\mathcal{H} \in PO(a,b,A,B,\delta,0)$ according to Definition 2.1, and set

$$t_1 = \sup\{t \in [0,T]\,;\, \forall\, s \in [0,t[\,:\; u(s) \in]A-\delta, B+\delta[\,\}\,. \tag{2.15}$$

By Definition 2.1 (v) we have

$$\mathcal{H}[u](t)(u(t)-B)^+ \;\geq\; 0\,, \quad \mathcal{H}[u](t)(u(t)-A)^- \;\leq\; 0 \tag{2.16}$$

for every $t \in [0,t_1[$. Testing Eq. (2.14) consecutively by $(u(t)-B)^+$, $(u(t)-A)^-$, we obtain

$$\frac{d}{dt}\left((u(t)-B)^+\right)^2 \;\leq\; 0\,, \quad \frac{d}{dt}\left((u(t)-A)^-\right)^2 \;\leq\; 0 \tag{2.17}$$

a. e. in $]0,t_1[$, hence $u(t) \in [A,B]$ for all $t \in [0,T[$. We conclude that $T = \infty$ and the proof is complete. ■

3 Hysteresis operators

3.1 The play and stop

We denote by $W^{1,\infty}_C(\mathbb{R}_+)$ the space of Lipschitz continuous functions $\lambda : \mathbb{R}_+ \to \mathbb{R}$ with compact support in $\mathbb{R}_+$, and introduce the set

$$\Lambda \;=\; \{\lambda \in W^{1,\infty}_C(\mathbb{R}_+)\,;\, |\lambda'(r)| \leq 1 \text{ a. e. }\;\}\,. \tag{3.1}$$

The theory of hysteresis operators in the form presented in [3, 6, 7, 13] is based on the following concept.

Definition 3.1 *For every* $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda \in \Lambda$, *and* $r > 0$ *we define the value* $\mathfrak{p}_r[\lambda, u] \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *of the* play operator $\mathfrak{p}_r : \Lambda \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *as the solution* $\mathfrak{p}_r[\lambda, u](t) = \xi_r(t)$ *of the variational inequality*

$$\left.\begin{array}{l} |u(t) - \xi_r(t)| \le r \qquad \forall\, t \ge 0\,, \\ \dot\xi_r(t)\,(u(t) - \xi_r(t) - y) \ \ge\ 0 \quad a.\,e. \qquad \forall\, |y| \le r\,, \\ \xi_r(0) = \min\{u(0) + r,\ \max\{u(0) - r,\ \lambda(r)\}\}\,. \end{array}\right\} \tag{3.2}$$

The definition can be extended to $r = 0$ consistently with (3.2) by putting $\mathfrak{p}_0[\lambda, u](t) = u(t)$ for each λ, u, and t. The set Λ of initial configurations for the one-parametric play system $\{\mathfrak{p}_r\,;\, r \ge 0\}$ constitutes an important element in hysteresis modelling, see [3, 7], and can be identified with the Preisach state space, see [8]. In particular, the function $r \mapsto \mathfrak{p}_r[\lambda, u](t)$ belongs to Λ for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda \in \Lambda$, and $t \ge 0$.

We also introduce the dual operator $\mathfrak{s}_r : \Lambda \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ called the *stop* by the formula

$$\mathfrak{s}_r[\lambda, u](t) \ :=\ u(t) - \mathfrak{p}_r[\lambda, u](t) \tag{3.3}$$

for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda \in \Lambda$, and $t \ge 0$.

Immediately from the variational inequality in Definition 3.1 we see that for inputs $u_0, u_1, u_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and initial configurations $\lambda_0, \lambda_1, \lambda_2 \in \Lambda$, the respective outputs $\xi^i_r := \mathfrak{p}_r[\lambda_i, u_i]$, $x^i_r := \mathfrak{s}_r[\lambda_i, u_i]$ for $i = 0, 1, 2$ have the properties

$$\dot\xi^0_r(t)\,\dot x^0_r(t) \ =\ 0 \quad \text{a.\,e.}\ , \tag{3.4}$$

$$(\dot\xi^1_r(t) - \dot\xi^2_r(t))(x^1_r(t) - x^2_r(t)) \ \ge\ 0 \quad \text{a.\,e.} \tag{3.5}$$

It follows from Inequality (3.5) that the stop and play are Lipschitz continuous as mappings $\Lambda \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$. The following more substantial result shows that they are Lipschitz continuous with respect to the sup-norm, and therefore can be extended to Lipschitz continuous mappings $\Lambda \times C(\mathbb{R}_+) \to C(\mathbb{R}_+)$.

Proposition 3.2 *Let* $u_1, u_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda$, $r > 0$ *be given, and put* $\xi^i_r := \mathfrak{p}_r[\lambda_i, u_i]$ *for* $i = 1, 2$. *Then for every* $t \ge 0$ *we have*

$$|\xi^1_r(t) - \xi^2_r(t)| \ \le\ \max\{|\lambda_1(r) - \lambda_2(r)|,\ |u_1 - u_2|_{[0,t]}\}\,. \tag{3.6}$$

This result has been proved for the first time in [6], an elementary proof can be found in [3] or [7]. We now prove another property of the play which will be useful in the sequel.

Lemma 3.3 *Let* $u \in C(\mathbb{R}_+)$, $\lambda \in \Lambda$, $r > 0$, $t \geq 0$ *be given, and let* $R > 0$ *be such that* $\lambda(r') = 0$ *for all* $r' \geq R$. *If*

$$u_{\max}(t) := \max_{s \in [0,t]} u(s) \geq R\,, \tag{3.7}$$

then

$$\mathfrak{p}_r[\lambda, u](s) \;\leq\; \max\{0,\, u_{\max}(t) - r\} \quad \forall\, s \in [0,t]\,, \tag{3.8}$$

$$\mathfrak{s}_r[\lambda, u](s) \;\geq\; \min\{u(s), u(s) - u_{\max}(t) + r\} \quad \forall\, s \in [0,t]\,. \tag{3.9}$$

Similarly, if

$$u_{\min}(t) := \min_{s \in [0,t]} u(s) \leq -R\,, \tag{3.10}$$

then

$$\mathfrak{p}_r[\lambda, u](s) \;\geq\; \min\{0,\, u_{\min}(t) + r\} \quad \forall\, s \in [0,t]\,, \tag{3.11}$$

$$\mathfrak{s}_r[\lambda, u](s) \;\leq\; \max\{u(s), u(s) - u_{\min}(t) - r\} \quad \forall\, s \in [0,t]\,. \tag{3.12}$$

Proof. We denote again $\xi_r = \mathfrak{p}_r[\lambda, u]$. By density of $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ in $C(\mathbb{R}_+)$ with respect to the metric (2.2), it suffices to assume that u belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ (we approximate u by piecewise linear interpolates with a fixed node at t, say). Suppose first that (3.7) holds, and set

$$V(s) \;:=\; \max\{\xi_r(s),\, u_{\max}(t) - r\} \tag{3.13}$$

for $s \in [0,t]$. From (3.2) it follows that $u(s) - \xi_r(s) = r$ whenever $\dot{\xi}_r(s) > 0$. Assuming $\dot{V}(s) > 0$ for some s, we therefore obtain $\xi_r(s) = u(s) - r \leq u_{\max}(t) - r$ which is a contradiction. Hence the function V is non-increasing in $[0,t]$, and we have

$$\xi_r(s) \;\leq\; \max\{\xi_r(0),\, u_{\max}(t) - r\} \qquad \forall\, s \in [0,t]\,. \tag{3.14}$$

By (3.2) we have

$$\xi_r(0) \leq \max\{\lambda(r),\, u(0) - r\} \leq \max\{\lambda(r),\, u_{\max}(t) - r\}\,, \tag{3.15}$$

and either

(i) $r \geq R$, $\lambda(r) = 0$, or

(ii) $r < R$, $\lambda(r) = \lambda(r) - \lambda(R) \leq R - r \leq u_{\max}(t) - r$.

In both cases, the assertion follows from (3.14) and (3.3). The argument in case (3.10) is analogous. ∎

Remark 3.4 From Lemma 3.3 it follows that for $a \le b$, $\delta \in]0, r]$, $h \in [0, r - \delta]$, $A \le \min\{a, -R, -r\}$ and $B \ge \max\{b, R, r\}$, the operator $\mathfrak{s}_r[\lambda, \cdot]$ is pointing outwards with bound h in the δ-neighbourhood of $[A, B]$ for initial values in [a,b] according to Definition 2.1(v), that is, $\mathfrak{s}_r[\lambda, \cdot] \in PO(a, b, A, B, \delta, h) \subset PO_h^{\text{weak}}(a, b)$.

This is no longer true for $\delta > r$. It suffices to consider some $t' > t \ge 0$, $B > 0$, $\varepsilon \in]0, (\delta - r)/2[$, and a function u which is decreasing on $[t, t']$, such that $u(t) = u_{\max}(t) = B + \delta - \varepsilon$, $u(t') = B + \varepsilon$. We have $\mathfrak{s}_r[\lambda, v](t) \le r$ and

$$\mathfrak{s}_r[\lambda, u](t') = \max\{\mathfrak{s}_r[\lambda, u](t) + u(t') - u(t), -r\} \le \max\{r - \delta + 2\varepsilon, -r\} < 0 \tag{3.16}$$

independently of λ and B, hence $\mathfrak{s}_r[\lambda, \cdot] \notin PO(a, b, A, B, \delta, 0)$. In particular, we have $\mathfrak{s}_r[\lambda, \cdot] \notin PO_0(a, b)$. Moreover, the same argument shows that the operator $\mathcal{H} := \mathfrak{s}_r[\lambda, \cdot]$ does not satisfy a possible counterpart of (2.13) in the form

$$H[u](t) \ge h \quad \text{for} \quad u(t) \in [B, B + \delta[\ , \quad H[u](t) \le -h \quad \text{for} \quad u(t) \in]A - \delta, A] \tag{3.17}$$

for any $\delta > 0$, $h \ge 0$. For operators with a nontrivial memory, the implication (2.6) $\Rightarrow$ (2.7) in the definition of the pointing outwards property generalizes this condition by taking also the past values of u into account. Some other examples where this generalization is also important will be shown e. g. in Remark 3.11 below.

Since it will be used later, we briefly recall here the concept of memory sequence of a function and its application in the representation of the play operator. With an arbitrary $\lambda \in \Lambda$, we associate a function $m_\lambda : \mathbb{R} \to \mathbb{R}_+$ by the formula

$$m_\lambda(v) = \min\{r \ge 0; |v - \lambda(r)| = r\}. \tag{3.18}$$

The function $r \mapsto r - |v - \lambda(r)|$ is non-decreasing. This immediately implies that $|v - \lambda(r)| > r$ for $r \in [0, m_\lambda(v)[$, $|v - \lambda(r)| \le r$ for $r \in [m_\lambda(v), +\infty[$; the function m_λ is increasing and left-continuous in $[\lambda(0), +\infty[$ and decreasing and right-continuous in $]-\infty, \lambda(0)]$, $m_\lambda(\lambda(0)) = 0$.

