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Asymptotic analysis of surface waves at vacuum/ porous medium interface: Low-frequency range

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Abstract

Existence and propagation of the surface waves at a free interface of a saturated porous medium are investigated in the low-frequency range. Similar to the high-frequency range, two types of surface waves are proven to be possible: the generalized Rayleigh wave, which exists always and propagates almost without attenuation and the Stoneley wave, which exists for a limited range of wave numbers and is strongly attenuated. Bifurcation behavior of both the Stoneley wave and the Biot slow bulk wave depending on wave number is revealed.

Introduction

Surface acoustic waves (SAW) at a plane interface of an isotropic elastic half-space were discovered by Lord Rayleigh [1]. Surface waves take different forms and exist in a broad frequency range governing more than 10 orders of magnitude. Current research extends from seismic waves in the infrasound region ($\sim 1 - 100\ Hz$) to interdigital transducers and laser-generated SAW pulses in the ultrasound region ($\sim 10 - 10^7\ kHz$) [2]. Surface waves are studied primarily within the scope of single-component models [3,4]. There are just a few papers concerning the surface waves in multicomponent media [5,6]. These works are based on the classical Biot model for fluid-saturated porous medium [7,8] and are devoted to investigation of surface modes in the high-frequency range in which, as the Biot theory predicts, the slow P2 bulk wave is propagatory.

The focus of this paper is on the research of existence and asymptotic behavior of the surface waves at a free interface of a saturated porous medium in the low-frequency range. In contrast to the widely used Biot's model, various phenomenological parameters of which it is difficult or impossible to measure, we rely on the more simple mathematical model of saturated poroelastic materials, proposed by K. Wilmanski [9-12]. This model leads to similar results as the classical Biot model. In particular, it also predicts the existence of three bulk waves in an unbounded fluid-saturated medium: shear, fast longitudinal (P1), and slow longitudinal (P2) waves. Detailed comparison of the models is presented in [13].

In our previous papers concerning the surface waves, which propagate along a free interface of a porous medium at high frequencies, it was proven the existence of two surface modes: the true Stoneley wave and the generalized Rayleigh wave [13,14]. It was shown that behavior of the surface modes depend crucially on the properties of the bulk waves. In the high-frequency range there are not peculiarities in propagation of both bulk and surface waves: velocities of bulk waves are almost constant and the true Stoneley and the generalized Rayleigh waves spread with speeds somewhat less than those of P2 and shear waves, respectively.

However in the low-frequency range the Biot slow wave (P2) is not always propagatory. Low-frequency limit of the Biot theory assumes that slow wave is highly dispersive and strongly attenuated below some critical frequency, which depends on the pores size in the skeleton and the viscosity of the fluid. This critical frequency is typically around $1 - 10\ kHz$ for water saturated porous materials of around 1 $Darcy$ permeability [15]. The Biot slow wave is characterized by the out-of-phase motions of the solid skeleton and pore fluid. This relative motion is very sensitive to the viscosity of the fluid and the dynamic permeability of the porous medium. >From experiments it is known, that the slow wave was observed only at ultrasonic frequencies in *artificial rocks* made of sintered glass beads [16] and in *natural granular soils* (Monterey sand) [17]. Although Biot's theory has been thoroughly studied during last 40 years, the question of why slow wave cannot be detected in low-permeability materials such as natural rocks is still open.

In this paper we prove analytically that the Biot slow wave is not propagatory below some critical wave number which depends on permeability of the media and viscosity of the fluid (see also [18]). This critical wave number is a bifurcation point, above which longitudinal wave of the second kind begins to propagate. Because of this complicated behavior of P2 mode, the properties of low-frequency surface modes should be different in comparison with a high-frequency range. In this paper we prove that in the low-frequency range, similar to P2 wave, the Stoneley surface mode possesses a bifurcation in the vicinity of some critical wave number. Also we prove an existence of the generalized Rayleigh wave.

In should be noted that we consider the propagation of elastic bulk waves through an infinite space in the absence of external forces, so that corresponding solutions are defined uniquely by the Cauchy data (initial value problem). Thus, one must set the wave number k to be real and define frequency $\omega = \omega(k)$, which can be complex, as a solution of dispersion equation.

1. Behavior of the Biot slow (P2) wave in the low-frequency range

Before we proceed to study the existence of the surface modes, let us examine propagation of the bulk waves through an unbounded fluid-filled porous medium. Specifically, we focus on the Biot slow (P2) wave.

1.1. Mathematical model

Let an infinite space Ω be occupied by a saturated porous medium. The set of balance equations describing the porous two-component medium has the following general form $(x \in \Omega,\ t \in [0, T])$ [9-12]:

Mass conservation equations

$$\frac{\partial \rho^F}{\partial t} + \mathrm{div}\big(\rho^F \mathbf{v}^F\big) = 0,$$
$$\frac{\partial \rho^S}{\partial t} + \mathrm{div}(\rho^S \mathbf{v}^S) = 0. \tag{1.1}$$

Here, ρ is the mass density, $\mathbf{v}$ is the velocity vector and indices F and S indicate fluid or solid phases, respectively.

Momentum conservation equations

$$\rho^F \left[\frac{\partial}{\partial t} + (v_j^F, \frac{\partial}{\partial x_j})\right] v_i^F - \frac{\partial}{\partial x_j} T_{ij}^F + \pi(v_i^F - v_i^S) = 0,$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v_j^S, \frac{\partial}{\partial x_j})\right] v_i^S - \frac{\partial}{\partial x_j} T_{ij}^S - \pi(v_i^F - v_i^S) = 0, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the inner product.

Balance equation for the change of porosity

$$\frac{\partial \Delta_n}{\partial t} + \big(v_i^S, \frac{\partial}{\partial x_i}\big)\Delta_n + n_E \mathrm{div}(\mathbf{v}^F - \mathbf{v}^S) = -\frac{\Delta_n}{\tau}, \tag{1.3}$$

where τ is the relaxation time of porosity, assumed to be constant. $\mathbf{T}^F$ and $\mathbf{T}^S$ are the partial stress tensors. Here a positive constant $\pi = \mu^f/\mathcal{K}$, μ^f is a viscosity of a liquid, $\mathcal{K}$ is a permeability of a porous medium.

Constitutive relations for linear poroelastic materials

$$\mathbf{T}^F = -p^F \mathbf{1} - \beta \Delta_n \mathbf{1}, \quad p^F = p_0^F + \kappa(\rho^F - \rho_0^F), \tag{1.4}$$

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S \mathrm{div}\mathbf{u}^S \mathbf{1} + 2\mu^S \mathrm{symgrad}\mathbf{u}^S + \beta \Delta_n \mathbf{1}, \tag{1.5}$$

where p^F is the pore pressure, p_0^F and ρ_0^F are the initial values of pore pressure and fluid mass density, respectively, κ is the constant compressibility coefficient of the fluid depending only on the equilibrium value of porosity n_E. $\Delta_n = n - n_E$ is the change of the porosity, and β denotes the coupling coefficient of the components. $\mathbf{T}_0^S$ denotes a constant reference value of the partial stress tensor in the skeleton, λ^S and μ^S are the Lamé constants of the skeleton, which depend only on n_E, and $\mathbf{u}^S$ is the displacement vector for the solid phase with

$$\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}. \tag{1.6}$$

1.2. Dimensionless variables and parameters

Let us rewrite the system of equation (1.1)-(1.6) in a dimensionless form. For this purpose we introduce the following dimensionless variables and parameters:

$$\hat{\rho}^F = \frac{\rho^F}{\rho_0^S}, \qquad \hat{\rho}^S = \frac{\rho^S}{\rho_0^S}, \qquad \hat{\mathbf{v}}^F = \frac{\mathbf{v}^F}{U_{||}^S}, \qquad \hat{\mathbf{v}}^S = \frac{\mathbf{v}^S}{U_{||}^S},$$

where ρ_0^S is the initial value of the skeleton mass density and $U_{||}^S = \sqrt{(\lambda^S + 2\mu^S)/\rho_0^S}$ is a velocity of a longitudinal wave in an unbounded elastic medium. Also one has

$$\hat{x} = \frac{x}{U_{||}^S \tau}, \qquad \hat{t} = \frac{t}{\tau}, \qquad \hat{\mathbf{u}} = \frac{\mathbf{u}}{U_{||}^S \tau}, \qquad \hat{p}^F = \frac{p^F}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\kappa} = \frac{\kappa}{(U_{||}^S)^2},$$

$$\hat{\pi} = \frac{\pi\tau}{\rho_0^S}, \qquad \hat{\beta} = \frac{\beta}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\lambda}^S = \frac{\lambda^S}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\mu}^S = \frac{\mu^S}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\alpha} = \alpha U_{||}^S.$$

After the change of variables and parameters the original system (1.1)-(1.6) keeps its form except of the right-hand side in the equation for the change of porosity. One gets there $-\Delta_n$. For typographical reasons we omit below the symbol $\hat{}$ characterizing dimensionless quantities.

