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Optimization of ordinary differential systems with hysteresis

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Abstract

We investigate general control problems governed by ordinary differential systems involving hysteresis operators. Our main hypotheses are of continuity type, and we discuss existence results, discretization methods, and approximation approaches.

1 Introduction

Many engineering systems contain nonlinear functional dependencies of hysteresis type. There are several books devoted to the study of such models, and we just quote the recent monographs by Visintin [16] and by Brokate and Sprekels [6] for a comprehensive introduction into the topic. Concerning the control of such systems basic references are the book of Brokate [2] and his articles [3], [4], [5], and the works of Smith [14], Banks, Smith and Wang [1].

In this paper, we analyze a controlled ordinary differential system with hysteresis:

$$z' = f(t,z,y,u) \quad \text{in } [0,T]\,, \tag{1.1}$$

$$z(0) = z_0 \in I\!R^N\,, \tag{1.2}$$

$$y(t) = W(S[z])(t)\,, \quad S[z](t) = g(z(t)) \text{ in } [0,T]\,, \tag{1.3}$$

$$u(t) \in U \quad \text{in } [0,T]\,. \tag{1.4}$$

Here $f:[0,T]\times I\!R^N \times I\!R \times I\!R^m \to I\!R^N$ and $g: I\!R^N \to I\!R$ are continuous mappings, $U \subset I\!R^m$ is a closed bounded convex set, and $W: C[0,T] \to C[0,T]$ is a hysteresis operator, i.e. rate-independent and with the Volterra property, Sprekels and Brokate [6]. More assumptions will be added as necessity appears.

To the relations (1.1)–(1.4) the following cost functional is associated

$$J(u) = \int_0^T L\left(y(t), z(t), u(t)\right)\, dt\,, \tag{1.5}$$

with $L: I\!R \times I\!R^N \times U \to I\!R$ continuous in y,z, convex and lower semicontinuous with respect to u.

Comparing with the study of Brokate [2], one difference is that we allow g or S to be nonlinear. Moreover, many of the results that we shall establish will use just continuity and will not require local Lipschitz assumptions on the data. Our investigation has the main motivation to provide a theoretically founded way towards the approximation and the numerical analysis of the control problem (1.1)–(1.5). In this respect, it seems that only the paper by Brokate [3] reports numerical experiments in a control problem with hysteresis. While in that work the optimality conditions are solved numerically, our approach uses a complete discretization of (1.1)–(1.5) and the computation of descent directions via the Clarke [8] generalized gradient. This is due to the lack of differentiability of hysteresis operators and still allows the application of bundle-type algorithms, Strodiot and Nguyen [15], Lemaréchal [11].

In Section 2 we give the formulation of the control problem, and we establish the existence of optimal controls under general assumptions. Section 3 introduces the fully discretized optimization problem and studies existence and approximation questions. It is shown that the mapping control $\mapsto$ state is Lipschitz under the given assumptions on f, g, W.

Section 4 uses an alternative formulation of the problem to analyze the variations and the directional derivatives. An algorithm and examples are also indicated.

The notations for spaces, norms, scalar products are standard.

2 Existence

We start with a Peano-type result for the Cauchy problem with hysteresis (1.1)–(1.3). We omit first the dependence on u, and we assume that $f : [0,T] \times I\!R^N \times I\!R \to I\!R^N$, $g : I\!R^N \to I\!R$, and $W : C[0,T] \to C[0,T]$ are continuous mappings and operators. Notice that W remains causal, i.e.

$$v_1|_{[0,t]} = v_2|_{[0,t]} \Rightarrow W(v_1)(t) = W(v_2)(t)\,, \quad t \in [0,T]\,, \tag{2.1}$$

and rate-independent, i.e.:

$$W(v \circ \varphi)(t) = W(v)(\varphi(t))\,, \quad \forall\, v \in C[0,T]\,, \tag{2.2}$$

for any admissible time transformation $\varphi : [0,T] \to [0,T]$, continuous, nondecreasing, and onto.

Theorem 2.1 *Under the above assumptions, the initial value problem* (1.1) – (1.3) *has at least one global solution* $z \in C^1([0,T]; I\!R^N)$ *if the following sublinearity hypotheses are fulfilled:*

$$|W(w)|_{C[0,T]} \le \alpha + \beta |w|_{C[0,T]}\,, \quad \forall\, w \in C[0,T]\,, \tag{2.3}$$

$$|f(t,z,y)|_{I\!R^N} \le \alpha + \beta |z|_{I\!R^N} + \gamma |y|\,, \tag{2.4}$$

$$|g(y)| \le \alpha + \gamma |y|\,, \quad \alpha, \beta, \gamma \in I\!R_+\,. \tag{2.5}$$

Proof. For the positive numbers a, b and $c = \alpha + \beta q$, $q = \max\{|g(z)|; |z - z_0|_{I\!R^N} \le b\}$, consider the set

$$\Delta = \{0 \le t \le a \,,\, |z - z_0|_{I\!R^N} \le b \,,\, |y| \le c\} \,.$$

Denote by $M = \max\{|f(t,z,y)|_{I\!R^N}; (t,z,y) \in \Delta\}$ and $\delta = \inf(a, \frac{b}{M}, \frac{c}{M})$.