According to [7, Section II.2] we now define the *memory sequence* $\{(t_j, r_j)\}$ *of* u *at time* t *with respect to the initial configuration* λ in the following way. For $\lambda \in \Lambda$, $u \in C(\mathbb{R}_+)$, and $t \ge 0$, we define the quantity

$$R_\lambda[u](t) = \max\{m_\lambda(u(\tau))\,;\, \tau \in [0, t]\}, \tag{3.19}$$

and set

$$\begin{cases} \bar{r} = R_\lambda[u](t), \\ \bar{t} = \max\{\tau \in [0, t]\,;\, m_\lambda(u(\tau)) = \bar{r}\}, \end{cases} \tag{3.20}$$

$$\begin{cases} t_0 = \bar{t},\ r_0 = \bar{r},\ j_0 = 0 & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) - \bar{r}, \\ t_1 = \bar{t},\ r_1 = \bar{r},\ j_0 = 1 & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) + \bar{r}, \end{cases} \tag{3.21}$$

and continue recursively by putting

$$\begin{cases} t_{2k+1} = \max\{\tau \in [t_{2k}, t]; u(\tau) = \max\{u(\sigma); \sigma \in [t_{2k}, t]\}\}, \ k = j_0, j_0+1, j_0+2, \dots, \\ t_{2k} = \max\{\tau \in [t_{2k-1}, t]; u(\tau) = \min\{u(\sigma); \sigma \in [t_{2k-1}, t]\}\}, \ k = 1, 2, \dots, \\ r_{j+1} = \frac{(-1)^j}{2}\left(u(t_{j+1}) - u(t_j)\right), \ j = j_0, j_0+1, j_0+2, \dots, \end{cases} \tag{3.22}$$

until $t_{2k+1} = t$ or $t_{2k} = t$. We will call j_0 the *starting index of the memory sequence* $\{(t_j, r_j)\}$, and $(\bar{t}, \bar{r}) = (t_{j_0}, r_{j_0})$ the *starting point of the memory sequence* $\{(t_j, r_j)\}$.

One of the following two possibilities occurs.

A. The sequence $\{(t_j, r_j)\}$ is infinite, $u(t) = \lim_{j\to\infty} u(t_j)$, $\lim_{j\to\infty} r_j = 0$;

B. The sequence $\{(t_j, r_j)\}$ is finite, $t = t_n$. In this case we put $r_j = 0$ for $j \ge n+1$.

The following result has been proved in Proposition II.2.5 of [7] and gives a memory representation of the system $\{\mathfrak{p}_r\,;\, r \ge 0\}$ of play operators.

Proposition 3.5 *Let $u \in C(\mathbb{R}_+)$, $\lambda \in \Lambda$, $r \ge 0$, and $t \ge 0$ be given, and let $\{(t_j, r_j)\}$ be the memory sequence defined in (3.21), (3.22). Then we have*

$$\mathfrak{p}_r[\lambda, u](t) = \begin{cases} \lambda(r) & \textit{for} \quad r \ge r_{j_0}, \\ u(t_j) + (-1)^j r & \textit{for} \quad r \in [r_{j+1}, r_j[, \ j = j_0, j_0+1, j_0+2, \dots, \\ u(t) & \textit{for} \quad r = 0. \end{cases} \tag{3.23}$$

3.2 Prandtl-Ishlinskii operator

In the classical setting, the *Prandtl-Ishlinskii operator* $\mathcal{F} : \Lambda \times C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is constructed according to [3, 6, 7, 13] by averaging the instantaneous values of the stops (or, alternatively, the plays) over the parameter range $r > 0$ by the Stieltjes integral formula

$$\mathcal{F}[\omega, u](t) \;=\; -\int_0^\infty \mathfrak{s}_r[\omega, u](t)\, d\eta(r) \tag{3.24}$$

for $u \in C(\mathbb{R}_+)$ and $\omega \in \Lambda$ (the reason for denoting here a generic element of Λ by ω instead of λ will become more clear later after Proposition 3.15 below), where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a given non-increasing function with $\eta(\infty) = 0$ which is to be identified from experimental data. In fact, in applications to plasticity, it is obtained by measuring the so-called *initial loading curve* $\sigma = \Phi(u)$ by letting u monotonically increase and decrease from 0 in a material with no previous memory (the memory being erased for instance by heating above the Curie temperature and cooling again). Assuming that $\Phi : \mathbb{R} \to \mathbb{R}$ is odd, increasing, and $\Phi|_{\mathbb{R}_+} : \mathbb{R}_+ \to \mathbb{R}_+$ is concave, we put $\eta(r) = \Phi'(r)$ for $r > 0$, that is,

$$\Phi(u) \;=\; \int_0^u \eta(r)\, dr \quad \text{for} \quad u \ge 0\,, \qquad \Phi(u) \;=\; -\Phi(-u) \quad \text{for} \quad u < 0\,. \tag{3.25}$$

A mathematical justification of the model via homogenization in connection with elasto-plastic oscillations has been suggested in [4]. The operator is Lipschitz continuous as an immediate consequence of Proposition 3.2 and the formula

$$\mathcal{F}[\omega, u](t) \;= \eta(0)u(t) + \int_0^\infty \mathfrak{p}_r[\omega, u](t)\, d\eta(r), \quad \forall\, u \in C(\mathbb{R}_+)\,,\ \omega \in \Lambda \tag{3.26}$$

which follows from (3.24) and (3.3). Indeed, for $u_1, u_2 \in C(\mathbb{R}_+)$ and $\omega_1, \omega_2 \in \Lambda$ we easily obtain for every $t \geq 0$ that

$$|\mathcal{F}[\omega_1, u_1](t) - \mathcal{F}[\omega_2, u_2](t)| \;\leq\; 2\eta(0)\, |u_1 - u_2|_{[0,t]} + \int_0^\infty |\omega_1(r) - \omega_2(r)|\, d\eta(r)\,. \tag{3.27}$$

Moreover, if $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then (3.5) yields $\dot u(t) \frac{d}{dt} \mathfrak{s}_r[\omega, u](t) = (\frac{d}{dt} \mathfrak{s}_r[\omega, u](t))^2$ a. e. From (3.24) we thus obtain in particular that $\mathcal{F}[\omega, u] \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and

$$\left(\frac{d}{dt}\, \mathcal{F}[\omega, u](t)\right)^2 \;\leq\; \eta(0)\, \frac{d}{dt}\, \mathcal{F}[\omega, u](t)\, \dot u(t) \;\leq\; \eta^2(0)\, \dot u^2(t) \qquad \text{a. e.} \tag{3.28}$$

This property is called *local monotonicity*.

The hysteresis energy dissipation will be discussed in more detail below in Section 4. Let us note here that the Prandtl-Ishlinskii operator (as well as any *clockwise convex* operator) admits a “second order energy inequality”

$$\frac{d}{dt}\, \mathcal{F}[\omega, u](t)\, \ddot u \geq \frac{d}{dt}\, \left(\frac{1}{2} \frac{d}{dt}\, \mathcal{F}[\omega, u](t)\, \dot u(t)\right) \tag{3.29}$$

in the sense of distributions whenever $\frac{d}{dt}\, \mathcal{F}[\omega, u] \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ and $\ddot u \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, see [7, 10] and Figure 1 below.

We will need the following estimate.

Proposition 3.6 *Let $u \in C(\mathbb{R}_+)$ and $\omega \in \Lambda$ be given, and let $R > 0$ and $t \geq 0$ be such that $\omega(r') = 0$ for all $r' \geq R$, and*

$$u_{\max}(t) := \max_{s \in [0,t]} u(s) \geq R\,, \quad u(t) \;\geq -u_{\max}(t)\,. \tag{3.30}$$

In terms of the function Φ defined by (3.25) we then have

$$\mathcal{F}[\omega, u](t) \geq \Phi(u_{\max}(t)) \;+\; 2\,\Phi\left(\frac{1}{2}(u(t) - u_{\max}(t))\right)\,. \tag{3.31}$$

Similarly, if

$$u_{\min}(t) := \min_{s \in [0,t]} u(s) \leq -R\,, \quad u(t) \;\leq -u_{\min}(t)\,, \tag{3.32}$$

then

$$\mathcal{F}[\omega, u](t) \leq \Phi(u_{\min}(t)) \;+\; 2\,\Phi\left(\frac{1}{2}(u(t) - u_{\min}(t))\right)\,. \tag{3.33}$$

Proof. By Lemma 3.3 we have in case (3.30) for every $r > 0$ that

$$\mathfrak{p}_r[\omega, u](t) \leq \max\{0,\, u_{\max}(t) - r\}\,. \tag{3.34}$$

On the other hand, the constraint $|\mathfrak{p}_r[\omega, u](t) - u(t)| \leq r$ is satisfied by definition, hence

$$\mathfrak{p}_r[\omega, u](t) \leq \min\{u(t) + r,\, \max\{0,\, u_{\max}(t) - r\}\}\,. \tag{3.35}$$

For $\hat{r} = \frac{1}{2}(u_{\max}(t) - u(t))$, formula (3.26) (note that η is non-increasing) implies that

$$\mathcal{F}[\omega, u](t) \;\geq\; \eta(0)\, u(t) + \int_0^{\hat{r}} (u(t) + r)\, d\eta(r) + \int_{\hat{r}}^{u_{\max}(t)} (u_{\max}(t) - r)\, d\eta(r)\,, \tag{3.36}$$

and it suffices to integrate by parts. Case (3.32) is completely analogous. ∎

3.3 Generalized Prandtl-Ishlinskii operator

We introduce here a class of generalized Prandtl-Ishlinskii operators under the following hypotheses.

Hypothesis 3.7 *We are given*

(i) *two non-decreasing locally Lipschitz continuous functions $g_0, g : \mathbb{R} \to \mathbb{R}$ such that g is odd, $\lim_{u\to\infty} g(u) = \infty$, $g'(u) > 0$ for almost every $u \in \mathbb{R}$,*

(ii) *an odd function $\varphi \in L^1(\mathbb{R})$ such that $\varphi(s) \geq 0$ for a. e. $s > 0$, and we set*

$$\eta(r) \;=\; \int_r^\infty \varphi(s)\, ds \qquad \textit{for} \quad r \geq 0\,.$$

Definition 3.8 *Let g_0, g, η be as in Hypothesis 3.7, and let $\omega \in \Lambda$ be given. Then the mapping $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by the formula*

$$\mathcal{H}[u](t) \;=\; g_0(u(t)) + \mathcal{F}[\omega, g(u)](t) \tag{3.37}$$

with $\mathcal{F}$ given by (3.24) is called a generalized Prandtl-Ishlinskii operator.

As a consequence of (3.27), we have the estimate

$$|\mathcal{H}[u_1](t) - \mathcal{H}[u_2](t)| \;\leq\; |g_0(u_1(t)) - g_0(u_2(t))| + 2\eta(0)\, |g(u_1) - g(u_2)|_{[0,t]}\,. \tag{3.38}$$

for every $u_1, u_2 \in C(\mathbb{R}_+)$ and every $t \geq 0$. In particular, $\mathcal{H}$ is locally Lipschitz according to Definition 2.1. Moreover, it is locally monotone for $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ in the sense that by (3.28) we have $\mathcal{H}[u] \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and

$$\left(\frac{d}{dt}\mathcal{H}[u](t)\right)^2 \;\leq\; \gamma(u(t))\, \frac{d}{dt}\mathcal{H}[u](t)\, \dot{u}(t) \;\leq\; \gamma^2(u(t))\, \dot{u}^2(t) \qquad \text{a. e.}\quad , \tag{3.39}$$

where $\gamma(u) = g_0'(u) + \eta(0)\, g'(u)$. For η and φ as in Hypothesis 3.7, we get for the function Φ defined in (3.25)

$$\Phi(u) \;=\; u\int_u^\infty \varphi(r)\,dr + \int_0^u r\varphi(r)\,dr \le \int_0^\infty r\varphi(r)\,dr \quad \forall\, u \ge 0, \tag{3.40}$$

$$\Phi'' \;=\; -\varphi \quad \text{a.e. on } \mathbb{R}. \tag{3.41}$$

We now prove the following crucial result.