1.3. Dispersion equation for the bulk waves

Let us investigate propagation of the bulk waves through the porous medium. We confine ourselves to the consideration of a 1D problem, i.e. we study the propagation of longitudinal waves only. In 1D case the system (1.1)-(1.6) takes the following form (for convenience strain tensor e^S has been introduced and we have assumed that $\beta = 0$):

$$\frac{\partial \rho^F}{\partial t} + \frac{\partial}{\partial x}\left(\rho^F v^F\right) = 0, \tag{1.7}$$

$$\frac{\partial \rho^S}{\partial t} + \frac{\partial}{\partial x}(\rho^S v^S) = 0. \tag{1.8}$$

$$\rho^F \left[\frac{\partial}{\partial t} + (v^F, \frac{\partial}{\partial x})\right] v^F + \kappa \frac{\partial \rho^F}{\partial x} + \pi(v^F - v^S) = 0, \tag{1.9}$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v^S, \frac{\partial}{\partial x})\right] v^S - (\lambda^S + 2\mu^S)\frac{\partial e^S}{\partial x} - \pi(v^F - v^S) = 0, \tag{1.10}$$

$$\frac{\partial e^S}{\partial t} = \frac{\partial v^S}{\partial x}, \tag{1.11}$$

$$\frac{\partial \Delta_n}{\partial t} + \left(v^S, \frac{\partial}{\partial x}\right)\Delta_n + n_E \frac{\partial}{\partial x}(v^F - v^S) = -\Delta_n, \tag{1.12}$$

Consider the propagation of the harmonic waves whose frequency is ω and wave number is k. Below we use the following dimensionless parameters: $\hat{\omega} = \omega\tau$ and $\hat{k} = kU_{||}^S\tau$ (the upper symbol $\hat{}$ is again omitted in further consideration). Substituting solutions in the form

$$\rho^F - \rho_0^F = R^F \exp\left(i\left(kx - \omega t\right)\right), \qquad \rho^S - \rho_0^S = R^S \exp\left(i\left(kx - \omega t\right)\right),$$

$$v^F = V^F \exp\left(i\left(kx - \omega t\right)\right), \qquad v^S = V^S \exp\left(i\left(kx - \omega t\right)\right), \tag{1.13}$$

$$e^S = E \exp\left(i\left(kx - \omega t\right)\right), \qquad \Delta_n = D \exp\left(i\left(kx - \omega t\right)\right)$$

into equation system (1.7)-(1.12) one gets the system of algebraic equations for the unknown amplitudes. Requesting that the determinant of this system must vanish yields the dispersion equation for longitudinal waves:

$$\mathcal{F}(k, \omega) = 0, \tag{1.14}$$

where

$$\mathcal{F}(k, \omega) = r(\omega^2 - c_f^2 k^2)(\omega^2 - k^2) + i\omega\pi\left((1+r)\omega^2 - k^2(1 + rc_f^2)\right), \tag{1.15}$$

$r = \rho_0^F/\rho_0^S$, $c_f = U^F/U_{||}^S$, and sound velocity in a fluid $U^F = \sqrt{\kappa}$.

It should be reminded here that similar to our previous research [13,14], we derive ω as a function of the real wave number $k \in R^1$. Thus, $\mathrm{Re}\omega/k$ defines the phase velocity of a wave and $\mathrm{Im}\omega$ gives its attenuation.

Our goal is to prove that solution $\omega_{P2}(k)$ of dispersion equation (1.14), corresponding to the Biot slow wave, possesses a bifurcation. It occures in some critical point k_{cr} (bifurcation point), in small neighborhood of which solution of equation (1.14) splits into several branches.

1.4. Bifurcation of the Biot slow wave

Let us rewrite equation (1.14) as

$$r(\tilde{\omega}^2 - c_f^2)(\tilde{\omega}^2 - 1) + i\tilde{\omega}\frac{1}{\tilde{k}}\left((1+r)\tilde{\omega}^2 - (1 + rc_f^2)\right) = 0, \tag{1.16}$$

where $\tilde{\omega} = \omega/k$ and $\tilde{k} = k/\pi$. Obviously, for the case $k \gg 1$ (high-frequency range) equation (1.16) has the roots (note that here $1/\tilde{k} \ll 1$ is assumed to be a small parameter)

$$\tilde{\omega}_{P1} = \pm 1 - \frac{i}{2}\frac{1}{\tilde{k}} \mp \frac{4 + r - rc_f^2}{8r(1 - c_f^2)}\frac{1}{\tilde{k}^2} + O\Big(\frac{1}{\tilde{k}^3}\Big) \tag{1.17}$$

and

$$\tilde{\omega}_{P2} = \pm c_f - \frac{i}{2r}\frac{1}{\tilde{k}} - \frac{1 - c_f^2(1 + 4r)}{8r^2(1 - c_f^2)(\pm c_f)}\frac{1}{\tilde{k}^2} + O\Big(\frac{1}{\tilde{k}^3}\Big), \tag{1.18}$$

which define the velocities and attenuations of forward and backward directed longitudinal waves of the first (P1) and second (P2) kinds, respectively. It is evident, that in the high frequency limit, phase velocities of P1 and P2 waves do not depend on frequency ω.

Next let us consider low-frequency range, when $k \ll 1$ and, consequently, $\tilde{k} \ll 1$. Solutions of equation (1.16) are sought in the following form:

$$\tilde{\omega} = \tilde{\omega}_0 + \tilde{k}\tilde{\omega}_1 + \tilde{k}^2\tilde{\omega}_2 + \dots \tag{1.19}$$

For the longitudinal P1 wave of forward and backward directions one obtains:

$$\tilde{\omega}_{P1} = \pm\sqrt{\frac{1 + rc_f^2}{1 + r}} - \tilde{k}\frac{ir(1 - c_f^2)^2}{2(1 + rc_f^2)(1 + r)^2}$$
$$\pm\tilde{k}^2\sqrt{\frac{1 + r}{1 + rc_f^2}}\frac{r^2(1 - c_f^2)^3(2(1 - r)(1 + rc_f^2) + 1 - c_f^2)}{8(1 + r)^4(1 + rc_f^2)^2} + O(\tilde{k}^3). \tag{1.20}$$

However for the P2 wave construction of asymptotic solution for the corresponding root of (1.16) is much more complicated. We prove later on that there exists some critical value of wave number k_{cr}, below which longitudinal wave of the second kind is not propagatory. Thus, asymptotic expansion of corresponding root of (1.16) has a different structure depending on whether wave number of P2 wave is smaller or bigger than its critical value k_{cr}.

Substitution of (1.19) into (1.16) yields for the forward directed P2 wave:

$$\tilde{\omega}^f_{P2} = -i\frac{rc_f^2}{1 + rc_f^2}\tilde{k} - i\frac{r^3c_f^4(1 + rc_f^4)}{(1 + rc_f^2)^4}\tilde{k}^3 + O(\tilde{k}^4). \tag{1.21}$$

Solution for the backward directed P2 wave is sought in the form

$$\tilde{\omega} = \frac{1}{\tilde{k}}\tilde{\omega}_0 + \tilde{\omega}_1 + \tilde{k}\tilde{\omega}_2 + \dots \tag{1.22}$$

and it leads to the expansion

$$\tilde{\omega}^b_{P2} = -i\frac{1 + r}{r}\frac{1}{\tilde{k}} + i\frac{r(r + c_f)}{(1 + r)^2}\tilde{k} + O(\tilde{k}^2). \tag{1.23}$$

Obviously, expansions (1.21), (1.23) consist of the imaginary terms only. The latter means that phase velocity of P2 wave is equal to zero, i.e. the wave is not propagatory. However, these expansions are valid only if wave number k is less then some critical value k_{cr}. In other words there exists a bifurcation point k_{cr} in small neighborhood of which solution of equation (1.14) splits into several branches. Let us prove this statement. Consider dispersion equation (1.14). It is easy to see that exact solution for P2 wave is given by:

$$k^2 = \frac{1}{2rc_f^2}\Big(r\omega^2(1+c_f^2) + i\pi\omega(1+rc_f^2)$$

$$+\sqrt{r^2\omega^4(1-c_f^2)^2 - \pi^2\omega^2(1+rc_f^2)^2 + 2ir\pi\omega^3(1-c_f^2)(1-rc_f^2)}\,\Big) \tag{1.24}$$

Proposition. There exists some critical value of wave number $k_{cr} \in R^+$ such that:

a) if $0 < k < k_{cr}$ then equation (1.24) has two pure imaginary roots $\omega_1(k)$ and $\omega_2(k)$, $\mathrm{Re}\omega_j(k) = 0$, $j = 1, 2$;

b) if $k = k_{cr}$ then equation (1.24) has one multiple pure imaginary root, i.e. $\omega_1(k) = \omega_2(k)$, $\mathrm{Re}\omega_j(k) = 0$, $j = 1, 2$;

c) if $k > k_{cr}$ then equation (1.24) has no pure imaginary roots.

