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Delayed loss of stability and excitation of oscillations in nonautonomous differential equations with retarded argument

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Abstract

Assume that zero is a stable equilibrium of an ODE $\dot{x} = f(x, \lambda)$ for parameter values $\lambda < \lambda_0$, and becomes unstable for $\lambda > \lambda_0$. If we suppose that $\lambda(t)$ varies slowly with t, then, under some conditions, the trajectories of the nonautonomous ODE $\dot{x} = f(x, \lambda(t))$ stay close to zero even long after $\lambda(t)$ has crossed the value λ_0. This phenomenon is called 'delayed loss of stability' and is well-known for ODEs . In this paper, we describe an analogous phenomenon for delay equations of the form $\dot{x}(t) = f(t, x(t-1))$.

Further, we point out a difference between delay equations and ODEs: The inhomogeneity h in the linear equation $\dot{x}(t) = cx(t-1) + h(t)$ inevitably leads to an excitation of the most unstable modes of oscillation of the homogeneous equation, even if all segments h_t are contained in a space of more rapidly decaying solutions for the homogeneous equation.

1 Introduction

Dynamical systems as mathematical models of real life processes depend on several parameters which are assumed to be fixed within some time period (see e.g. [6], [16], [1]). The influence of a parameter λ on the behavior of a dynamical system is studied within the framework of bifurcation theory. Suppose now that a relevant system parameter λ changes very slowly in time, for example, because of an aging process. In the model equation, one can then replace the parameter λ by $\lambda(\varepsilon t)$, where $\varepsilon > 0$ is a small. (The new equation is then nonautonomous.) The so-called dynamic bifurcation theory is concerned with the investigation of the corresponding changes of the system behavior [1]. A special phenomenon, known as *delayed loss of stability*, can lead to dramatic consequences (e.g. thermal explosion [11]). For ordinary differential equations (ODEs), this effect is well-known and has been studied from different points of view [22], [5], [12], [9], [8], [19], [20], [2],[3], [4], [17], [18], [21].

Let us illustrate the phenomenon by considering the simple linear equation

$$\dot{y}(t) = k(\varepsilon t)\, y(t), \tag{1.1}$$

where $\varepsilon > 0$ is small, such that the coefficient $k(\varepsilon t)$ in equation (1.1) changes slowly in time. Setting $\varepsilon t = \tau, \quad y(t) = y(\tau/\varepsilon) = x(\tau)$, we get from (1.1)

$$\varepsilon \frac{dx}{d\tau} = k(\tau)x. \tag{1.2}$$

Concerning the function k, we suppose

(A) $\quad k : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, strictly increasing, and there exist numbers $\tau_- < \tau_0 < \tau_+$ such that

$$(1.3) \qquad k(\tau) < 0 \text{ for } \tau < \tau_0, \; k(\tau) > 0 \text{ for } \tau > \tau_0, \quad \int_{\tau_-}^{\tau_+} k(\tau) d\tau = 0.$$

The so-called associated system to (1.2) reads

$$(1.4) \qquad \frac{dx}{d\sigma} = k(\tau) x(\sigma),$$

where τ in the right-hand side has to be considered as a parameter and σ is the independent variable. From hypothesis (A) it follows that the equilibrium $x = 0$ of the associated equation (1.4) is stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$, that is, it changes its stability at $\tau = \tau_0$.

The solution $x(\cdot, \tau_-, x_-)$ of equation (1.1) satisfying $x(\tau_-, \tau_-, x_-) = x_-$ is explicitly given by

$$x(\tau, \tau_-, x_-) = x_- \exp \left\{ \frac{1}{\varepsilon} \int_{\tau_-}^{\tau} k(s) ds \right\}.$$

We see that if k satisfies assumption (A), then $x(\tau, \tau_-, x_-)$ is exponentially decaying for $\tau_- < \tau < \tau_0$, and stays near $x = 0$ also for some time interval $\tau_0 < \tau < \hat{\tau}$ with $\hat{\tau} < \tau_+$, during which $x = 0$ is an unstable equilibrium of (1.4).

The main goal of this paper is to describe a similar effect for differential-delay equations, where we restrict ourselves to simplest cases. In Section 2 we study the linear inhomogeneous equation

$$(1.5) \qquad \dot{x}(t) = a(t) x(t-1) + h(t),$$

assuming that the function a takes values in $[-3\pi/4, -\pi/4]$ and changes slowly. It is well known that for constant a and $h = 0$, the zero solution of equation (1.5) is stable for $a \in (-\pi/2, 0)$, and unstable if $a < -\pi/2$. Contrary to the ODE case, the exponential rate of growth or decay is not directly given by a, but has to be estimated. We provide such estimates in Section 2, and we derive a variation-of-constants formula for the case of nonconstant a and $h \neq 0$. In Section 3 we use this formula to express solutions of

$$\dot{x}(t) = g(t, x(t-1))$$

(with nonlinear g) on successive time intervals I_i by solutions of the equation

$$\frac{dx}{dt} = c_i x(t-1),$$

with constants c_i which are values of $\partial_2 g(\cdot, 0)$ on I_i. In Theorem 3.2, we obtain estimates that express the phenomenon of delayed loss of stability for differential-delay equations. In Section 4 we treat the equation

$$\dot{x}(t) = (-\pi/4 - \varepsilon t) \arctan(x(t-1))$$

as an example. Here, we study the initial value problem with the initial segment identically 1, and estimate the time until the solution is close enough to zero by a method that is not based on linearization. Theorem 3.2 is then applicable to the motion close to zero, and we obtain a lower bound for the time until the solution reaches absolute value 1 again.

Complementary to the results on delayed loss of stability, which express similar behavior of delay equations and ODEs, Section 5 exhibits a substantial difference between both types of equations. Namely, the additive term $h(t)$ in the equation $\dot{x}(t) = c \cdot x(t-1) + h(t)$ inevitably has an influence on the development of all 'components' of solutions (in terms of expansion into eigenfunctions of the homogeneous equation). For a linear constant coefficient system of ODEs, the perturbation h can, of course, be chosen such that it influences only specific components.

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Notation. For bounded functions φ on $[-1, 0]$, the sup-norm is denoted by $|\varphi|$. Generally, we use the symbol $|| \ ||_\infty$ for the sup-norm of bounded functions on some domain.

Let $\mathbf{C}$ denote the space of continuous functions on $[-1, 0]$ with the max-norm. Assume that $G : \mathbb{R} \times \mathbf{C} \to \mathbb{R}$ is continuous, locally Lipschitz continuous with respect to the second argument, and satisfies a linear growth condition

$$|G(t, \varphi)| \leq L(t)(1 + |\varphi|) \quad (t \in \mathbb{R}, \ \varphi \in \mathbf{C})$$

with $L : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ continuous. Then, for $\varphi \in \mathbf{C}$ and $\tau \in \mathbb{R}$, there is a unique continuous function $x^{G,\varphi,\tau} : [\tau - 1, \infty] \longrightarrow \mathbb{R}$ such that

$$x_\tau^{G,\varphi,\tau} = \varphi, \quad \dot{x}^{G,\varphi,\tau}(t) = G(t, x_t^{G,\varphi,\tau}) \text{ for } t \geq \tau.$$

(At $t = \tau$, the derivative is to be read as right-side derivative.) The symbol x_t, as usual, denotes the segment of the function x at time t, that is, $x_t(\theta) = x(t+\theta)$, $-1 \leq \theta \leq 0$.

We shall need solutions of *linear* equations also for discontinuous initial values; let $\mathbf{J}$ denote the space of functions $\varphi : [-1, 0] \longrightarrow \mathbb{R}$ which are continuous on $[-1, 0)$, but possibly have a jump discontinuity at 0 (i.e., $\lim_{t \longrightarrow 0, t<0} \varphi(t)$ exists). We use the sup-norm $| \ |$ also on this space, and we introduce the weaker norm $| \ |_*$ on $\mathbf{J}$ defined by

$$|\psi|_* := |\psi(0)| + \int_{-1}^{0} |\psi(s)| \, ds.$$

2 Linear equations

First we consider linear equations of the type

$$(a) \qquad \dot{x}(t) = a(t)x(t-1).$$

Proposition 2.1 *Let $\tau, T \in \mathbb{R}$, $\tau < T$, and let $a : [\tau, T] \longrightarrow \mathbb{R}$ be continuous.*

a) For $\psi \in \mathbf{J}$ and $s \in [\tau, T]$ there exists a unique solution $x^{a,\psi,s} : [s-1, T] \longrightarrow \mathbb{R}$ of the initial value problem $\dot{x}(t) = a(t)x(t-1)$, $x_s = \psi$.

b) The map

$$F : (\mathbf{J}, |\,|_*) \times \left\{(s,t) \in [\tau, T]^2 \;\middle|\; s \le t\right\} \ni (\psi, s, t) \mapsto x^{a,\psi,s}(t) \in \mathbb{R}$$

is continuous.

c) If $a \in C^1$ and $T \ge \tau + 3$, then for $t \ge \tau + 3$, $t \le T$ the segment $x_t^{a,\psi,\tau}$ is C^2.

Proof: Ad a): This assertion follows from successive integration: On $[\tau, \tau+1]$, one has $x^{a,\psi,\tau}(t) = \psi(0) + \int_\tau^t a(s)\psi(s-\tau-1)ds$, the segment $x_{\tau+1}^{a,\psi,\tau}$ is contained in $\mathbf{C}$, and so on.

Ad b): It follows from [13], Theorem 2.2, p. 43 that the map

$$F_1 : (\mathbf{C}, |\,|) \times \left\{(t_1, t_2) \in [\tau, T] \;\middle|\; t_1 \le t_2\right\} \ni (\varphi, t_1, t_2) \mapsto x^{a,\varphi,t_1}(t_2) \in \mathbb{R}$$

is continuous. If $t, s \in [\tau, T]$ with $t > s+1$ then

$$x^{a,\psi,s}(t) = x^{a, x_{s+1}^{a,\psi,s}, s+1}(t),$$

and we have $x_{s+1}^{a,\psi,s} \in \mathbf{C}$. In order to prove the asserted continuity of F at points (ψ, s, t) with $t > s+1$, it suffices to show that the map

$$F_2 : (\mathbf{J}, |\,|_*) \times \left\{s \in [\tau, T] \;\middle|\; s+1 \le T\right\} \ni (\psi, s) \mapsto x_{s+1}^{a,\psi,s} \in \mathbf{C}$$

is continuous, since we have

$$F(\psi, s, t) = x^{a,\psi,s}(t) = F_1[F_2(\psi, s), s+1, t] \text{ if } t > s+1.$$

Assume $\tilde{s} \in [\tau, T-1]$ and $\psi, \tilde{\psi} \in \mathbf{J}$. It follows from the differential equation that (2.1.1)

$$|x_{\tilde{s}+1}^{a,\psi,\tilde{s}} - x_{\tilde{s}+1}^{a,\tilde{\psi},\tilde{s}}| \le |\psi(0) - \tilde{\psi}(0)| + ||a||_\infty \int_{-1}^0 |\psi(r) - \tilde{\psi}(r)|\, dr \le (1 + ||a||_\infty)|\psi - \tilde{\psi}|_*.$$

Further, for $\theta \in [-1,0]$,

$$|x_{\tilde{s}+1}^{a,\psi,\tilde{s}}(\theta) - x_{s+1}^{a,\psi,s}(\theta)| =$$

$$|\psi(0) + \int_{\tilde{s}}^{\tilde{s}+1+\theta} a(r)\psi(r-\tilde{s}-1)\,dr - \psi(0) - \int_{s}^{s+1+\theta} a(r)\psi(r-s-1)\,dr|$$

$$\leq 2|\tilde{s}-s| \cdot ||a||_\infty |\psi| + \int_{[\max\{s,\tilde{s}\},\min\{\tilde{s},s\}+1+\theta]} |a(r)[\psi(r-\tilde{s}-1) - \psi(r-s-1)]|\,dr,$$

where the integral is to be read as zero if $\max\{s,\tilde{s}\} \geq \min\{\tilde{s},s\}+1+\theta$. (Note that in the opposite case the arguments of ψ are contained in $[-1,0]$.) Denote the continuous extension of $\psi_{|[-1,0)}$ to $[-1,0]$ (which in general differs from ψ) by $\hat{\psi}$. We may replace ψ by $\hat{\psi}$ in the last integral, and uniform continuity of $\hat{\psi}$ implies that there exists a function $\omega_\psi : [0,\infty) \longrightarrow [0,\infty)$ with $\omega_\psi(0) = 0$, which is continuous at 0 and such that

$$\forall t_1, t_2 \in [-1,0]: \ |\hat{\psi}(t_1) - \hat{\psi}(t_2)| \leq \omega_\psi(|t_1 - t_2|).$$

We conclude that, if $s, \tilde{s} \in [\tau, T-1]$, one has

$$|x_{\tilde{s}+1}^{a,\psi,\tilde{s}} - x_{s+1}^{a,\psi,s}| \leq 2|\tilde{s}-s| \cdot ||a||_\infty |\psi| + ||a||_\infty \omega_\psi(|\tilde{s}-s|). \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2), continuity of F_2 (with respect to $|\,|_*$ in the first argument) follows.