Theorem 3.9 *Let Hypothesis 3.7 hold, let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined in (3.25), and let $h \ge 0$ be given.*

a) *If*

$$\inf_{\delta>0} \limsup_{r\to+\infty} \left(g_0(r) + \Phi(g(r)) + 2\Phi\left(\frac{1}{2}(g(r) - g(r+\delta))\right)\right) \;>\; h\,, \tag{3.42}$$

$$\sup_{\delta>0} \liminf_{r\to-\infty} \left(g_0(r) + \Phi(g(r)) + 2\Phi\left(\frac{1}{2}(g(r) - g(r-\delta))\right)\right) \;<\; -h\,, \tag{3.43}$$

then $\mathcal{H} \in PO_h(a,b)$ for every $-\infty < a \le b < \infty$.

b) *If*

$$\lim_{r\to+\infty} (g_0(r) + \Phi(g(r))) \;>\; h\,, \tag{3.44}$$

$$\lim_{r\to-\infty} (g_0(r) + \Phi(g(r))) \;<\; -h\,, \tag{3.45}$$

then $\mathcal{H} \in PO_h^{\mathrm{weak}}(a,b)$ for every $-\infty < a \le b < \infty$.

Remark 3.10 Typical cases, where conditions (3.42) – (3.43) are satisfied, are for instance

$$\lim_{r\to\pm\infty} (g_0(r) + \Phi(r)) \;=\; \pm\infty\,, \quad \sup\{g'(r)\,;\, r \in \mathbb{R}\} \;<\; \infty\,, \tag{3.46}$$

or

$$\pm \lim_{r\to\pm\infty} (g_0(r) + \Phi(r)) \;>\; h\,, \quad \lim_{r\to\infty} g'(r) \;=\; 0\,. \tag{3.47}$$

In the case (3.46) we have $\mathcal{H} \in PO_\infty(a,b)$ for every $a \le b$.

Remark 3.11 The Prandtl-Ishlinskii operator $\mathcal{F}[\omega,\cdot] : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by (3.24) also belongs to the class of generalized Prandtl-Ishlinskii operators and corresponds to the choice $g_0 \equiv 0$, $g(u) = u$. If $\int_0^\infty r\varphi(r)\,dr = \infty$, then (3.40) yields that the initial–loading curve is unbounded, and, by (3.46), we have $\mathcal{F}[\omega,\cdot] \in PO_\infty(a,b)$ for all $a \le b$. On the other hand, if $\int_0^\infty r\varphi(r)\,dr =: \bar\Phi < \infty$, then by (3.40) we have $\lim_{r\to\infty} \Phi(r) = \bar\Phi$, and using Theorem 3.9 b), we obtain only $\mathcal{F}[\omega,\cdot] \in PO_h^{\mathrm{weak}}(a,b)$ for small h and for all

$a \leq b$. Repeating the construction from Remark 3.4 with $\delta > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small (inequality (3.16) holds indeed for all $\delta > 2\varepsilon > 0$), we deduce from (3.24), (3.25), (3.40), and Hypothesis 3.7 (ii) that

$$\begin{aligned}\mathcal{F}[\omega, u](t') &\leq -\int_0^\infty \max\{r - \delta + 2\varepsilon, -r\}\, d\eta(r) = \int_0^\infty \max\{r - \delta + 2\varepsilon, -r\}\varphi(r)\, dr \\ &= \bar{\Phi} - 2\Phi(\delta/2 - \varepsilon) < 0\,.\end{aligned}$$

Hence, $\mathcal{F}[\omega, \cdot] \notin PO(a, b, A, B, \delta, 0)$ for large $\delta > 0$, and therefore $\mathcal{F}[\omega, \cdot] \notin PO_0(a, b)$. Moreover, for all $\delta > 0$, we see that (3.17) does not hold for $\mathcal{H} := \mathcal{F}[\omega, \cdot]$.

Proof of Theorem 3.9. Let $a \leq b$ be given, and let $R > 0$ be such that $\omega(r) = 0$ for $r \geq R$. To prove part a) of the theorem, we have to show that for every $\delta > 0$ there exist $A \leq a \leq b \leq B$ such that $\mathcal{H} \in PO(a, b, A, B, \delta, h)$, that is, the implication (2.6) $\Rightarrow$ (2.7) holds. Let $\delta > 0$ be arbitrarily given. Recalling (3.42), (3.43), and Hypothesis 3.7, we find $A \leq a$ and $B \geq b$ such that $g(A) < -R$, $g(B) > R$, and

$$g_0(B) + \Phi(g(B)) + 2\Phi\left(\frac{1}{2}(g(B) - g(B + \delta))\right) \;\geq\; h\,, \tag{3.48}$$

$$g_0(A) + \Phi(g(A)) + 2\Phi\left(\frac{1}{2}(g(A) - g(A - \delta))\right) \;\leq\; -h\,. \tag{3.49}$$

Let $u \in C(\mathbb{R}_+)$ and $t \geq 0$ be such that $u(t) > B$ and (2.6) holds. By (3.31) we have

$$\begin{aligned}\mathcal{H}[u](t) &\geq g_0(u(t)) + \Phi(g(u_{\max}(t))) + 2\Phi\left(\frac{1}{2}(g(u(t)) - g(u_{\max}(t)))\right) \qquad (3.50)\\ &\geq g_0(B) + \Phi(g(B)) + 2\Phi\left(\frac{1}{2}(g(B) - g(B + \delta))\right) \;\geq\; h\,.\end{aligned}$$

Similarly, if $u(t) < A$ and (2.6) holds, then we obtain from (3.33) that

$$\begin{aligned}\mathcal{H}[u](t) &\leq g_0(u(t)) + \Phi(g(u_{\min}(t))) + 2\Phi\left(\frac{1}{2}(g(u(t)) - g(u_{\min}(t)))\right) \qquad (3.51)\\ &\leq g_0(A) + \Phi(g(A)) + 2\Phi\left(\frac{1}{2}(g(A) - g(A - \delta))\right) \;\leq\; -h\,,\end{aligned}$$

hence (2.7) holds and the proof of part a) of Theorem 3.9 is complete.

To prove part b), we have to find suitable constants $A \leq a$, $B \geq b$, and $\delta > 0$ such that $\mathcal{H} \in PO(a, b, A, B, \delta, h)$. Applying (3.44), (3.45), and Hypothesis 3.7, we choose $A \leq a$, $B \geq b$ such that $g(A) < -R$, $g(B) > R$, and

$$g_0(B) + \Phi(g(B)) > h\,, \quad g_0(A) + \Phi(g(A)) < -h\,. \tag{3.52}$$

Since g is locally Lipschitz continuous, we find some $\delta > 0$ such that

$$g_0(B) + \Phi(g(B)) - h \geq \eta(0)\,(g(B+\delta) - g(B)) \,, \tag{3.53}$$
$$g_0(A) + \Phi(g(A)) + h \leq \eta(0)\,(g(A-\delta) - g(A)) \,. \tag{3.54}$$

Since η in non-increasing on $\mathbb{R}_+$, we now see by (3.25) that (3.48) and (3.49) are satisfied. Arguing as above we show that $\mathcal{H} \in PO(a, b, A, B, \delta, h)$, and Theorem 3.9 is proved. ■

The following three lemmas will be used in Proposition 3.15 below which allows us to control the "hysteresis" component of $\mathcal{H}$ in the next section.

Lemma 3.12 *Let $\omega \in \Lambda$ with $\omega(s) = 0$ for $s \geq R$, and g satisfying Hypothesis 3.7 be given. Then there exist uniquely defined functions $s^* : \mathbb{R}_+ \to \mathbb{R}_+$ and $\lambda \in \Lambda$ such that*

$$s^*(r) + \omega(s^*(r)) = g(r + \lambda(r)) \,, \quad s^*(r) - \omega(s^*(r)) = g(r - \lambda(r)) \quad \forall\, r \geq 0 \,. \tag{3.55}$$

The function s^* is continuous and increasing and satisfies $s^*(0) = 0$, $s^*(r) = g(r)$ for $r \geq g^{-1}(R)$. For the function λ it holds $\lambda(r) = 0$ for $r \geq g^{-1}(R)$.

If moreover there exist $v \in \mathbb{R}$ and an interval $[s_1, s_2] \subset \mathbb{R}_+$ such that

$$\omega(s) = g(v) + s \quad \forall\, s \in [s_1, s_2] \,, \tag{3.56}$$

then for $v_i^+ := g^{-1}(g(v) + 2s_i)$, $i = 1, 2$, we have

$$\lambda(r) = v + r \quad \forall\, r \in [r_1, r_2] \,, \quad r_i = \frac{1}{2}(v_i^+ - v) \,, \quad i = 1, 2 \,. \tag{3.57}$$

Similarly, if

$$\omega(s) = g(v) - s \quad \forall\, s \in [s_1, s_2] \,, \tag{3.58}$$

then for $v_i^- := g^{-1}(g(v) - 2s_i)$, $i = 1, 2$, we have

$$\lambda(r) = v - r \quad \forall\, r \in [r_1, r_2] \,, \quad r_i = \frac{1}{2}(v - v_i^-) \,, \quad i = 1, 2 \,. \tag{3.59}$$

Proof. By Hypothesis 3.7, g is invertible and its inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ is odd, increasing, continuous, with $\lim_{s \to +\infty} g^{-1}(s) = +\infty$. The function $g^* : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula

$$g^*(s) = \frac{1}{2}\left(g^{-1}(s + \omega(s)) + g^{-1}(s - \omega(s))\right) \tag{3.60}$$

is continuous, increasing (note that both functions $s \mapsto s + \omega(s)$, $s \mapsto s - \omega(s)$ are non-decreasing, and $g^*(s_1) = g^*(s_2)$ for $s_1 \leq s_2$ implies $s_1 \pm \omega(s_1) = s_2 \pm \omega(s_2)$, hence $s_1 = s_2$), $g^*(0) = 0$, $g^*(s) = g^{-1}(s)$ for $s \geq R$. Now, we define

$$s^*(r) := (g^*)^{-1}(r) \,, \quad \lambda(r) := g^{-1}(s^*(r) + \omega(s^*(r))) - r \quad \text{for } r \in \mathbb{R}_+ \,, \tag{3.61}$$

and observe that (3.55) holds. Therefore, both functions $r \mapsto r + \lambda(r)$ and $r \mapsto r - \lambda(r)$ are non-decreasing, hence $\lambda \in \Lambda$, $\lambda(r) = 0$ for $r \geq g^{-1}(r)$, and we have thus constructed functions λ and s^* with the desired properties.

To prove the uniqueness, we assume that we have functions $\overline{s}^* : \mathbb{R}_+ \to \mathbb{R}_+$ and $\overline{\lambda} \in \Lambda$ such that (3.55) holds with s^* and λ replaced by $\overline{s}^*$ and $\overline{\lambda}$, respectively. Using now (3.60), we get for every $r \geq 0$

$$g^*(\overline{s}^*(r)) = \frac{1}{2}\left(g^{-1}\left(g(r + \overline{\lambda}(r))\right) + g^{-1}\left(g(r - \overline{\lambda}(r))\right)\right) = r. \tag{3.62}$$

Recalling (3.61), we see that $\overline{s}^* = s^*$. Applying now (3.55) for $\overline{\lambda}$ and also for λ, and using the invertibility of g, we see that $\overline{\lambda} = \lambda$, which completes the uniqueness proof.