This critical wave number is defined asymptotically and is given by (for complete Proof of Proposition see Appendix):

$$k_{cr} \approx c_f\Big(1 + \frac{1}{2rc_f^2}\Big)\pi. \tag{1.25}$$

Corresponding critical frequency is equal to:

$$\omega_{cr} = -i\pi\Omega_{cr}, \qquad \Omega_{cr} \approx \frac{1}{2r} + 2c_f^2(1 + 3rc_f^2 - 2c_f^2). \tag{1.26}$$

Consequently, if $k \le k_{cr}$ than the Biot slow wave does not propagate. It is fully attenuated mode (see (1.21)). If wave number of P2 wave is bigger than critical value k_{cr}, then this mode becomes to be propagatory. Namely, for any small parameter ε and wave number

$$k = k_{cr}\Big(1 + \varepsilon^2 k_2\Big) + O(\varepsilon^2) \tag{1.27}$$

frequency of P2 wave is defined as

$$\omega_{P2} = \omega_{cr} + \varepsilon\omega_1 + O(\epsilon^2) \tag{1.28}$$

with

$$\omega_1 = 2k_{cr}\sqrt{\frac{k_2}{\mathcal{A}}}, \tag{1.29}$$

$$\mathcal{A} = \frac{1+c_f^2}{c_f^2} + \frac{1-c_f^2}{c_f^2 g(\Omega_{cr})\sqrt{g(\Omega_{cr})}}\Big(-r^3(1-c_f^2)^3\Omega_{cr}^3 + 3r^2(1-c_f^2)^2(1-rc_f^2)\Omega_{cr}^2$$
$$-3r(1-c_f^2)(1+r^2c_f^4)\Omega_{cr} + (1-rc_f^2)(1+rc_f^2)^2\Big) > 0. \qquad (1.30)$$

Therefore, phase velocity of forward and backward directed P2 wave is defined by $\pm\mathrm{Re}(\omega_{P2})/k$, where $k = k_{cr}\Big(1+\varepsilon^2 k_2\Big) + O(\epsilon^3)$ (see Fig.1).

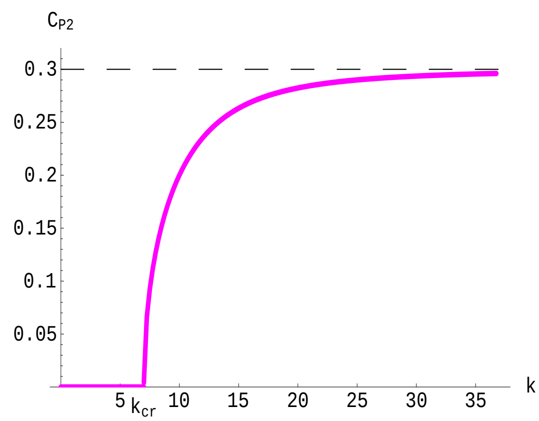

Figure 1: Phase velocity of P2 wave: $r = 0.1,\ c_f = 0.3,\ k_{cr} \approx 7$

Numerical example. Formula (1.25) shows clearly that critical wave number depends on parameter π. Thus, corresponding critical wavelength's dependence on permeability $\mathcal{K}$ is through a direct proportionality. To obtain estimates of critical wavelength, we take the following typical values of parameters [19]: $\rho_0^F = 0.2 \cdot 10^3\ \frac{kg}{m^3}$, $\rho_0^S = 2.0 \cdot 10^3\ \frac{kg}{m^3}$, $U^F = 0.9 \cdot 10^3\ \frac{m}{s}$, $U^S = 3.0 \cdot 10^3\ \frac{m}{s}$. Also $\tau = 4 \cdot 10^{-6}\ s$. For $\pi = 10^9\ \frac{kg}{m^3 \cdot s}$, $\pi = 10^8\ \frac{kg}{m^3 \cdot s}$, and $\pi = 10^7\ \frac{kg}{m^3 \cdot s}$ critical wavelength is equal to $0.22\ cm$, $2.22\ cm$, and $22.21\ cm$, respectively. Corresponding critical frequency for $\pi = 10^9\ \frac{kg}{m^3 \cdot s}$ (water saturated porous material of permeability $\mathcal{K} \sim 1\ Darcy$) is about $20\ kHz$. Thus, we conclude that the Biot slow wave becomes to be propagatory with rather short wavelength and, consequently, it cannot be detected in the low-frequency range of interest in seismology $(1 - 100\ Hz)$.

Obviously, complicated behavior of the Biot slow wave at low frequencies should have an influence on the propagation conditions for the surface modes. Indeed, as we prove below, in contrast to the high-frequency range, in which both the true Stoneley wave and the generalized Rayleigh wave exist always at a free interface of a porous solid [13,14], at low frequencies the Stoneley mode exists for a limited range of wave numbers. Similar to the bulk P2 wave it possesses a bifurcation. Moreover, characteristic features of high- and low-frequency surface modes are completely different. Next we study existence and propagation of the surface waves at an interface between vacuum and porous medium in the low-frequency range.

2. Problem statement

Consider two semi-infinite spaces, Ω^- and Ω^+, having a common interface Γ. Let the region Ω^- be occupied by a saturated porous medium and the region Ω^+ be occupied by the vacuum. Balance equations (1.1)-(1.3) describe the porous two-component medium ($x \in \Omega^-$, $t \in [0,T]$). Let us linearize the system (1.1)-(1.3) about some equilibrium state. The simplest case arises when in the equilibrium state the fields have the following constant values: $\rho^F = \rho_0^F$, $\rho^S = \rho_0^S$, $\mathbf{v}^F = \mathbf{0}$, $\mathbf{v}^S = \mathbf{0}$ and $\Delta_n = 0$. After the introduction of the displacement vector for the fluid phase $\mathbf{u}^F$ and linearization, the system (1.1)-(1.3) takes the following form:

$$\frac{\partial \rho^F}{\partial t} + r\mathrm{div}\frac{\partial \mathbf{u}^F}{\partial t} = 0, \tag{2.1}$$

$$\frac{\partial \rho^S}{\partial t} + \mathrm{div}\frac{\partial \mathbf{u}^S}{\partial t} = 0, \tag{2.2}$$

$$r\frac{\partial^2 \mathbf{u}^F}{\partial t^2} + \mathrm{grad}(p^F + \beta\Delta_n) + \pi\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{2.3}$$

$$\frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \mu^S\Delta\mathbf{u}^S - (\lambda^S + \mu^S)\mathrm{graddiv}\mathbf{u}^S - \beta\mathrm{grad}\Delta_n - \pi\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{2.4}$$

$$\frac{\partial \Delta_n}{\partial t} + n_E\mathrm{div}\frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = -\Delta_n. \tag{2.5}$$

The general problem of propagation of elastic waves through a bounded space is complicated. We confine ourselves to the consideration of a 2D problem (xy plane). This assumption does not limit the generality for the plane boundary Γ. We investigate surface waves on the interface of a porous medium which occupies the semi-infinite space $y > 0$ (region Ω^-) and is bounded by the vacuum, which fills the semi-infinite space $y < 0$ (region Ω^+).