It remains to prove continuity of F at points (ψ, s, t) with $s \leq t \leq s+1$. Assume $\psi \in \mathbf{J}$ and $s, t \in [\tau, T]$, $s \leq t \leq s+1$.

First case: $s < t < s+1$. There exists $\delta_1 > 0$ such that $|s-\tilde{s}| + |t-\tilde{t}| < \delta_1$ implies $\tilde{s} < \min\{t,\tilde{t}\} \leq \max\{t,\tilde{t}\} < \tilde{s}+1$. For such $\tilde{s}$ and $\tilde{t}$, and $\tilde{\psi} \in \mathbf{J}$, we have with $\tilde{\theta} := \tilde{t} - (\tilde{s}+1)$, $\theta := t - (s+1)$ the estimate

$$\begin{aligned} |x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\psi,s}(t)| &= |x_{\tilde{s}+1}^{a,\tilde{\psi},\tilde{s}}(\tilde{\theta}) - x_{s+1}^{a,\psi,s}(\theta)| \\ &\leq |x_{\tilde{s}+1}^{a,\tilde{\psi},\tilde{s}} - x_{s+1}^{a,\psi,s}| + |x_{s+1}^{a,\psi,s}(\tilde{\theta}) - x_{s+1}^{a,\psi,s}(\theta)| \\ &\leq |F_2(\tilde{\psi},\tilde{s}) - F_2(\psi,s)| + ||a||_\infty |\psi|(|t-\tilde{t}| + |s-\tilde{s}|). \end{aligned} \tag{2.1.3}$$

Continuity of F_2 now shows continuity of F at (ψ, s, t).

Second case: $s = t$. Then $x^{a,\psi,s}(t) = \psi(0)$, and for $\tilde{\psi} \in \mathbf{J}$, $|\tilde{\psi} - \psi|_* \leq 1$ and $\tilde{s}, \tilde{t} \in [\tau, T]$ with $\tilde{s} \leq \tilde{t}, |\tilde{t}-t| \leq 1/2, |\tilde{s}-s| \leq 1/2$, one has $\tilde{t} \leq \tilde{s}+1$ and

$$\begin{aligned} |x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\psi,s}(t)| &\leq |x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\tilde{\psi},\tilde{s}}(\tilde{s})| + |\tilde{\psi}(0) - \psi(0)| \\ &\leq ||a||_\infty \int_{\tilde{s}}^{\tilde{t}} |\tilde{\psi}(r-\tilde{s}-1)|\,dr + |\tilde{\psi}-\psi|_* \\ &\leq ||a||_\infty \left[|\tilde{\psi}-\psi|_* + \int_{\tilde{s}}^{\tilde{t}} |\psi(r-\tilde{s}-1)|\,dr \right] + |\tilde{\psi}-\psi|_* \\ &\leq (||a||_\infty + 1)|\tilde{\psi}-\psi|_* + ||a||_\infty |\psi|(|\tilde{t}-t| + |s-\tilde{s}|). \end{aligned}$$

This shows continuity of F at (ψ, s, t).

Third case: $t = s+1$. Consider $\tilde{s}, \tilde{t} \in [\tau, T]$, $\tilde{s} \leq \tilde{t}$, and $\tilde{\psi} \in \mathbf{J}$, with $|\tilde{s}-s|+|\tilde{t}-t| \leq 1$ and $|\tilde{\psi} - \psi|_* \leq 1$. If $\tilde{t} \leq \tilde{s}+1$ then we obtain (similar to (2.1.3))

(2.1.4)
$$\begin{aligned}|x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\psi,s}(t)| &= |F_2(\tilde{\psi},\tilde{s})(\tilde{t}-(\tilde{s}+1)) - F_2(\psi,s)(0)| \\ &\leq |F_2(\tilde{\psi},\tilde{s}) - F_2(\psi,s)| + |F_2(\psi,s)(\tilde{t}-(\tilde{s}+1)) - F_2(\psi,s)(0)| \\ &\leq |F_2(\tilde{\psi},\tilde{s}) - F_2(\psi,s)| + ||a||_\infty |\psi|(|t-\tilde{t}| + |s-\tilde{s}|).\end{aligned}$$

If $\tilde{t} > \tilde{s}+1$ then $\tilde{t} - (\tilde{s}+1) \leq |\tilde{t}-t| + |\tilde{s}-s| \leq 1$, so the differential equation shows

$$|x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\tilde{\psi},\tilde{s}}(\tilde{s}+1)| \leq |F_2(\tilde{\psi},\tilde{s})| \cdot ||a||_\infty (|\tilde{t}-t| + |\tilde{s}-s|),$$

and

$$|x^{a,\tilde{\psi},\tilde{s}}(\tilde{s}+1) - x^{a,\psi,s}(t)| = |x^{a,\tilde{\psi},\tilde{s}}(\tilde{s}+1) - x^{a,\psi,s}(s+1)| \leq |F_2(\tilde{\psi},\tilde{s}) - F_2(\psi,s)|,$$

so we obtain

(2.1.5) $$|x^{a,\tilde{\psi},\tilde{s}}(\tilde{t}) - x^{a,\psi,s}(t)| \leq |F_2(\tilde{\psi},\tilde{s})| \cdot ||a||_\infty (|\tilde{t}-t| + |\tilde{s}-s|) + |F_2(\tilde{\psi},\tilde{s}) - F_2(\psi,s)|.$$

Continuity of F_2 at (ψ, s) and estimates (2.1.4) and (2.1.5) now show that F is continuous at (ψ, s, t) also in the third case.

Ad c): Assume now $a \in C^1$, and $T \geq \tau + 3$, and let $\psi \in \mathbf{J}$. We know that the restriction of $x^{a,\psi,\tau}$ to $[\tau+1, T]$ is C^1. For $t \geq \tau+2$, we have $t-1 \geq \tau+1$ and $\dot{x}^{a,\psi,\tau}(t) = a(t)x^{a,\psi,\tau}(t-1)$. Hence $\ddot{x}^{a,\psi,\tau}(t)$ exists and equals $\dot{a}(t)x^{a,\psi,\tau}(t-1) + a(t)a(t-1)x^{a,\psi,\tau}(t-2)$. (At $t = \tau+2$, we mean the second derivative from the right.) The last expression is continuous in t on $[\tau+2, T]$ (only continuous from the right at $\tau+2$), so we obtain that the restriction of $x^{a,\psi,\tau}$ to $[\tau+2, T]$ is C^2. The assertion follows. □

Our aim is to express solutions of equation (a) with slowly varying coefficient by solutions of the constant coefficient equation

$$(c) \qquad \dot{x}(t) = c \cdot x(t-1) \quad (c \in \mathbb{R}).$$

It is known that the zero solution of equation (c) is stable for $c \in (-\pi/2, 0)$ and becomes unstable for $c < -\pi/2$. We first provide more detailed information on equation (c) for values of c around $-\pi/2$. For $c \in \mathbb{R}$, let $\Sigma_c \subset \mathbb{C}$ denote the set of zeroes of the characteristic function $\lambda \mapsto \lambda - c \cdot \exp(-\lambda)$ associated to equation (c).

Proposition 2.2 *For $c \in (-\infty, -e^{-1})$, the set Σ_c has the form*

$$\Sigma_c = \left\{\lambda_k(c) \mid k \in \mathbb{N}_0\right\} \cup \left\{\overline{\lambda_k(c)} \mid k \in \mathbb{N}_0\right\},$$

where $\lambda_k(c) = \rho_k(c) + i\omega_k(c)$, $\overline{\lambda_k(c)} = \rho_k(c) - i\omega_k(c) \quad (k \in \mathbb{N}_0)$, and $\omega_k(c) \in (2k\pi, (2k+1)\pi)$. The following properties hold:

a) $\rho_k(c) > \rho_{k+1}(c) \quad (k \in \mathbb{N}_0)$*, so that* $\rho_0(c) = \max \mathrm{Re}\Sigma_c$.

b) $\rho_0(-\pi/2) = 0$.

c) For $c \in [-3\pi/4, -\pi/4]$, $\rho_0'(c)$ *exists, and* $\rho_0'(-\pi/2) = \dfrac{-2\pi}{4+\pi^2}$. *Further,* $\omega_0(c) \in (\pi/4, \pi)$*, and*

$$-\frac{4(\pi+2)}{\pi^2} \le \rho_0'(c) \le -\frac{4(\pi-2)}{3\pi^2}, \text{ and}$$

$$\rho_0(c) \le \begin{cases} -|c+\pi/2|\dfrac{4(\pi-2)}{3\pi^2}, & \text{if } c > -\pi/2, \\ |c+\pi/2|\dfrac{4(\pi+2)}{\pi^2}, & \text{if } c \le -\pi/2. \end{cases}$$

d) $|\rho_0(c)| \le (\pi+2)/\pi \le 2$ *for* $c \in [-3\pi/4, -\pi/4]$.

Proof: The assertions on Σ_c and property a) follow from Theorem 5 in [23]. Writing $\lambda = \rho + i\omega$, the characteristic equation $\lambda = c\exp(-\lambda)$ is equivalent to the equations

$$\rho = ce^{-\rho}\cos\omega, \quad \omega = -ce^{-\rho}\sin\omega.$$

Note that $\sin\omega = 0$ would imply $\omega = 0$, but we know already that for $c < -e^{-1}$ there exist no real roots of the characteristic equation. Hence, we can restrict ourselves to the case $\sin\omega \neq 0$, and we obtain from the above two equations $\omega = -c\exp(\omega\cot\omega)\sin\omega$. Setting

$$\chi(\omega) := \frac{\omega}{\sin\omega}\exp(-\omega\cot\omega) \text{ for } \omega \in \mathbb{R} \setminus \left\{k\pi \mid k \in \mathbb{Z}\right\},$$

the last equation is equivalent to

$$\chi(\omega) = -c. \tag{2.2.1}$$

The function χ is discussed in [23]. One has

$$\chi'(\omega) = \frac{\chi(\omega)}{\omega}[(1-\omega\cot\omega)^2 + \omega^2], \tag{2.2.2}$$

χ and χ' are positive on $(0,\pi)$, with $\chi(\omega) \longrightarrow e^{-1}$ as $\omega \longrightarrow 0$, and $\chi(\omega) \longrightarrow \infty$ as $\omega \longrightarrow \pi$, $\omega < \pi$. For $c \in (-\infty, -e^{-1})$, the number $\omega_0(c)$ is the unique solution of equation (2.2.1) in $(0,\pi)$, and $\rho_0(c) = -\omega_0(c)\cot\omega_0(c) = \log\dfrac{-c\sin\omega_0(c)}{\omega_0(c)}$, so we have

$$\rho_0(c) = \log(-c) + \log\sin\omega_0(c) - \log\omega_0(c). \tag{2.2.3}$$

Obviously $\chi(\pi/2) = \pi/2$, so $\omega_0(-\pi/2) = \pi/2$ and $\rho_0(-\pi/2) = 0$. Properties a) and b) are proved.

Ad c): It follows from the inverse function theorem and from (2.2.3) that ω_0 and ρ_0 are differentiable functions on $(-\infty, -e^{-1})$, in particular, on $[-3\pi/4, -\pi/4]$. Using (2.2.2) we obtain for $c \in (-\infty, -e^{-1})$

$$\begin{aligned}\omega_0'(c) &= -\frac{1}{\chi'(\omega_0(c))} = -\frac{\omega_0(c)}{\chi(\omega_0(c))[(1-\omega_0(c)\cot\omega_0(c))^2 + \omega_0(c)^2]} \\ &= \frac{\omega_0(c)}{c[(1-\omega_0(c)\cot\omega_0(c))^2 + \omega_0(c)^2]},\end{aligned}$$

and from (2.2.3) we get

$$\begin{aligned}\rho_0'(c) &= \frac{1}{c} + \omega_0'(c)(\cot\omega_0(c) - \frac{1}{\omega_0(c)}) = \frac{1}{c} + \frac{\omega_0(c)\cot\omega_0(c) - 1}{c[(1-\omega_0(c)\cot\omega_0(c))^2 + \omega_0(c)^2]} \\ &= \frac{1}{c}\left(1 + \frac{\omega_0(c)\cot\omega_0(c) - 1}{[(1-\omega_0(c)\cot\omega_0(c))^2 + \omega_0(c)^2]}\right).\end{aligned}$$

In particular, we see that $\rho_0'(-\pi/2) = \dfrac{-2}{\pi}\left(1 + \dfrac{-1}{1+\pi^2/4}\right) = \dfrac{-2\pi}{4+\pi^2}$, which is the first assertion of c). Note now that $\chi(\pi/4) = \frac{\pi/4}{\sqrt{2}/2}\exp(-\pi/4) = \frac{\pi\sqrt{2}}{4}\exp(-\pi/4) < \frac{\pi}{4}\frac{\sqrt{2}}{1+\pi/4} < \pi/4$, so $\omega_0(-\pi/4) > \pi/4$. It follows that

$$\omega_0([-3\pi/4, -\pi/4]) \subset (\pi/4, \pi). \tag{2.2.4}$$

Further, for all $\omega > 0$ and $u \in \mathbb{R}$, one has $|\dfrac{u}{u^2+\omega^2}| \leq \dfrac{1}{2\omega}$. With (2.2.4) we conclude that

$$\left|\frac{\omega_0(c)\cot\omega_0(c) - 1}{[(1-\omega_0(c)\cot\omega_0(c))^2 + \omega_0(c)^2]}\right| \leq \frac{1}{2\omega_0(c)} \leq \frac{2}{\pi}.$$

With the above expression for $\rho_0'(c)$, we now obtain that $\rho_0'(c) \in \frac{1}{c}[1-2/\pi, 1+2/\pi]$ for $c \in [-3\pi/4, -\pi/4]$, so for these c one has $(1+2/\pi)(-4/\pi) \leq \rho_0'(c) \leq (1 - 2/\pi)(-4/3\pi)$, or

$$-\frac{4(\pi+2)}{\pi^2} \leq \rho_0'(c) \leq -\frac{4(\pi-2)}{3\pi^2}.$$

The estimates on $\rho_0(c)$ in part c) follow by integration.

