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Abstract

The one-dimensional heat equation driven by a particular white noise term is studied. From initial conditions with compact support, solutions retain this compact support and die out in finite time. The long term behavior of solutions from certain initial conditions can be described by a system of wavefronts whose positions move approximately as Brownian motions and such that two wavefronts annihilate when they collide.

1 Description of results

We consider in this paper the equation

$$\dot{u} = (1/2)\Delta u + |u(1-u)|^{1/2}\dot{W} \tag{1}$$

where $\dot{W}$ is a space time white noise on $[0,\infty) \times \mathrm{R}$. Solutions are processes $(u_t(x) : t \geq 0, x \in \mathrm{R})$, jointly continuous in (t,x), which satisfy a weak form of (1) (see Walsh [11]). Throughout and without further comment we consider only solutions for which $u_t(x) \in [0,1]$ for all t, x.

Equation (1) has a dual and in section 2 we give a formula for the moments $E(u_t(x_1)\ldots u_t(x_n))$ as the expectation of a certain functional of an n-dimensional Brownian motion. This will imply uniqueness in law for solutions of (1) and the strong Markov property. In section 3 we calculate some moment bounds needed in later sections. The existence of solutions follows from an approximation argument as in Shiga [10] or Reimers [6]. This construction also allows the coupling of solutions which is the basis of comparison arguments needed in future sections. The symmetry of equation (1) is also used repeatedly: if u is a solution started at f then $1-u$ is a solution started at $1-f$.

In section 4 we prove a compact support property. Define for $f : \mathrm{R} \to [0,1]$

$$R(f) = \sup(x : f(x) > 0), \quad L(f) = \inf(x : f(x) < 1).$$

We show that if $-\infty < L(u_0) < R(u_0) < \infty$ then $-\infty < L(u_t) < R(u_t) < \infty$ for all time. We call the region of the solution lying between $L(u_t)$ and $R(u_t)$ a wavefront. The paths $t \to R(u_t), t \to L(U_t)$ are right continuous with left limits.

In section 5 we show that solutions that initially have compact support will die out in finite time. We also obtain some weak control on the width of the wavefront showing that $\sup_{s \leq t} R(u_t) - L(u_t)$ grows slower than $O(t^{1/2})$.

In section 6 we consider initial conditions with a single wavefront. We show that, under Brownian rescaling, the motion of the position of the wavefront $(t \to R(u_{n^2t})/n)$ converges to that of a Brownian motion as $n \to \infty$.

Finally in section 7 we consider solutions started from initial conditions $\sum_{i=1}^{n}(-1)^i f_i$ where $-\infty < L(f_i) < R(f_i) < \infty$. The solution consists of intervals where it equals 0 or 1 seperated by wavefronts. When two wavefronts collide they annihilate each other precisely because solutions with compact support die out. Using this we show that (again under Brownian rescaling) the motion of the wavefronts approaches the motions of a system of annihilating Brownian motions.

We remark that the proofs require the exact moment formulas obtained in section 2 and do not immediately generalise to similar equations for which in general no moment formulae exist.

Notation: For a function $f : [0,\infty) \times \mathrm{R} \to \mathrm{R}$ we write $f', \Delta f$ for spatial derivatives and $\dot{f}$ for the partial derivative in time.

For $f, g : \mathrm{R} \to \mathrm{R}$ we write (f, g) for the integral $\int f(x)g(x)dx$ whenever this is defined. We use the same notation (μ, f) when μ is a measure on R.

$\mathcal{C}$ will be the space of continuous functions on R with values in $[0, 1]$ and with the topology of uniform convergence on compacts. M will be the space of Radon measures on R with the vague topology: $\mu_n \to \mu$ if and only if $(\mu_n, \phi) \to (\mu, \phi)$ for all $\phi \in C_c$ the space of continuous functions on R with compact support. On $C([0, \infty), \mathcal{C})$ (or $C([0, \infty), M)$), the space of continuous functions from $[0, \infty)$ into $\mathcal{C}$ (respectively M), we write $(w_t : t \geq 0)$ (respectively $(\mu_t : t \geq 0)$) for the coordinate maps and $\mathcal{W}_t$ (respectively $\mathcal{M}_t$) for the filtration they generate. We shall write Q^f for the law of a solution to (1) started at f on either of the above canonical path spaces.

$\underline{B} = (B_1, \ldots, B_n)$ will be a Brownian motion with $\underline{B}(0) = \underline{x} = (x_1, \ldots, x_n)$ under $P_{\underline{x}}$. We shall write $L_t^a(X)$ for the local time at a of the semimartingale X.

$||f||_p$ denotes the L^p norm for $p \in [0, \infty]$. We write $p_t(x)$ for the Brownian transition density and P_t for the Brownian semigroup. Throughout we have the convention that $\inf(\emptyset) = +\infty$.

2 Uniqueness

To state the moment formula below we use the following notation: We write $L_{i,j}(t)$ for the local time of $B_i - B_j$ at zero and define

$$L_n(t) = \sum_{i,j=1,\, i\neq j}^{n} L_{i,j}(t).$$

We also use the following notation for any n-vector $\underline{x}$:

$$\begin{aligned} \underline{x}^{(j)} &= (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathrm{R}^{n-1} \\ \underline{x}^{(i,j)} &= (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_j, x_i, x_{j+1}, \ldots, x_n) \in \mathrm{R}^n. \end{aligned}$$

Proposition 2.1 *Let u be a solution to (1) started at f. Write $m_n(t,\underline{x})$ for the moment $E(u_t(x_1)\ldots u_t(x_n))$ and set $m_0(t,\underline{x}) = 0$. Then for $n \geq 1$*

$$\begin{aligned} m_n(t,\underline{x}) &= E_{\underline{x}}(f(B_1(t))\ldots f(B_n(t))e^{-L_n(t)/4}) \qquad (2) \\ &\quad +(1/4) \sum_{i,j=1,\, i\neq j}^{n} E_{\underline{x}}\Big(\int_0^t e^{-L_n(s)/4} m_{n-1}(t-s, \underline{B}^{(i)}(s))dL_{i,j}(s)\Big) \end{aligned}$$

Remark. The proof shows (see equation (3)) that the moments $m_n(t,\underline{x})$ satisfy the mild form of a heat equation with certain singular forcing terms that act only on the subspaces $x_i = x_j, i \neq j$. Then (2) is the Feynman-Kac formula for the solution.