On the interface $y = 0$, separating the porous medium and the vacuum, the following linearized boundary conditions, which are consequences of the general conditions [13], have to be satisfied:

1) the total stress vector must vanish

$$\Big(\frac{\partial u_1^S}{\partial y} + \frac{\partial u_2^S}{\partial x}\Big)|_{y=0} = 0, \tag{2.6}$$

$$\Big(\lambda^S \,\mathrm{div}\,\mathbf{u}^S + 2\mu^S\frac{\partial u_2^S}{\partial y} - \kappa(\rho^F - \rho_0^F)\Big)|_{y=0} = 0, \tag{2.7}$$

2) the relative normal velocity must be equal to zero, i.e. the pores at the interface are completely closed

$$\frac{\partial (u_2^F - u_2^S)}{\partial t}|_{y=0} = 0. \tag{2.8}$$

Our goal is to prove that the boundary value problem (2.1)-(2.8) has solutions in the form of surface waves, i.e. solutions which decrease sufficiently fast as $|y| \to \infty$. For this purpose we will investigate the propagation of a harmonic wave whose frequency is ω, wave number is k, and its amplitude depends on y. The frequency ω is sought as a function of the real wave number $k \in R^1$. Thus, $\mathrm{Re}(\omega/k)$ defines the phase velocity of waves, while $\mathrm{Im}(\omega)$ defines the attenuation. Below we study the propagation of the surface waves in the low-frequency range.

3. Surface waves at a free interface of a porous medium

3.1. Construction of solution

Solution in the region Ω^- (porous medium half-space) is sought in the following form [13,14]:

$$\mathbf{u}^F = \mathrm{grad}\varphi^F + \mathrm{rot}\mathbf{\Psi}^F, \quad \mathbf{u}^S = \mathrm{grad}\varphi^S + \mathrm{rot}\mathbf{\Psi}^S, \tag{3.1}$$

where $\mathbf{\Psi}^F = (0, 0, \psi^F)$ and $\mathbf{\Psi}^S = (0, 0, \psi^S)$. Concequently, in the explicit form one has

$$u_1^F = \frac{\partial \varphi^F}{\partial x} + \frac{\partial \psi^F}{\partial y}, \quad u_2^F = \frac{\partial \varphi^F}{\partial y} - \frac{\partial \psi^F}{\partial x},$$

$$u_1^S = \frac{\partial \varphi^S}{\partial x} + \frac{\partial \psi^S}{\partial y}, \quad u_2^S = \frac{\partial \varphi^S}{\partial y} - \frac{\partial \psi^S}{\partial x}.$$

Here unknown potentials are sought as

$$\varphi^F = A^F(y) \exp\left(i\left(kx - \omega t\right)\right), \quad \varphi^S = A^S(y) \exp\left(i\left(kx - \omega t\right)\right),$$

$$\psi^F = B^F(y) \exp\left(i\left(kx - \omega t\right)\right), \quad \psi^S = B^S(y) \exp\left(i\left(kx - \omega t\right)\right). \tag{3.2}$$

Simultaneously,

$$\rho^F - \rho_0^F = A_\rho^F(y) \exp\left(i\left(kx - \omega t\right)\right), \quad \rho^S - \rho_0^S = A_\rho^S(y) \exp\left(i\left(kx - \omega t\right)\right),$$

$$\Delta_n = A_\Delta \exp\left(i\left(kx - \omega t\right)\right). \tag{3.3}$$

Substitution of (3.1) into (2.1)-(2.5) and the following insertion of expressions (3.2), (3.3) result in three equations for the unknown amplitudes $A^F(y)$, $A^S(y)$ and $B^S(y)$

$$\left(c_f^2\left(\frac{d^2}{d\,y^2} - k^2\right) + \omega^2\right)A_F + \left(\frac{\beta\omega n_E}{r\,(i+\omega)}\left(\frac{d^2}{dy^2} - k^2\right) + \frac{i\pi\omega}{r}\right)\left(A^F - A^S\right) = 0, \tag{3.4}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \omega^2\right) A^S - \left(\frac{\beta\omega n_E}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\pi\omega\right)\left(A^F - A^S\right) = 0, \tag{3.5}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \frac{\omega^2}{c_s^2} - \frac{i\pi\omega^2 r}{c_s^2\left(\omega r + i\pi\right)}\right) B^S = 0, \tag{3.6}$$

and in four algebraic relations for $B^F(y)$, $A_\Delta(y)$, $A_\rho^S(y)$, and $A_\rho^F(y)$ as follows:

$$B^F = \frac{i\pi}{\omega r + i\pi} B^S, \tag{3.7}$$

$$A_\Delta = -\frac{m_E\omega}{i+\omega}\left(\frac{d^2}{d\,y^2} - k^2\right)(A^F - A^S), \tag{3.8}$$

$$A_\rho^S = -\left(\frac{d^2}{d\,y^2} - k^2\right)A^S, \tag{3.9}$$

$$A_\rho^F = \frac{r\omega^2}{c_f^2} A^F + \frac{1}{c_f^2}\left(\frac{\beta\omega n_E}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\pi\omega\right)\left(A^F - A^S\right) = 0. \tag{3.10}$$

Here

$$c_s = U_\perp^S / U_{||}^S < 1, \qquad U_\perp^S = \sqrt{\mu^S/\rho_0^S}.$$

Next let us prove the existence of solutions for the system (3.4)-(3.5) and for equation (3.6) that decay with y. First consider (3.6). The solution has the following form

$$B^S = C_s \exp(\pm\gamma_s\, y) \tag{3.11}$$

with

$$\gamma_s = \sqrt{k^2 - \frac{\omega^2}{c_s^2} + \frac{i\pi\omega^2 r}{c_s^2\left(\omega r + i\pi\right)}}. \tag{3.12}$$

We define

Condition 1,

$$\mathrm{Re}\Big[k^2 - \frac{\omega^2}{c_s^2} + \frac{i\pi\omega^2 r}{c_s^2\left(\omega r + i\pi\right)}\Big] > 0. \tag{3.13}$$

As we will show below, this condition is indeed fulfilled by all surface waves, which are proven to be possible on the free interface of a porous medium. It is also quite natural. Namely, a similar condition in the classical theory of elasticity yields the conclusion that the phase velocity of a surface wave should be less than the velocity of a shear wave. Then, the square root in (3.12) is defined as $\sqrt{1} = 1$ and in order to get a bounded solution we choose

$$B^S = C_s \exp(-\gamma_s\, y). \tag{3.14}$$

We proceed to prove the existence of solution for the system (3.4)-(3.5). The solution is sought in the form

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C_j \begin{pmatrix} R_j^F \\ R_j^S \end{pmatrix} \exp(\pm\gamma_j\, y). \tag{3.15}$$

Substituting (3.15) into (3.4),(3.5), one obtains the eigenvalue problem

$$\begin{pmatrix} d_1^F(j) & d_1^S(j) \\ d_2^F(j) & d_2^S(j) \end{pmatrix} \begin{pmatrix} R_j^F \\ R_j^S \end{pmatrix} = \mathbf{0}, \tag{3.16}$$

where

$$d_1^F(j) = \Big(rc_f^2 + \frac{\beta\, n_E \omega}{\omega + i}\Big)\Big(\frac{\gamma_j^2}{k^2} - 1\Big) + \frac{\omega}{k}\big(r\frac{\omega}{k} + i\frac{\pi}{k}\big),$$

$$d_1^S(j) = -\frac{\beta\, n_E \omega}{\omega + i}\Big(\frac{\gamma_j^2}{k^2} - 1\Big) - i\frac{\pi\omega}{k^2},$$

$$d_2^F(j) = -\frac{\beta\, n_E \omega}{\omega + i}\Big(\frac{\gamma_j^2}{k^2} - 1\Big) - i\frac{\pi\omega}{k^2},$$

$$d_2^S(j) = \Big(1 + \frac{\beta\, n_E \omega}{\omega + i}\Big)\Big(\frac{\gamma_j^2}{k^2} - 1\Big) + \frac{\omega}{k}\big(\frac{\omega}{k} + i\frac{\pi}{k}\big), \tag{3.17}$$

for which eigenvalues γ_j and eigenvectors $(R_j^F, R_j^S)^T$ have to be found. Obviously, γ_j are defined from the condition that the determinant of the matrix of (3.16) must vanish. Consequently, one can derive eigenvectors $(R_j^F, R_j^S)^T$.