Clearly f is uniformly continuous in Δ, that is $|f(t,z,y) - f(\tilde{t},\tilde{z},\tilde{y})|_{I\!R^N} < \varepsilon$ for any $\varepsilon > 0$ if $(t,z,y), (\tilde{t},\tilde{z},\tilde{y}) \in \Delta$ and $|t - \tilde{t}| < \eta(\varepsilon)$, $|z - \tilde{z}|_{I\!R^N} < \eta(\varepsilon)$, $|y - \tilde{y}| < \eta(\varepsilon)$. Denote by $h_\varepsilon = \inf\left(\eta(\varepsilon), \dfrac{\eta(\varepsilon)}{M}\right)$ and take the division $t_j = j h_\varepsilon, j \in I\!N$, of $[0,\delta]$.

We consider the polygonal functions (the Picard iterations with Euler polygonal lines):

$$\begin{aligned} \varphi_\varepsilon(t) &= \varphi_\varepsilon(t_j) + (t - t_j)\, f(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j)), \quad t_j < t \le t_{j+1}\,, \\ \varphi_\varepsilon(0) &= z_0\,, \\ y_\varepsilon(t_j) &= W_f\,(S(\varphi_\varepsilon))\,(t_j)\,. \end{aligned} \tag{2.6}$$

Due to (2.1), relation (2.6) makes sense. Recall that

$$W_f\,(S(\varphi_\varepsilon))\,(t_j) = \tilde{W}_f\,(S(z_0), S\,(\varphi_\varepsilon(t_1))\,, \ldots, S\,(\varphi_\varepsilon(t_j))) \tag{2.7}$$

with $W_f : C[0,T] \to I\!R$ being the generating functional of W, Brokate and Sprekels [6]. Here, $\mathcal{S}$ is the set of all finite strings of real numbers and $\tilde{W}_f$ the application induced on $\mathcal{S}$ by W_f (see Section 3).

Note that (2.2) plays an essential role in this construction. We have the following inequality:

$$\begin{aligned} |\varphi_\varepsilon(t_{j+1}) - z_0|_{I\!R^N} &\le |\varphi_\varepsilon(t_j) - z_0|_{I\!R^N} + h_\varepsilon\,|f\,(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j))| \\ &\le |\varphi_\varepsilon(t_j) - z_0|_{I\!R^N} + M\,h_\varepsilon\,. \end{aligned} \tag{2.8}$$

Here, we argue by induction: Assuming that $(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j)) \in \Delta$ and $|\varphi_\varepsilon(t_j) - z_0|_{I\!R^N} \le j\,M\,h_\varepsilon$, relation (2.8) gives that $|\varphi_\varepsilon(t_{j+1}) - z_0| \le (j+1)M\,h_\varepsilon \le M\,\delta \le b$.

By (2.3), (2.7), and (2.8), we have

$$|y_\varepsilon(t_j)| \le \alpha + \beta \max_{0 \le i \le j} |S\,(\varphi_\varepsilon(t_i))|_{I\!R^N} \le \alpha + \beta\,q = c\,. \tag{2.9}$$

Inequalities (2.8), (2.9) show that $(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j)) \in \Delta$ in all steps (2.6) such that $j\,h_\varepsilon \in [0,\delta]$. Then, $|f(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j))|_{I\!R^N} \le M$, and (2.6) gives directly that

$$|\varphi_\varepsilon(t) - \varphi_\varepsilon(s)|_{I\!R^N} \le M|t - s| \le \eta(\varepsilon)\,, \forall\, \varepsilon > 0\,, \forall\, t,s \in [0,\delta]\,. \tag{2.10}$$

Then $\varphi_\varepsilon \to \varphi$ uniformly in $[0,\delta]$, on a subsequence, and $\varphi \in C([0,\delta]; I\!R^N)$. If $\tilde{y}_\varepsilon(t) = W(S(\varphi_\varepsilon))(t)$, then $\tilde{y}_\varepsilon(t_j) = y_\varepsilon(t_j)$ and $\tilde{y}_\varepsilon \to y = W(S(\varphi))$ in $C[0,\delta]$ due to the continuity of S, W and on the same subsequence.

We also have that

$$\varphi_\varepsilon'(t) \;=\; f(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j)) \text{ if } t \in]t_j, t_{j+1}[\tag{2.11}$$

while, by (2.10),

$$|f\,(t_j, \varphi_\varepsilon(t_j), y_\varepsilon(t_j)) \;-\; f\,(t, \varphi_\varepsilon(t), y_\varepsilon(t))|_{I\!R^N} \;\le\; \varepsilon\,, \quad t \in [t_j, t_{j+1}] \tag{2.12}$$

by the uniform continuity of f in Δ.