Proof. For fixed t, x

$$(u_s, p_{t-s}(x-\cdot)) = (f, p_t(x-\cdot)) + \int_0^s \int p_{t-s}(x-y)|u_s(y)(1-u_s(y))|^{1/2} dW_{y,s}$$

for $s \in [0,t)$ and both sides converge to $u_t(x)$ as $s \to t$. Ito's formula then gives for $s \in [0,t)$

$$\begin{aligned} & (u_s, p_{t-s}(x_1-\cdot))\ldots(u_s, p_{t-s}(x_n-\cdot)) \\ = \;& (f, p_t(x_1-\cdot))\ldots(f, p_t(x_n-\cdot)) + M_s \\ & +(1/2) \sum_{i,j=1, i\neq j}^{n} \int_0^s \int p_{t-s}(x_i-y)p_{t-s}(x_j-y)u_s(y)(1-u_s(y)) \\ & \qquad \cdot \prod_{k\neq i,j} (u_s, p_{t-s}(x_k-\cdot))dy\, ds \end{aligned}$$

where M_s is a martingale. Taking expectations and letting $s \to t$ gives

$$\begin{aligned} & m_n(t,\underline{x}) - (f, p_t(x_1-\cdot))\ldots(f, p_t(x_n-\cdot)) \qquad (3) \\ = \;& (1/2) \sum_{i,j=1, i\neq j}^{n} \int_0^t \int_{\mathrm{R}^{n-1}} p_{t-s}(x_j - z_i) \\ & \cdot \prod_{k\neq j} p_{t-s}(x_k - z_k))(m_{n-1}(s, \underline{z}^{(j)}) - m_n(s, \underline{z}^{(i,j)}))d\underline{z}^{(j)}\, ds. \end{aligned}$$

Let $\tilde{m}_0(t,\underline{x}) = 0$ and define $\tilde{m}_n(t,\underline{x})$ inductively by using equation (2). We shall show below that $(\tilde{m}_n(t,\underline{x}) : n = 1,2,\ldots)$ also satisfie (3) and that the two solutions agree. In several steps we use the observation that for $h \geq 0$ measurable

$$\begin{aligned}
&E_{\underline{x}}\Big(\int_0^t h(B_i(s))dL_{i,j}(s)\Big)\\
= \;& 2\int_0^t \int p_s(x_i - y)p_s(x_j - y)h(y)dy\,ds.
\end{aligned}$$

We split the right hand side of (3) (with m_n replaced by $\tilde{m}_n$) into three parts. The first part:

$$\begin{aligned}
&-(1/2)\sum_{i,j=1,i\neq j}^{n}\int_0^t\int_{\mathrm{R}^{n-1}} p_{t-s}(x_j - z_i)\prod_{k\neq j} p_{t-s}(x_k - z_k)\\
&\qquad E_{\underline{z}^{(i,j)}}(f(B_1(s))\ldots f(B_n(s))e^{-L_n(s)/4})d\underline{z}^{(j)}\,ds\\
= \;& -(1/2)\sum_{i,j=1,i\neq j}^{n}\int_0^t\int_{\mathrm{R}^{n-1}} p_s(x_j - z_i)\prod_{k\neq j} p_s(x_k - z_k)\\
&\qquad E_{\underline{z}^{(i,j)}}(f(B_1(t-s))\ldots f(B_n(t-s))e^{-L_n(t-s)/4})d\underline{z}^{(j)}\,ds\\
= \;& -(1/4)\sum_{i,j=1,i\neq j} E_{\underline{x}}(f(B_1(t))\ldots f(B_n(t))\int_0^t e^{-(L_n(t)-L_n(s))/4}dL_{i,j}(s))\\
= \;& E_{\underline{x}}(f(B_1(t))\ldots f(B_n(t))(e^{-L_n(t)/4} - 1)). \qquad (4)
\end{aligned}$$

The second part:

$$\begin{aligned}
&-(1/2)\sum_{i,j=1,i\neq j}^{n}\int_0^t\int_{\mathrm{R}^{n-1}} p_{t-s}(x_j - z_i)\prod_{k\neq j} p_{t-s}(x_k - z_k)\\
&(1/4)\sum_{l,m=1,\ l\neq m}^{n} E_{\underline{z}^{(i,j)}}\Big(\int_0^s e^{-L_n(r)/4}\tilde{m}_{n-1}(s-r,\underline{B}^{(l)}(r))dL_{l,m}(r)\Big)d\underline{z}^{(j)}\,ds\\
= \;& -(1/16)\sum_{i,j=1,i\neq j}\ \sum_{l,m=1,\ l\neq m}^{n}\\
&\quad E_{\underline{x}}\Big(\int_0^t\int_s^t e^{-(L_n(r)-L_n(s))/4}\tilde{m}_{n-1}(t-r,\underline{B}^{(l)}(r))dL_{l,m}(r)dL_{i,j}(s)\Big)\\
= \;& -(1/16)E_{\underline{x}}\Big(\int_0^t\int_0^r e^{-(L_n(r)-L_n(s))/4}\\
&\qquad \sum_{i,j=1,i\neq j} dL_{i,j}(s)\sum_{l,m=1,\ l\neq m}^{n}\tilde{m}_{n-1}(t-r,\underline{B}^{(l)}(r))dL_{l,m}(r)\Big)
\end{aligned}$$

$$= \quad (1/4)E_{\underline{x}}\Big(\int_0^t (e^{-L_n(r)/4}-1) \sum_{l,m=1,\, l\neq m}^{n} \tilde{m}_{n-1}(t-r,\underline{B}^{(l)}(r))dL_{l,m}(r)\Big) \qquad (5)$$