Assume now that (3.56) holds. We have $s_i + \omega(s_i) = 2s_i + g(v) = g(v_i^+)$, $s_i - \omega(s_i) = -g(v)$, hence $g^*(s_i) = r_i$ for $i = 1, 2$. For $r \in [r_1, r_2]$ we thus have $s^*(r) \in [s_1, s_2]$, and (3.55) yields that $g(r - \lambda(r)) = s^*(r) - \omega(s^*(r)) = -g(v)$, hence $\lambda(r) = v + r$. The argument in case (3.58) is analogous. ∎

The connection between the memory sequences of a function u and of $g(u)$ is established in the following lemma:

Lemma 3.13 *Let $\omega \in \Lambda$ with $\omega(s) = 0$ for $s \geq R$, g satisfying Hypothesis 3.7, $u \in C(\mathbb{R}_+)$, and $t \geq 0$ be given. Let λ, s^* be associated with ω according to Lemma 3.12. Let $\{(t_j, r_j)\}$ be the memory sequence of u at time t with respect to the initial configuration λ and let j_0 be the starting index of this memory sequence. Let the sequence $\{s_j\}$ be defined by*

$$\left\{ \begin{array}{l} s_{j_0} := s^*(r_{j_0}), \\ s_{j+1} := (-1)^j \frac{1}{2}(g(u(t_{j+1})) - g(u(t_j))), \quad j = j_0, j_0+1, j_0+2, \ldots . \end{array} \right. \tag{3.63}$$

Then $\{(t_j, s_j)\}$ is the memory sequence of $g(u)$ at time t with respect to the initial configuration ω and j_0 is the starting index of this memory sequence.

Proof. Let $(\bar{t}, \bar{r})$ be the starting point of the memory sequence $\{(t_j, r_j)\}$ and $\bar{s} := s_{j_0}$. Combining the properties of m_λ, m_ω, and s^*, we conclude that $s^*(m_\lambda(v)) = m_\omega(g(v))$ for all $v \in \mathbb{R}$. Since s^* is increasing, this yields by (3.20) that $(\bar{t}, \bar{s})$ is the starting point of the memory sequence of $g(u)$ at time t with respect to the initial configuration ω. Because of (3.55), we have $g(u(\bar{t})) = \omega(\bar{s}) - \bar{s}$ if and only if $u(\bar{t}) = \lambda(\bar{s}) - \bar{s}$ and, on the other hand, $g(u(\bar{t})) = \omega(\bar{s}) + \bar{s}$ if and only if $u(\bar{t}) = \lambda(\bar{s}) + \bar{s}$. Recalling (3.21), we see that j_0 is the starting index of the memory sequence of $g(u)$ at time t with respect to the initial configuration ω. We now apply (3.22) and the monotonicity of g for the remaining part of the sequence to complete the proof. ∎

We now apply the above construction to the time-dependent case.

Lemma 3.14 *Let $\omega \in \Lambda$ with $\omega(s) = 0$ for $s \geq R$, g satisfying Hypothesis 3.7, $u \in C(\mathbb{R}_+)$, and $t \geq 0$ be given. Let λ be associated with ω according to Lemma 3.12. Let $\tilde{\omega} : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula*

$$\tilde{\omega}(s) \;=\; \mathfrak{p}_s[\omega, g(u)](t) \quad \textit{for} \quad s \geq 0\,, \tag{3.64}$$

and let $\tilde{\lambda}$ be associated with $\tilde{\omega}$ as in Lemma 3.12. Then for every $r \geq 0$ we have

$$\tilde{\lambda}(r) \;=\; \mathfrak{p}_r[\lambda, u](t)\,. \tag{3.65}$$

Proof. Let $\{(t_j, r_j)\}$ be the memory sequence of u at time t with respect to the initial configuration λ, and let j_0 be its starting index. Recalling now Proposition 3.5, we see that it suffices to verify the identity

$$\tilde{\lambda}(r) = \begin{cases} \lambda(r) & \text{for} \quad r \geq r_{j_0}\,, \\ u(t_j) + (-1)^j r & \text{for} \quad r \in [r_{j+1}, r_j[\quad j = j_0, j_0+1, j_0+2, \dots\,, \\ u(t) & \text{for} \quad r = 0\,. \end{cases} \tag{3.66}$$

By Lemma 3.13, the memory sequence $\{(t_j, s_j)\}$ of $g(u)$ at time t with respect to the initial configuration ω has the form (3.63). Applying Proposition 3.5 to $\mathfrak{p}_s[\omega, g(u)](t)$ we obtain

$$\tilde{\omega}(s) = \begin{cases} \omega(s) & \text{for} \quad s \geq s_{j_0}\,, \\ g(u(t_j)) + (-1)^j s & \text{for} \quad s \in [s_{j+1}, s_j[\,, \quad j = j_0, j_0+1, j_0+2, \dots\,, \\ g(u(t)) & \text{for} \quad s = 0. \end{cases} \tag{3.67}$$

We associate with $\tilde{\omega}$ the function $\tilde{s}^* : \mathbb{R}_+ \to \mathbb{R}_+$ as in Lemma 3.12, that is,

$$\tilde{s}^*(r) + \tilde{\omega}(\tilde{s}^*(r)) = g(r + \tilde{\lambda}(r))\,, \quad \tilde{s}^*(r) - \tilde{\omega}(\tilde{s}^*(r)) = g(r - \tilde{\lambda}(r)) \quad \forall\, r \geq 0\,. \tag{3.68}$$

Let $g^* = (s^*)^{-1}$ be as in (3.60). We may set analogously

$$\tilde{g}^*(s) = \frac{1}{2}\left(g^{-1}(s + \tilde{\omega}(s)) + g^{-1}(s - \tilde{\omega}(s))\right) = (\tilde{s}^*)^{-1}(s) \;\; \text{for} \quad s \geq 0\,. \tag{3.69}$$

For $s \geq s_{j_0}$ we have by (3.55) and (3.67) – (3.68) that $s \pm \omega(s) = g(g^*(s) \pm \lambda(g^*(s))) = g(\tilde{g}^*(s) \pm \tilde{\lambda}(\tilde{g}^*(s)))$, hence

$$g^*(s) = \tilde{g}^*(s) \quad \text{for} \quad s \geq s_{j_0}\,, \qquad \lambda(r) = \tilde{\lambda}(r) \quad \text{for} \quad r \geq r_{j_0} = \tilde{g}^*(s_{j_0})\,. \tag{3.70}$$

For $j \geq j_0$ and $s \in [s_{j+1}, s_j[$ we have by (3.69) and (3.67) that

$$\begin{aligned} \tilde{g}^*(s) \;&=\; \frac{1}{2}\left(g^{-1}\left((1 + (-1)^j)s + g(u(t_j))\right) + g^{-1}\left((1 - (-1)^j)s - g(u(t_j))\right)\right) \\ &=\; \frac{1}{2}\left(g^{-1}\left(2s + (-1)^j g(u(t_j))\right) - (-1)^j u(t_j)\right)\,, \end{aligned} \tag{3.71}$$

and from (3.63) and (3.22) it follows that

$$\tilde{g}^*(s_{j+1}) \;=\; r_{j+1} \qquad \forall\, j = j_0, j_0+1, j_0+2, \dots\,. \tag{3.72}$$

Furthermore, (3.67) – (3.69) yield that

$$g(u(t_j)) = \tilde{\omega}(s) - (-1)^j s = g(\tilde{\lambda}(\tilde{g}^*(s)) - (-1)^j \tilde{g}^*(s)),$$

hence $\tilde{\lambda}(\tilde{g}^*(s)) - (-1)^j \tilde{g}^*(s) = u(t_j)$ for $s \in [s_{j+1}, s_j[$, that is,

$$\tilde{\lambda}(r) \;=\; u(t_j) + (-1)^j r \quad \text{for} \quad r \in [r_{j+1}, r_j[\ , \tag{3.73}$$

which is nothing but the corresponding case considered in (3.66). Putting now $r = 0$ in (3.68) and using (3.67) we obtain $\tilde{s}^*(0) = 0$ and $g(\tilde{\lambda}(0)) = g(u(t))$, hence $\tilde{\lambda}(0) = u(t)$. Recalling (3.70) and (3.73) we see that (3.66) holds, and the proof is complete. ∎

Let g, φ as in Hypothesis 3.7 be given. For $r \ge 0$ and $v \in \mathbb{R}$ we define

$$\gamma(r,v) \;=\; \varphi\left(\frac{1}{2}(g(r+v) + g(r-v))\right) g'(r+v)\, g'(r-v)\,, \tag{3.74}$$

$$P(r,v) \;=\; \int_0^v \gamma(r,z)\, dz\,. \tag{3.75}$$

We now introduce the operator

$$\mathcal{P}[\lambda, u](t) = \int_0^\infty P(r, \mathfrak{p}_r[\lambda, u](t))\, dr\,, \tag{3.76}$$

which belongs to the class of *Preisach operators*, see [7, Section II.3]. We make use of the following representation formula similar to [7, Exercise II.3.15].

Proposition 3.15 *Let $\omega \in \Lambda$, g, η, and φ satisfying Hypothesis 3.7 be given. Let λ be associated with ω according to Lemma 3.12 and let $\gamma, P, \mathcal{P}$ be as in (3.74)–(3.76). Then for every $u \in C(\mathbb{R}_+)$ we have*

$$\mathcal{F}[\omega, g(u)] \;=\; \eta(0)\, g(u) - \mathcal{P}[\lambda, u]\,. \tag{3.77}$$

Proof. Let $t \ge 0$ be fixed, and let $\tilde{\omega}$, $\tilde{\lambda}$, and $\tilde{s}^*$ be as in Lemma 3.14 and (3.68). Then

$$\tilde{s}^*(r) \;=\; \frac{1}{2}(g(r + \tilde{\lambda}(r)) + g(r - \tilde{\lambda}(r))), \tag{3.78}$$

$$\tilde{\omega}(\tilde{s}^*(r)) \;=\; \frac{1}{2}(g(r + \tilde{\lambda}(r)) - g(r - \tilde{\lambda}(r))) \tag{3.79}$$

for all $r \ge 0$.