Ad d): It follows from b) and c) that for $c \in [-3\pi/4, -\pi/4]$ one has

$$|\rho_0(c)| \leq \frac{\pi}{4}\frac{4(\pi+2)}{\pi^2} = \frac{\pi+2}{\pi} \leq 2.$$

□

It is known that for $c < -e^{-1}$ and $\rho > \rho_0(c)$, there exists $K > 0$ such that all solutions $x^{c,\varphi,\tau}$ of equation (c) satisfy an estimate of the form
$|x^{c,\varphi,\tau}(t)| \leq K\exp(\rho(t-\tau))|\varphi|$ for $t \geq \tau$. (Compare, e.g., Cor. 6.1, p. 215 of [13], and the definition of the constant K given in the proof of Lemma 6.2, p. 213 of the same reference.) Analogous results hold for much more general linear equations. We now derive a similar estimate with an explicit value for K, and with $\rho = \rho_0(c)$, for the special case of equation (c).

Proposition 2.3 *Set* $K := [4+15\cdot(3\pi/4)+24(3\pi/4)^2]e^4$*, and let* $c \in [-3\pi/4, -\pi/4]$ *and* $t, s \in \mathbb{R},\ t \geq s$.

a) For $\psi \in C^2([-1,0],\mathbb{R})$*, one has* $|x^{c,\psi,s}(t)| \leq (4|\psi| + 7|\psi'| + 13|\psi''|)e^{\rho_0(c)(t-s)}$.

b) For $\varphi \in \mathbf{J}$*, one has* $|x^{c,\varphi,s}(t)| \leq |\varphi| K \exp[\rho_0(c)(t-s)]$.

Proof: Since equation (c) is autonomous, it suffices to prove the assertions for the case $s = 0$. For $t > 0$, we have for $\varphi \in \mathbf{C}$ the series expansion

$$x^{c,\varphi,0}(t) = \sum_{\mu\in\Sigma_c} (\mathrm{pr}_\mu \varphi) \exp(\mu t),$$

where $\mathrm{pr}_\mu\varphi = \dfrac{1}{1+\mu}[\varphi(0) + \mu \int_{-1}^{0} e^{-\mu s}\varphi(s)ds]$ (see [23], Theorem 6, or [14], Lemma 6.8).

Claim: For $\psi \in C^2([-1,0],\mathbb{R})$ and for all $\mu \in \Sigma_c$, one has

$$|\mathrm{pr}_\mu\psi| \leq \frac{(3\pi/4)|\psi| + 4|\psi'| + e^2|\psi''|}{|\mu(1+\mu)|}.$$

Proof. If $\mu \in \Sigma_c$ then $\mu = ce^{-\mu}$, so $e^\mu = c/\mu$. Using partial integration twice, we calculate

$$\begin{aligned}
\mu \int_{-1}^{0} e^{-\mu s}\psi(s)ds &= [-e^{-\mu s}\psi(s)]_{-1}^{0} + \int_{-1}^{0} e^{-\mu s}\psi'(s)ds \\
&= -\psi(0) + e^\mu\psi(-1) + \int_{-1}^{0} e^{-\mu s}\psi'(s)ds \\
&= -\psi(0) + e^\mu\psi(-1) + [\frac{-1}{\mu}e^{-\mu s}\psi'(s)]_{-1}^{0} + \frac{1}{\mu}\int_{-1}^{0} e^{-\mu s}\psi''(s)\,ds \\
&= -\psi(0) + \frac{c}{\mu}\psi(-1) + \frac{1}{\mu}[\frac{c}{\mu}\psi'(-1) - \psi'(0)] + \frac{1}{\mu}\int_{-1}^{0} e^{-\mu s}\psi''(s)\,ds.
\end{aligned}$$

It follows that

$$|\mathrm{pr}_\mu\psi| \leq \frac{1}{|1+\mu|}\Big[\frac{|c|}{|\mu|}|\psi| + \frac{1}{|\mu|}(\frac{|c|}{|\mu|} + 1)|\psi'| + \frac{1}{|\mu|}\int_{-1}^{0} |e^{-\mu s}|\,ds \cdot |\psi''|\Big].$$

Since $c \in [-3\pi/4, \pi/4]$, we know from Proposition 2.2 that

$$\Sigma_c = \Big\{\rho_k(c) \pm i\omega_k(c) \;\Big|\; k \in \mathbb{N}_0\Big\},$$

that $\omega_k(c) \geq 2k\pi$ for $k \geq 1$, and that $\omega_0(c) \geq \pi/4$. In particular, $|c|/|\mu| \leq (3\pi/4)/(\pi/4) = 3$ for $\mu \in \Sigma_c$. Further, it follows from Proposition 2.2,d) that for $s \in [-1,0]$ we have $|e^{-\mu s}| \leq e^{|\rho_0(c)|} \leq e^2$. Thus we obtain

$$|\mathrm{pr}_\mu\psi| \leq \frac{1}{|1+\mu|}\Big[\frac{3\pi/4}{\mu}|\psi| + \frac{1}{|\mu|}4|\psi'| + \frac{e^2}{|\mu|}|\psi''|\Big].$$

The claim is proved. For $\psi \in C^2([-1,0],\mathbb{R})$, we have

$$\begin{aligned}\sum_{\mu\,\in\,\Sigma_c} |\mathrm{pr}_\mu \psi| &\leq 2 \cdot \sum_{\substack{\mu\,\in\,\Sigma_c,\\ \mathrm{Im}\mu > 0}} |\mathrm{pr}_\mu \psi| \\
&\leq 2\sum_{k=0}^{\infty} [(3\pi/4)|\psi| + 4|\psi'| + e^2|\psi''|] \frac{1}{|\omega_k(c)|(|\omega_k(c)|+1)} \\
&\leq 2(\frac{3\pi}{4}|\psi| + 4|\psi'| + e^2|\psi''|)\{\frac{1}{\pi/4(1+\pi/4)} + \sum_{k=1}^{\infty}\frac{1}{(2k\pi)^2}\} \\
&\leq 2(\frac{3\pi}{4}|\psi| + 4|\psi'| + e^2|\psi''|)\{\frac{1}{(3/4)\cdot(7/4)} + \frac{1}{4\pi^2}\frac{\pi^2}{6}\} \\
&\leq 2(\frac{3\pi}{4}|\psi| + 4|\psi'| + e^2|\psi''|)\frac{17}{21} \leq \frac{19/2}{2}\frac{17}{21}|\psi| + \frac{8\cdot 17}{21}|\psi'| + \frac{15\cdot 17}{21}|\psi''| \\
&\leq 4|\psi| + 7|\psi'| + 13|\psi''|.\end{aligned}$$

Now we obtain from the series expansion, and from $|e^{\mu t}| \leq e^{\rho_0(c)t}$ for $\mu \in \Sigma_c$, that

$$|x^{c,\psi,0}(t)| \leq \sum_{\mu\,\in\,\Sigma_c} |\mathrm{pr}_\mu \psi| e^{\rho_0(c)t} \leq (4|\psi| + 7|\psi'| + 13|\psi''|)e^{\rho_0(c)t}.$$

Assertion a) is proved.

Ad b): We know from Proposition 2.2,d) that $|\rho_0(c)| \leq 2$, and hence we have

$$e^{|\rho_0(c)|} \leq e^2. \tag{2.3.1}$$

Let $\varphi \in \mathbf{J}$. For $t \in [0,1]$, we have (using (2.2.1))

$$\begin{aligned}|x^{c,\varphi,0}(t)| &\leq |\varphi|(1+|c|t) \leq |\varphi|(1+|c|)e^{-\rho_0(c)t}e^{\rho_0(c)t} \\ &\leq |\varphi|(1+|c|)e^2 e^{\rho_0(c)t}.\end{aligned} \tag{2.3.2}$$

Moreover, one has $x_1^{c,\varphi,0} \in C^1$, although $\dot{x}^{c,\varphi,0}$ may have a jump discontinuity at 1. Similarly, we have for $t \in [1,2]$

$$|x^{c,\varphi,0}(t)| \leq |\varphi|(1+|c|)^2 e^{-\rho_0(c)t}e^{\rho_0(c)t} \leq |\varphi|(1+|c|)^2 e^4 e^{\rho_0(c)t}. \tag{2.3.3}$$

Set $\psi := x_2^{c,\varphi,0}$; then $\psi \in C^2$, since $x_1^{c,\varphi,0} \in C^1$, and we have

$$|\psi| \leq (1+|c|)^2|\varphi|, \quad |\psi'| \leq |c| \cdot |x_1^{c,\varphi,0}| \leq |c|(1+|c|)|\varphi|, \quad |\psi''| \leq c^2|\varphi|. \tag{2.3.4}$$

Using part a), and inequality (2.3.1) for the last step, we obtain for $t \geq 2$

$$\begin{aligned}|x^{c,\varphi,0}(t)| = |x^{c,\psi,2}(t)| &= |x^{c,\psi,0}(t-2)| \\
&\leq (4|\psi| + 7|\psi'| + 13|\psi''|)e^{\rho_0(c)(t-2)} \\
&\leq [4(1+|c|)^2|\varphi| + 7|c|(1+|c|)|\varphi| + 13|c|^2|\varphi|]e^{\rho_0(c)(t-2)} \\
&\leq [4 + 15|c| + 24|c|^2]e^4 e^{\rho_0(c)t}|\varphi|.\end{aligned}$$

We see from (2.3.2) and (2.3.3) that this estimate also holds for $t \in [0,2]$. The assertion of b) now follows from $|c| \leq 3\pi/4$. $\square$

Next, we want to express solutions of equation (a), where the coefficient function a is slowly varying, by solutions of a constant coefficient equation. For this purpose, and also for the treatment of nonlinear equations in Section 3, we need a variation-of-constants formula. As a preparation, we study the nonhomogeneous linear equation

$$(a,h) \qquad \dot{x}(t) = a(t)x(t-1) + h(t).$$

We assume that a and h are continuous on an interval $[\tau, T]$. For $t \in [\tau, T]$ we define a segment $\hat{h}(t) \in \mathbf{J}$ by setting

$$\hat{h}(t)(\theta) := \begin{cases} h(t), \, \theta = 0 \\ \quad 0, \, \theta \in [-1, 0). \end{cases}$$

Note that $|\hat{h}(t) - \hat{h}(s)| = |h(t) - h(s)|$ for $s, t \in [\tau, T]$, so that the map $\hat{h} : [\tau, T] \longrightarrow (\mathbf{J}, |\,|), \quad t \mapsto \hat{h}(t)$ is continuous.

Recall the notation $x^{a,\psi,s}$ for the solution of $\dot{y}(t) = a(t)y(t-1)$ starting with ψ at time s. We now see from continuity of $\hat{h}$ and from Proposition 2.1, b) that, for $t \in [\tau, T]$, the function $[\tau, T] \ni s \mapsto x^{a,\hat{h}(s),s}(t)$ is continuous. In particular, the integral $\int_\tau^t x^{a,\hat{h}(s),s}(t)ds$ exists.

We can now prove a variation-of-constants formula which is suitable for our purposes. It is expressed in terms of values of functions, not as an equality in the space $\mathbf{C}$. Formulas of the latter type are given for a general class of *autonomous* equations in [7], e.g., formula (2.16), p. 63. However, it is not so obvious what these formulas (respectively, their generalizations to the nonautonomous case) mean concretely in our situation. Therefore, and to make the presentation self-contained, we decided to include a proof.

Lemma 2.4 (Variation of constants) *For $\psi \in \mathbf{C}$, the solution $x^{a,h,\psi,\tau}$ of equation (a,h) with $x_\tau^{a,h,\psi,\tau} = \psi$ satisfies*

$$x^{a,h,\psi,\tau}(t) = x^{a,\psi,\tau}(t) + \int_\tau^t x^{a,\hat{h}(s),s}(t)ds \text{ for } t \geq \tau.$$

Proof: Set $y(t) := \int_\tau^t x^{a,\hat{h}(s),s}(t)ds$ for $t \geq \tau$, and $y(t) := 0$ for $t \in [\tau - 1, \tau]$.