For any $\lambda > 0$, again the uniform continuity of f in Δ gives that

$$|f\,(t, \varphi_\varepsilon(t), y_\varepsilon(t_j)) \;-\; f\,(t, \varphi_\varepsilon(t), \tilde{y}_\varepsilon(t))|_{I\!R^N} \;\le\; \lambda\,, \quad t \in [t_j, t_{j+1}] \tag{2.13}$$

if $\varepsilon < \varepsilon(\lambda)$, due to the equicontinuity of the sequence $\{\tilde{y}_\varepsilon\}$ and to $\tilde{y}_\varepsilon(t_j) = y_\varepsilon(t_j)$.

Relations (2.11)–(2.13) allow to pass to the limit and to see that φ, y give a solution of (1.1)–(1.3) in $[0,\delta]$ and that $\varphi \in C^1[0,\delta]$.

Under assumptions (2.3)–(2.5), it is well-known that the local solution is in fact a global one, Brokate and Sprekels [6], p. 126. □

Remark. By **Theorem 2.1**, uniqueness may be not true for the state system (1.1)–(1.3). The control problem (1.1)–(1.5) has to be understood as a minimization over pairs: to each control we associate all the possible states. This is well-known in the setting of optimal control theory of ODEs, Cesari [7], or in the case of singular control problems for PDEs, Lions [12].

We assume that the continuity of $L(\cdot,\cdot,u) : I\!R \times I\!R^N \to I\!R_+$ is uniform with respect to $u \in U$. The mapping f is affine with respect to u, i.e.

$$f(t,z,y,u) \;=\; f_1(t,z,y) \;+\; f_2(t,z,y)\,u \tag{2.14}$$

where $f_1 : [0,T] \times I\!R^N \times I\!R \to I\!R^N, f_2 : [0,T] \times I\!R^N \times I\!R \to I\!R^{m\times N}$ are continuous sublinear mappings (like in (2.4)).

As U is bounded, convex and closed, the admissible controls are in $L^\infty([0,T]; I\!R^m)$, and the mapping (2.14) will not satisfy the continuity requirements of **Theorem 2.1** (with respect to t, via u). However, the argument from its proof can be repeated when u is continuous, and a simple approximation argument may be used for $u \in L^\infty([0,T]; I\!R^m)$. The state $z \in W^{1,\infty}([0,T]; I\!R^N)$ in this case.

Theorem 2.2 *Under the above assumptions, the optimal control problem* (1.1) – (1.5) *has at least one optimal triplet* $[u^*, z^*, y^*] \in L^\infty([0,T]; U) \times W^{1,\infty}([0,T]; I\!R^N) \times C[0,T]$.

Proof. Let $[u_n, z_n, y_n]$ be a minimizing sequence. Then $\{u_n\}$ is bounded in $L^\infty([0,T];\mathbb{R}^m)$. By (2.4), (2.14) and (1.1), (1.2), we have

$$|z_n(\hat{t})|_{\mathbb{R}^N} \le |z_0|_{\mathbb{R}^N} + \int_0^{\hat{t}} |f(t, z_n(t), y_n(t), u_n(t))|_{\mathbb{R}^N}\, dt$$

$$\le |z_0|_{\mathbb{R}^N} + c\int_0^{\hat{t}} (\alpha + \beta|z_n(t)|_{\mathbb{R}^N} + \gamma|y_n(t)|)\, dt\,.$$

We may consider the truncated functions $\tilde{y}_n, \tilde{z}_n$ to the interval $[0,t]$, and we still have $\tilde{y}_n(t) = y_n(t) = W(S(\tilde{z}_n))(t)$. Then (2.3) applies, and we get:

$$|z_n(\hat{t})|_{\mathbb{R}^N} \le |z_0|_{\mathbb{R}^N} + c\int_0^{\hat{t}} \left(\alpha + \beta|z_n(t)|_{\mathbb{R}^N} + \alpha + \beta \sup_{s\in[0,t]} |z_n(s)|_{\mathbb{R}^N}\right) dt$$

$$\le |z_0|_{\mathbb{R}^N} + 2\,c\,\alpha\,\hat{t} + 2\,c\,\beta\int_0^{\hat{t}} \sup_{s\in[0,t]} |z_n(s)|_{\mathbb{R}^N}\, dt\,.$$

Consequently, by taking the supremum in both sides of the inequality above, we obtain

$$\sup_{t\in[0,\hat{t}]} |z_n(t)|_{\mathbb{R}^N} \le |z_0|_{\mathbb{R}^N} + 2\,c\,\alpha\,T + 2\,c\,\beta\int_0^{\hat{t}} \sup_{s\in[0,t]} |z_n(s)|_{\mathbb{R}^N}\, dt\,,$$

and Gronwall's lemma shows that $\{z_n\}$ is bounded in $C([0,T];\mathbb{R}^N)$. By (1.1), (1.2), (2.14) we see that $\{z_n\}$ is bounded in $W^{1,\infty}([0,T];\mathbb{R}^N)$. And (2.3), (2.5), (1.3) give that $\{y_n\}$ is bounded in $C[0,T]$.