The third part:

$$(1/2)\sum_{i,j=1,i\neq j}^{n}\int_0^t\int_{\mathrm{R}^{n-1}} p_{t-s}(x_j-z_i)$$
$$\cdot\prod_{k\neq j} p_{t-s}(x_k-z_k)\tilde{m}_{n-1}(s,\underline{z}^{(j)})d\underline{z}^{(j)}\,ds$$
$$= \quad (1/4)E_{\underline{x}}\Big(\int_0^t \sum_{i,j=1,i\neq j}^{n} \tilde{m}_{n-1}(t-r,\underline{B}^{(j)}(r))dL_{i,j}(r)\Big). \qquad (6)$$

Combining (4,5,6) shows that $(\tilde{m}_n : n = 1,2,\ldots)$ also solve (3). Note that $m_1(t,x) = \tilde{m}(t,x) = P_t f(x)$ and that m_n are bounded. An induction argument shows that $\tilde{m}_n$ are also bounded. Then a Gronwall argument (and induction on n) shows that (3) has a unique bounded solution and hence that $m_n = \tilde{m}_n$ for all n proving the result. •

Existence of solutions can be obtained by an approximation argument, see Shiga [10] or Reimers [6]. We chose the method of Reimers as it quickly gives us the following coupling construction. Reimers considers only deterministic initial conditions but, as he points out, his construction easily deals with random initial data.

Proposition 2.2 *Let $f(x), \bar{f}(x) \in [0,1]$ for $x \in \mathrm{R}$ be measurable variables on $(\Omega', \mathcal{G}', P')$ with $x \to f(x), x \to \bar{f}(x)$ continuous. Then there is an extension $(\Omega, \mathcal{G}_t, P) = (\Omega' \times \Omega'', \mathcal{G}' \times \mathcal{G}_t'', P' \times P'')$, a $\mathcal{G}_t$ adapted white noise $W_{t,x}$ and $\mathcal{G}_t$ adapted processes $(u_t(x), \bar{u}_t(x) : t \geq 0, x \in \mathrm{R})$ such that*

a) $(t,x) \to u_t(x), (t,x) \to \bar{u}_t(x)$ are jointly continuous and $t \to u_t, t \to \bar{u}_t$ are continuous as maps from $[0,\infty)$ to C almost surely.

b) $u, \bar{u}$ are solutions to (1) started at $f, \bar{f}$ with respect to $W_{t,x}$.

c) $(\omega : u_t(x) \leq \bar{u}_t(x),\ \forall t \geq 0, x \in \mathrm{R}) = (\omega : \bar{f}(x) \leq f(x),\ \forall x \in \mathrm{R})$.

(Here we are regarding $f, \bar{f}$ as having been extended to Ω in the obvious way).

Proof. Reimers solves a discrete space and time equation as follows. Let $x_n^k = k2^{-n}, t_n^j = (1/4)j2^{-2n}$. Let $\xi_{j,k}$ be an array of I.I.D variables with $P(\xi_{j,k} = 1) = P(\xi_{j,k} = -1) = 1/2$. Then solve the discrete equation $u_0(x_n^k) = f(x_n^k)$ and

$$\begin{aligned} u_{t_n^{j+1}}(x_n^k) &= u_{t_n^j}(x_n^k) + (1/4)(u_{t_n^j}(x_n^{k+1}) - 2u_{t_n^j}(x_n^k) + u_{t_n^j}(x_n^{k-1})) \\ &\quad + 2^{-(n/2)-1}\sigma(u_{t_n^j}(x_n^k))\xi_{j,k}. \end{aligned} \qquad (7)$$

$\bar{u}$ also solves (7) but with $\bar{u}_0(x_n^k) = \bar{f}(x_n^k)$. We shall choose σ shortly. Reimers avoids a limiting argument by using a little non-standard analysis. Fixing an infinite n, he checks that u is S-continuous and that the formula, for $t \geq 0, x \in \mathrm{R}$

$$u_t(x) = \mathrm{st}(u_{t_n^j}(x_n^k)) \quad \text{for some } t_n^j \approx t, x_n^k \approx x$$

defines a solution to (1) with respect to a certain white noise. This construction works provided σ is a uniform lifting of the function $|u(1-u)|^{1/2}$. We choose the particular lifting $\sigma(u) = |u(1-u)|^{1/2} \wedge \alpha|u(1-u)|$ where $\alpha = 2^{n/2}$. Hence $|\sigma(u) - \sigma(\bar{u})| \leq 2^{n/2}|u - \bar{u}|$. We now check simply by induction using (7) that if $u_{t_n^j}(x_n^k) \leq \bar{u}_{t_n^j}(x_n^k)$ for all k then $u_{t_n^{j+1}}(x_n^k) \leq \bar{u}_{t_n^{j+1}}(x_n^k)$ for all k. Indeed

$$\begin{aligned}
& u_{t_n^{j+1}}(x_n^k) - \bar{u}_{t_n^{j+1}}(x_n^k) \\
= \; & (1/4)(u_{t_n^j}(x_n^{k+1}) - \bar{u}_{t_n^j}(x_n^{k+1})) + (1/4)(u_{t_n^j}(x_n^{k-1}) - \bar{u}_{t_n^j}(x_n^{k-1})) \\
& +(1/2)(u_{t_n^j}(x_n^k) - \bar{u}_{t_n^j}(x_n^k)) + 2^{-(n/2)-1}(\sigma(u_{t_n^j}(x_n^k)) - \sigma(\bar{u}_{t_n^j}(x_n^k))) \\
\leq \; & (1/2)(u_{t_n^j}(x_n^k) - \bar{u}_{t_n^j}(x_n^k)) - 2^{-(n/2)-1}|\sigma(u_{t_n^j}(x_n^k)) - \sigma(\bar{u}_{t_n^j}(x_n^k))| \\
\leq \; & 0.
\end{aligned}$$