(2.4.1) Claim: $y|_{[\tau,T]}$ is differentiable, and $\dot{y}(t) = a(t)y(t-1) + h(t)$ for $t \in [\tau, T]$.
Proof: Let $t \in [\tau, T)$ be given, and let $\delta \in (0,1)$ be such that $t + \delta \in [\tau, T]$. Then

$$
\begin{aligned}
& y(t+\delta) - y(t) - [a(t)y(t-1) + h(t)]\delta \\
&= \int_\tau^{t+\delta} x^{a,\hat{h}(s),s}(t+\delta)ds - \int_\tau^t x^{a,\hat{h}(s),s}(t)ds - [\ldots]\delta \\
&= \int_t^{t+\delta} x^{a,\hat{h}(s),s}(t+\delta)ds + \\
&\quad + \int_\tau^t [x^{a,\hat{h}(s),s}(t+\delta) - x^{a,\hat{h}(s),s}(t)]ds - a(t)y(t-1)\delta - h(t)\delta \\
&= \int_t^{t+\delta} [x^{a,\hat{h}(s),s}(t+\delta) - h(t)]ds + \\
&\quad + \int_\tau^t [x^{a,\hat{h}(s),s}(t+\delta) - x^{a,\hat{h}(s),s}(t)]ds - a(t)y(t-1)\delta.
\end{aligned}
$$

We abbreviate the first term with T_1, and the terms after the plus sign with T_2. Since $\delta < 1$, we have for $s \in [t, t+\delta]$ that $t + \delta \in [s, s+1]$ and hence $x^{a,s,\hat{h}(s)}(t+\delta) = h(s)$. It follows that

$$
|T_1| \leq \int_t^{t+\delta} |h(s) - h(t)|ds. \tag{2.4.2}
$$

We now consider the term T_2 for the case $t - 1 \geq \tau$. For $s \in [\tau, t]$ one has

$$
x^{a,s,\hat{h}(s)}(t+\delta) - x^{a,s,\hat{h}(s)}(t) = \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr,
$$

so

$$
\begin{aligned}
T_2 &= \int_\tau^t \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr\,ds - \delta \cdot a(t) \int_\tau^{t-1} x^{a,s,\hat{h}(s)}(t-1)ds \\
&= \int_\tau^{t-1+\delta} \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr\,ds + \int_{t-1+\delta}^t \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr\,ds \\
&\quad - \delta a(t) \int_\tau^{t-1} x^{a,s,\hat{h}(s)}(t-1)ds.
\end{aligned}
$$

In the second last term, we have $s \geq t - 1 + \delta$, so $r - 1 \leq t + \delta - 1 \leq s$, and hence $x^{a,s,\hat{h}(s)}(r-1) = 0$. Thus we obtain

$$
\begin{aligned}
T_2 &= \int_\tau^{t-1+\delta} \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr\,ds - \delta a(t) \int_\tau^{t-1} x^{a,s,\hat{h}(s)}(t-1)ds \\
&= \int_\tau^{t-1} \int_t^{t+\delta} [a(r)x^{a,s,\hat{h}(s)}(r-1) - a(t)x^{a,s,\hat{h}(s)}(t-1)]dr\,ds \\
&\quad + \int_{t-1}^{t-1+\delta} \int_t^{t+\delta} a(r)x^{a,s,\hat{h}(s)}(r-1)dr\,ds.
\end{aligned}
$$

In the second term above, $\delta \leq 1$ implies that $r-1 \leq t+\delta-1 \leq s+\delta \leq s+1$, so $x^{a,s,\hat{h}(s)} = h(s)$. It follows that the second term can be estimated by $\delta^2\|h\|_\infty \cdot \|a\|_\infty$, where the norms denote the max-norm on $[\tau, T]$. Hence we have

$$(2.4.3)\quad |T_2| \leq \int_\tau^{t-1} \int_t^{t+\delta} |a(r)x^{a,s,\hat{h}(s)}(r-1) - a(t)x^{a,s,\hat{h}(s)}(t-1)|dr\, ds + \delta^2\|h\|_\infty\|a\|_\infty.$$

Let now $\varepsilon > 0$. Continuity of h and (2.4.2) imply that there exists $\delta_1 \in (0,1)$ such that for $\delta \in (0, \delta_1]$ one has $|T_1| \leq \delta \cdot \varepsilon/3$. There exists $\delta_2 \in (0, \delta_1]$ with $\delta^2\|h\|_\infty\|a\|_\infty \leq \delta\varepsilon/3$ for $\delta \in (0, \delta_2]$. In view of continuity of the involved functions (see Remark 2.1 b), there exists $\delta_3 \in (0, \delta_2]$ such that the left term in (2.4.3) can be estimated by

$$\int_\tau^{t-1} \int_t^{t+\delta} \frac{\varepsilon}{3(T-\tau)} dr\, ds \leq \delta\varepsilon/3, \text{ if } \delta \in [0, \delta_3].$$

Together, we have shown that for $\delta \in (0, \delta_3]$ one has

$$(2.4.4)\qquad |T_1 + T_2| \leq |T_1| + |T_2| \leq 3\delta \cdot \varepsilon/3 = \delta \cdot \varepsilon,$$

in case $t - 1 \geq \tau$.

We now treat the case $\delta \in [0, 1)$ and $t - 1 < \tau$. In this case, $y(t-1) = 0$, and for δ such that also $t + \delta < \tau + 1$, we have for $s \in [\tau, t]$ that

$$x^{a,s,\hat{h}(s)}(t+\delta) - x^{a,s,\hat{h}(s)}(t) = \hat{h}(s) - \hat{h}(s) = 0,$$

so $T_2 = 0$. It is now clear that (2.4.4) also holds in this case for all sufficiently small δ. We have now proved that y is differentiable from the right on $[\tau, T)$, with right derivative given by $t \mapsto a(t)y(t-1) + h(t)$. It is clear that the latter function is uniformly continuous on $[\tau, T)$. It can be seen from estimate (2.4.4) that differentiability of y to the right is uniform w.r. to $t \in [\tau, T)$. The claim now follows from the proposition given after this lemma.

We conclude the proof of the lemma: With y as above, the right hand side $r(t)$ of the asserted equality satisfies $r(t) = x^{a,\psi,\tau}(t) + y(t)$, so

$$\dot{r}(t) = a(t)x^{a,\psi,\tau}(t-1) + a(t)y(t-1) + h(t) = a(t)r(t-1) + h(t) \text{ for } t \in [\tau, T].$$

Uniqueness of solutions and $r_\tau = \psi$ now imply that

$$x^{a,h,\psi,\tau}(t) = r(t) \text{ for } t \in [\tau - 1, T],$$

which is the assertion. $\square$

The proposition below was a technical auxiliary in the proof of Lemma 2.4.

Proposition 2.5 *Let $z : [\tau, T] \longrightarrow \mathbb{R}$ be continuous, and uniformly differentiable from the right on $[\tau, T)$ with uniformly continuous right derivative $z'^{,+} : [\tau, T) \longrightarrow \mathbb{R}$. Then $z \in C^1([\tau, T], \mathbb{R})$.*

Proof: Let $\varepsilon > 0$ and $t \in [\tau, T]$. There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1]$ and all $s \in [\tau, T - \delta]$ one has

$$\left| \frac{z(s+\delta) - z(s)}{\delta} - z'^{,+}(s) \right| < \frac{\varepsilon}{2}.$$

It follows that for all such δ and for $s \in [\tau + \delta, T]$ one has

$$\left| \frac{z(s) - z(s-\delta)}{\delta} - z'^{,+}(s-\delta) \right| < \varepsilon/2. \tag{2.5.1}$$

Since $z'^{,+}$ is uniformly continuous on $[\tau, T)$, it has a uniformly continuous extension to $[\tau, T]$, which we also denote by $z'^{,+}$. There exists $\delta_2 \in (0, \delta_1]$ such that for $\delta \in (0, \delta_2]$ and $s \in [\delta, T]$ one has

$$|z'^{,+}(s-\delta) - z'^{,+}(s)| < \varepsilon/2. \tag{2.5.2}$$

Combining (2.5.1) und (2.5.2) we get

$$\left| \frac{z(s) - z(s-\delta)}{\delta} - z'^{,+}(s) \right| < \varepsilon \text{ for } s \in [\tau + \delta, T],$$

which shows that z also has a left derivative on $(\tau, T]$, which coincides with $z'^{,+}$. □

3 Nonlinear nonautonomous equations

We consider equations of the type

$$\dot{x}(t) = g(t, x(t-1)), \tag{g}$$

where we assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and has two continuous derivatives w.r. to the second argument. Further, we assume that for all t one has $g(t, 0) = 0$, and that $|\partial_2^2 g|$ has a finite supremum which we denote by $\|\partial_2^2 g\|$.

For a bounded function a on an interval $[s, t]$, we use the notation

$$V_a(s,t) := \sup_{\tau \in [s,t]} a(\tau) - \inf_{\tau \in [s,t]} a(\tau).$$

Using Lemma 2.4, we can now obtain an estimate on solutions of nonautonomous and nonlinear equations.

Lemma 3.1 *Let $\varphi \in \mathbf{C}, T \geq 1, s \in \mathbb{R}$, and let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be as above. Set $a(t) := \partial_2 g(t, 0)$ for $t \in \mathbb{R}$. Assume that $c \in a([s, s+T]) \cap [-3\pi/4, -\pi/4]$. Set $V := V_a(s, s+T)$, and, with K from Proposition 2.3, set*

$$K_V := \max\{K, 1 + 3\pi/4 + V\},\ L_V := K_V e^2.$$

Let $x : [s-1,\infty) \to \mathbb{R}$ be the solution of equation (g) with $x_s = \varphi$, and assume that $\tau \in [s, s+T]$ and $\xi \geq 0$ are such that $|x_t| \leq \xi$ for all $t \in [s,\tau]$. Then, for all $t \in [s,\tau]$, one has

$$|x_t| \leq L_V|\varphi| \exp[(\rho_0(c) + L_V V + L_V\|\partial_2^2 g\|\xi/2) \cdot (t-s)].$$

Proof: For $t \in [s, s+T]$, there exists $r_t \in (0,1)$ such that

$$\begin{aligned}
&g(t, x(t-1)) = \\
&= \partial_2 g(t,0)x(t-1) + [\partial_2^2 g(t, r_t x(t-1))/2]x(t-1)^2 \\
&= cx(t-1) + (\partial_2 g(t,0) - c)x(t-1) + [\partial_2^2 g(t, r_t x(t-1))/2]x(t-1)^2 \\
&= cx(t-1) + (a(t) - c)x(t-1) + [\partial_2^2 g(t, r_t x(t-1))/2]x(t-1)^2.
\end{aligned}$$

Thus, with $h(t) := (a(t)-c)x(t-1) + [\partial_2^2 g(t, r_t x(t-1))/2]x(t-1)^2$ for $t \in [s, s+T]$, one has for these t

$$\dot{x}(t) = cx(t-1) + h(t).$$

Further, for $t \in [s,\tau]$ one has

$$|\hat{h}(t)| \leq V|x(t-1)| + (\|\partial_2^2 g\|/2)x(t-1)^2 \leq [V + \|\partial_2^2 g\|\xi/2] \cdot |x(t-1)|.$$

It follows from Lemma 2.4 and from Proposition 2.3,b) that, for $t \in [s,\tau]$,

$$\begin{aligned}
&|x(t)| = \\
&= |x^{c,\varphi,s}(t) + \int_s^t x^{c,\hat{h}(\sigma),\sigma}(t)\, d\sigma| \\
&\leq K|\varphi| \exp[\rho_0(c)(t-s)] + K\int_s^t \exp[\rho_0(c)(t-\sigma)]|\hat{h}(\sigma)|\, d\sigma \\
&\leq K\left\{|\varphi| \exp[\rho_0(c)(t-s)] + [V + \|\partial_2^2 g\|\xi/2]\int_s^t \exp[\rho_0(c)(t-\sigma)]|x(\sigma-1)|\, d\sigma\right\}.
\end{aligned}$$

Set $W := [V + (\|\partial_2^2 g\|\xi/2)]$. If now $t \in [s+1,\tau]$ (in case $s+1 \leq \tau$) and $\theta \in [-1,0]$ then

$$\begin{aligned}
&|x(t+\theta)| \leq \\
&\leq K\left\{|\varphi| \exp[\rho_0(c)(t+\theta-s)] + W\int_s^{t+\theta} \exp[\rho_0(c)(t+\theta-\sigma)]|x(\sigma-1)|\, d\sigma\right\} \\
&\leq K\exp(|\rho_0(c)|)\left\{|\varphi| \exp[\rho_0(c)(t-s)] + W\int_s^{t+\theta} \exp[\rho_0(c)(t-\sigma)]|x_\sigma|\, d\sigma\right\} \\
&\leq K\exp(|\rho_0(c)|)\left\{|\varphi| \exp[\rho_0(c)(t-s)] + W\int_s^{t} \exp[\rho_0(c)(t-\sigma)]|x_\sigma|\, d\sigma\right\}.
\end{aligned}$$

Hence, for $t \in [s+1,\tau]$, it follows trivially that with
$K_V := \max\{K, 1 + 3\pi/4 + V\}$ we have
(3.1.1)

$$|x_t| \leq K_V \exp(|\rho_0(c)|)\left\{|\varphi| \exp[\rho_0(c)(t-s)] + W\int_s^t \exp[\rho_0(c)(t-\sigma)]|x_\sigma|\, d\sigma\right\}.$$

For $t \in [s, s+1] \cap [s, \tau]$, we obtain (using the differential equation, and the definition of K_V) that

$$\begin{aligned} |x_t| &\leq |\varphi| + \int_{-1}^{0} (|c| + V)|\varphi| \, ds = |\varphi|(1 + |c| + V) \\ &\leq |\varphi|(1 + 3\pi/4 + V) \leq K_V |\varphi|. \end{aligned}$$

The right hand side of (3.1.1) is, for $t \in [s, s+1]$, obviously bounded below by $K_V|\varphi|$. Hence, (3.1.1) holds also for $t \in [s, s+1]$.