In what follows we consider the simplified case when $\beta = 0$. The assumption on the vanishing coefficient β means that we neglect a static coupling between components. The vanishing of the determinant of the matrix of system (3.16) yields a biquadratic equation for the unknown functions γ_j:

$$\Big(\frac{\gamma_j^2}{k^2} - 1\Big)^2 + \frac{\omega}{k}\left(\frac{\omega}{k}\Big(1 + \frac{1}{c_f^2}\Big) + i\frac{\pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right)\Big(\frac{\gamma_j^2}{k^2} - 1\Big)$$
$$+\frac{1}{c_f^2}\frac{\omega^3}{k^3}\left(\frac{\omega}{k} + i\frac{\pi}{k}\Big(1 + \frac{1}{r}\Big)\right) = 0. \tag{3.18}$$

One can prove, that there exist two roots, γ_1 and γ_2, such that [13,14]:

$$\gamma_{1,2} = k\sqrt{1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1+\frac{1}{c_f^2}\Big) \mp \frac{1}{2}\mathrm{Re}\,\delta + \frac{i}{2}\left(\mp\mathrm{Im}\,\delta - \frac{\omega}{k}\frac{\pi}{k}\Big(1+\frac{1}{rc_f^2}\Big)\right)}, \tag{3.19}$$

where

$$\delta = \sqrt{\frac{\omega^4}{k^4}\Big(1-\frac{1}{c_f^2}\Big)^2 - \frac{\omega^2}{k^2}\frac{\pi^2}{k^2}\Big(1+\frac{1}{rc_f^2}\Big)^2 + 2i\frac{\omega^3}{k^3}\frac{\pi}{k}\Big(1-\frac{1}{rc_f^2}\Big)\Big(1-\frac{1}{c_f^2}\Big)}.$$

The corresponding eigenvectors are given by

$$(R_1^F,\ R_1^S) = \Big(R_1^F,\ \frac{i\frac{\pi\omega}{k^2}}{\frac{\gamma_1^2}{k^2} - 1 + \frac{\omega}{k}\Big(\frac{\omega}{k} + i\frac{\pi}{k}\Big)}R_1^F\Big)$$

and

$$(R_2^F,\ R_2^S) = \Big(\frac{i\frac{\pi\omega}{k^2}}{rc_f^2\Big(\frac{\gamma_2^2}{k^2} - 1\Big) + \frac{\omega}{k}\Big(r\frac{\omega}{k} + i\frac{\pi}{k}\Big)}R_2^S,\ R_2^S\Big). \tag{3.20}$$

In this paper we investigate the low-frequency range, i.e. $k \ll 1$ is assumed to be a small parameter. Let us rewrite equation (3.18) as follows:

$$\begin{aligned}&\left(\tilde\gamma_j^2 - 1\right)^2 + \tilde\omega^2\Big(1+\frac{1}{c_f^2}\Big)\left(\tilde\gamma_j^2 - 1\right) + \frac{1}{c_f^2}\tilde\omega^4\\ &+i\tilde\omega\frac{1}{\tilde k}\left(\Big(1+\frac{1}{rc_f^2}\Big)\left(\tilde\gamma_j^2 - 1\right) + \frac{1}{c_f^2}\tilde\omega^2\Big(1+\frac{1}{r}\Big)\right) = 0,\end{aligned} \tag{3.21}$$

where $\tilde\gamma_j = \gamma_j/k$, $j = 1, 2$, $\tilde\omega = \omega/k$, and $\tilde k = k/\pi$.

Next we prove that similar to the high-frequency limit [13,14] equation (3.21) has two roots corresponding to the longitudinal waves of first and second kind. However here construction of solutions for (3.21) is more complicated since in limit problem ($k = 0$, i.e. $\tilde k = 0$) the equation reduces to the second order. Thus, in contrast to the high-frequency range, here these two roots have different order. This fact is an evidence of the phenomenon of hierarchy of P1 and P2 waves. Consequently, one of the solutions (namely that one corresponding to P2 wave) has the structure of a boundary layer. The idea of the boundary layer is that the higher-order terms of (3.21) dominate the behavior of solution in the boundary layer. First root of (3.21) is sought in the form:

$$\tilde\gamma_1^2 - 1 = \frac{1}{\tilde k}Z_0 + Z_1 + \tilde k Z_2 + \dots \tag{3.22}$$

Substitution of (3.22) into (3.21) results in the expression:

$$\tilde{\gamma}_1^2 - 1 = -i\tilde{\omega}\frac{1}{\tilde{k}}\Big(1 + \frac{1}{rc_f^2}\Big) - \frac{\tilde{\omega}^2}{c_f^2}\frac{c_f^2 + \frac{1}{rc_f^2}}{1 + \frac{1}{rc_f^2}} + O\Big(\tilde{k}\Big), \tag{3.23}$$

i.e.

$$\tilde{\gamma}_1 = \sqrt{1 - i\tilde{\omega}\frac{1}{\tilde{k}}\Big(1 + \frac{1}{rc_f^2}\Big) - \frac{\tilde{\omega}^2}{c_f^2}\frac{c_f^2 + \frac{1}{rc_f^2}}{1 + \frac{1}{rc_f^2}}} + O\Big(\sqrt{\tilde{k}}\Big). \tag{3.24}$$

We define

Condition 2,

$$\mathrm{Re}\Big[1 - i\tilde{\omega}\frac{1}{\tilde{k}}\Big(1 + \frac{1}{rc_f^2}\Big) - \frac{\tilde{\omega}^2}{c_f^2}\frac{c_f^2 + \frac{1}{rc_f^2}}{1 + \frac{1}{rc_f^2}}\Big] \gg 1, \tag{3.25}$$

which means that $\tilde{\gamma}_1 = O(\sqrt{1/\tilde{k}})$ and it is referred to as the boundary-layer solution. The latter means that the part of solution (3.15), corresponding to γ_1, is expected to decay exponentially near $y = 0$ and the boundary layer occurs there. (We will prove below whether this condition is indeed fulfilled by the surface waves, which are proven to be possible on the free interface of a porous medium.)

The corresponding eigenvector is given by

$$\begin{pmatrix} R_1^F \\ R_1^S \end{pmatrix} = \begin{pmatrix} 1 \\ -rc_f^2 \end{pmatrix} + \tilde{k}\begin{pmatrix} 1 \\ -rc_f^2 \end{pmatrix} + O\Big(\tilde{k}^2\Big). \tag{3.26}$$

Second root of (3.21) is sought as

$$\tilde{\gamma}_2^2 - 1 = Z_0 + \tilde{k}Z_1 + \dots \tag{3.27}$$

and has the form:

$$\tilde{\gamma}_2^2 - 1 = -\tilde{\omega}^2\frac{1 + r}{1 + rc_f^2} + O\Big(\tilde{k}\Big). \tag{3.28}$$

Thus,

$$\tilde{\gamma}_2 = \sqrt{1 - \tilde{\omega}^2\frac{1 + r}{1 + rc_f^2}} + O\Big(\sqrt{\tilde{k}}\Big). \tag{3.29}$$

The corresponding eigenvector is given by

$$\begin{pmatrix} R_2^F \\ R_2^S \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{k}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O\Big(\tilde{k}^2\Big). \tag{3.30}$$

We define

Condition 3,

$$\mathrm{Re}\Big[1-\tilde{\omega}^2\frac{1+r}{1+rc_f^2}\Big]>0. \tag{3.31}$$

As we will show below, this condition is indeed fulfilled by all surface waves.

Remark. Expansions (3.24) and (3.26), corresponding to P2 wave, are not valid in a neighborhood of the bifurcation point k_{cr}.

Thus, a bounded solution to (3.4)-(3.6) exists and has the form

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C_1 \begin{pmatrix} R_1^F \\ R_1^S \end{pmatrix} \exp(-\gamma_1\, y) + C_2 \begin{pmatrix} R_2^F \\ R_2^S \end{pmatrix} \exp(-\gamma_2\, y),$$

$$B^S = C_s \exp(-\gamma_s\, y). \tag{3.32}$$

Here, the constants C_1, C_2, and C_s are still unknown. In order to derive a system of equations for C_1, C_2, and C_s and to get a dispersion relation for the definition of the velocities of the surface waves, one should substitute solution (3.32) into the boundary conditions.

It should be noted that in contrast to the case of high-frequency range, that has been investigated earlier (see [13,14]) and where independent expressions for radicals $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have been obtained, here these radicals are related because of coupling of P1 and P2 waves in the low-frequency domain of propagation.