This proves part c) of the proposition. The same method shows that when $f, \bar{f} \in [0,1]$ then $u, \bar{u} \in [0,1]$. Reimers shows that $u_t(x) - P_t f(x)$ are uniformly Hölder continuous on compacts and this implies the continuity needed in part a). •

Proposition 2.3 *The law Q^f on $(C([0,\infty), \mathcal{C}), \mathcal{W}, \mathcal{W}_t)$ of a solution to (1) started at f is unique. The family $(Q^f : f \in \mathcal{C})$ is strong Markov.*

Proof. Since the moments $E(u_t(x_1) \dots u_t(x_n))$ are determined and since the solutions are bounded by 1 then the distribution of u_t is determined. The extension to finite dimensional distributions and the strong Markov property are standard. For instance one can check that the law of any solution to (1) is a solution to a standard martingale problem and that, as above, the one dimensional distributions are unique. Then one may appeal to Ethier and Kurtz [2] theorem 4.4.2. •

3 Moments

Lemma 3.1 *a)*
$$E(u_t(x)(1-u_t(y))) = E_{(x,y)}(f(B_1(t))(1-f(B_2(t)))e^{-L_2(t)/4})$$

b)
$$\begin{aligned} & E(u_t(x)(1-u_t(y))u_t(z)) \\ = \; & E_{(x,y,z)}(f(B_1(t))(1-f(B_2(t)))f(B_3(t))e^{-L_3(t)/4}) \\ & +(1/2)E_{(x,y,z)}\int_0^t e^{-L_3(s)/4}h_1(t-s,B_1(s),B_2(s))dL_{1,3}(s) \end{aligned}$$

where $h_1(s,x,z) = E(u_s(x)(1-u_s(z)))$*,*

c)
$$\begin{aligned} & E(u_t(x)(1-u_t(y))u_t(z)(1-u_t(w))) \\ = \; & E_{(x,y,z,w)}(f(B_1(t))(1-f(B_2(t)))f(B_3(t))(1-f(B_4(t)))e^{-L_4(t)/4}) \\ & +(1/2)E_{(x,y,z,w)}\int_0^t e^{-L_4(s)/4}h_2(t-s,B_1(s),B_2(s),B_3(s))dL_{2,4}(s) \\ & +(1/2)E_{(x,y,z,w)}\int_0^t e^{-L_4(s)/4}h_3(t-s,B_1(s),B_2(s),B_4(s))dL_{1,3}(s) \end{aligned}$$

where h_2, h_3 are defined by $h_2(s,x,y,z) = E(u_s(x)(1-u_s(y))u_s(z))$ *and* $h_3(s,x,y,w) = E(u_s(x)(1-u_s(y))(1-u_s(w)))$.

This lemma can be proved by the same method as in Proposition 2.1.

Lemma 3.2 *Suppose that $-\infty < L(f) < R(f) < \infty$ and that u is a solution to (1) started at f.*

a) $E(\int u_t(x)(1-u_t(x))dx) \to 1$ *as* $t \to \infty$*,*

b) $E(\int_{-\infty}^{\infty}\int_{-\infty}^{z}(1-u_t(y))u_t(z)dy\,dz) \le C(f)t^{1/2}\log^{1/2}(t)$ *for all* $t \ge e$*.*

c) $E((\int u_t(x)(1-u_t(x))dx)^2) \le C(f)$ *for all* $t \ge 0$*.*

Proof. a) From lemma 3.1 a) we have
$$\begin{aligned} & E(\int u_t(x)(1-u_t(x))dx) \\ = \; & \int E_{(x,x)}(f(B_1(t))(1-f(B_2(t)))e^{-L_{1,2}(t)/2})dx \\ = \; & E_{\underline{0}}\int f(x+B_1(t))(1-f(x+B_2(t)))dx e^{-L_{1,2}(t)/2}. \end{aligned}$$

We now break this into two parts, replacing f by $f_0(x) = \mathrm{I}(x \le 0)$ in one part and estimating the error by doing this in the other part.

$$
\begin{aligned}
& E_{\underline{0}} \int f_0(x+B_1(t))(1-f_0(x+B_2(t)))\,dx e^{-L_{1,2}(t)/2} \\
= \; & E_{\underline{0}}((B_2(t)-B_1(t))_+ e^{-L_{1,2}(t)/2}) \\
= \; & E_{\underline{0}}(\int_0^t (B_1(s)-B_2(s))_+ e^{-L_{1,2}(s)/2}(-1/2)dL_{1,2}(s) \\
& + E_{\underline{0}} \int_0^t e^{-L_{1,2}(s)/2} \mathrm{I}(B_1(s)-B_2(s) \ge 0) d(B_1-B_2)(s)) \\
& + E_{\underline{0}} \int_0^t e^{-L_{1,2}(s)/2}(1/2)dL_{1,2}(s)) \qquad \text{(integration by parts)} \\
= \; & E_{\underline{0}}(1-e^{-L_{1,2}(t)/2}) \to 1 \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