We denote $u_n \to \bar{u} \in U$ weakly* in $L^\infty([0,T];\mathbb{R}^m)$ and $z_n \to \bar{z}$ in $C([0,T];\mathbb{R}^N)$, on a subsequence. Then, the continuity of W, S gives that $y_n \to \bar{y}$ in $C[0,T]$ with $\bar{y} = W(S(\bar{z}))$.

Moreover, (2.4) and (2.14), together with the Lebesgue theorem, ensures that

$$\int_0^{\hat{t}} f(t, z_n(t), y_n(t), u_n(t))\, dt \to \int_0^{\hat{t}} f(t, \bar{z}(t), \bar{y}(t), \bar{u}(t))\, dt$$

which shows that $[\bar{u}, \bar{z}, \bar{y}]$ satisfies (1.1)–(1.4), i.e. it is admissible for the control problem.

Notice that

$$\left|\int_0^T L(z_n(t), y_n(t), u_n(t))\, dt - \int_0^T L(\bar{z}(t), \bar{y}(t), u_n(t))\, dt\right| \to 0$$

as the continuity of $L(\cdot,\cdot,u_n)$ is uniform with respect to u_n. We also have that

$$\liminf_{n\to\infty} \int_0^T L(\bar z(t),\bar y(t),u_n(t))\,dt \ge \int_0^T L(\bar z(t),\bar y(t),\bar u(t))\,dt$$

by the convexity and lower semicontinuity with respect to u of L and the weak* convergence of u_n.

The two last convergence properties show that $[\bar u,\bar z,\bar y]$ is optimal for the problem (1.1)–(1.5), and the proof is finished. □

3 Discretization

Consider an equidistant partition $\{t_i\}_{i=\overline{0,k}}$ of $[0,T]$ of step-size $\Delta t>0$. The discretized control problem is

$$\text{Min}\left\{\sum_{j=1}^{k+1} L(y_j,z_j,u_j)\,\Delta t\right\} \qquad (\mathbf{P_k})$$

subject to $u_{i+1}\in U$, $i=\overline{0,k}$ and

$$z_{i+1} = z_i + \Delta t\, f(t_{i+1},z_{i+1},y_{i+1},u_{i+1}), \qquad (3.1)$$

$$y_{i+1} = \tilde W_f\,(S(z_0),S(z_1),\ldots,S(z_{i+1}))\,. \qquad (3.2)$$

Here, $\tilde W_f(s)=W_f(\pi_A(s))$, $\forall\ s\in\mathcal S$, the set of all finite strings of real numbers, and z_0 is given. The application $\pi_A(s)$ is the piecewise linear interpolation operator with equidistant nodes in $[0,T]$ corresponding to the number of components of $s\in\mathcal S$, and W_f is the generating functional associated to W, Brokate and Sprekels [6]. The definition of $\tilde W_f$ is essentially based on (2.2). We have the following result.

Proposition 3.1 *If $f,g,\tilde W_f$ are continuous in their arguments and satisfy* (2.3) – (2.5)*, then for every $\{u_{i+1}\}_{\overline{0,k}}\in U^{k+1}$, the equations* (3.1), (3.2) *have at least one solution $\{z_{i+1}\}_{\overline{0,k}}\in[\mathbb R^N]^{k+1}$, $\{y_{i+1}\}_{\overline{0,k}}\in\mathbb R^{k+1}$.*

Proof. The argument is iterative, for every i. Assuming the solution defined at level i is known, then $z_{i+1}\in\mathbb R^N$ and $y_{i+1}\in\mathbb R$ are obtained as a fixed point of the continuous application (in finite dimensional spaces) defined by the right-hand side of (3.1).

If B_{i+1} is a "big" closed ball around 0, in $\mathbb R^N$, containing $z_0,z_1,\ldots,z_i$ in the interior of $\frac12 B_{i+1}$, then (2.3)–(2.5) show that $z_i+\Delta t\,f(t_{i+1},z,y,u_{i+1})\in B_{i+1}$ for Δt "small" if $z\in B_{i+1}$ and $y=\tilde W_f(S(z_0),S(z_1),\ldots,S(z_i),S(z))$. Then, Brouwer's fixed point theorem, Kelley [10], provides at least one solution z_{i+1} of (3.1), and $y_{i+1}=\tilde W_f(S(z_0),S(z_1),\ldots,S(z_i),S(z_{i+1}))$.