For $(r,v) \in \mathbb{R}_+ \times \mathbb{R}$ we define an auxiliary function

$$\tilde{f}(r,v) \;=\; \int_0^v \Phi\left(\frac{1}{2}(g(r+z) + g(r-z))\right) dz\,. \tag{3.80}$$

Then we deduce by invoking (3.41) that the formula

$$\tilde{f}_{rr}(r,v) - \tilde{f}_{vv}(r,v) \;=\; -P(r,v)\,, \tag{3.81}$$

holds for all $(r,v) \in \mathbb{R}_+ \times \mathbb{R}$ whenever g is smooth. Approximating a general function g satisfying Hypothesis 3.7 (i) by smooth functions and passing to the limit we obtain (3.81) almost everywhere. Using the integral formula (3.26) for $\mathcal{F}$ and (3.64), and integrating by parts, we obtain

$$\mathcal{F}[\omega, g(u)](t) \;=\; \eta(0)g(u(t)) + \int_0^\infty \tilde{\omega}(s)\, d\eta(s) \;=\; -\int_0^\infty \eta(s)\, \tilde{\omega}'(s)\, ds\,, \tag{3.82}$$

and the substitution $s = \tilde{s}^*(r)$ from (3.78) – (3.79) yields that

$$\begin{aligned}
\mathcal{F}[\omega, g(u)](t) \;&=\; -\frac{1}{2}\int_0^\infty \eta\left(\frac{1}{2}(g(r+\tilde{\lambda}(r)) + g(r-\tilde{\lambda}(r)))\right) \\
&\qquad \times \left(g'(r+\tilde{\lambda}(r))(1+\tilde{\lambda}'(r)) - g'(r-\tilde{\lambda}(r))(1-\tilde{\lambda}'(r))\right) dr \\
&=\; -\int_0^\infty \left(\tilde{f}_{vv}(r,\tilde{\lambda}(r)) + \tilde{\lambda}'(r)\, \tilde{f}_{vr}(r,\tilde{\lambda}(r))\right) dr \\
&=\; -\int_0^\infty \left(P(r,\tilde{\lambda}(r)) + \frac{d}{dr}\tilde{f}_r(r,\tilde{\lambda}(r))\right) dr
\end{aligned} \tag{3.83}$$

and the assertion follows easily. ■

4 Thermodynamic consistency

According to the discussion in [3, 7], we say that the generalized Prandtl-Ishlinskii operator $\mathcal{H}$ defined in (3.37) is *thermodynamically consistent* if there exists an internal energy operator $\mathcal{U} : W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ (a *clockwise admissible potential* in the terminology of [3]) such that for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ we have

$$\mathcal{U}[u](t) \;\geq\; 0 \quad \forall\, t \geq 0\,, \tag{4.1}$$

$$\mathcal{H}[u](t)\,\frac{d}{dt}u(t) - \frac{d}{dt}\mathcal{U}[u](t) \;\geq\; 0 \quad \text{for a.e. } t > 0\,. \tag{4.2}$$

In the sequel, we consider fixed ω, g, η, λ, γ, and $\mathcal{P}$ as in Proposition 3.15. For $r \geq 0$ and $v \in \mathbb{R}$ we introduce auxiliary functions

$$V(r,v) \;=\; \int_0^v z\,\gamma(r,z)\, dz\,, \tag{4.3}$$

$$G(v) \;=\; \int_0^v g(z)\, dz\,, \tag{4.4}$$

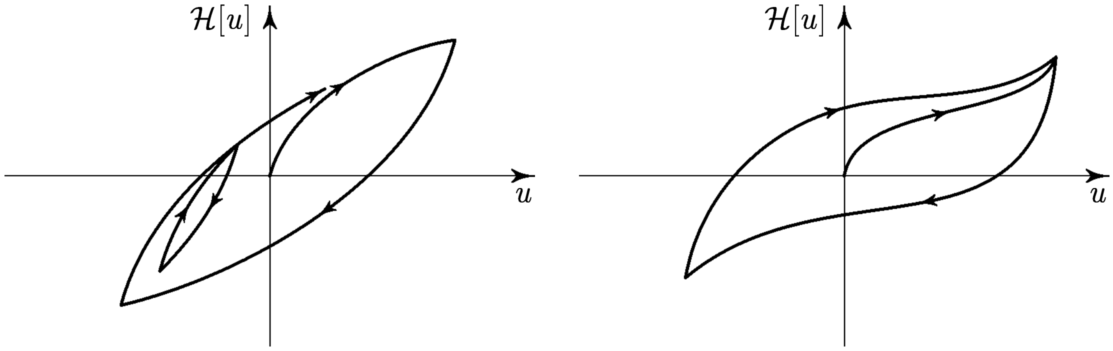

Figure 1: Clockwise convex and clockwise admissible diagrams.

and a Preisach operator

$$\mathcal{V}[\lambda, u](t) = \int_0^\infty V(r, \mathfrak{p}_r[\lambda, u](t))\, dr. \tag{4.5}$$

Proposition 3.15 enables us to check that the operator $\mathcal{W}$ defined by the identity

$$\mathcal{W}[\lambda, u](t) \;=\; \eta(0)\, G(u(t)) - \mathcal{P}[\lambda, u](t)\, u(t) + \mathcal{V}[\lambda, u](t) \tag{4.6}$$

is a good candidate for the internal energy operator for $\mathcal{F}[\omega, g(u)]$. Indeed, we have

$$\begin{aligned} \mathcal{F}[\omega, g(u)](t)\, \frac{d}{dt}\, u(t) - \frac{d}{dt}\, \mathcal{W}[\lambda, u](t) \;&=\; \int_0^\infty \gamma(r, \xi_r(t))\, x_r(t)\, \dot{\xi}_r(t)\, dr \\ &=\; \int_0^\infty \gamma(r, \xi_r(t))\, r \left|\dot{\xi}_r(t)\right|\, dr \;\geq\; 0\,, \end{aligned} \tag{4.7}$$

where $x_r(t) = \mathfrak{s}_r[\lambda, u](t)$, $\xi_r(t) = \mathfrak{p}_r[\lambda, u](t)$. Putting

$$G_0(v) \;=\; \int_0^v g_0(z)\, dz \tag{4.8}$$

we see from (3.37) and (4.7) that the energy inequality (4.2) will be satisfied provided we define

$$\mathcal{U}[u](t) \;:=\; G_0(u(t)) + \mathcal{W}[\lambda, u](t) + K_1 \tag{4.9}$$

with a constant $K_1 > 0$. Condition (4.1) will hold for K_1 sufficiently large if for instance both $G_0(u)$ and $\mathcal{W}[\lambda, u]$ are a priori bounded from below independently of u. A necessary and sufficient condition for the boundedness of G_0 from below reads

$$\int_{-\infty}^0 g_0(r)\, dr \;<\; \infty\,, \quad \int_0^\infty g_0(r)\, dr \;>\; -\infty\,. \tag{4.10}$$

In particular, if $\pm \lim_{r\to\pm\infty} g_0(r) > k \geq 0$, then there exists a constant $K_2 \geq 0$ such that

$$G_0(v) \;\geq\; k|v| - K_2 \quad \forall\, v \in \mathbb{R}\,. \tag{4.11}$$

Example 4.4 below shows that the existence of a lower bound for $\mathcal{W}[\lambda, u]$ is much less trivial in general, and a detailed analysis is still to be done. On the other hand, we will also see that a possible non-existence of a lower bound for $\mathcal{W}[\lambda, u]$ may in some cases be compensated by the term $G_0(u)$.

We now state and prove a necessary and sufficient condition for the thermodynamic consistency of the model.

Proposition 4.1 *Let Hypothesis 3.7 hold, and let γ and G be as in (3.74) and (4.4). For $R \ge u \ge 0$ we define the set*

$$\Omega_{R,u} \;=\; \{(r,v) \in \mathbb{R}_+ \times \mathbb{R}\,;\, 0 < r < R\,,\ 0 < v < \min\{u, R-r\}\}\,, \tag{4.12}$$

and the function

$$\Psi(u,R) \;=\; \eta(0)\, G(u) + \iint_{\Omega_{R,u}} (v-u)\,\gamma(r,v)\, dr\, dv\,. \tag{4.13}$$

Let $\mathcal{W}$ and G_0 be of the form (4.6) and (4.8). Then there exists a constant $K \ge 0$ such that for every $\lambda \in \Lambda$, $\bar{u} \in C(\mathbb{R}_+)$, and $t \ge 0$ we have

$$G_0(\bar{u}(t)) + \mathcal{W}[\lambda, \bar{u}](t) \;\ge\; -K \tag{4.14}$$

if and only if

$$\liminf_{R\to\infty} \Psi(u,R) \;>\; -\infty \quad \forall\, u \ge 0\,, \tag{4.15}$$

$$\liminf_{u\to\pm\infty} \Big(G_0(u) + \liminf_{R\to\infty} \Psi(|u|, R)\Big) \;>\; -\infty\,. \tag{4.16}$$

Note that the set $\Omega_{R,u}$ from (4.12) corresponds to the trapezoidal domain on Figure 2.

Proof. We start by considering the "if" part and assume that (4.15) and (4.16) are satisfied. Since (3.74) and Hypothesis 3.7 yield that γ is non-negative, we see that the integrand in (4.13) is non-positive on $\Omega_{R,u}$. Hence, we see that for fixed u the mapping $R \mapsto \Psi(u,R)$ is non–increasing. Therefore, because of (4.15) and (4.16), we have some constant $K \ge 0$ such that

$$G_0(u) + \Psi(|u|, R) \ge -K, \quad \forall\, R \ge |u|, u \in \mathbb{R}\,. \tag{4.17}$$

We fix $\bar{u} \in C(\mathbb{R}_+)$, $\lambda \in \Lambda$, and $t \ge 0$, and set $u := \bar{u}(t)$, $\tilde{\lambda}(r) := \mathfrak{p}_r[\lambda, \bar{u}](t)$. By (4.6), (3.76), and (4.5), we have

$$\mathcal{W}[\lambda, \bar{u}](t) \;=\; \eta(0)\, G(\bar{u}(t)) + \int_0^\infty \Big(V(r, \tilde{\lambda}(r)) - u\, P(r, \tilde{\lambda}(r))\Big)\, dr\,. \tag{4.18}$$

Assume for instance that $u \ge 0$, the other case is analogous. Recalling (3.75) and (4.3), we see that the function $\kappa(r,v) = V(r,v) - u\,P(r,v)$ has the property

$$\frac{\partial}{\partial v}\kappa(r,v) \;=\; (v-u)\,\gamma(r,v)\,,$$

hence for a fixed r, $\kappa(r,v)$ is non-increasing on $]-\infty,u]$ and non-decreasing on $[u,\infty[$. On the other hand, we have $\tilde\lambda \in \Lambda$, and therefore $\tilde\lambda(r) \le (R-r)^+$ for some $R \ge u$ and every $r > 0$. This implies that

$$\kappa(r,\tilde\lambda(r)) \;\ge\; \kappa(r,\min\{u,(R-r)^+\}) \tag{4.19}$$

for every $r > 0$, hence we conclude by using (4.13) that

$$\mathcal{W}[\lambda,\bar u](t) \;\ge\; \Psi(\bar u(t), R). \tag{4.20}$$

The assertion (4.14) now follows from (4.17), and the "if" part is proved.

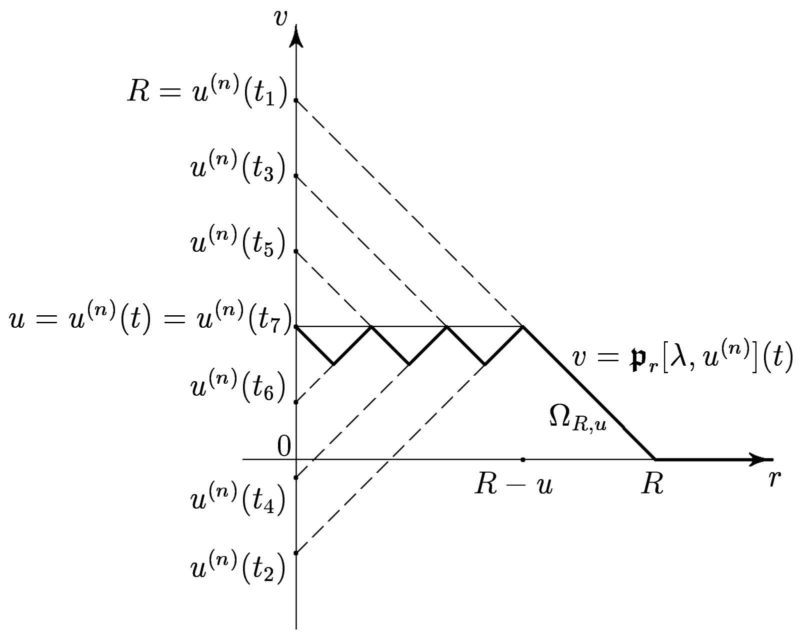

Figure 2: Construction of a lower bound for the internal energy.