The error is

$$
\begin{aligned}
& |E_{\underline{0}} \int f(x+B_1(t))(1-f(x+B_2(t))) \\
& \qquad -f_0(x+B_1(t))(1-f_0(x+B_2(t)))dx\, e^{-L_{1,2}(t)/2}| \\
\le \; & 2E_{\underline{0}} \int |f(x+B_1(t)) - f_0(x+B_1(t))| dx\, e^{-L_{1,2}(t)/2} \\
\le \; & 2(|R(f)| + |L(f)|) E(e^{-L_{1,2}(t)/2}) \to 0 \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

b) We shall write $W(t) = (B_1(t)-B_2(t))/2^{1/2}$. We also use below the change of variables $u = (z-y)/2^{1/2}, v = (z+y)/2^{1/2}$. Taking $t \ge e$ we have

$$
\begin{aligned}
& E(\int_{-\infty}^{\infty} \int_{-\infty}^{z} (1-u_t(y))(u_t(z)) dy\, dz) \\
= \; & E_{\underline{0}} \int \int \mathrm{I}(y \le z)((1-f(y+B_1(t)))f(z+B_2(t)) \\
& \qquad \exp(-L_t^{z-y}(B_1-B_2)/2) dy\, dz \\
\le \; & E_{\underline{0}} \int \int \mathrm{I}(y \le z, y+B_1(t) \ge L(f), z+B_2(t) \le R(f)) \\
& \qquad \exp(-L_t^{z-y}(B_1-B_2)/2) dy\, dz \\
\le \; & E_{\underline{0}} \int \int \mathrm{I}(u \ge 0, (v-u)/2^{1/2} + B_1(t) \ge L(f)) \\
& \qquad \mathrm{I}((v+u)/2^{1/2} + B_2(t) \le R(f)) \exp(-L_t^{2^{1/2}u}(B_1-B_2)/2 du\, dv \\
= \; & E_{\underline{0}} \int_0^{\infty} (2^{1/2}(R(f)-L(f)) + 2(W(t)-u))_+ \exp(-L_t^0(W-u)/2^{1/2}) du \\
\le \; & E_{\underline{0}} \int_0^{\infty} 2^{1/2}(R(f)-L(f)) \mathrm{I}(2(W(t)-u) \ge -2^{1/2}(R(f)-L(f))) du
\end{aligned}
$$

$$+E_{\underline{0}}\int_0^\infty 2(W(t)-u)_+\exp(-L_t^0(W-u)/2^{1/2})du$$

$$\begin{aligned}
&\le C(f)E_{\underline{0}}(1+W_+)+E_{\underline{0}}\int_0^\infty\int_0^t \exp(-L_s^0(W-u)/2^{1/2})dL_s^0(W-u)du \\
&\le C(f)t^{1/2}+E_{\underline{0}}\int_0^\infty 2^{1/2}(1-\exp(-L_t^0(W-u)/2^{1/2}))du \\
&\le C(f)t^{1/2}\log^{1/2}(t)+\int_{2t^{1/2}\log^{1/2}(t)}^\infty E_{\underline{0}}L_t^u(W)du \\
&\le C(f)t^{1/2}\log^{1/2}(t)+\int_{2t^{1/2}\log^{1/2}(t)}^\infty\int_0^t p_s(u)ds\,du \\
&\le C(f)t^{1/2}\log^{1/2}(t)+Ct^{1/2}\int_{2t^{1/2}\log^{1/2}(t)}^\infty e^{-u^2/2t}du \\
&\le C(f)t^{1/2}\log^{1/2}(t).
\end{aligned}$$

c) From lemma 3.1 c) we have

$$E((\int u_t(x)(1-u_t(x))dx)^2)$$

$$= \int\int E_{(x,x,x+y,x+y)}$$

$$\Big((f(B_1(t))(1-f(B_2(t)))f(B_3(t))(1-f(B_4(t)))e^{-L_4(t)/4}) \tag{8}$$

$$(1/2)\int_0^t e^{-L_4(s)/4}h_2(t-s,B_1(s),B_2(s),B_3(s))dL_{2,4}(s) \tag{9}$$

$$(1/2)\int_0^t e^{-L_4(s)/4}h_3(t-s,B_1(s),B_2(s),B_4(s))dL_{1,3}(s)\Big)\,dx\,dy. \tag{10}$$

We shall show that (8,9,10) are all bounded uniformly in time.

Considering first the term (8). The error in replacing $f(B_1(t))$ by $f_0(B_1(t))$ (where $f_0(x)=\mathrm{I}(x\le 0)$ again) is bounded by

$$\begin{aligned}
&\int\int E_{(x,x,x+y,x+y)}(\mathrm{I}(|B_1(t)|\le |R(f)|\vee|L(f)|)f(B_3(t))(1-f(B_4(t))) \\
&\qquad e^{-(L_{1,2}(t)+L_{3,4}(t))/2})dxdy \\
\le\ & (|R(f)|+|L(f)|)E_{(0,0)}(e^{-L_{1,2}(t)/2}) \\
&\qquad \int E_{(0,0,y,y)}(f(B_3(t))(1-f(B_4(t)))e^{-L_{3,4}(t)/2})dy \\
\le\ & C(f)E_{(0,0)}(e^{-L_{1,2}(t)/2})\to 0 \quad \text{as } t\to\infty.
\end{aligned}$$