The important remark in this argument is that, for $z \in B_{i+1}$, then $|z|_{I\!R^N} \le r_{i+1}$ (the radius), and (2.3)–(2.5) generate a constant $\Gamma > 0$, independent of i, such that $|f(t_{i+1}, z, y, u_{i+1})|_{I\!R^N} \le \Gamma r_{i+1}$. Then $\Delta t = \frac{T}{k} < (2\Gamma)^{-1}$ will be a satisfactory choice of Δt in the above argument, which is also independent of i. □

Corollary 3.2 *Under the above assumptions, if L is convex and lower semicontinuous in u and continuous in y, z uniformly with respect to u, then the discrete control problem* $(\mathbf{P}_k)$ *has at least one optimal n-tuple* $[(u_j^k), (z_j^k), (y_j^k)]_{j=\overline{1,k+1}}$ *in* $U^{k+1} \times (I\!R^N)^{k+1} \times I\!R^{k+1}$.

The argument is similar to that used in the proof of **Theorem 2.2**, and we omit it.

Obviously $\{u_i^k\}$ are bounded in $I\!R^m$ for any k and for $i = \overline{1, k+1}$. We examine the boundedness properties of $\{z_i^k\}$, $\{y_i^k\}$.

By (2.4), (3.1), and the boundedness of U, we get

$$|z_{i+1}^k|_{I\!R^N} \le |z_i^k|_{I\!R^N} + C\,\Delta t\left(1 + |z_{i+1}^k|_{I\!R^N} + |y_{i+1}^k|\right). \tag{3.3}$$

We also have, by (3.2), (2.3), (2.5), that

$$|y_{i+1}^k| \le C\left(1 + \max_{0 \le j \le i+1} |z_j^k|_{I\!R^N}\right). \tag{3.4}$$

Combining (3.3) and (3.4), and taking the maximum with respect to the indices, we obtain

$$\max_{0 \le j \le i+1} |z_j^k|_{I\!R^N} \le \max_{0 \le j \le i} |z_j^k|_{I\!R^N} + C\,\Delta t\left(1 + 2 \max_{0 \le j \le i+1} |z_j^k|_{I\!R^N}\right). \tag{3.5}$$

Here C is an "absolute" constant depending just on α, β, γ from (2.3)–(2.5) and on the bound of $\{u_i^k\}$ in $I\!R^m$.

By summing (2.5) with respect to i, we can infer that

$$\max_{0 \le j \le i+1} |z_j^k|_{I\!R^N} \le \sum_{l=0}^{i} C\,\Delta t\left(1 + 2 \max_{0 \le j \le l+1} |z_j^k|_{I\!R^N}\right). \tag{3.6}$$

If Δt is "small", the discrete Gronwall inequality shows that $\{z_i^k\}$ are bounded in $I\!R^N$ with respect to k and to $i = \overline{0, k+1}$. Inequality (3.4) gives the same for $\{y_i^k\}$ in $I\!R$.

Let us now construct the functions φ_k as in the proof of **Theorem 2.1**. In particular, we have:

$$\varphi_k'(t) = f\left(t_{i+1}, \varphi_k(t_{i+1}), y_k(t_{i+1}), u_k(t_{i+1})\right), \quad t \in]t_{i+1}, t_{i+2}]. \tag{3.7}$$

The mapping $y_k(t) = W(S(\varphi_k))(t)$ and clearly $\{\varphi_k\}$ is bounded in $W^{1,\infty}([0,T]; I\!R^N)$, $\{y_k\}$ is bounded in $C[0,T]$, and u_k (piecewise constant interpolation of u_i^k) is bounded in $L^\infty([0,T]; I\!R^m)$.

We thus have $\varphi_k \to \hat{\varphi}$ uniformly in $C([0,T];\mathbb{R}^N)$, on a subsequence. By the continuity of S and of W, we get $y_k \to \hat{y} = W(S\,\hat{\varphi})$ in $C[0,T]$. We also may assume $u_k \to \hat{u}$, on the same subsequence, weakly* in $L^\infty([0,T];\mathbb{R}^m)$.

Now, compute the difference

$$D = f\left(t, \varphi_k(t), u_k(t)\right) - f\left(t_{i+1}, \varphi_k(t_{i+1}), y_k(t_{i+1}), u_k(t_{i+1})\right)\ ,\ t \in]t_{i+1}, t_{i+2}]\,,$$

and take into account that $u_k(t) \equiv u_k(t_{i+1})$ in this interval, and (2.14). The uniform continuity of f_1, f_2 and the above uniform convergences show that $|D| \leq \varepsilon$ if k is big enough. One can pass to the limit to see that $\hat{\varphi}, \hat{y}, \hat{u}$ is an admissible pair for the original control problem, i.e. it satisfies (1.1)–(1.4). We also have:

Theorem 3.3 *The triplet* $[\hat{u}, \hat{\varphi}, \hat{y}]$ *is optimal for the problem* $(1.1) - (1.5)$.