Now, setting $y(t) := \exp[-\rho_0(c)t]|x_t|$ for $t \in [s, \tau]$, we obtain from (3.1.1) and from (2.3.1) that

$$\begin{aligned} y(t) &\leq K_V \exp(|\rho_0(c)|) \left\{ |\varphi| \exp[-\rho_0(c)s] + W \int_s^t y(\sigma) \, d\sigma \right\} \\ &\leq K_V e^2 \left\{ |\varphi| \exp[-\rho_0(c)s] + W \int_s^t y(\sigma) \, d\sigma \right\} \\ &= L_V \left\{ |\varphi| \exp[-\rho_0(c)s] + W \int_s^t y(\sigma) \, d\sigma \right\}. \end{aligned}$$

It follows from Gronwall's Lemma that for $t \in [s, \tau]$ one has

$$y(t) \leq L_V |\varphi| \exp[-\rho_0(c)s] \exp[L_V W(t-s)].$$

Hence we conclude

$$\begin{aligned} |x_t| &\leq L_V |\varphi| \exp[\rho_0(c)(t-s)] \exp[L_V W(t-s)] \\ &= L_V |\varphi| \exp[(\rho_0(c) + L_V V + L_V \|\partial_2^2 g\| \xi/2) \cdot (t-s)]. \end{aligned}$$

□

We are now prepared for the proof of a delayed loss of stability estimate for nonlinear nonautonomous equations of type (g). Again, we restrict attention to the case where $\partial_2 g(\cdot, 0)$ takes values in $[-3\pi/4, -\pi/4]$. Recall the definition of $V_a(s,t)$ for $s \leq t$.

Theorem 3.2 *Let $t_- \in \mathbb{R}$, let g be as above, and assume that the function defined by $a(t) := \partial_2 g(t, 0)$ takes values in $[-3\pi/4, -\pi/4]$. Assume that there exists $T \geq 1$ and $V \geq 0$ such that one has for all $s \geq t_-$*

(3.2.1) $$V_a(s, s+T) \leq V.$$

Let $\varphi \in \mathbf{J}$, and let $x : [t_- - 1, \infty) \to \mathbb{R}$ be the solution of equation (g) with $x_{t_-} = \varphi$. Assume that $t_+ \geq t_-$ and $\xi \geq 0$ are such that

$$\forall t \in [t_-, t_+] : \; |x_t| \leq \xi.$$

Define L_V as in Lemma 3.1, and set

$$C := C(V, T, \xi) := L_V V + L_V \|\partial_2^2 g\| \xi/2 + \log(L_V)/T.$$

Finally, for $t, s \in \mathbb{R}, \; t \geq s \geq t_-$, set

$$u(t,s) := \exp[\int_s^t (\rho_0(a(r)) + C) \, dr.$$

a) Then one has for all $t \in [t_-, t_+]$

$$|x_t| \leq |\varphi| L_V u(t, t_-).$$

b) With $c_- := \dfrac{4(\pi - 2)}{3\pi^2}$, $c_+ := \dfrac{4(\pi + 2)}{\pi^2}$, *the following estimates hold:*

$$\text{If } t, s \in [t_-, t_+],\ s \leq t,\ \text{ and } a(\cdot) \geq -\pi/2 \text{ on } [s, t] \text{ then}$$
$$u(t, s) \leq \exp[\int_s^t (-c_-|a(s) + \pi/2| + C)\, ds].$$
$$\text{If } t, s \in [t_-, t_+],\ s \leq t,\ \text{ and } a(\cdot) \leq -\pi/2 \text{ on } [s, t] \text{ then}$$
$$u(t, s) \leq \exp[\int_s^t (c_+|a(s + \pi/2| + C)\, ds].$$

Remarks. 1. The first estimate in b) implies (not necessarily monotonous) decay of $|x_t|$, as long as $a(s) \geq -\pi/2$ and $c_-|a(s) + \pi/2| \geq C$. One can expect the second inequality to hold only if the term $\|\partial_2^2 g\|\xi/2$ is small enough, i.e., if the solution x takes sufficiently small values. This is natural since the decay is an effect of the linearization at zero. If one wants to obtain decay for 'large' initial values φ, it is necessary to combine the estimate of Theorem 3.2, a) with different methods, as we do in the example in Section 4.

2. If one obtains $|x_{t_0}| < |\varphi|$ for some $t_0 \in [t_-, t_+]$, then the second inequality in b) can be used to give a lower estimate for the time until $|x_t|$ reaches $|\varphi|$ again.

Proof: [Proof of Theorem 3.2.] Set $\tilde{C} := \tilde{C}(V, \xi) := L_V V + L_V \|\partial_2^2 g\|\xi/2$. For $t \geq t_-$, set $\eta(t) := \exp[\int_{t_-}^t (\rho_0(a(s)) + \tilde{C})\, ds] = \exp[\int_{t_-}^t (\rho_0(a(s))\, ds] \exp[\tilde{C}(t - t_-)]$. Consider φ and x as in the theorem.

Claim: If $t \in [t_- + (j-1)T, t_- + jT]$ for some $j \in \mathbb{N}$, and $t \leq t_+$, then

$$|x_t| \leq |\varphi| L_V^j \eta(t).$$

Proof. (Induction on j.) The case $j = 1$: Assume $t \in [t_-, t_- + T]$. From the mean value theorem, there exists $\tau = \tau(t) \in [t_-, t]$ such that

$$\int_{t_-}^t \rho_0(a(s)) ds = (t - t_-)\rho_0(a(\tau)).$$

Applying Lemma 3.1 with $s := t_-, \tau := T, c := a(\tau)$, one obtains

$$\begin{aligned}
|x_t| &\leq L_V |\varphi| \exp[(\rho_0(c) + L_V V + L_V \|\partial_2^2 g\|\xi/2)(t - t_-)] \\
&= L_V |\varphi| \exp[\int_{t_-}^t (\rho_0(a(s)) + \tilde{C})\, ds] \\
&= |\varphi| L_V \eta(t),
\end{aligned}$$

which is the assertion for $j = 1$.

Assume now that the assertion holds for some $j \in \mathbb{N}$, and that $t \in [t_- + jT, t_- + (j+1)T]$, $t \le t_+$. Set $\psi := x_{t_- + jT}$. Then the induction hypotheses gives $|\psi| \le |\varphi| L_V^j \eta(t_- + jT)$. From the case $j = 1$, applied with $t_- + jT$ in place of t_-, one obtains for the solution $y : [t_- + jT - 1, \infty) \to \mathbb{R}$ of equation (g) with $y_{t_- + jT} = \psi$ that

$$|y_t| \le |\psi| L_V \exp[\int_{t_- + jT}^{t} (\rho_0(a(s)) + \tilde{C})\, ds].$$

Together with the estimate on $|\psi|$, we conclude

$$\begin{aligned} |x_t| = |y_t| &\le |\varphi| L_V^j \eta(t_- + jT) L_V \exp[\int_{t_- + jT}^{t} (\rho_0(a(s)) + \tilde{C})\, ds] \\ &= |\varphi| L_V^{j+1} \eta(t). \end{aligned}$$

The claim is proved.

Now let $t \in [t_-, t_+]$, and set $j := \min\left\{ n \in \mathbb{N} \;\middle|\; t_- + nT > t \right\}$. Then $t_- + (j-1)T \le t < t_- + jT$, and from the above claim we get $|x_t| \le |\varphi| L_V^j \eta(t)$. Note that

$$L_V^{j-1} = \exp[\frac{(j-1)T \log(L_V)}{T}] \le \exp[(t - t_-)\frac{\log(L_V)}{T}] = \exp[\int_{t_-}^{t} \frac{\log(L_V)}{T} ds].$$

Recalling the definition of η, and noting that $\tilde{C} + \log(L_V)/T = C$, we obtain

$$|x_t| \le |\varphi| L_V \exp[\int_{t_-}^{t} (\rho_0(a(s)) + \tilde{C} + \log(L_V)/T)\, ds] = |\varphi| L_V u(t, t_-),$$

that is, assertion a). Assertion b) follows from the estimates on ρ_0 from Proposition 2.2,c). □

4 An example

For $\varepsilon \in (0, 0.01]$, we set

$$g(t, x) := (-\pi/4 - \varepsilon t) \arctan(x),$$

and we consider the solution $x : [-1, \infty) \longrightarrow \mathbb{R}$ of equation (g) with the constant function equal to 1 as initial segment. (The dependence of all objects on ε is not denoted.) Note that $a(t) := \partial_2 g(t, 0)$ satisfies $a(t) \in [-3\pi/4, -\pi/4]$ as long as $t \in [0, \pi/(2\varepsilon)]$. Further, for these t and for $y \in \mathbb{R}$, one has

$$|\partial_2^2 g(t, y)| \le |-3\pi/4| \sup_{z \in \mathbb{R}} |2z/(1+z^2)^2| \le 2 \cdot 3\pi/4 \le 5.$$

(It is inessential that these properties do not hold for t outside the interval $[0, \pi/(2\varepsilon)]$, in which we will be interested.)

Proposition 4.1 *The solution x is slowly oscillating, that is: There exists a sequence $(z_1, z_2, ...)$ in $\mathbb{R}$ such that $0 < z_1 < z_2 < ...$, and such that the z_i are precisely the zeroes of x, and $z_{i+1} - z_i > 1$ for all $i \in \mathbb{N}$. The extrema of x on $(0, \infty)$ occur at the times $\mu_i := z_i + 1 \in (z_i, z_{i+1})$, so we have*

$$z_1 < \mu_1 = z_1 + 1 < z_2 < \mu_2 = z_2 + 1 < \ldots .$$

Further, one has $z_1 \le 2$.

Proof: Assume that x has no zero on some interval of the form $[t_0, \infty)$. Then the negative feedback property $g(t, y)y < 0$ $(t > 0, y \in \mathbb{R} \setminus \{0\})$ implies that $x(t) \longrightarrow 0$ $(t \longrightarrow \infty)$, so there exists $t_1 \ge t_0$ with $|x(\cdot)| \le 0.1$ on $[t_1, \infty)$. Now setting $\alpha(t) := \int_0^1 \partial_2 g(t, sx(t-1))\, ds$, the function x satisfies $\dot{x}(t) = \alpha(t)x(t-1)$ for $t \ge t_1$, and for these t one has

$$\alpha(t) \le -(\pi/4) \min_{|y| \le 0.1} \arctan'(y) = -(\pi/4) \cdot 100/101 < -\exp(-1).$$

We can now apply Theorem 8 in [10] (with $n := 1, r := 1, \eta(t, -1) := 0, \eta(t, \theta) := \alpha(t)$ for $\theta \in (-1, 0]$, and with $q(t, \theta) := -\eta(t, \theta)$); in particular, the last inequality shows that Condition (A4) of that theorem is satisfied. It follows that x has infinitely many zeroes on $[t_1, \infty)$, in contradiction to our assumption.

We know now that x must have infinitely many zeroes. It follows from the fact that the segment x_0 has no zero, and from the fact that the zero-counting Liapunov functional used in [15] does not increase in time, that x is slowly oscillating (see [15], Theorem 2.1). The assertion about extrema is now clear, in view of the differential equation.

We now prove $z_1 \le 2$: On $[0, 1]$, we have

$$\dot{x}(t) = (-\pi/4 - \varepsilon t) \arctan(1) \le (-\pi/4)(\pi/4) = -\pi^2/16,$$

and hence $x(1) \le 1 - \pi^2/16$. On the other hand, for $t \in [0, 1]$, one obtains (using $\varepsilon \le 0.01$) that $\dot{x}(t) \ge (-\pi/4 - \varepsilon)(\pi/4) \ge -10/16 = -5/8$, so $x(t) \ge 1 - (5/8)t$ for these t. It follows from $|\arctan(y)| \ge |(\pi/4)y|$ if $|y| \le 1$ that for $t \in [1, 2]$ one has

$$\dot{x}(t) \le -(\pi/4)(\pi/4)[1 - (5/8)(t-1)].$$

Hence, integrating, we obtain

$$\begin{aligned} x(2) &\le 1 - (\pi^2/16) - (\pi^2/16)[1 - (5/8)(1/2)] = 1 - (\pi^2/16) - (\pi^2/16) \cdot (11/16) \\ &= 1 - (27\pi^2/256) \le 1 - 27 \cdot 9.5/256 = 1 - 256.5/256 < 0, \end{aligned}$$

and consequently x has a first zero z_1 in [1,2]. □

Set $m_i := |x(\mu_i)|$ for $i \in \mathbb{N}$; then $m_i = \max_{t \in [z_i, z_{i+1}]} |x(t)|$. We first focus attention on the time interval $(0, \pi/(16\varepsilon)]$. Let $J \in \mathbb{N}$ be such that the extrema of x in this interval occur at the times $\mu_1, ..., \mu_J$. The following estimate exploits the fact that for t in $[0, \pi/(16\varepsilon)]$ one has $|g(t, y)| \le q|y|$ $(y \in \mathbb{R})$ with some $q \in (0, 1)$.