To prepare the proof of the "only if" statement, we first check that the estimate (4.20) is optimal in the sense that for each $u \ge 0$, $R > u$ sufficiently large, $t > 0$, and $n \in \mathbb{N}$ there exists a function $u^{(n)} \in C(\mathbb{R}_+)$ with $u^{(n)}(t) = u$ such that

$$\lim_{n\to\infty} \mathcal{W}[\lambda, u^{(n)}](t) = \Psi(u,R)\,. \tag{4.21}$$

We consider some $R > u$ such that $\lambda(r) = 0$ for every $r \ge R$. To see that (4.21) holds, we fix any partition $0 = t_1 < t_2 < \ldots < t_{2n+1} = t$ and put $u_1 = R$, $u_{2k} = u - \frac{n-k+1}{n}(R-u)$, $u_{2k+1} = u + \frac{n-k}{n}(R-u)$ for $k = 1,\ldots,n$. The function $u^{(n)}$ can be defined by the formula $u^{(n)}(t_j) = u_j$ for $j = 1,\ldots,2n+1$, linearly interpolated in each interval $[t_j,t_{j+1}]$ and arbitrarily extended to $[t,\infty[$. Using the representation formula in Proposition 3.5 with $\bar r = r_1 = R$, $r_j = \frac{2n+2-j}{2n}(R-u)$ for $j = 2,\ldots,2n+2$, we obtain

$\mathfrak{p}_r[\lambda, u^{(n)}](t) = (R-r)^+$ for $r \geq r_2 = R - u$, and $0 \leq u - \mathfrak{p}_r[\lambda, u^{(n)}](t) \leq r_j - r_{j+1}$ for $r \in [r_{j+1}, r_j[$, $j = 2, \dots, 2n+1$, hence

$$0 \leq \min\{u, (R-r)^+\} - \mathfrak{p}_r[\lambda, u^{(n)}](t) \leq \frac{R-u}{2n}\,, \tag{4.22}$$

and (4.21) follows, see Figure 2.

Assuming that (4.14) holds for every $\bar{u}$, λ, and t, we now obtain (4.15) and (4.16) directly from (4.21), and the proof is complete. ∎

For practical purposes, it would be convenient to establish reasonable criteria in terms of g_0, g, and φ from Hypothesis 3.7 to ensure that (4.15) and (4.16) hold. Before doing this, we derive the following useful formulas.

Lemma 4.2 *Let Hypothesis 3.7 hold, let Φ be given by (3.25), and let Ψ be given by (4.13). Then for all $R \geq u \geq 0$ we have*

$$\begin{aligned}\Psi(u,R) &= \int_0^{R-u} \left(\Phi(g(r)) - \Phi\left(\frac{1}{2}(g(u+r) - g(u-r))\right)\right) dr \\ &\quad + \int_{R-2u}^{R} \left(\Phi\left(g\left(\frac{R+r}{2}\right)\right) - \Phi\left(\frac{1}{2}(g(R) + g(r))\right)\right) dr \\ &\quad + \int_{R-u}^{R} (\Phi(g(R)) - \Phi(g(r)))\, dr\,.\end{aligned} \tag{4.23}$$

If moreover there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{R>0} \int_{R-2u}^{R} (g(R) - g(r))\, dr \;\leq\; \psi(u) \qquad \forall\, u \geq 0\,, \tag{4.24}$$

then

$$\inf_{R \geq u} \Psi(u,R) = \int_0^{\infty} \left(\Phi(g(r)) - \Phi\left(\frac{1}{2}(g(u+r) - g(u-r))\right)\right) dr. \tag{4.25}$$

Proof. Let us introduce the function

$$\Gamma(r,v) \;:=\; \Phi\left(\frac{1}{2}(g(v+r) - g(v-r))\right) \qquad \text{for } (r,v) \in \mathbb{R}_+ \times \mathbb{R}\,. \tag{4.26}$$

If $g \in C^2(\mathbb{R})$ then $\Gamma_{rr} - \Gamma_{vv} = -\gamma(r,v)$, and Green's Theorem yields

$$\begin{aligned}\iint_{\Omega_R} (v-u)\, \gamma(r,v)\, dr\, dv \;&=\; \iint_{\Omega_R} \Big(((u-v)\,\Gamma_r)_r - ((u-v)\,\Gamma_v + \Gamma)_v\Big)\, dr\, dv \\ &= \int_0^R (u\,\Gamma_v + \Gamma)(r,0)\, dr \;-\; \int_0^{R-u} \Gamma(r,u)\, dr \;-\; \int_0^u ((u-v)\,\Gamma_r)(0,v)\, dv \\ &\quad + \int_0^u \Big(((u-v)\,\Gamma_r) - (u-v)\,\Gamma_v - \Gamma\Big)(R-v,v)\, dv\end{aligned} \tag{4.27}$$

$$
\begin{aligned}
&= \int_0^R \Phi(g(r))\,dr - \int_0^{R-u} \Phi\left(\frac{1}{2}(g(u+r) - g(u-r))\right)\,dr - \eta(0)\int_0^u (u-v)\,g'(v)\,dv \\
&\quad - \int_0^u \left(\frac{\partial}{\partial v}\left((u-v)\Phi\left(\frac{1}{2}(g(R) - g(2v-R))\right)\right) + 2\,\Phi\left(\frac{1}{2}(g(R) - g(2v-R))\right)\right)\,dv \\
&= \int_0^R \Phi(g(r))\,dr + u\,\Phi(g(R)) - \eta(0)\,G(u) \\
&\quad - \int_0^{R-u} \Phi\left(\frac{1}{2}(g(u+r) - g(u-r))\right)\,dr \;-\; \int_{R-2u}^{R} \Phi\left(\frac{1}{2}(g(R) + g(r))\right)\,dr\,,
\end{aligned}
$$

and (4.23) follows. For general g, we get Eq. (4.23) by considering again smooth approximations of g.

To obtain (4.25), we use again the fact, similarly as in the proof of Proposition 4.1, that the function $R \mapsto \Psi(u,R)$ is non-increasing on $[u,\infty[$, hence $\inf_{R\geq u}\Psi(u,R) = \lim_{R\to\infty}\Psi(u,R)$. Since we assume that (4.24) holds and that $\eta(\infty) = 0$, we see that the second and the third integral in (4.23) tend to 0 as $R\to\infty$. Passing to the limit as $R\to\infty$ in (4.23), we obtain precisely (4.25). ■

We now state some sufficient conditions in terms of the functions g and Φ which guarantee the existence of an a priori lower bound for the internal energy.

Corollary 4.3 *Assume in addition to Hypothesis 3.7 that* at least one *of the following three conditions holds.*

(i) g *is concave in* $\mathbb{R}_+$,

(ii) g *has the form* $g(r) = ar + \alpha(r)$ *with* $\lim_{r\to\infty}\alpha(r) = 0$, $\int_0^\infty |\alpha(r)|\,dr < \infty$, *and with a constant* $a > 0$,

(iii) $\lim_{u\to\infty}\Phi(u) = \bar{\Phi} < \infty$, *and* $\int_0^\infty (\bar{\Phi} - \Phi(g(r)))\,dr < \infty$.

Then there exists a constant $K \geq 0$ *such that for every* $u \in C(\mathbb{R}_+)$, $\lambda\in\Lambda$, *and* $t\geq 0$ *we have* $\mathcal{W}[\lambda,u](t) \geq -K$.

Proof. In order to obtain a uniform lower bound for $\mathcal{W}$, it suffices to prove, according to Proposition 4.1, that the function Ψ defined in (4.13) is bounded from below.

In case (i) we have $\frac{1}{2}(g(R)+g(r)) \leq g(\frac{R+r}{2})$, $\frac{1}{2}(g(u+r) - g(u-r)) \leq g(r)$ (note that g is odd!), hence, by (4.23),

$$
\Psi(u,R) \geq \int_{R-u}^{R} (\Phi(g(R)) - \Phi(g(r)))\,dr \;\geq\; 0\,. \tag{4.28}
$$

In case (ii) we notice that condition (4.24) is satisfied. Using now (4.25) and (3.25), we see that

$$
\Psi(u,R) \geq -\eta(0)\int_0^\infty \left|\alpha(r) - \frac{1}{2}(\alpha(u+r) - \alpha(u-r))\right|\,dr \;\geq\; -\frac{5}{2}\eta(0)\int_0^\infty |\alpha(r)|\,dr\,.
$$

The case (iii) is easy as well. We have $\frac{1}{2}(g(u+r)-g(u-r)) \leq g(R)$ for $0 \leq r \leq R-u$, $\frac{1}{2}(g(R)+g(r)) \leq g(R)$ for $0 \leq r \leq R$, hence, by (4.23),

$$\Psi(u,R) \geq -\int_0^R (\Phi(g(R)) - \Phi(g(r)))\, dr \geq -\int_0^\infty (\bar{\Phi} - \Phi(g(r)))\, dr. \tag{4.29}$$

■

We now show two typical examples where the hypotheses of Corollary 4.3 are not fulfilled and $\mathcal{W}$ is unbounded from below. In one of these examples it is nevertheless possible to find a function g_0 in such a way that the total internal energy is bounded from below.

Example 4.4 According to Corollary 4.3, counterexamples will be constructed for functions g which are *convex* in $\mathbb{R}_+$. This will enable us for $R \geq 2u$ to split the first integral on the right-hand side of (4.23) into $R-u > r > u$ and $u \geq r > 0$. In the latter case we have

$$\begin{aligned}
\frac{1}{2}(g(u+r)-g(u-r)) - g(r) &= \frac{1}{2}\int_{u-r}^{u+r} g'(s)\, ds - g(r) \geq \frac{1}{2}\int_0^{2r} g'(s)\, ds - \int_0^r g'(s)\, ds \\
&= \frac{1}{2}\int_0^r (g'(s+r) - g'(s))\, ds \geq 0\,,
\end{aligned}$$

and analogously for $R-2u < r < R$ we have

$$\frac{1}{2}(g(R)+g(r)) \geq g\left(\frac{R+r}{2}\right).$$

The above considerations and (4.23) imply that for a convex function g we have

$$\begin{aligned}
\Psi(u,R) \leq &\int_u^{R-u} \left(\Phi(g(r)) - \Phi\left(\frac{1}{2}(g(r+u)+g(r-u))\right)\right) dr \\
&+ \int_{R-u}^R (\Phi(g(R)) - \Phi(g(r)))\, dr\,.
\end{aligned} \tag{4.30}$$

A. We set $\Phi(u) = \log(1+u)$ for $u \geq 0$, $g(r) = \Phi^{-1}(r) = e^r - 1$ for $r \geq 0$. For any $R > 2u > 0$ we have by (4.30) that

$$\begin{aligned}
\Psi(u,R) &\leq \int_u^{R-u} r\,(1-\log(\cosh(u)))\, dr + \int_{R-u}^R (R-r)\, dr \\
&\leq \frac{1}{2}u^2 - (\log(\cosh(u)) - 1)\left(\frac{1}{2}R^2 - uR\right),
\end{aligned} \tag{4.31}$$

hence Ψ is unbounded from below, and even (4.15) does not hold.