Proof. The admissibility of $[\hat{u}, \hat{\varphi}, \hat{y}]$ is established above. The optimality of $[\hat{u}, \hat{\varphi}, \hat{y}]$ can be obtained by considering $[u^*, y^*, z^*]$ as given by **Theorem 2.2** and discretizing u^* (or some regularization of it). Denoting this by u_k^*, the discrete cost corresponding to it in $(\mathbf{P_k})$ will be greater or equal to the one associated to u_k. One can pass to the limit in the corresponding equations and in this inequality, which ends the proof. □

Lemma 3.4 *If* $W : C[0,T] \to C[0,T]$ *is Lipschitz of rank* $C > 0$ *and* $s_1 = (v_0, v_1, \ldots, v_l)$, $s_2 = (w_0, w_1, \ldots, w_l) \in \mathcal{S}$ *have the same number of components, then*

$$\left|\tilde{W}_f(v_0, v_1, \ldots, v_l) - \tilde{W}_f(w_0, w_1, \ldots, w_l)\right| \leq C \max_{0 \leq j \leq l} |v_j - w_j|\,. \tag{3.8}$$

Proof.

$$\begin{aligned} & \left|\tilde{W}_f(v_0, \ldots, v_l) - \tilde{W}_f(w_0, \ldots, w_l)\right| = \left|W_f(\pi_A(s_1)) - W_f(\pi_A(s_2))\right| \\ = \ & \left|W(\pi_A(s_1))(T) - W(\pi_A(s_2))(T)\right| \leq C \left|\pi_A(s_1) - \pi_A(s_2)\right|_{C[0,T]} \\ \leq \ & C \max_{0 \leq j \leq l} |v_j - w_j|\,. \end{aligned}$$

□

Remark. Under regularity/Lipschitz assumptions on f, g, L, **Lemma 3.4** shows that the functional dependence from $\{u_i\} \in U^{k+1}$ to the cost is a Lipschitzian dependence. Thus, the Clarke [8] generalized gradient may be used to write the optimality conditions for $(\mathbf{P_k})$ and to devise descent algorithms. More will be said about this below.

Proposition 3.5 *Assume that* f, g *are real Lipschitz mappings and that* W *is a Lipschitz operator in* $C[0,T]$. *Then the correspondence* $\{u_i\} \mapsto \{z_i\}$ *defined by* (3.1), (3.2) *is Lipschitz from* $(\mathbb{R}^m)^{k+1}$ *to* $(\mathbb{R}^N)^{k+1}$.

Proof. Consider another control vector $\{l_i\} \subset U^{k+1}$, and denote by $\{w_i\}$ the solution of (3.1), and by $x_i = \tilde{W}_f(S(z_0), S(w_1), \ldots, S(w_i))$, corresponding to $\{l_i\}$ and to the same initial condition z_0.

Then we have

$$\left| \frac{z_{i+1} - z_i}{\Delta t} - \frac{w_{i+1} - w_i}{\Delta t} \right|_{I\!R^N}$$

$$\begin{aligned} &= |f(t_{i+1}, z_{i+1}, y_{i+1}, u_{i+1}) - f(t_{i+1}, w_{i+1}, x_{i+1}, l_{i+1})|_{I\!R^N} \\ &\leq C \{ |z_{i+1} - w_{i+1}|_{I\!R^N} + |u_{i+1} - l_{i+1}|_{I\!R^m} + |y_{i+1} - x_{i+1}| \} ; \end{aligned} \tag{3.9}$$

$$|y_{i+1} - x_{i+1}| \leq \left| \tilde{W}_f (S(z_0), S(z_1), \ldots, S(z_{i+1})) \right.$$
$$\left. - \tilde{W}_f (S(z_0), S(w_1), \ldots, S(w_{i+1})) \right| \leq C \max_{1 \leq j \leq i+1} |z_j - w_j|_{I\!R^N} , \tag{3.10}$$

due to **Lemma 3.4** and to the Lipschitz assumptions on f, g.

We can rewrite (3.9), (3.10) as

$$|z_{i+1} - w_{i+1}|_{I\!R^N} \leq |z_i - w_i|_{I\!R^N}$$
$$+ C \Delta t \left\{ |z_{i+1} - w_{i+1}|_{I\!R^N} + |u_{i+1} - l_{i+1}|_{I\!R^m} + \max_{1 \leq j \leq i+1} |z_j - w_j|_{I\!R^N} \right\} .$$

Acting as in the proof of **Theorem 3.3** (see (3.5)), we get the desired Lipschitz dependence via the discrete Gronwall inequality and with respect to the l^∞ finite dimensional norm. Since all the norms are equivalent in finite dimensional spaces, the proof is finished. □

Remark. If the differentiable mapping $\tilde{S} : (I\!R^N)^{i+1} \to I\!R^{i+1}$, $\tilde{S}(z_0, z_1, \ldots, z_{i+1}) = (S(z_0), S(z_1), \ldots, S(z_{i+1}))$ has a surjective Jacobian, then the chain rule is valid for the Clarke generalized gradient "∂" of the composed mapping $\tilde{W}_f(\tilde{S})$, Clarke et al. [9], **Theorem 3.2**:

$$\partial(\tilde{W}_f \circ \tilde{S})(\cdot) = \left[\tilde{S}'(\cdot) \right]^* \partial \tilde{W}_f(\tilde{S}(\cdot)) , \tag{3.11}$$

and one can write the equation in variations for (3.1), (3.2) and the first-order optimality conditions for $(\mathbf{P_k})$.