3.2. Dispersion relation for small wave numbers $k \ll k_{cr}$

By substituting (3.32) into the boundary conditions (2.6)-(2.8) one obtains the following system of equations for unknown constants C_1, C_2, C_s:

$$\tilde{\gamma}_1 C_1 R_1^S + \tilde{\gamma}_2 C_2 R_2^S + \frac{i}{2}\Big(\tilde{\gamma}_s^2+1\Big) C_s = 0, \tag{3.33}$$

$$\begin{gathered}(\tilde{\gamma}_1^2-1) C_1 R_1^S + (\tilde{\gamma}_2^2-1) C_2 R_2^S \\ +2c_s^2(C_1R_1^S + C_2R_2^S) + 2ic_s^2\tilde{\gamma}_s C_s - \Big(\tilde{\omega}^2 r + i\tilde{\omega}\frac{1}{\tilde{k}}\Big)(C_1R_1^F + C_2R_2^F) \\ +i\tilde{\omega}\frac{1}{\tilde{k}}\Big(C_1R_1^S + C_2R_2^S\Big) = 0,\end{gathered} \tag{3.34}$$

$$\tilde{\gamma}_1 C_1 (R_1^F - R_1^S) + \tilde{\gamma}_2 C_2 (R_2^F - R_2^S) - iC_s\Big(1 - \frac{i}{\tilde{k}\tilde{\omega}r + i}\Big) = 0. \tag{3.35}$$

Here unknown constants $C_1,\ C_2$, and C_s are sought as follows:

$$C_1 = \tilde{k}^{3/2} C_{1,0} + \tilde{k}^2 C_{1,1} + \tilde{k}^{5/2} C_{1,2} + \dots,$$

$$C_2 = C_{2,0} + \tilde{k}^{1/2} C_{2,1} + \tilde{k} C_{2,2} + \dots, \tag{3.36}$$

$$C_s = C_{s,0} + \tilde{k}^{1/2} C_{s,1} + \tilde{k} C_{s,2} + \dots$$

Also we have

$$\tilde{\omega} = \tilde{\omega}_0 + \tilde{k}^{1/2} \tilde{\omega}_1 + \tilde{k} \tilde{\omega}_2 + \dots \tag{3.37}$$

and, consequently, radicals $\tilde{\gamma}_1,\ \tilde{\gamma}_2$ and $\tilde{\gamma}_s$ have to be expanded as well:

$$\tilde{\gamma}_1 = \frac{1}{\tilde{k}^{1/2}} \tilde{\gamma}_1^{(0)} + \tilde{\gamma}_1^{(1)} + \tilde{k}^{1/2} \tilde{\gamma}_1^{(2)},$$

$$\tilde{\gamma}_2 = \tilde{\gamma}_2^{(0)} + \tilde{k}^{1/2} \tilde{\gamma}_2^{(1)} + \tilde{k} \tilde{\gamma}_2^{(2)}, \tag{3.38}$$

$$\tilde{\gamma}_s = \tilde{\gamma}_s^{(0)} + \tilde{k}^{1/2} \tilde{\gamma}_s^{(1)} + \tilde{k} \tilde{\gamma}_s^{(2)}.$$

By substituting expressions (3.36)-(3.38) into (3.33)-(3.35) in the lowest order approximation one obtains the following system of equations:

$$\tilde{\gamma}_2^{(0)} C_{2,0} + \frac{i}{2} \Big((\tilde{\gamma}_s^0)^2 + 1 \Big) C_{s,0} = 0, \tag{3.39}$$

$$\Big((\tilde{\gamma}_2^{(0)})^2 - 1 + 2c_s^2 - \tilde{\omega}_0^2 r \Big) C_{2,0} + 2ic_s^2 \tilde{\gamma}_s^0 C_{s,0} = 0, \tag{3.40}$$

$$\tilde{\gamma}_1^0 (1 + rc_f^2) C_{1,0} - \tilde{\omega}_0 r C_{s,0} = 0, \tag{3.41}$$

where

$$\tilde{\gamma}_1^0 = \sqrt{-i\tilde{\omega}_0 \Big(1 + \frac{1}{rc_f^2} \Big)},$$

$$\tilde{\gamma}_2^0 = \sqrt{1 - \tilde{\omega}_0^2 \frac{1 + r}{1 + rc_f^2}}, \tag{3.42}$$

$$\tilde{\gamma}_s^0 = \sqrt{1 - \tilde{\omega}_0^2 \frac{1+r}{c_s^2}}.$$

Requesting that the determinant of this system must vanish yields the dispersion equation. It has the form

$$\mathcal{P}(\tilde{\omega}_0) \equiv \Big(-\tilde{\omega}_0^2 \frac{1+r}{1+rc_f^2} + 2c_s^2 - \tilde{\omega}_0^2 r \Big) \Big(1 - \tilde{\omega}_0^2 \frac{1+r}{2c_s^2} \Big)$$

$$-2c_s^2 \sqrt{1 - \tilde{\omega}_0^2 \frac{1+r}{1+rc_f^2}} \sqrt{1 - \tilde{\omega}_0^2 \frac{1+r}{2c_s^2}} = 0. \tag{3.43}$$

Let us prove that dispersion equation (3.43) has a unique root, corresponding to the generalized Rayleigh surface wave. It lies within the interval (c_f, c_s) and satisfies conditions (3.13), (3.31). Evidently, that in case $r = \rho_0^F/\rho_0^S \to 0$ (limit passage to elastic medium) equation (3.43) is degenerated into the classical Rayleigh equation:

$$\mathcal{P}_R(\tilde{\omega}) = \left(2 - \frac{\tilde{\omega}^2}{c_s^2} \right)^2 - 4\sqrt{1 - \tilde{\omega}^2} \sqrt{1 - \tilde{\omega}^2/c_s^2}. \tag{3.44}$$

Let us consider r to be a small parameter $\varepsilon \equiv r$ that is indeed fulfilled by virtue of physical meaning: $r < 1$. Asymptotic expansion of the root is sought as follows:

$$\tilde{\omega}_0 = \Omega_0 + \varepsilon \Omega_1 + \dots \tag{3.45}$$

It is easy to show that the leading part Ω_0 of (3.45) satisfies the Rayleigh equation $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$, where c_R is the speed of the classical Rayleigh wave in an elastic half-space. For the definition of the next term Ω_1 of expansion (3.45) the following equation is obtained:

$$\Omega_1 \frac{\mathcal{P}_R(\tilde{\omega})}{d\tilde{\omega}}|_{\tilde{\omega}=c_R} = \Big(1 - \frac{c_R^2}{2c_s^2} \Big) \Big(3c_R^2 - 2c_s^2 c_f^2 \Big)$$

$$+\sqrt{1 - c_R^2} \sqrt{1 - c_R^2/c_s^2} \Big(1 - \frac{c_R^2}{c_s^2 - c_R^2} - \frac{c_R^2 - c_f^2}{1 - c_R^2} \Big). \tag{3.46}$$

Finally one has

$$\tilde{\omega}_{0,R'} = c_R + \varepsilon \Omega_1 + O(\epsilon^2), \tag{3.47}$$

where Ω_1 is determined by (3.46). Obviously, in the low-frequency range the generalized Rayleigh wave propagates almost without attenuation (leading term and the next term in expansion (3.47) are real). This is because of the fact that the Biot slow wave does not propagate with small wave numbers $k \le k_{cr}$ and, consequently, at low frequencies wave properties of a porous medium are very similar to those of an elastic solid. Thus, asymptotic behavior of the generalized Rayleigh wave in the low-frequency range resembles the behavior of the classical Rayleigh wave in an elastic half-space.

3.3. Dispersion relation for wave numbers in the vicinity of bifurcation point k_{cr}

As it was proven in preceding sections of the paper, the Biot slow wave does not propagate for wave numbers $k \le k_{cr}$. As a consequence, the only surface mode, which appear in this range of wave numbers, is the generalized Rayleigh wave. Next we investigate an existence of the surface modes in small neighborhood of the bifurcation point k_{cr}, where $k > k_{cr}$ and P2 bulk wave is propagatory.