Proposition 4.2 *For $t \in [0, z_{J+1}]$ one has $|x(t)| \leq 1$.*
Further, with $q := 5\pi/16$, one has

$$m_{i+1} \leq q m_i \quad \text{if } i \in \{1, \ldots, J-1\}.$$

Proof: Note that $|\arctan(y)| \leq |y|$ for $y \in \mathbb{R}$. As long as $t \leq \pi/(16\varepsilon)$, we thus have

$$|g(t,y)| \leq (\pi/4 + \pi/16)|y| = 5\pi/16|y| = q|y|.$$

Since $|x(\cdot)| \leq 1$ on $[0, z_1]$, we have $m_1 \leq q < 1$. Further, if $i \in \{1, \ldots, J-1\}$, we obtain (using $z_{i+1} - z_i > 1$) that

$$m_{i+1} = \Big|\int_{z_{i+1}}^{z_{i+1}+1} g(s, x(s-1))\, ds\Big| \leq \int_{z_{i+1}-1}^{z_{i+1}} q|x(s)|ds \leq q m_i.$$

Together with $|x(\cdot)| \leq 1$ on $[0, \mu_1]$ and the fact that $|x(\cdot)|$ decreases on $[\mu_J, z_{J+1}]$, it follows that $|x(\cdot)| \leq 1$ on $[0, z_{J+1}]$. □

Next, we give a decay estimate for the case that $\mu_i - \mu_{i-1}$ is 'large'.

Proposition 4.3 *Assume $i \in \{2, \ldots, J+1\}$ and $\mu_{i-1} + 1 \leq z_i$. Then for all $j \in \mathbb{N}_0$ with $\mu_{i-1} + j \leq z_i$ one has*

$$|x(\mu_{i-1} + j)| \leq q^{j-1} m_{i-1}.$$

Proof: The estimate is trivial for $j = 0$. Since $|x(\cdot)|$ decreases on $[\mu_{i-1}, z_i]$ and $m_{i-1} \leq 1$, we have $|x(\cdot)| \leq 1$ on $[\mu_{i-1}, z_i]$, and $|x(\mu_{i-1} + 1)| \leq m_{i-1}$. Hence the assertion holds for $j = 1$. Now if $j \in \mathbb{N}$ and $[\mu_{i-1} + j, \mu_{i-1} + j + 1] \subset [\mu_{i-1}, z_i]$, we obtain (using $|\arctan(y)| \geq (\pi/4)|y|$ if $|y| \leq 1$, and the monotonicity of $|x(\cdot)|$ on $[\mu_{i-1} + j - 1, \mu_{i-1} + j]$) that

$$\begin{aligned}
|x(\mu_{i-1} + j + 1)| &= |x(\mu_{i-1} + j) + \int_{\mu_{i-1}+j}^{\mu_{i-1}+j+1} g(s, x(s-1))\, ds| \\
&\leq |x(\mu_{i-1} + j)| - \min_{s \in [\mu_{i-1}+j-1, \mu_{i-1}+j]} |g(s+1, x(s)| \\
&\leq |x(\mu_{i-1} + j)| - (\pi/4)|\arctan(x(\mu_{i-1} + j))| \\
&\leq |x(\mu_{i-1} + j)| - (\pi^2/16)|x(\mu_{i-1} + j)| \\
&= [(16 - \pi^2)/16]|x(\mu_{i-1} + j)| \leq q|x(\mu_{i-1} + j)|.
\end{aligned}$$

For $j \in \mathbb{N}$ with $\mu_{i-1} + j \leq z_i$, it follows inductively that

$$|x(\mu_{i-1} + j)| \leq q^{j-1}|x(\mu_{i-1} + 1)| \leq q^{j-1} m_{i-1}.$$

□

Proposition 4.2 above relates the value m_i to the index i, but not to the time μ_i at which it occurs. This is achieved in the next result.

Proposition 4.4 *With the negative number* $\lambda := \log(q)/4$*, one has*

$$m_j \le q^{-1} \exp(\lambda \mu_j) \quad (j = 1, \ldots, J).$$

Proof: Let $i \in \{2, ..., J\}$. If $z_i - \mu_{i-1} \le 2$ then $\mu_i - \mu_{i-1} \le 3 < 4$, and

$$m_i/m_{i-1} \le q = \exp(4\lambda) \le \exp(\lambda(\mu_i - \mu_{i-1})). \tag{4.4.1}$$

Consider now the case $z_i - \mu_{i-1} > 2$. Then, setting

$$j_1 := \max\left\{j \in \mathbb{N} \;\middle|\; \mu_{i-1} + j + 1 \le z_i\right\},$$

we obtain from Proposition 4.3 that

$$|x(\mu_{i-1} + j_1)| \le q^{j_1 - 1} m_{i-1}.$$

Note that $[z_i - 1, z_i] \subset [\mu_{i-1} + j_1, z_i]$, and hence $|x(t)| \le |x(\mu_{i-1} + j_1)|$ for $t \in [z_i - 1, z_i]$. We infer from the differential equation that

$$m_i \le q|x(\mu_{i-1} + j_1)| \le q q^{j_1 - 1} m_{i-1} = q^{j_1} m_{i-1}.$$

Now, from the definition of j_1,

$$\mu_i - \mu_{i-1} = z_i + 1 - \mu_{i-1} \le \mu_{i-1} + j_1 + 3 - \mu_{i-1} = j_1 + 3 \le 4j_1,$$

and thus

$$m_i/m_{i-1} \le q^{j_1} = \exp(\lambda \cdot 4j_1) \le \exp(\lambda(\mu_i - \mu_{i-1})),$$

and we see that (4.4.1) also holds in the second case.

We conclude that for $j \in \{1, \ldots, J\}$ one has

$$\begin{aligned} m_j &= m_1 \prod_{i=2}^{j} (m_i/m_{i-1}) \le m_1 \prod_{i=2}^{j} \exp(\lambda(\mu_i - \mu_{i-1}) \\ &= m_1 \exp[\lambda(\mu_j - \mu_1)] = m_1 \exp(-\lambda\mu_1) \exp(\lambda\mu_j), \end{aligned}$$

where the product is to be read as 1 if $j = 1$. Since $m_1 \le 1$ (Proposition 4.2) and $\mu_1 = z_1 + 1 \le 3 < 4$ (Proposition 4.1), it follows that $m_j \le \exp(-4\lambda)\exp(\lambda\mu_j) = q^{-1}\exp(\lambda\mu_j)$. □

We can now obtain an exponential decay estimate for x (which is not based on linearization at zero) for the time interval $[0, \pi/(16\varepsilon)]$.

Corollary 4.5 *For* $t \in [0, \pi/(16\varepsilon)]$*, one has* $|x_t| \le 2q^{-3} \exp(\lambda t)$*.*

Proof: 1. For $t \in [0, \mu_1]$, one has $|x_t| \le 1$, and $\mu_1 \le 3 < 4$ implies

$$2q^{-3} \exp(\lambda t) \ge 2q^{-3} \exp(4\lambda) = 2q^{-2} > 1,$$

so the assertion is true for these t.

2. Let $t \in [\mu_1, \pi/(16\varepsilon)]$. There exists $i \in \{2, \dots, J+1\}$ with $t \in [\mu_{i-1}, \mu_i]$.

Case 1: $t \le \mu_{i-1} + 2$. We have $|x(s)| \le m_{i-1}$ for $s \in [z_{i-1}, z_i]$, and

$$|\dot{x}(s)| \le |(-\pi/4 - \pi/16)| m_{i-1} = q m_{i-1} \le m_{i-1}$$

for $s \in [z_i, t]$, if $t \ge z_i$. In this case, $t - z_i \le \mu_{i-1} + 2 - z_i \le 2$, so $|x(s)| \le 2m_{i-1}$ for $s \in [z_i, t]$. With Proposition 4.4, it follows that

$$\begin{aligned}|x_t| &\le 2m_{i-1} \le 2q^{-1}\exp(\lambda\mu_{i-1}) = 2q^{-1}\exp(\lambda(\mu_{i-1} - t))\exp(\lambda t) \\ &\le 2q^{-1}\exp(-2\lambda)\exp(\lambda t) \le 2q^{-2}\exp(\lambda t).\end{aligned}$$

Case 2: $t > \mu_{i-1} + 2$. Then $z_i = \mu_i - 1 \ge t - 1 \ge \mu_{i-1} + 1$. Setting $j_1 := \max\left\{ j \in \mathbb{N} \;\middle|\; \mu_{i-1} + j \le \min\{t, z_i\} \right\}$, we obtain from Proposition 4.3 that

$$|x(\mu_{i-1} + j_1)| \le q^{j_1 - 1} m_{i-1}.$$

Subcase 2a: $t \le z_i$. Then $j_1 \ge 2$, and $t - (\mu_{i-1} + j_1) \le 1$, and it follows from Proposition 4.3 and Proposition 4.4 that

$$\begin{aligned}|x_t| &\le |x_{\mu_{i-1}+j_1}| = |x(\mu_{i-1} + j_1 - 1)| \le q^{j_1-2} m_{i-1} \\ &\le q^{j_1-2} q^{-1}\exp(\lambda\mu_{i-1}) = q^{-1}\exp[4\lambda(j_1 - 2)]\exp(\lambda\mu_{i-1}) \\ &\le q^{-1}\exp[\lambda(j_1 - 2)]\exp(\lambda\mu_{i-1}) = q^{-1}\exp(\lambda t)\exp[\lambda(j_1 - 2 + \mu_{i-1} - t)] \\ &\le q^{-1}\exp(-3\lambda)\exp(\lambda t) \le q^{-2}\exp(\lambda t).\end{aligned}$$

Subcase 2b: $t > z_i$. Then $t \in [z_i, \mu_i] = [z_i, z_i + 1]$. From Subcase 2a, applied to z_i, we obtain $|x_{z_i}| \le q^{-2}\exp(\lambda z_i)$. For $s \in [z_i, t]$, one has $\dot{x}(s) \le |(-\pi/4 - \pi/16)||x_{z_i}| \le q q^{-2}\exp(\lambda z_i)$. It follows that

$$\begin{aligned}|x_t| &\le q^{-2}\exp(\lambda z_i) = q^{-2}\exp(\lambda t)\exp(\lambda(z_i - t)) \\ &\le q^{-2}\exp(-\lambda)\exp(\lambda t) \le q^{-3}\exp(\lambda t).\end{aligned}$$

From Part 1 and the different cases of Part 2, the asserted estimate is obtained.

□

Combining the above estimate with the ones which were obtained from linearization at zero in Theorem 3.2, we can now provide a lower estimate on the time that passes until $|x(t)| > 1$ again.

With c_- and c_+ from Theorem 3.2, we set $c_1 := \lambda\pi/16 - c_+\pi^2/16 - 5c_-\pi^2/512$ and $c_2 := c_+\pi/4 + c_-\pi/32$. Note that $c_1 < 0 < c_2$.

Corollary 4.6 *There exists $\varepsilon_0 \in (0, 0.01]$ such that for $\varepsilon \in (0, \varepsilon_0]$ the function $t \mapsto |x_t|$ decreases to values below ε on the interval $[0, \pi/4\varepsilon]$, and then reaches the value $\sqrt{\varepsilon}$ again not before the time $|c_1/2c_2\varepsilon|$. (In particular, the value 1 is not reached before this time.)*

Proof: With K from Proposition 2.3, we set $L := Ke^2$. There exists $\varepsilon_0 \in (0, 0.01]$ such that for $\varepsilon \in (0, \varepsilon_0]$ the following estimates hold.

$$2q^{-3}\exp(\lambda\pi/16\varepsilon) \leq \varepsilon \tag{4.6.1}$$

$$5L\sqrt{\varepsilon} \leq c_-\pi/32 \tag{4.6.2}$$

$$L2q^{-3}\exp[(\lambda\pi/16 - c_-\pi^2/512)/\varepsilon)] \leq \varepsilon \tag{4.6.3}$$

$$|\log(q^3\sqrt{\varepsilon}/2L)| \leq |c_1|/2\varepsilon. \tag{4.6.4}$$

Let now $\varepsilon \in (0, \varepsilon_0]$. We set $T := T(\varepsilon) := 1/\sqrt{\varepsilon}$. We then have for all $s \in \mathbb{R}$

$$V_a(s, s+T) = \varepsilon T = \sqrt{\varepsilon} \leq 1.$$

It follows that with $V := \sqrt{\varepsilon}$ and with K_V, L_V as in Lemma 3.1, one has $K_V = K$ and $L_V = Ke^2 = L$. Further, we have $\log(L_V)/T = \sqrt{\varepsilon}\log(L)$.