Similarly f maybe replaced by f_0 throughout (8) at no loss. Doing this leads to

$$\int\int E_{(0,0,y,y)}(e^{-L_4(t)/4}\mathrm{I}(x+B_1(t)\le 0, x+B_2(t)\ge 0)$$

$$
\begin{aligned}
& \mathrm{I}(x + B_3(t) \le 0, x + B_4(t) \ge 0))dxdy \\
= & \int E_{(0,0,y,y)}(((B_1(t) \wedge B_3(t)) - (B_2(t) \vee B_4(t))_+ e^{-L_4(t)/4})dy.
\end{aligned}
$$

We now write for brevity E for $E_{(0,0,y,y)}$. Expanding this using Ito's formula we get

$$
\begin{aligned}
& (1/2)E(\int_0^t e^{-L_4(s)/4} dL_s^0((B_1 \wedge B_3) - (B_2 \vee B_4))) \\
& +E(\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \wedge B_3(s) \ge B_2(s) \vee B_4(s)) \\
& \qquad\qquad d((B_1(s) \wedge B_3(s)) - (B_2(s) \vee B_4(s))) \\
& -(1/4)E(\int_0^t e^{-L_4(s)/4}((B_1(s) \wedge B_3(s)) - (B_2(s) \vee B_4(s))_+ dL_4(s)) \\
= & (1/2)E(\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \ge B_3(s), B_2(s) \ge B_4(s)) dL_{2,3}(s)) \qquad (11) \\
& +(1/2)E(\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \le B_3(s), B_2(s) \le B_4(s)) dL_{1,4}(s))) \qquad (12) \\
& +(1/2)E(\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \ge B_3(s), B_2(s) \le B_4(s)) dL_{3,4}(s)) \qquad (13) \\
& +(1/2)E(\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \le B_3(s), B_2(s) \ge B_4(s)) dL_{1,2}(s))) \qquad (14) \\
& -(1/2)E\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \wedge B_3(s) \ge B_2(s) \vee B_4(s)) dL_{1,3}(s)) \\
& -(1/2)E\int_0^t e^{-L_4(s)/4} \mathrm{I}(B_1(s) \wedge B_3(s) \ge B_2(s) \vee B_4(s)) dL_{2,4}(s)) \\
& -(1/2)E\int_0^t e^{-L_4(s)/4}(B_1(s) - (B_2(s) \vee B_4(s)))_+ dL_{1,3}(s) \\
& -(1/2)E\int_0^t e^{-L_4(s)/4}(B_1(s) \wedge B_3(s)) - B_2(s))_+ dL_{2,4}(s). \qquad (15)
\end{aligned}
$$

Note that only the first four terms are non-negative. We bound the term (11) by

$$
\begin{aligned}
& (1/2)E(\int_0^t e^{-(1/2)(L_{1,2}(s)+L_{3,4}(s)+L_{2,3}(s))} dL_{2,3}(s)) \\
\le & E(\exp(-(1/2)(L_{1,2}(\tau_{2,3}) + L_{3,4}(\tau_{2,3}))))
\end{aligned}
$$

where $\tau_{i,j} = \inf(t : B_i(t) = B_j(t))$. The term (12) may be bounded symmetrically. For $y \ge 0$ we bound the term (13) by

$$
(1/2)E(\int_0^t e^{-(1/2)(L_{1,2}(s)+L_{3,4}(s))} \mathrm{I}(B_1(s) \ge B_3(s)) dL_{2,4}(s))
$$

$$\leq \quad E(\exp(-(1/2)(L_{1,2}(\tau_{1,3}) + L_{3,4}(\tau_{1,3})))).$$

A similar bound (i.e. by interchanging the roles of some of the Brownian motions) holds when $y \leq 0$ and also for the term (14). The following lemma shows that all these terms and hence (8) have bounded integrals in y, completing the first step.

Lemma 3.3 *Let* $\sigma = \inf(t \geq 0 : |B_1(t)| \geq |y|^{3/4}/2)$.
a) $E_{(0,0)}(\exp(-(1/2)L_{1,2}(\sigma)))$ *is square integrable in* y *on* R.
b) $E_{(0,0,y,y)}(\exp(-(1/2)(L_{1,2}(\tau_{2,3}) + L_{3,4}(\tau_{2,3}))))$ *is integrable in* y *over* R.
c) $E_{(0,0,y,y)}(\exp(-(1/2)(L_{1,2}(\tau_{1,3}) + L_{3,4}(\tau_{2,4}))))$ *is integrable in* y *over* R.

Proof. a) We bound the expectation in the region $|y| \geq 1$. Define $\rho = \inf(t \geq 0 : |B_1(t) - B_2(t)| \geq |y|^{2/3}/2)$ and $\tilde{\rho} = \inf(t \geq 0 : |B_1(t) + B_2(t)| \geq |y|^{3/4}/2)$. Then $\sigma \geq \rho \wedge \tilde{\rho}$ so that

$$\begin{aligned} & E_{(0,0)}(\exp(-(1/2)L_{1,2}(\sigma))) \\ \leq \quad & E_{(0,0)}(\exp(-(1/2)L_{1,2}(\rho))) + P_{(0,0)}(\tilde{\rho} \leq \rho). \end{aligned} \tag{16}$$

Letting $W = (B_1 - B_2)/2^{1/2}$ then $\rho = \inf(t \geq 0 : |W(t)| \geq |y|^{2/3}/2^{3/2})$ and

$$\begin{aligned} & E_{(0,0)}(\exp(-(1/2)L_{1,2}(\rho))) \\ = \quad & E_{(0,0)}(\exp(-2^{-1/2}L^0_\rho(W)) \\ \leq \quad & C|y|^{-2/3} \end{aligned}$$

(using Revuz and Yor [7] exercise VI.4.9) giving a square integrable bound. Define

$$g(x,z) = P_{(x,z)}(\inf(t : |B_1(t)| \geq |y|^{3/4}/2^{3/2}) \leq \inf(t : |B_2(t)| \geq |y|^{2/3}/2^{3/2})).$$