Requesting that the determinant of the system (3.33)-(3.35) must vanish yields the following dispersion equation, which holds true for any k:

$$\begin{aligned}
&\tilde{\gamma}_1 \left(i\mathcal{R}_1 + \frac{1}{2}(\tilde{\gamma}_s^2+1)(1-i\mathcal{R}_1)\frac{r\tilde{\omega}+i\frac{\pi}{k}}{r\tilde{\omega}} \right) \Bigg((\tilde{\gamma}_2^2-1) + 2c_s^2 \\
&\quad -2c_s^2\tilde{\gamma}_s\tilde{\gamma}_2(1-i\mathcal{R}_2)\frac{r\tilde{\omega}+i\frac{\pi}{k}}{r\tilde{\omega}} - \tilde{\omega}\left(r\tilde{\omega}+i\frac{\pi}{k}\right)i\mathcal{R}_2 + i\tilde{\omega}\frac{\pi}{k} \Bigg) \\
&-\tilde{\gamma}_2 \left(1 - \frac{1}{2}(\tilde{\gamma}_s^2+1)(1-i\mathcal{R}_2)\frac{r\tilde{\omega}+i\frac{\pi}{k}}{r\tilde{\omega}} \right) \Bigg((\tilde{\gamma}_1^2-1)i\mathcal{R}_1 + 2c_s^2 i\mathcal{R}_1 \\
&\quad +2c_s^2\tilde{\gamma}_s\tilde{\gamma}_1(1-i\mathcal{R}_1)\frac{r\tilde{\omega}+i\frac{\pi}{k}}{r\tilde{\omega}} - \tilde{\omega}\left(r\tilde{\omega}+i\frac{\pi}{k}\right) - \tilde{\omega}\mathcal{R}_1\frac{\pi}{k} \Bigg) = 0.
\end{aligned} \tag{3.48}$$

Here

$$\mathcal{R}_1 = \frac{\frac{\pi}{k}\tilde{\omega}}{\tilde{\gamma}_1{}^2 - 1 + \tilde{\omega}(\tilde{\omega}+i\frac{\pi}{k})}, \quad \mathcal{R}_2 = \frac{\frac{\pi}{k}\tilde{\omega}}{rc_f^2(\tilde{\gamma}_2{}^2-1) + \tilde{\omega}(r\tilde{\omega}+i\frac{\pi}{k})}. \tag{3.49}$$

Taking into account (1.25)-(1.28), consider the expansions in the vicinity of k_{cr}

$$k = \frac{\pi}{2rc_f}\left(1 + \varepsilon^2 k_2 + \cdots\right) \tag{3.50}$$

and

$$\tilde{\omega} = -i\varepsilon + \tilde{\omega}_1\varepsilon^2 + \cdots, \tag{3.51}$$

where small parameter $\varepsilon \equiv c_f$. Radicals (3.12) and (3.29) remain to be valid for any k. Thus, one has

$$\tilde{\gamma}_2 = 1 + \frac{1}{2}\left(1+r\right)\varepsilon^2 + O(\varepsilon^3) \tag{3.52}$$

and

$$\tilde{\gamma}_s = 1 + \frac{1}{2c_s^2}\Big(1 - 2r\Big)\varepsilon^2 + O(\varepsilon^3). \tag{3.53}$$

However, expansion (3.24), which corresponds to P2 wave, is not true in a neighborhood of the bifurcation point k_{cr}. Solution for $\tilde{\gamma}_1$ in the vicinity of k_{cr} is sought as follows:

$$\tilde{\gamma}_1 = \Gamma_1\varepsilon + \Gamma_2\varepsilon^2 + \cdots \tag{3.54}$$

By substituting (3.50)-(3.54) into (3.21) and (3.48), one obtains from the lowest approximations:

$$\Gamma_1 = \frac{r(4c_s^2 - 1)\Big(2(1+r) - \frac{1}{2c_s^2}(1-2r)\Big)}{(1+r)(c_s^2 - 1)} \tag{3.55}$$

and

$$\tilde{\omega}_1^2 = 2\Big(k_2 - 2r\Big). \tag{3.56}$$

Finally, one gets

$$\tilde{\omega} = -i\varepsilon + \sqrt{2\Big(k_2 - 2r\Big)}\varepsilon^2 + O(\varepsilon^3). \tag{3.57}$$

This root corresponds to the Stoneley surface wave and satisfies conditions (3.13), (3.31). Obviously, real solution for $\tilde{\omega}_1$ exists if expression in right-hand side of (3.56) is positive. Therefore, similar to P2 bulk wave, the Stoneley mode has bifurcation behavior in neighborhood of the bifurcation point

$$k_{cr} \approx \frac{\pi}{2rc_f}\Big(1 + c_f^2 k_2\Big), \tag{3.58}$$

where

$$k_2 = 2r. \tag{3.59}$$

Thus, if $k_2 \leq 2r$, i.e. $k \leq k_{cr}$, than the Stoneley wave does not propagate; it is fully attenuated mode. Otherwise, if $k_2 > 2r$, i.e. $k > k_{cr}$, it begins to emerge. In the same way as P2 bulk wave, the Stoneley surface mode is strongly attenuated (leaky mode). Its velocity is very close to the speed of P2 wave.

4. Conclusions

The results presented in the paper concern propagation of the Biot slow wave through an unbounded saturated porous medium and the surface waves which appear at a free interface of saturated porous media in the low-frequency range. The asymptotic behavior of both the Biot slow wave and the surface waves is very different in comparison with the high-frequency limit.

It was proven that the Biot slow wave has a bifurcation behavior depending on its wave number. Bifurcation occures in neighborhood of the critival value k_{cr} (see (1.25)), so that P2 wave becomes to be propagatory with wave numbers k bigger then k_{cr}. Formula (1.25) shows that slow wave behavior is dominated by permeability of a medium. One consequence of this result is the fact that in fluid-filled geological materials of low permeability the Biot slow wave does not propagate at sesmic frequencies.

Complicated behavior of P2 wave at low frequencies causes considerable changes in the properties of the surface modes which appear at a free interface of a porous solid. Similar to the high-frequency range two types of surface modes were proven to be possible: the Stoneley wave and the generalized Rayleigh wave. However, asymptotic behavior of these waves is very distinct at low frequencies.

The generalized Rayleigh wave exists always (for any wave number) and, contrary to the high-frequency limit, propagates almost without attenuation. The asymptotic analysis showed that its phase velocity is close to the speed of the classical Rayleigh wave. Furthermore, asymptotic behavior of the generalized Rayleigh wave in the low-frequency range resembles the behavior of the classical Rayleigh wave in an elastic half-space. The letter is because of the fact that the Biot slow wave does not propagate with small wave numbers $k \leq k_{cr}$ and, consequently, at low frequencies wave properties of a porous medium are very similar to those of an elastic solid.

Another surface mode, which appear at a free interface of a porous medium, is the Stoneley wave. Bifurcation behavior of the Biot slow wave dictates that the Stoneley wave must also possess a bifurcation. Indeed, this surface mode exists for a limited range of wave numbers. If its wave number $k \leq k_{cr}$ then the Stoneley wave does not propagate. If $k > k_{cr}$ then it begins to emerge with phase velocity very close to the velocity of P2 wave. By contrast to the high-frequency limit, the Stoneley wave is strongly attenuated at low frequencies (leaky wave). It radiates a part of its energy into interior of the medium.

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Appendix: Proof of Proposition

Applying the change $\omega = -i\pi\Omega,\ \Omega \geq 0$, equation (1.24) can be rewritten as

$$F_1(\Omega) = F_2(\Omega), \tag{A1}$$

where

$$F_1(\Omega) = \Omega\sqrt{\Omega^2 r^2 (1 - c_f^2)^2 - 2r\Omega(1 - c_f^2)(1 - rc_f^2) + (1 + rc_f^2)^2}, \tag{A2}$$

$$F_2(\Omega) = 2rc_f^2\tilde{k}^2 + \Omega^2 r(1+c_f^2) - \Omega(1+rc_f^2), \tag{A3}$$

and, as above, $\tilde{k} = k/\pi$. It should be noted here that function under the square root $g(\Omega) = \Omega^2 r^2(1-c_f^2)^2 - 2r\Omega(1-c_f^2)(1-rc_f^2) + (1+rc_f^2)^2$ is always positive. Consider equation (A1). First let us investigate behavior of functions $F_1(\Omega)$, $F_2(\Omega)$ as $\Omega \to \infty$. Obviously,

$$\frac{F_1(\Omega)}{\Omega^2} \sim r(1-c_f^2) \text{ and } \frac{F_2(\Omega)}{\Omega^2} \sim r(1+c_f^2) \tag{A4}$$

i.e. $F_2(\Omega)$ is steeper than $F_1(\Omega)$. Consequently, if $\tilde{k} = 0$ then function $(F_2 - F_1)(\Omega)$ has two real roots: $\Omega = 0$ and some Ω_*, so that $(F_2 - F_1)(\Omega) < 0$ in $(0,\ \Omega_*)$.