B. In the situation of Corollary 4.3 (iii), the integral condition is also necessary. Consider for some $0 < \alpha < 1/2$ functions Φ, g defined for $u \geq 1$, $r \in \mathbb{R}$ by the formula

$$\Phi(u) = 2 - u^{-\alpha}, \quad g(r) = r|r|,$$

with Φ concave in $[0,1]$. For $R > 2u > 2$ we obtain from (4.30) similarly as in the previous example that

$$\Psi(u,R) \ \leq \ \int_u^{R-u} \left((r^2+u^2)^{-\alpha} - r^{-2\alpha}\right)\, dr + \int_{R-u}^{R} \left(r^{-2\alpha} - R^{-2\alpha}\right)\, dr\,. \tag{4.32}$$

We substitute $R = \varrho u$, $r = us$, and obtain for $\varrho > 2$, $u > 1$ that

$$\Psi(u,\varrho u) \ \leq \ u^{1-2\alpha}\left(\int_{\varrho-1}^{\varrho} \left(s^{-2\alpha} - \varrho^{-2\alpha}\right)\, ds - \int_1^{\varrho-1} \left(s^{-2\alpha} - (s^2+1)^{-\alpha}\right)\, ds\right). \tag{4.33}$$

We have $\lim_{\varrho\to\infty} \int_{\varrho-1}^{\varrho} (s^{-2\alpha} - \varrho^{-2\alpha})\, ds = 0$, hence for ϱ sufficiently large we conclude that

$$\lim_{u\to\infty} \Psi(u,\varrho u) \ = \ -\infty\,. \tag{4.34}$$

We see that Ψ itself is unbounded from below. One the other hand, setting

$$S_0(u) \ = \ \int_0^u \left(\Phi(g(r)) - \Phi\left(\frac{1}{2}(g(u+r) - g(u-r))\right)\right) dr \quad \text{ for } \quad u \geq 0\,,$$

we obtain from (4.23) for $R \geq 2u$ that

$$\Psi(u,R) \geq S_0(u) - \int_u^{R-u} \left(\Phi\left(r^2+u^2\right) - \Phi\left(r^2\right)\right) dr - \int_{R-2u}^{R} \left(\Phi\left(R^2\right) - \Phi\left((R-u)^2\right)\right) dr\,.$$

For a fixed u, the integrals on the right-hand side of the above inequality remain bounded as $R \to \infty$ due to the elementary inequality $\Phi(b) - \Phi(a) \leq \alpha(b-a)a^{-\alpha-1}$ for all $1 \leq a \leq b$. We thus obtain (4.15) and (4.16) provided G_0 grows sufficiently fast as $u \to \pm\infty$.

5 Equations of visco-elasto-plasticity

As an application of the above theory, we consider the equation

$$\varrho u_{tt} - \sigma_x \ = \ f(x,t)\,, \tag{5.1}$$

where

$$\sigma \ = \ \mu u_{xt} + E u_x + \hat{\sigma}\,, \tag{5.2}$$

with given constants $\varrho > 0$, $\mu > 0$, $E \geq 0$, coupled with the boundary conditions

$$u(0,t) = \sigma(1,t) = 0\,, \tag{5.3}$$

and initial conditions

$$u(x,0) = u^0(x)\,, \quad u_t(x,0) = u^1(x)\,, \tag{5.4}$$

with data $u^0, u^1 \in W^{2,2}(0,1)$, $f \in L^2_{\mathrm{loc}}(0,\infty; L^1(0,1))$ satisfying some further conditions which will be specified later.

We assume that $\hat{\sigma}$ is given by means of an operator $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ according to the formula

$$\hat{\sigma}(x,t) = \mathcal{H}[u_x(x,\cdot)](t) \tag{5.5}$$

under the following hypotheses which are typically satisfied if $\mathcal{H}$ is a generalized Prandtl-Ishlinskii operator (3.37), see Subsection 3.3 and Section 4.

Hypothesis 5.1 $\mathcal{H} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ *is a causal locally Lipschitz operator such that*

(i) $E\mathcal{I} + \mathcal{H} \in PO_h(a,b)$ *for some* $h \geq 0$, *and* $a \leq b$, *where* $\mathcal{I} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ *is the identity operator.*

(ii) $\mathcal{H}$ *maps* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *into* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *and there exists a non-decreasing function* $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for every* $\varepsilon \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *we have*

$$(\mathcal{H}[\varepsilon]_t(t))^2 \;\leq\; \zeta(|\varepsilon|_{[0,t]})\,\mathcal{H}[\varepsilon]_t(t)\,\varepsilon_t(t) \;\leq\; \zeta^2(|\varepsilon|_{[0,t]})\,\varepsilon_t^2(t) \qquad \textit{a. e.} \tag{5.6}$$

(iii) *There exists a constant* $k \geq 0$ *and an internal energy operator* $\mathcal{U} : W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *such that*

$$\mathcal{U}[\varepsilon](t) \geq k\,|\varepsilon(t)| \qquad \forall\, t \geq 0 \;\; \forall\, \varepsilon \in C(\mathbb{R}_+)\,, \tag{5.7}$$

$$\mathcal{U}[\varepsilon]_t \leq \mathcal{H}[\varepsilon]\,\varepsilon_t \qquad \textit{a. e.} \;\; \forall\, \varepsilon \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)\,. \tag{5.8}$$

Existence of solutions to equations of the type (5.1) – (5.2) with a hysteresis operator $\mathcal{H}$ has been investigated e. g. in [2]. Here, we are interested in the long-time behaviour as $t \to \infty$.

The causality of and the local Lipschitz property of $\mathcal{H}$ entail that there exists a locally Lipschitz continuous function $I_{\mathcal{H}} : \mathbb{R} \to \mathbb{R}$ such that for every $\varepsilon \in C(\mathbb{R}_+)$ we have $\mathcal{H}[\varepsilon](0) = I_{\mathcal{H}}(\varepsilon(0))$.

We denote by $|\cdot|_p$ the norm in $L^p(0,1)$ for $1 \leq p \leq \infty$. We will systematically use the embedding inequalities

$$|v|_2 \;\leq\; |v|_\infty \;\leq\; |v_x|_1 \;\leq\; |v_x|_2 \tag{5.9}$$

for every $v \in W^{1,2}(0,1)$ such that $v(0) = 0$ or $v(1) = 0$.

We now can state the main result of this section.

Theorem 5.2 *Let* $\mathcal{H}$ *be an operator satisfying Hypothesis 5.1, and let the data* $u^0, u^1 \in W^{2,2}(0,1)$, $f \in L^2_{\mathrm{loc}}(0,\infty; L^1(0,1))$ *be such that*

$$u^0(0) = 0\,,\;\; \mu\, u^1_x(1) + E u^0_x(1) + I_{\mathcal{H}}(u^0_x(1)) = 0\,, \tag{5.10}$$

$$u^0_x(x) \in [a,b] \qquad \forall\, x \in [0,1]\,, \tag{5.11}$$

the right-hand side f *is of the form*

$$f(x,t) = f_1(x,t) + f_2(x)\,, \tag{5.12}$$

where $f_1, (f_1)_t = f_t \in L^2(0,\infty; L^1(0,1))$, $f_2 \in L^1(0,1)$ *are functions satisfying*

$$|f(t)|_1 \le h \quad \forall\, t \ge 0\,, \tag{5.13}$$

$$\textit{either} \quad E > 0 \quad \textit{or} \quad E = 0 \quad \textit{and} \quad |f_2|_1 \le k\,, \tag{5.14}$$

where k, h, a, b *are the constants from Hypothesis 5.1. Then the system (5.1) – (5.5) admits a unique solution* $u \in L^\infty(\,]0,1[\,\times\,]0,\infty[\,) \cap C([0,1]\times[0,\infty[\,)$ *such that the functions* $u_x, \hat\sigma$ *belong to* $L^\infty(\,]0,1[\,\times\,]0,\infty[\,) \cap C([0,1]\times[0,\infty[\,)$, *the functions* $u_t, u_{tt}, u_{xt}, u_{xtt}, \sigma_x$ *belong to* $L^2(\,]0,1[\,\times\,]0,\infty[\,)$, *and we have*

$$\lim_{t\to\infty} \int_0^1 \left(u_{tt}^2 + u_{xt}^2\right)(x,t)\,dx \;=\; 0\,. \tag{5.15}$$

Proof. Following [1, 11], we consider the auxiliary problem

$$\varrho p_t - \mu p_{xx} \;=\; (E\mathcal{I} + \mathcal{H})[\varepsilon] + F(x,t)\,, \tag{5.16}$$

$$q_t \;=\; -(E\mathcal{I} + \mathcal{H})[\varepsilon] - F(x,t)\,, \tag{5.17}$$

$$\varepsilon \;=\; \frac{1}{\mu}(\varrho p + q)\,, \tag{5.18}$$

where $F(x,t) = -\int_x^1 f(\xi,t)\,d\xi$, with boundary conditions

$$p_x(0,t) = p(1,t) = 0\,, \tag{5.19}$$

and initial conditions

$$p(x,0) = p^0(x) := -\int_x^1 u^1(\xi)\,d\xi\,, \qquad q(x,0) = q^0(x) := \mu u_x^0(x) - \varrho p^0(x)\,. \tag{5.20}$$

This is an easy system with a causal locally Lipschitz right-hand side. The existence of a unique global strong solution follows from a standard compactness argument based on a suitable (Galerkin, say) approximation provided we derive a uniform L^∞ bound for the argument ε of the nonlinearity. We proceed formally, having implicitly in mind the corresponding sequence of approximate problems.

By $C_1, C_2, \ldots$ we denote any constants depending only on the data and, especially, independent of t.

We define the function

$$u(x,t) = u^0(x) + \int_0^t p_x(x,\tau)\,d\tau\,. \tag{5.21}$$

Then we have

$$\varepsilon_t = p_{xx} = u_{xt} \ \text{a. e.,} \qquad u_x(x,0) = \varepsilon(x,0)\,, \qquad \text{hence} \ \ \varepsilon = u_x\,. \tag{5.22}$$

Testing Eq. (5.16) by $-p_{xx}$ and integrating by parts we obtain

$$\frac{d}{dt}\left(\frac{\varrho}{2}\,|p_x(t)|_2^2 + \frac{E}{2}\,|\varepsilon(t)|_2^2\right) + \int_0^1 (\mathcal{H}[\varepsilon]\,\varepsilon_t)(x,t)\,dx + \mu|p_{xx}(t)|_2^2 \ = \ \int_0^1 f(x,t)\,p_x(x,t)\,dx\,. \tag{5.23}$$

The energy inequality (5.8), (5.12), and (5.21) yield

$$\begin{aligned}\frac{d}{dt}\left(\frac{\varrho}{2}\,|p_x(t)|_2^2 + \frac{E}{2}\,|\varepsilon(t)|_2^2 + |\mathcal{U}[\varepsilon](t)|_1 - \int_0^1 f_2(x)\,u(x,t)\,dx\right) + \mu|p_{xx}(t)|_2^2 \\ \leq \ \int_0^1 f_1(x,t)\,p_x(x,t)\,dx\,.\end{aligned} \tag{5.24}$$

By integration of (5.24) over t and using (5.7) and (5.9) for $v = u$ we obtain

$$\frac{\varrho}{2}\,|p_x(t)|_2^2 + \frac{E}{2}\,|\varepsilon(t)|_2^2 + k\,|\varepsilon(t)|_1 + \mu\int_0^t |p_{xx}(\tau)|_2^2\,d\tau \leq\ C_1 + |f_2|_1|\varepsilon(t)|_1 + \int_0^t |f_1(\tau)|_2\,|p_x(\tau)|_2\,d\tau\,, \tag{5.25}$$

where

$$C_1 = \frac{\varrho}{2}\,|p_x^0|_2^2 + \frac{E}{2}\,|u_x^0|_2^2 + |\mathcal{U}[\varepsilon](0)|_1 + \left|\int_0^1 f_2(x)\,u_x^0(x)\,dx\right|\,. \tag{5.26}$$

Recalling (5.14) and (5.9) for $v = p_x$, we conclude that

$$\sup_{t\geq 0}\left(|p_x(t)|_2^2 + E\,|\varepsilon(t)|_2^2 + \int_0^t |p_{xx}(\tau)|_2^2\,d\tau\right) \leq\ C_2\,, \tag{5.27}$$

and in particular,

$$\sup_{t\geq 0}\left(|p(t)|_\infty + \left(\int_0^t |\varepsilon_t(\tau)|_2^2\,d\tau\right)^{1/2}\right) \leq\ C_3\,. \tag{5.28}$$