We set $\xi := \sqrt{\varepsilon}$; then the constant $C = C(V, T, \xi)$ from Theorem 3.2 satisfies

$$C \leq L\sqrt{\varepsilon} + 5L\sqrt{\varepsilon}/2 + \log(L)\sqrt{\varepsilon} \leq 5L\sqrt{\varepsilon}.$$

From Corollary 4.5 and from (4.6.1), we obtain that

$$|x_{\pi/16\varepsilon}| \leq 2q^{-3}\exp(\lambda\pi/16\varepsilon) \leq \varepsilon < \xi.$$

Now we set $t_- := \pi/16\varepsilon$, and $t_+ := \min\Big\{\inf\Big\{t > t_- \mid |x_t| > \xi\Big\}, \pi/2\varepsilon\Big\}$, and we apply Theorem 3.2. It follows that with $u(t, t_-)$ defined as in that theorem, one has

$$\forall\, t \in [t_-, t_+]: \ |x_t| \leq L2q^{-3}\exp(\lambda\pi/16\varepsilon)u(t, t_-).$$

Next, we estimate $u(t, s)$ for t, s in different time intervals. Note that for $t \in \mathbb{R}$, one has $|a(t) + \pi/2| = |-\pi/4 + \pi/2 - \varepsilon t| = |\pi/4 - \varepsilon t|$. Thus, for $t \in [\pi/16\varepsilon, 3\pi/16\varepsilon]$, we have $|a(t) + \pi/2| \geq \pi/16$. It follows from Theorem 3.2, b) and from (4.6.2) that, for these t,

$$\begin{aligned} u(t, \pi/16\varepsilon) &\leq \exp[\int_{\pi/16\varepsilon}^{t} (-c_-\pi/16 + C)\, ds] \\ &\leq \exp[\int_{\pi/16\varepsilon}^{t} (-c_-\pi/16 + 5L\sqrt{\varepsilon})\, ds] \\ &\leq \exp[\int_{\pi/16\varepsilon}^{t} (-c_-\pi/32)\, ds] = \exp[(-c_-\pi/32)(t - \pi/16\varepsilon)]. \end{aligned} \tag{4.6.5}$$

In particular, the function $t \mapsto u(t, t_-)$ decays on $[\pi/16\varepsilon, 3\pi/16\varepsilon]$.

With $t_0 := \pi/4\varepsilon$, we have $a(t_0) = -\pi/2$. For $t \in [3\pi/16\varepsilon, t_0]$, one has

$$\rho_0(a(t)) + C \le C \le 5L\sqrt{\varepsilon},$$

and for these t one has from the definition of $u(\cdot,\cdot)$ and from (4.6.2) that

$$\begin{aligned} u(t, 3\pi/16\varepsilon) &\le \exp[5L\sqrt{\varepsilon}(t - 3\pi/16\varepsilon)] \le \exp[5L\sqrt{\varepsilon}\pi/16\varepsilon] \\ &\le \exp[c_-\pi^2/512\varepsilon]. \end{aligned} \tag{4.6.6}$$

Combining (4.6.5) and (4.6.6), we see that

$$\begin{aligned} |x_{t_0}| &\le L2q^{-3}\exp[\lambda\pi/16\varepsilon]u(t_0, 3\pi/16\varepsilon)u(3\pi/16\varepsilon, \pi/16\varepsilon) \\ &\le L2q^{-3}\exp[\lambda\pi/16\varepsilon + c_-\pi^2/512\varepsilon - c_-(\pi/32)(\pi/8\varepsilon)] \\ &= L2q^{-3}\exp[(\lambda\pi/16 - c_-\pi^2/512)/\varepsilon]. \end{aligned}$$

Now (4.6.3) shows that $|x_{t_0}| \le \varepsilon < \xi$, in particular, $t_+ > t_0$.

Finally, for $t \in [t_0, t_+]$ we have $|a(t) + \pi/2| \le \pi/4$. Using Part b) of Theorem 3.2, together with the inequalities $C \le 5L\sqrt{\varepsilon}$ and (4.6.2), one sees that

$$u(t, t_0) \le \exp[(c_+\pi/4 + C)(t - t_0)] \le \exp[(c_+\pi/4 + c_-\pi/32)(t - \pi/4\varepsilon)]. \tag{4.6.7}$$

Combining the estimates (4.6.5), (4.6.6) and (4.6.7), we conclude that for $t \in [t_0, t_+]$ one has

$$\begin{aligned} &|x_t| \le \\ &L2q^{-3}\exp[(\lambda\pi/16 - c_-\pi^2/512)/\varepsilon - (c_+\pi/4 + c_-\pi/32)(\pi/4\varepsilon) + (c_+\pi/4 + c_-\pi/32)t] \\ &= L2q^{-3}\exp[(\lambda\pi/16 - c_+\pi^2/16 - 5c_-\pi^2/512)/\varepsilon + (c_+\pi/4 + c_-\pi/32)t] \\ &= L2q^{-3}\exp[c_1/\varepsilon + c_2 t]. \end{aligned}$$

First case: $t_+ < \pi/2\varepsilon$. Then

$$L2q^{-3}\exp[c_1/\varepsilon + c_2 t_+] \ge \xi = \sqrt{\varepsilon}, \text{ so } t_+ \ge [\log(q^3\sqrt{\varepsilon}/2L) - c_1/\varepsilon]/c_2.$$

Using (4.6.4), we infer $t_+ \ge -c_1/2c_2\varepsilon$. Thus, $|x_t|$ reaches the value $\xi = \sqrt{\varepsilon}$ again not earlier than this time.

Second case: $t_+ = \pi/2\varepsilon$. Then the function $t \mapsto |x_t|$ is bounded by $\sqrt{\varepsilon}$ on the interval $[t_0, \pi/2\varepsilon]$. From the expressions for c_1 and c_2, it is not difficult to see that $|c_1/2c_2| < \pi/2$. Hence, the assertion also holds in the second case. □

Remark. The estimate in Corollary 4.6 is, of course, quantitatively correct only in the sense that it predicts a 'growth' time of order $1/\varepsilon$. Further, the upper bound 0.01 for ε, which we used above, is only of technical nature. As an illustration, we show a numerically obtained plot of the solution x for $\varepsilon = 0.03$ on the time interval $[0, 60]$ in Figure 4.1. (The equation was solved using Simpson's rule.)

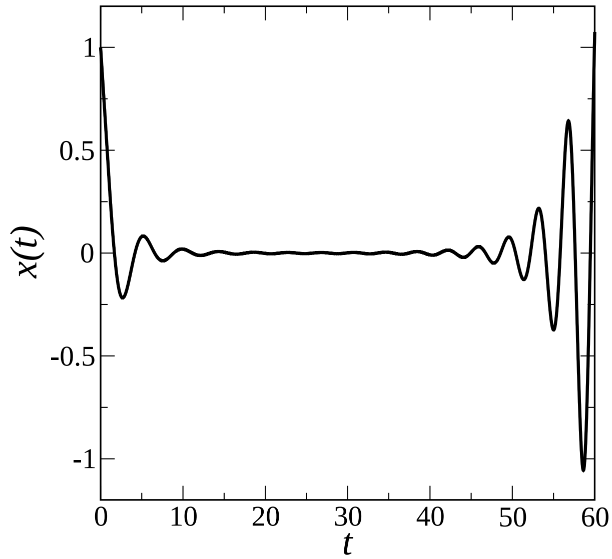

Figure 1: Numerical solution of $\dot{x}(t) = (-\pi/4 - \varepsilon t)\arctan(x(t-1))$ with constant initial segment equal to 1 and $\varepsilon = 0.03$.

5 Additive nonautonomous perturbations

The results of Sections 3 and 4 are not unexpected, in view of their analogy to the case of ODEs.

In the last section, we want to demonstrate an effect in nonautonomous delay equations which is specific for delay equations and does not occur in ODEs. Briefly, the effect is as follows: If a (linear) autonomous delay equation has a leading pair of eigenvalues on the imaginary axis, then *every* nonzero inhomogeneous term $... + h(t)$ on the right side of the equation inevitably has at least a transient influence on these oscillation modes. Contrary to the ODE case, it is not possible to choose h such that it interacts only with the remaining, exponentially decaying modes. This observation is a rather easy consequence of the variation-of-constants formula (Lemma 2.4), but has to our knowledge not been emphasized in the literature so far. We turn to the detailed description now, starting with a simple observation. For $c \in \mathbb{C}$, set

$$\mathbf{D}_c := \Big\{\psi \in C^0([-1,0],\mathbb{C}) \;\big|\; \psi \text{ is } C^1,\ \psi'(0) = c\psi(-1)\Big\}.$$

Remark 5.1 *If $c \in \mathbb{C}$ and $x : [-1,0] \longrightarrow \mathbb{C}$ satisfies $\forall t \geq 0:\ x_t \in \mathbf{D}_c$, then x is C^1 and $\dot{x}(t) = cx(t-1)$ for $t \geq 0$.*

Proof: All segments x_t ($t \geq 0$) are C^1, which implies that x is C^1. For $t \geq 0$, it follows from $x_t \in \mathbf{D}_c$ that $\dot{x}(t) = (x_t)'(0) = cx_t(-1) = cx(t-1)$. $\square$

Consider now the harmonic oscillator equation

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -\frac{\pi^2}{4}x(t).$$

Setting $y(t) := -\frac{2}{\pi}v(t)$, one obtains

$$\dot{x}(t) = -\frac{\pi}{2}y(t), \quad \dot{y}(t) = \frac{\pi}{2}x(t),$$

or, setting $w := x + iy$, $\nu := i\pi/2$, the more convenient complex form

$$\dot{w}(t) = \nu w(t). \tag{5.1}$$

We now turn to the case of delay equations of the form

$$\dot{x}(t) = -\frac{\pi}{2}x(t-1) + h(t), \tag{5.2}$$

with continuous $h : \mathbb{R} \longrightarrow \mathbb{R}$. For $h = 0$, equation (5.2) reduces to

$$\dot{x}(t) = -\frac{\pi}{2}x(t-1). \tag{5.3}$$

We explain why the last equation may be regarded as an analog of the harmonic oscillator, together with a sequence of exponentially decaying oscillators (and no coupling between any two of them):

Equation (5.3) has a leading pair of characteristic values $\lambda_0 = i\pi/2, \overline{\lambda_0} = -i\pi/2$, with corresponding solutions $t \mapsto e^{\pm it\pi/2}$. The remaining characteristic values (elements of $\Sigma_{-\pi/2}$, in the notation of Section 2) $\lambda_1 = \rho_1 \pm i\omega_1, \lambda_2 = \rho_2 \pm i\omega_2, ...$ satisfy $\rho_{k+1} < \rho_k < 0$ for $k \in \mathbb{N}$.

Now we add to the unperturbed harmonic oscillator (5.1) a second complex equation $\dot{z}(t) = \nu_1 z(t)$ with $\mathrm{Re}(\nu_1) < 0$, and then to the obtained system a perturbation $h(t)$ which lies entirely in the $z-$space. Thus we obtain the system

$$\begin{aligned} \dot{w}(t) &= \nu w(t), \\ \dot{z}(t) &= \nu_1 z(t) + h(t), \end{aligned} \tag{5.4}$$

and it is clear that the perturbation has no effect on the evolution of the w-component.

Our aim is now to point out that this is not so for equation (5.3), even if one chooses $h(t) = e^{\lambda_k t}$, where λ_k is one of the characteristic values with negative real part. More precisely, we can state the following.

Theorem 5.2 *a) Let $\Lambda \subset \{\lambda_0, \overline{\lambda_0}, \lambda_1, \overline{\lambda_1}, ...\}$ be a finite subset, and set*

$$U := \sum_{\lambda \in \Lambda} \mathbb{C} \cdot e^{\lambda \cdot}\big|_{[-1,\,0]} \subset C^0([-1,0],\mathbb{C}).$$

If $x : [-1, 0] \longrightarrow \mathbb{C}$ satisfies $\forall t \geq 0 : \; x_t \in U$ then x is C^1 and $\dot{x}(t) = -\frac{\pi}{2}x(t-1)$ for $t \geq 0$. Thus, x can satisfy equation (5.2) only with $h = 0$.

b) Let $h \in C^1([0,\infty),\mathbb{R})$ be such that h' is bounded, that $|h(s)| \longrightarrow 0$ $(s \longrightarrow \infty)$, and that the integrals $\int_0^\infty |h(s)|ds$ and $\int_0^\infty |h'(s)|ds$ converge. The solution of equation (5.2) with initial segment $x_0 = 0$ has an asymptotic oscillation amplitude given by

$$W_\infty := \left| \frac{2}{1+i\pi/2} \int_0^\infty e^{-is\pi/2} h(s)ds \right|.$$

c) In particular, if $\varepsilon, \mu \in \mathbb{C}$, and $\mathrm{Re}(\mu) < 0$, and $h(t) = \varepsilon e^{\mu t}$, one has

$$W_\infty = \left| 2 \frac{\varepsilon}{(1+i\pi/2)(\mu - i\pi/2)} \right| \neq 0.$$

If $\mu = \lambda_k$ for some $k \in \{1,2,...\}$, then this is true even though all segments h_t $(t \geq 0)$ belong to the space with exponentially decaying solutions for the unperturbed equation (5.3).