Then the second term in (16) is $g(0,0)$ and g is harmonic on the rectangle $|x| \leq |y|^{3/4}/2^{3/2}, |z| \leq |y|^{2/3}/2^{3/2}$ and has the obvious boundary conditions. Then by the maximum principle $g(x,z)$ is bounded above by the harmonic function

$$\tilde{g}(x,z) = 2\cosh(2^{3/2}\pi x/3|y|^{2/3})\cos(2^{3/2}\pi z/3|y|^{2/3})(\cosh(\pi|y|^{1/12}/3))^{-1}.$$

Hence $g(0,0) \leq 2(\cosh(\pi|y|^{1/12}/3))^{-1}$ which is again square integrable completing the proof of part a).

b) Again we consider $|y| \geq 1$. Define $\sigma_i = \inf(t \geq 0 : |B_i(t) - B_i(0)| \geq |y|^{3/4}/2)$ and $\tilde{\sigma}_i = \inf(t \geq 0 : |B_i(t) - B_i(0)| \geq |y|/2)$. Then $\tau_{2,3} \geq \sigma_2 \wedge \tilde{\sigma}_3$ and $\tau_{2,3} \geq \sigma_3 \wedge \tilde{\sigma}_2$ so that

$$\begin{aligned} & E_{(0,0,y,y)}(\exp(-(1/2)(L_{1,2}(\tau_{2,3}) + L_{3,4}(\tau_{2,3})))) \\ \leq \quad & E_{(0,0,y,y)}(\exp(-(1/2)(L_{1,2}(\sigma_2) + L_{3,4}(\sigma_3)))) \\ & + E_{(0,0,y,y)}((\tilde{\sigma}_3 \leq \sigma_2) \cup (\tilde{\sigma}_2 \leq \sigma_3)). \end{aligned}$$

The first term on the right hand side can be factored by independence and is then bounded using part a). The second term on the right hand side can be bounded as in the proof of part a). A similar argument also proves part c).•

We now consider the term (9). From lemma 3.1 b)

$$\begin{aligned}
& h_2(t-s, B_1(s), B_2(s), B_3(s)) \\
= \;& E_{(B_1(s),B_2(s),B_3(s))}(\\
& \quad f(W_1(t-s))(1-f(W_2(t-s)))f(W_3(t-s))e^{-L_3(t-s)/4} \\
& +(1/2)\int_0^{t-s} e^{-L_3(r)/4} h_1(t-s-r, W_1(r), W_2(r)) dL_r(W_1 - W_3)) \\
= \;& E_{(x,x,x+y)}(f(B_1(t))(1-f(B_2(t)))f(B_3(t))e^{-(L_3(t)-L_3(s))/4}|\mathcal{F}_s) \qquad (17) \\
& +(1/2)E_{(x,x,x+y)}(\int_s^t e^{-(L_3(r)-L_3(s))/4} h_1(t-r, B_1(r), B_2(r)) dL_{1,3}|\mathcal{F}_s) \qquad (18)
\end{aligned}$$

Substituting (17) into the expectation in (9) gives

$$\begin{aligned}
& (1/2)E_{(x,x,x+y,x+y)}(f(B_1(t))(1-f(B_2(t)))f(B_3(t)) \\
& \quad \int_0^t e^{-(L_4(s)+L_3(t)-L_3(s))/4} dL_{2,4}(s)). \qquad (19)
\end{aligned}$$

Repacing $f(B_1(t))$ by $f_0(B_1(t))$ in this expression gives an error of at most

$$(1/2)E_{(x,x,x+y,x+y)}(\mathrm{I}(|B_1(t)| \leq |R(f)| \vee |L(f)|) \int_0^t e^{-L_4(s)/4} dL_{2,4}).$$

This has an integral over (x, y) that is uniformly bounded in t as we have seen in the treatment of (11). Similarly f may be replaced by f_0 throughout (19) at the cost of at most a constant. This leads to

$$\begin{aligned}
& (1/2)E_{(x,x,x+y,x+y)}(f_0(B_1(t))(1-f_0(B_2(t)))f_0(B_3(t)) \\
& \quad \int_0^t e^{-(L_4(s)+L_3(t)-L_3(s))/4} dL_{2,4}(s)) \\
= \;& E_{(0,0,-y,-y)}((1/2)\mathrm{I}(x \leq B_1(t), x \geq B_2(t), x \leq B_3(t))e^{-L_3(t)/4} \\
& \quad \int_0^t e^{-(L_4(s)-L_3(s))/4} dL_{2,4}(s)).
\end{aligned}$$

We now perform the integral over x to obtain (writing E for $E_{(0,0,-y,-y)}$).

$$\begin{aligned}
& (1/2)E(((B_1(t) \wedge B_3(t)) - B_2(t))_+ e^{-L_3(t)/4} \int_0^t e^{-(L_4(s)-L_3(s))/4} dL_{2,4}(s)) \\
= \;& (1/2)E(\int_0^t ((B_1(s) \wedge B_3(s)) - B_2(s))_+ e^{-L_4(s)/4} dL_{2,4}(s))
\end{aligned}$$

$$+(1/2)E(\int_0^t \int_0^s e^{-(L_4(r)-L_3(r))/4} dL_{2,4}(r)$$
$$d(((B_1(s) \wedge B_3(s)) - B_2(s))_+ e^{-L_3(s)/4})).$$