To derive the L^∞-bound for ε, we set

$$\delta = \frac{1}{\mu}(2\varrho C_3 + 1)\,, \tag{5.29}$$

and using Hypothesis 5.1 (i) we find suitable constants $A \leq a \leq b \leq B$ such that $E\mathcal{I} + \mathcal{H} \in PO(a,b,A,B,\delta,h)$. Let $x \in [0,1]$ and $t_1 > 0$ be arbitrarily chosen such that

$$\varepsilon(x,t) \in\,]A-\delta, B+\delta[\quad \forall\, t \in [0,t_1]\,. \tag{5.30}$$

Assume that for some $t < t_1$ we have

$$q(x,t)\ >\ \mu B + \varrho C_3\,. \tag{5.31}$$

Then

$$\varepsilon(x,t) = \frac{1}{\mu}(\varrho p(x,t) + q(x,t)) > B + \frac{\varrho}{\mu}(C_3 - |p(x,t)|) \geq B\,,$$

hence by hypothesis

$$(E\mathcal{I} + \mathcal{H})[\varepsilon(x,\cdot)](t) \ \geq \ h\,. \tag{5.32}$$

Eq. (5.17) then yields $q_t(x,t) \leq -h - F(x,t) \leq 0$, and we conclude that the inequality

$$\frac{1}{2}\frac{d}{dt}\left((q(x,t) - \mu B - \varrho C_3)^+\right)^2 \ = \ q_t(x,t)(q(x,t) - \mu B - \varrho C_3)^+ \ \leq \ 0 \tag{5.33}$$

holds for a. e. $0 < t < t_1$, hence, by (5.11) and (5.28), we have $q(x,t) \leq \mu B + \varrho C_3$ for all $t \in [0, t_1]$. From (5.18) it follows that

$$\varepsilon(x,t) \ \leq \ B + \frac{2\varrho}{\mu} C_3 \ = \ B + \delta - \frac{1}{\mu}\,. \tag{5.34}$$

Similarly, assuming

$$q(x,t) \ < \ \mu A - \varrho C_3 \tag{5.35}$$

instead of (5.31) we repeat the above argument and obtain

$$\varepsilon(x,t) \ \geq \ A - \delta + \frac{1}{\mu}\,. \tag{5.36}$$

We therefore have

$$|\varepsilon(x,t)| \ \leq \ C_4 := \max\{|A - \delta + 1/\mu|, |B + \delta - 1/\mu|\} \tag{5.37}$$

for every $x \in [0,1]$ in the whole interval of existence, hence the solution is global and (5.37) holds for all $(x,t) \in [0,1] \times \mathbb{R}_+$.

We easily check using (5.22) that u defined by (5.21) is a solution to (5.1) – (5.5). Passing again through the approximations, we can consider the time derivative of (5.1) and test it consecutively by u_{tt} and u_t. Thus, we derive the identities

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \left(\frac{\varrho}{2}\, u_{tt}^2 + \frac{E}{2}\, u_{xt}^2\right)(x,t)\,dx + \int_0^1 \left(\mu u_{xtt}^2 + \hat{\sigma}_t\, u_{xtt}\right)(x,t)\,dx \qquad (5.38)\\
&\qquad = \int_0^1 (f_t\, u_{tt})(x,t)\,dx\,,\\
&\frac{d}{dt}\int_0^1 \left(\varrho u_t u_{tt} + \frac{\mu}{2}\, u_{xt}^2\right)(x,t)\,dx + \int_0^1 \left(E u_{xt}^2 + \hat{\sigma}_t\, u_{xt}\right)(x,t)\,dx \qquad (5.39)\\
&\qquad = \int_0^1 \left(\varrho u_{tt}^2 + f_t\, u_t\right)(x,t)\,dx
\end{aligned}$$

for almost all $t > 0$. We now set

$$\kappa \ = \ \frac{\mu}{2\varrho}\,, \tag{5.40}$$

$$\mathcal{E}(t) \ = \ \int_0^1 \left(\frac{\varrho}{2}\, u_{tt}^2 + \frac{E + \kappa\mu}{2}\, u_{xt}^2 + \kappa\varrho\, u_t u_{tt}\right)(x,t)\,dx\,. \tag{5.41}$$

Mutliplying (5.39) by κ and adding the resulting equation to (5.38), we conclude that

$$\frac{d}{dt}\,\mathcal{E}(t)+\int_0^1\left(\frac{\mu}{2}u_{xtt}^2+\hat{\sigma}_t\,u_{xtt}+\kappa E u_{xt}^2+\kappa\,\hat{\sigma}_t\,u_{xt}\right)(x,t)\,dx\ \le\ \int_0^1(|f_t|(|u_{tt}|+\kappa|u_t|))(x,t)\,dx\,. \tag{5.42}$$

Recalling (5.5), (5.6), (5.22), and (5.37), and defining $\alpha := \zeta(C_4)\ge 0$, we observe that

$$\hat{\sigma}_t^2\ \le\ \alpha\hat{\sigma}_t\,u_{xt}\le \alpha^2 u_{xt}\qquad \text{a. e.}\quad . \tag{5.43}$$

We now fix constants $\beta>0\,,\ L\ge 0$ such that

$$\beta+\kappa\alpha>\frac{\alpha^2}{2\mu}\,,\qquad L+\kappa E\ =\ 2\beta\,, \tag{5.44}$$

and find $\gamma_1>0$ such that (5.43) and Young's inequality yield

$$\begin{aligned} &L u_{xt}^2+\frac{\mu}{2}u_{xtt}^2+\hat{\sigma}_t\,u_{xtt}+\kappa E u_{xt}^2+\kappa\,\hat{\sigma}_t\,u_{xt} \\ &\qquad\ge\ \beta u_{xt}^2+\frac{\mu}{2}u_{xtt}^2+\hat{\sigma}_t\,u_{xtt}+\frac{1}{\alpha^2}(\beta+\kappa\alpha)\hat{\sigma}_t^2\ge\gamma_1(u_{xtt}^2+u_{xt}^2)\,. \end{aligned} \tag{5.45}$$

On the other hand, by (5.40), (5.41), and Young's inequality, there exist constants $\gamma_2>\gamma_3>0$ such that

$$\gamma_3\,\mathcal{E}(t)\ \le\ \int_0^1\left(u_{tt}^2+u_{xt}^2\right)(x,t)\,dx\ \le\ \gamma_2\,\mathcal{E}(t) \tag{5.46}$$

for every $t>0$. Using (5.42), (5.45), (5.46), we thus obtain

$$\frac{d}{dt}\,\mathcal{E}(t)+c\,\mathcal{E}(t)\ \le\ K\int_0^1(f_t^2+u_{xt}^2)(x,t)\,dx \tag{5.47}$$

for some $c>0$ and $K\ge L$. From (5.27) it follows that

$$\int_0^\infty\int_0^1 u_{xt}^2(x,t)\,dx\,dt\ \le\ C_2\,, \tag{5.48}$$

hence the function $\chi(t)=K\int_0^1(f_t^2+u_{xt}^2)(x,t)\,dx$ belongs to $L^1(0,\infty)$.
Multiplying (5.47) by e^{ct}, and integrating over time, we obtain

$$\mathcal{E}(t)\ \le\ e^{-ct}\mathcal{E}(0)+\int_0^t e^{c(s-t)}\chi(s)\,ds\ \le\ e^{-ct}\mathcal{E}(0)+e^{-ct/2}\int_0^{t/2}\chi(s)\,ds+\int_{t/2}^t\chi(s)\,ds\,, \tag{5.49}$$

hence $\mathcal{E}(t)\to 0$ at $t\to\infty$ and the proof is complete. ■

Remark 5.3 If $f_1\equiv 0\,,\ E>0$, and if α is sufficiently small (this may include a condition on an upper bound for $|u_x|$), then we may choose $L=0$ in (5.44) and obtain (5.47) with $K=0$. In this case, the decay is exponential.

Exponential decay is obtained also in the case where the operator $\mathcal{H}$ is clockwise convex, where a second order energy inequality of the type (3.29) enables us to obtain additional estimates from (5.38).

Due to the memory stored in the hysteresis operator, we cannot expect that u_x would asymptotically vanish as $t \to \infty$. A more detailed discussion on this subject can be found in Section III.2 of [7].

We see that Hypothesis 5.1(i) is satisfied independently of $E \geq 0$ if $\mathcal{H} \in PO_h(a,b)$ for some $h \geq 0$ and $a \leq b$. On the other hand, if $\mathcal{H}$ is any causal operator with linear growth, say, $|\mathcal{H}[u](t)| \leq k + \ell\,|u|_{[0,t]}$ for every $u \in C(\mathbb{R}_+)$ and $t \geq 0$ with some constants $k > 0$, $\ell \geq 0$, then $E\mathcal{I} + \mathcal{H} \in PO_\infty(a,b)$ for every $E > \ell$ and every $[a,b] \subset \mathbb{R}$. This follows from Definition 2.1 (v) if for arbitrary $h \geq 0$ and $\delta > 0$ we choose $-A = B = \max\{-a, b, (h + k + \ell\delta)/(E - \ell)\}$.

References

[1] Andrews G.: On the existence of solutions to the equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$. *J. Differential Equations* **35** (1980), 200–231.

[2] Brokate, M.; Sprekels, J.: Existence and optimal control of mechanical processes with hysteresis in viscous solids. *IMA J. Appl. Math.* **43**, 219–229.

[3] Brokate, M.; Sprekels, J.: *Hysteresis and phase transitions*. Appl. Math. Sci., Vol. 121, Springer-Verlag, New York, 1996.

[4] Franců, J.; Krejčí, P.: Homogenization of scalar wave equations with hysteresis. *Cont. Mech. & Ther.* **11** (1999), 371–391.

[5] Klein, O.: Asymptotic behaviour for a phase-field model with hysteresis in one-dimensional thermo-visco-plasticity. Preprint 734, Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Berlin, 2002, submitted to *Appl. Math.*

[6] Krasnosel'skii, M. A.; Pokrovskii, A. V.: *Systems with hysteresis*. Springer-Verlag, Heidelberg, 1989 (Russian edition: Nauka, Moscow, 1983).

[7] Krejčí, P.: *Hysteresis, convexity and dissipation in hyperbolic equations.* Gakuto Int. Series Math. Sci. & Appl., Vol. 8, Gakkōtosho, Tokyo, 1996.

[8] Krejčí, P.: Resonance in Preisach systems. *Appl. Math.* **45** (2000), 439–468.

[9] Krejčí, P.; Sprekels, J.: Weak stabilization of solutions to PDEs with hysteresis in thermovisco-elasto-plasticity. In: *EQUADIFF 9 - Proceedings* (R.P. Agarwal, F. Neuman, J. Vosmanský eds.), Masaryk Univ., Brno 1998, 81–96.

[10] Krejčí, P.; Sprekels, J.; Zheng, S.: Asymptotic behaviour for a phase-field system with hysteresis. *J. Differential Equations* **175** (2001), 88–107.

[11] Pego, R. L.: Phase transitions in one–dimensional nonlinear viscoelasticity: admissibility and stability. *Arch. Rational Mech. Anal.* **97** (1987), 353–394.

[12] Racke, R.; Zheng, S.: Global existence and asymptotic behavior in nonlinear thermoviscoelasticity. *J. Differential Equations* **134** (1997), 46–67.

[13] Visintin, A. : *Differential models of hysteresis*. Springer-Verlag, New York, 1994.