Proof: Ad a): Note that with $\mathbf{D}_{-\pi/2}$ defined as before Remark 5.1, we have $U \subset \mathbf{D}_{-\pi/2}$, since each $\lambda \in \Lambda$ satisfies $\lambda = -(\pi/2)e^{-\lambda}$. It follows from Remark 5.1 that x is C^1 and that $\dot{x}(t) = -(\pi/2)x(t-1)$ for $t \geq 0$.

Ad b): From Lemma 2.4 we obtain that, for $t \geq 1$, the solution of equation (5.2) with $x_0 = 0$ is given by

$$\begin{aligned} x(t) &= \int_0^t x^{c,\hat{h}(s),s}(t)\, ds \\ &= \int_0^{t-1} x^{c,\hat{h}(s),s}(t)\, ds + \int_{t-1}^t x^{c,\hat{h}(s),s}(t)\, ds. \end{aligned} \tag{5.2.1}$$

Call the first and second term in the last sum $A(t)$ and $B(t)$, respectively. Let $\sigma := \Sigma_{-\pi/2} = \{\lambda_0, \overline{\lambda_0}, \lambda_1, \overline{\lambda_1}, ...\}$ denote the set of characteristic values of equation (5.4), and for $c \in \mathbb{R}$ let c^* denote the constant function on $[-1,0]$ with value c.

In the term $A(t)$ of (5.2.1) (with $t \geq 1$), the integrand is for all $s < t-1$ equal to the convergent series

$$\sum_{\nu\in\sigma} \mathrm{pr}_\nu[h(s)^*]e^{\nu(t-1-s)}$$

(compare the beginning of the proof of Proposition 2.3), and

$$\mathrm{pr}_\nu[h(s)^*] = h(s)\frac{1}{1+\nu}[1+\nu\int_{-1}^0 e^{-\nu s}\, ds] = h(s)\frac{e^\nu}{1+\nu}.$$

For $s \in [0, t-1]$, we get from $\mathrm{Re}(\nu) \leq 0$ $(\nu \in \sigma)$ and from the characteristic equation that $|e^{\nu(t-s)}| \leq |e^\nu| = |\frac{-\pi/2}{\nu}|$, and hence

$$|\frac{h(s)e^{\nu(t-s)}}{1+\nu}| \leq ||h||_\infty \frac{\pi}{2|\nu(1+\nu)|}.$$

Now $\sum_{\nu\in\sigma} \dfrac{1}{\nu(1+\nu)}$ converges, since $|\lambda_k| = |\rho_k + i\omega_k| \geq |\omega_k| > 2k\pi$ $(k \in \mathbb{N})$, see Proposition 2.2. Thus we have uniform convergence on $[0, t-1]$ of the series in the following integral, and we obtain

$$\begin{aligned} A(t) &= \int_0^{t-1} \sum_{\nu\in\sigma} \frac{h(s)}{1+\nu} e^{\nu(t-s)}\, ds \\ &= \sum_{\nu\in\sigma} \frac{1}{1+\nu} \int_0^{t-1} e^{\nu(t-s)} h(s)\, ds. \end{aligned}$$

We denote the last integral by $I_\nu(t)$.

Claim: For $\nu \in \sigma$ with $\mathrm{Re}(\nu) < 0$ (i.e., $\nu \in \{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, ...\}$) one has for $t \geq 2$

$$|I_\nu(t)| \leq \frac{1}{|\nu|}[|h(t-1)| + |h(0)|e^{\rho_1 t} + ||h'||_\infty \frac{t}{2} e^{\rho_1 t/2} + \int_{t/2}^\infty |h'(s)|\, ds].$$

Proof.

$$\begin{aligned} |\int_0^{t-1} e^{\nu(t-s)} h(s)\, ds| &= |e^{\nu t}\{[-\frac{1}{\nu} e^{-\nu s} h(s)]_0^{t-1} + \frac{1}{\nu}\int_0^{t-1} e^{-\nu s} h'(s)\, ds\}| \\ &\leq \frac{1}{|\nu|}[|e^{\nu}||h(t-1)| + |e^{\nu t}||h(0)| + |\int_0^{t-1} e^{\nu(t-s)} h'(s)\, ds|]. \end{aligned}$$

The last integral can be estimated by

$$||h'||_\infty \int_0^{t/2} |e^{\nu(t-s)}|\, ds + \int_{t/2}^{t-1} |h'(s)|\, ds \leq ||h'||_\infty \frac{t}{2} |e^{\nu t/2}| + \int_{t/2}^\infty |h'(s)|\, ds.$$

Using that $\nu = \rho_k + i\omega_k$ for some $k \in \mathbb{N}$, and that $\rho_k \leq \rho_1 < 0$, one obtains the claim.

Abbreviating the square bracket in the above claim by $C(t)$, we have $|I_\nu(t)| \leq \frac{1}{|\nu|} C(t)$, and the properties of h imply that $C(t) \longrightarrow 0$ $(t \longrightarrow \infty)$. Now

$$A(t) = \sum_{\nu\in\sigma, \mathrm{Re}(\nu)<0} \frac{1}{1+\nu} I_\nu(t) + \sum_{\nu\in\sigma, \mathrm{Re}(\nu)=0} \frac{1}{1+\nu} I_\nu(t),$$

and the absolute value of the left sum can be estimated by $\sum_{\nu\in\sigma,\mathrm{Re}(\nu)<0} \frac{1}{|1+\nu|}\frac{1}{|\nu|} C(t)$, which converges to zero as $t \longrightarrow \infty$. Recall that $\left\{\nu \in \sigma \;\middle|\; \mathrm{Re}(\nu) = 0\right\} = \{\lambda_0, \overline{\lambda_0}\}$. Thus, setting

$$A_0(t) := \left[\frac{1}{1+\lambda_0} \int_0^{t-1} e^{-\lambda_0 s} h(s)\, ds\right] e^{\lambda_0 t} + \overline{\left[\frac{1}{1+\lambda_0} \int_0^{t-1} e^{-\lambda_0 s} h(s)\, ds\right] e^{\lambda_0 t}},$$

we have

$$A(t) - A_0(t) \longrightarrow 0 \; (t \longrightarrow \infty).$$

In the term $B(t)$ of (5.2.1), the integrand equals $h(s)$, so we have

$$B(t) = \int_{t-1}^{t} h(s)ds.$$

The integrability condition on h implies

$$B(t) \longrightarrow 0 \quad (t \longrightarrow \infty).$$

Together, the asymptotic amplitude of oscillation is determined by the term $A_0(t)$. Since $\lambda_0 = i\pi/2$ and since the oscillation amplitude of a function of the form $ae^{i\omega t} + \overline{ae^{-i\omega t}}$ $(\omega \in \mathbb{R})$ is equal to $2|a|$, we conclude that

$$W_\infty = 2 \left| \frac{\int_0^\infty e^{-is\pi/2} h(s)\, ds}{1 + i\pi/2} \right|,$$

as asserted.

Ad c): For h as in c), one has

$$\int_0^\infty \varepsilon e^{(\mu-\lambda_0)s}\, ds = \frac{-\varepsilon}{\mu - \lambda_0} = \frac{-\varepsilon}{\mu - i\pi/2},$$

so $W_\infty = 2 \left| \dfrac{\varepsilon}{(1+i\pi/2)(\mu - i\pi/2)} \right|$. The remaining statements are clear. □

Comment. In view of Remark 5.1, one might (in analogy to ODEs) try to find solutions $X : \mathbb{R} \longrightarrow C^0([-1,0],\mathbb{C})$ of equations of the form

(5.5) $$\dot{X}(t) = (-\pi/2)X(t-1) + H(t) \quad (t \in \mathbb{R}),$$

which are defined on all of $\mathbb{R}$ and have values in some closed subspace S of the space $C^0([-1,0],\mathbb{C})$ (e.g., in one of the eigenspaces associated with a characteristic value and its conjugate, or a finite sum of such spaces). The function H in equation (5.5) would then be continuous from $\mathbb{R}$ to S. Such equations and solutions exist: For example, take as $X : \mathbb{R} \longrightarrow S$ any C^1 curve in S, and read equation (5.5) as a definition of $H : \mathbb{R} \longrightarrow S$. Such a function H does then typically *not* consist of segments of one function $h : \mathbb{R} \longrightarrow \mathbb{R}$, i.e.,

$$\nexists\, h : \mathbb{R} \longrightarrow \mathbb{R} : \; H(t) = h_t \; (t \in \mathbb{R}).$$

In other words, the inhomogeneity in equation (5.5) is not of the type that would typically be encountered in applications.

References

[1] R. Aris, *Mathematical modeling. A chemical engineer's perspective*, Process Systems Engineering **1**, Academic Press, San Diego, CA, 1999.

[2] E. Benoît, ed., *Dynamic Bifurcation*, Lecture Notes in Math. 1493, Springer-Verlag, New York, 1991.

[3] N. Berglund and K. R. Schneider, *Control of dynamic bifurcations*, Lect. Notes Control Inf. Sci. 246 (1999), 75-93.

[4] N. Berglund, *Control of dynamic Hopf bifurcations*, Nonlinearity **13** (2000), 225-248.

[5] J. L. Callot, F. Diener, and M. Diener, *Le probleme de la "chasse au canar"*, C. R. Acad. Sci., Paris, Sér. A **286** (1978), 1059-1061.

[6] E. Cumberbatch and A. Fitt (eds.), *Mathematical modelling. Case studies from industry*, Cambridge University Press, Cambridge, 2001.

[7] O. Diekmann, S. A. van Gils, S. N. Verduyn-Lunel, and H.-O. Walther, *Delay Equations*, Applied Mathematical Sciences 110, Springer-Verlag, New York, 1995.

[8] W. Eckhaus, *Relaxation oscillations including a standard chase on French ducks*, in Asymptotic Analysis II, F. Verhulst (ed.), Lecture Notes in Math. **985** (1983), 449-494.

[9] T. Erneux and P. Mandel, *The slow passage through a steady state bifurcation: Delay and memory effects*, J. Stat. Phys. **48** (1987), 1059-1069.

[10] J. M. Ferreira and I. Györi, *Oscillatory Behavior in Linear Retarded Functional Differential Equations*, J. Math. Anal. Appl. **128** (1987), 347-361.

[11] G. N. Gorelov and V. A. Sobolev, *Mathematical modeling of critical phenomena in thermal explosion theory*, Combust. Flame **87** (1991), 203-210.

[12] R. Haberman, *Slowly varying jump and transition phenomena associated with algebraic bifurcation problems*, SIAM J. Appl. Math. **37** (1979), 69-106.

[13] J. K. Hale and S. M. Verduyn-Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences 99, Springer Verlag, New York, 1993.

[14] B. Lani-Wayda, *Wandering solutions of sine-like delay equations*, Mem. Amer. Math. Soc. Vol. **151**, No. 718, American Mathematical Society, Providence, RI, 2001.

[15] J. Mallet-Paret and G. R. Sell, *Systems of Differential Delay Equations: Floquet Multipliers and Discrete Lyapunov Functions*, J. Differential Equations **125** (1996), 385-440.

[16] Ph. K. Maini and H. G. Othmer (eds.), *Mathematical models for biological pattern formation*, The IMA Volumes in Mathematics and its Applications **121**, Springer Verlag, New York, 2000.

[17] N. N. Nefedov and K. R. Schneider, *Delayed exchange of stabilities in singularly perturbed systems*, Preprint **270** of the Weierstrass Institute for Applied Mathematics and Stochastics, Berlin 1996.

[18] N. N. Nefedov and K. R. Schneider, *Forced canards*, Progress in Nonlinear Science, Vol. I Mathematical Problems of Nonlinear Dynamics, 139-151, Nizhny Novgorod, Russia, 2002.

[19] A. I. Neishtadt, *On delayed stability loss under dynamic bifurcations I* (in Russian), Differ. Uravn. **23** (1987), 2060-2067.

[20] A. I. Neishtadt, *On delayed stability loss under dynamic bifurcations II* (in Russian), Differ. Uravn. **24** (1988), 226-233.

[21] V. F. Butuzov, N. N. Nefedov and K. R. Schneider, *Singularly Perturbed Problems in Case of Exchange of Stabilities*, Report Nr. 21, Weierstrass Institute for Applied Analysis and Stochastics, Berlin 2002, 150 p.

[22] M. A. Shishkova, *Study of a system of differential equations with a small parameter at the highest derivatives* (in Russian), Dokl. Akad. Nauk SSSR **209** (1973), 576-579.

[23] E. M. Wright, *A non-linear difference-differential equation*, J. Reine Angew. Math. **194** (1955), 66-87.