4 Approximation

In this section, in order to fix ideas, we shall assume that the mapping L is quadratic and independent of y. We shall perform a further approximation of the problem $(\mathbf{P_k})$ by the penalization of (3.2) into the cost. We do not regularize the hysteresis operator W, as in Brokate [2]. Roughly speaking, we shall interpret y as a supplementary/artificial control, and (3.2) as a mixed control-state constraint.

Since in the theory of hysteresis operators, Brokate and Sprekels [6], the piecewise monotonicity of mappings plays an important role, our penalization method uses just the positive part function, $(\cdot)_+$, which is monotone.

Therefore, we introduce the penalized cost functional

$$
\begin{aligned}
&\mathrm{Min}\left\{\frac{1}{2}\sum_{j=1}^{k+1}|z_j - z_d(t_j)|^2_{I\!R^N}(t_{j+1}-t_j) \;+\; \frac{1}{2}\sum_{j=1}^{k+1}|u_j|^2_{I\!R^m}(t_{j+1}-t_j)\right. \\
&\left. + \frac{1}{\varepsilon}\max_{j=\overline{1,k+1}}\left[\left(y_j - \tilde{W}_f\left(S(z_0),\ldots,S(z_j)\right)\right)_+ ;\; \left(\tilde{W}_f\left(S(z_0),\ldots,S(z_j)\right) - y_j\right)_+\right]\right\}
\end{aligned}
\tag{4.1}
$$

subject to $u_j \in U$, $y_j \in I\!R$, (3.1), and with z_0 given as an initial condition.

The approximation properties of the penalized problem (4.1) with respect to $(\mathbf{P_k})$, when $\varepsilon \to 0$, are standard, and we do not discuss this here.

Moreover, under usual differentiability and Lipschitz assumptions on f in (3.1) the mapping $\{y_i\} \in I\!R^{k+1}$, $\{u_i\} \in (I\!R^m)^{k+1} \mapsto \{z_i\} \in (I\!R^N)^{k+1}$ is differentiable and Lipschitz for the corresponding finite dimensional norms.

Consequently, we may view (4.1) as the minimization of a Lipschitzian real mapping ($\tilde{W}_f$ is just Lipschitz), depending on $\{u_i\}$ and $\{y_i\}$, and under the constraint $u_i \in U$, convex, closed, bounded subset in $I\!R^m$.

We have already seen that the Clarke generalized gradient of $\tilde{W}_f(\tilde{S}(\cdot))$ may be computed via the chain rule (3.11). This may be done directly with respect to $\{u_i\}$ and $\{y_i\}$, since the dependence (3.1) of $\{z_i\}$ on these variables, denoted as $\mathcal{A} : I\!R^{k+1} \times (I\!R^m)^{k+1} \to (I\!R^N)^{k+1}$, may be assumed C^1, and having a surjective Jacobian.

Let us denote shortly by $\mathcal{C} = \tilde{W}_f \circ \tilde{S} \circ \mathcal{A} : I\!R^{k+1} \times (I\!R^m)^k \to I\!R$, the superposition Lipschitzian mapping. It is to be noticed that the composition of $\mathcal{C}$ with $(\cdot)_+$, appearing in (4.1), does not fulfil the assumptions of the chain rules indicated in Clarke et al. [9], Ch. 2.4. In particular, the mapping $\mathcal{C}$ is not regular, in general (i.e. the generalized Clarke directional derivative may not coincide with the usual directional derivative). However, the mapping $(\cdot)_+$ is regular (since it is convex), has positive gradients and, clearly, a very simple structure. A direct computation may be used to establish the following result.

Proposition 4.1 *If* $y_i - \mathcal{C}(\{u_i\},\{y_i\}) > 0$, *and if* $[\{v_i\},\{x_i\}]$ *is some variation of* $[\{u_i\},\{y_i\}]$, *respectively, then*

$$
\limsup_{\substack{[\{\tilde{u}_i\},\{\tilde{y}_i\}] \to [\{u_i\},\{y_i\}] \\ \lambda \downarrow 0}} \frac{(\tilde{y}_i + \lambda x_i - \mathcal{C}([\{\tilde{u}_i\},\{\tilde{y}_i\}] + \lambda[\{v_i\},\{x_i\}]))_+ - (\tilde{y}_i - \mathcal{C}(\{\tilde{u}_i\},\{\tilde{y}_i\}))_+}{\lambda}
$$

$$
= \limsup_{\substack{[\{\tilde{u}_i\},\{\tilde{y}_i\}] \to [\{u_i\},\{y_i\}] \\ \lambda \downarrow 0}} \frac{\tilde{y}_i + \lambda x_i - \mathcal{C}([\{\tilde{u}_i\},\{\tilde{y}_i\}] + \lambda[\{v_i\},\{x_i\}]) - \tilde{y}_i + \mathcal{C}(\{\tilde{u}_i\},\{\tilde{y}_i\})}{\lambda}.
$$