Next we calculate stationary points for $F_1(\Omega)$ and $F_2(\Omega)$ and inflation points for $F_1(\Omega)$. One can easily check, that function $F_1(\Omega)$ has two stationary points, namely $\Omega_1^{(1)} \approx (1+(1+7r)c_f^2)/(2r)$ and $\Omega_1^{(2)} \approx (1+(1-5r)c_f^2)/r$ and function $F_2(\Omega)$ has one stationary point $\Omega_2 = (1+rc_f^2)/(2r(1+c_f^2))$ such that:

$$\Omega_2 < \Omega_1^{(1)} < \Omega_1^{(2)}. \tag{A5}$$

Function $F_1(\Omega)$ has unique inflation point

$$\Omega_{inf} \approx \frac{1 - rc_f^2 - \sqrt[3]{2}\sqrt[3]{r}\sqrt[3]{c_f^2}(1 - \sqrt[3]{2}\sqrt[3]{r}\sqrt[3]{c_f^2})}{r(1-c_f^2)} \tag{A6}$$

and it being known that $\Omega_1^{(1)} < \Omega_{inf} < \Omega_1^{(2)}$ as well as that $F_1(\Omega)$ is concave if $\Omega < \Omega_i$ and $F_1(\Omega)$ is convex if $\Omega > \Omega_i$. Presented analysis allows us to conclude that there exists unique point of tangency of functions $F_1(\Omega)$ and $F_2(\Omega)$ (see Fig.2).

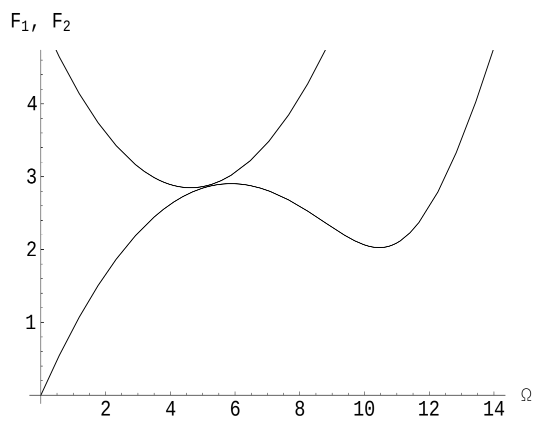

Figure 2: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} =\ \tilde{k}_{cr}$

Thus, using that $F_1^{'}(\Omega) = F_2^{'}(\Omega)$ one can define point of tangency, i.e. critical value Ω_{cr}:

$$\Omega_{cr} \approx \frac{1}{2r} + 2c_f^2(1 + 3rc_f^2 - 2c_f^2), \qquad (A7)$$

which is positive by virtue of physical sense. Corresponding critical value of wave number is defined from equation (A1) and is given by:

$$\tilde{k}_{cr} \approx c_f\Big(1 + \frac{1}{2rc_f^2}\Big). \qquad (A8)$$

Therefore, it was proven that there exist some critical real value $k_{cr} = \tilde{k}_{cr}\pi$ for which equation (1.24) has one multiple imaginary root $\omega_{cr} = -i\pi\Omega_{cr}$.

Next we prove that if $k < k_{cr}$ then equation (1.24) has two pure imaginary roots and if $k > k_{cr}$ then equation (1.24) has no imaginary roots. Consider the expansions

$$k = k_{cr}\Big(1 \pm \epsilon k_1 \pm \epsilon^2 k_2 + \dots\Big)$$

and

$$\omega = \omega_{cr} + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots, \qquad (A9)$$

where ϵ is a small parameter. Substitution of (A9) into (1.24) yields the bifurcation equation. >From its $O(\epsilon)$ approximation it follows that $k_1 = 0$. From the next $O(\epsilon^2)$ approximation one has:

$$\pm k_2 = \frac{1}{4}\frac{\omega_1^2}{k_{cr}^2}\mathcal{A} \qquad (A10)$$

with

$$\mathcal{A} = \frac{1 + c_f^2}{c_f^2} + \frac{1 - c_f^2}{c_f^2 g(\Omega_{cr})\sqrt{g(\Omega_{cr})}}\Big(- r^3(1 - c_f^2)^3\Omega_{cr}^3$$

$$+3r^2(1 - c_f^2)^2(1 - rc_f^2)\Omega_{cr}^2 - 3r(1 - c_f^2)(1 + r^2c_f^4)\Omega_{cr} + (1 - rc_f^2)(1 + rc_f^2)^2\Big) > 0. \quad (A11)$$

It is obvious, that for given $k_2 > 0$ equation (A10) has two real solutions for ω_1 if plus sign is chosen in its left-hand side. The letter means that we consider expansion $k = k_{cr} + \epsilon^2 k_2 + \dots$ and $k > k_{cr}$. Consequently, equation (A1) has no solution (see Fig.3).

Vice versa, if $k = k_{cr} - \epsilon^2 k_2 + \dots < k_{cr}$ then for given $k_2 > 0$ equation (A10), as well as equation (1.24), has two imaginary roots (see Fig.4). Thus, **Proposition** was proven.

Remark. One can also prove **Proposition** applying the same procedure to the dispersion equation (1.14). Taking into account that $\mathcal{F}(k_{cr}, \omega_{cr}) = 0$ and $\mathcal{F}_\omega^{'}(k_{cr}, \omega_{cr}) = 0$, one can define critical values k_{cr} and ω_{cr}. Next one has to substitute expansions (A9) into (1.14). As above one obtains at $O(\epsilon)$ approximation that $k_1 = 0$. From the next $O(\epsilon^2)$ approximation one gets:

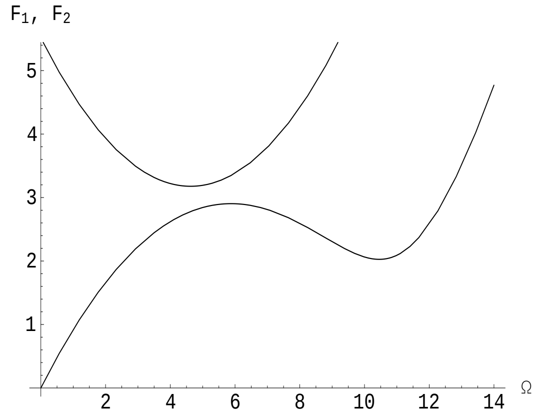

Figure 3: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} > \ \tilde{k}_{cr}$

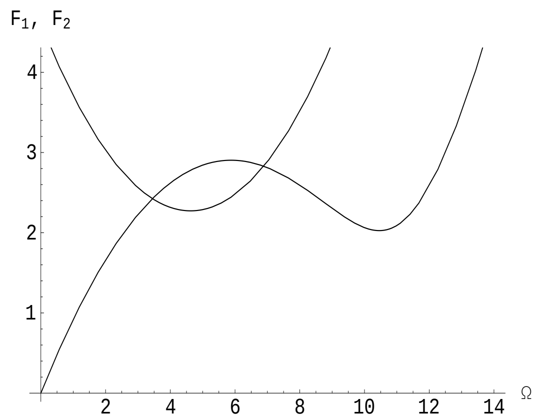

Figure 4: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} < \ \tilde{k}_{cr}$

$$\pm k_2 = \frac{1}{2}\frac{\omega_1^2}{k_{cr}^2}\mathcal{A}_1 \tag{A12}$$

with

$$\mathcal{A}_1 = \frac{-6r\Omega_{cr}^2 + 3(1+r)\Omega_{cr} - r(1+c_f^2)\tilde{k}_{cr}^2}{-r(1+c_f^2)\Omega_{cr}^2 + (1+rc_f^2)\Omega_{cr} - 2rc_f^2\tilde{k}_{cr}^2} > 0. \tag{A13}$$

Analogously to (A10), equation (A12) has two real solutions for ω_1 if for given $k_2 > 0$ plus sign is chosen in its left-hand side.

Therefore we conclude that P2 wave is not propagatory if its wave number is less than critical value k_{cr}. Otherwise, the frequency of P2 wave is given by

$$\omega_{P2} = \omega_{cr} + \epsilon\omega_1 + O(\epsilon^2) \tag{A14}$$

with

$$\omega_1 = 2k_{cr}\sqrt{\frac{k_2}{\mathcal{A}}}. \tag{A15}$$

Consequently, phase velocity of forward and backward directed P2 wave is defined by $\pm\mathrm{Re}(\omega_{P2})/k$, where $k = k_{cr} + \epsilon^2 k_2 + O(\epsilon^3)$.

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