If $y_i - \mathcal{C}(\{u_i\},\{y_i\}) < 0$, *the above* $\limsup$ *is null.*

Remark. By Clarke et al. [9], p. 79, if $\tilde{S}$ and $\mathcal{A}$ are C^1 with surjective Jacobians, then the last lim sup in **Proposition 4.1** coincides with

$$\limsup_{\substack{w \to \tilde{S}(\mathcal{A}(\{u_i\},\{y_i\})) \\ \lambda \downarrow 0}} \left[-\frac{\tilde{W}_f\left(w + \lambda(\tilde{S}\circ\mathcal{A})'(\{v_i\},\{x_i\})\right) - \tilde{W}_f(w)}{\lambda} \right] + x_i \,. \tag{4.2}$$

Here, $(\tilde{S}\circ\mathcal{A})'$ denotes the Jacobi matrix of the composed mapping. Relation (4.2) and **Proposition 4.1** indicate how to compute the Clarke generalized directional derivative of the cost functional (4.1). For the max-operation appearing in (4.1), one has to take the maximum of the lim sup computed as above.

The main example that we consider concerns the case when W is the so-called *play operator*. We introduce the real mapping (for some given $r > 0$)

$$f_r(v, w) = \max\{v - r\,,\, \min\{v + r\,,\, w\}\}\,, \quad \forall\, v, w \in I\!R\,. \tag{4.3}$$

Taking into account the discretized problems (3.2) or (4.1), we define directly the mapping $\tilde{W}_f$ on $\mathcal{S}$. This can be done inductively, Brokate and Sprekels [6], p. 39:

$$\tilde{W}_f(v_0) = f_r(v_0, w)\,, \quad \forall\, v_0 \in I\!R\,, \tag{4.4}$$

$$\tilde{W}_f(v_0,\ldots,v_i) = f_r(v_i, \tilde{W}_f(v_0,\ldots,v_{i-1}))\,, \quad \forall\, v_0,\ldots,v_i \in I\!R\,. \tag{4.5}$$

Here $w \in I\!R$ is fixed (one can take $w = 0$) and has the significance of an initial (or memory) condition imposed on the operator $\tilde{W}_f$.

Proposition 4.2 *The nonlinear functional $\tilde{W}_f$ is piecewise linear on $I\!R^i$, for any given i.*

This is an immediate consequence of (4.3)–(4.5). We underline that piecewise linear functionals are neither regular in the sense of Clarke [8], nor weakly semismooth in the sense of Mifflin [13]. This shows the difficulties related to optimization problems involving hysteresis operators, even in the simple case of the play operator.

However, **Proposition 4.2** and (3.11) show that it is possible to compute numerically the Clarke generalized gradient $\partial\tilde{W}_f(\tilde{S}\circ\mathcal{A}(\,\cdot\,))$, in any point. The observation is that $\partial\tilde{W}_f(\,\cdot\,)$ is piecewise constant in $I\!R^i$, for any i. Therefore, a finite number of operations will suffice to compute the vectors whose closed convex hull gives the generalized gradient, in any point of interest.

It is known that $d(\,\cdot\,) = \mathrm{proj}_{\partial\tilde{W}_f(\tilde{S}\circ\mathcal{A}(\,\cdot\,))}\{0\}$ is a descent direction if it is nonzero, Clarke [8]. This may be tested by the computations indicated in **Proposition 4.1**. If it is zero, then a stationary point has been achieved.

The following conceptual algorithm may be used:

Algorithm 4.3

1. Let $\{u_i\}, \{y_i\}$ be given.

2. Compute the generating vectors of $\partial J(\{u_i\},\{y_i\})$.

3. Compute d.

4. If $d=0$, then STOP.

5. If $d\neq 0$, then

$$[\{u_i\},\{y_i\}] \to [\{u_i\},\{y_i\}] - \rho\, d\,, \quad \rho > 0\,.$$

6. Compute $J([\{u_i\},\{y_i\}] - \rho\, d)$.

7. GO TO 2.

Here J is the cost defined by (4.1). We note that in Step 5, a line search has to be performed. In general, there is no convergence ensured for **Algorithm 4.3** as the weak semismoothness of J is not valid, Strodiot and Nguyen [15]. Therefore, in practice, a number of steps has to be prescribed or some numerical convergence tests have to be introduced.

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