The first term on the right hand side exactly cancels with (15). We expand the second term further:

$$\begin{aligned}
&(1/4)E(\int_0^t \int_0^s e^{-(L_4(r)-L_3(r))/4} dL_{2,4}(r) e^{-L_3(s)/4} dL_s^0((B_1 \wedge B_3) - B_2))) \\
&-(1/8)E(\int_0^t \int_0^s e^{-(L_4(r)-L_3(r))/4} dL_{2,4}(r) \\
&\qquad ((B_1(s) \wedge B_3(s)) - B_2(s))_+ e^{-L_3(s)/4} dL_3(s)) \qquad (20) \\
&-(1/4)E(\int_0^t \int_0^s e^{-(L_4(r)-L_3(r))/4} dL_{2,4}(r) e^{-L_3(s)/4} dL_{1,3}(s)).
\end{aligned}$$

We may further bound the one positive term here by

$$\begin{aligned}
&(1/4)E(\int_0^t \int_0^s e^{-(L_4(r)-L_3(r))/4} dL_{2,4}(r) e^{-L_3(s)/4} dL_3(s)) \\
= \;& E(\int_0^t e^{-L_4(s)} dL_{2,4}(s))
\end{aligned}$$

which has a bounded integral over y exactly as before.

To finish the bound on (9) it remains to consider the term (18). Substituting (18) into the expectation in (9) we obtain

$$\begin{aligned}
&(1/4)E_{(x,x,x+y,x+y)}(\int_0^t e^{-L_4(s)/4} \int_s^t e^{-(L_3(r)-L_3(s))/4} \\
&\qquad h_1(t-r, B_1(r), B_2(r)) dL_{1,3}(r) dL_{2,4}(s)) \qquad (21)
\end{aligned}$$

From lemma 3.1 a) we have

$$\begin{aligned}
& h_1(t-r, B_1(r), B_2(r)) \\
= \;& E_{(B_1(r),B_2(r))}(f(W_1(t-r))(1-f(W_2(t-r)))e^{-L_2(t-r)/4}) \\
= \;& E_{(x,x)}(f(B_1(t))(1-f(B_2(t)))e^{-(L_2(t)-L_2(r))/4} | \mathcal{F}_r).
\end{aligned}$$

Substituting this (21) gives

$$\begin{aligned}
&(1/4)E_{(x,x,x+y,x+y)}(f(B_1(t))(1-f(B_2(t))) \\
&\int_0^t e^{-L_4(s)/4} \int_s^t e^{-(L_3(r)-L_3(s)+L_2(t)-L_2(r))/4} dL_{1,3}(r) dL_{2,4}(s)).
\end{aligned}$$

Replacing f by f_0 here gives an error of at most

$$(1/4)E_{(x,x,x+y,x+y)}(\mathrm{I}(|x+B_1(t)| \le |R(f)| \vee |L(f)|) \int_0^t e^{-L_4(s)/4} dL_{2,4}(s)).$$

As above this has an integral in (x, y) that is bounded in t. So we may consider

$$
\begin{aligned}
& (1/4)E_{(x,x,x+y,x+y)}(f_0(B_1(t))(1 - f_0(B_2(t))) \\
& \quad \int_0^t e^{-L_4(s)/4} \int_s^t e^{-(L_3(r)-L_3(s)+L_2(t)-L_2(r))/4} dL_{1,3}(r) dL_{2,4}(s)) \\
= \; & (1/4)E_{(0,0,-y,-y)}(\mathrm{I}(B_2(t) \leq x \leq B_1(t)) \\
& \quad \int_0^t e^{-L_4(s)/4} \int_s^t e^{-(L_3(r)-L_3(s)+L_2(t)-L_2(r))/4} dL_{1,3}(r) dL_{2,4}(s)).
\end{aligned}
$$

Integrating over x gives (again writing E for $E_{(0,0,-y,-y)}$)

$$
\begin{aligned}
& (1/4)E((B_1(t) - B_2(t))_+ \\
& \int_0^t e^{-L_4(s)/4} \int_s^t e^{-(L_3(r)-L_3(s)+L_2(t)-L_2(r))/4} dL_{1,3}(r) dL_{2,4}(s)) \\
= \; & (1/4)E((B_1(t) - B_2(t))_+ e^{-L_2(t)/4} \\
& \quad \int_0^t \int_0^r e^{-(L_4(s)+L_3(r)-L_3(s)-L_2(r))/4} dL_{2,4}(s) dL_{1,3}(r)) \\
= \; & (1/4)E(\int_0^t \int_0^r e^{-(L_4(s)+L_3(r)-L_3(s))/4} (B_1(r) - B_2(r))_+ dL_{2,4}(s) dL_{1,3}(r)) \qquad (22) \\
& +(1/8)E(\int_0^t \int_0^q \int_0^r e^{-(L_4(s)+L_3(r)-L_3(s)-L_2(r)+L_2(q))/4} \\
& \qquad dL_{2,4}(s) dL_{1,3}(r) dL_{1,2}(q)). \qquad (23)
\end{aligned}
$$

The first term on the right hand side of (23) exactly cancels with (20). Bounding the s-integral in the second term gives the upper bound

$$
\begin{aligned}
& (1/4)E(e^{-(1/2)L_{3,4}(\tau_{2,4})} \int_0^t \int_0^q e^{-(L_3(r)-L_2(r)+L_2(q))/4} dL_{1,3}(r) dL_{1,2}(q)) \\
\leq \; & (1/2)E(e^{-(1/2)L_{3,4}(\tau_{2,4})} \int_0^t \mathrm{I}(q \geq \tau_{1,3}) e^{-L_2(q)/4} dL_{1,2}(q)) \\
\leq \; & E(e^{-(1/2)L_{3,4}(\tau_{2,4})} e^{-(1/2)L_{1,2}(\tau_{1,3})})
\end{aligned}
$$

which is integrable over y by lemma 3.3. This finishes the bound for the term (9). The term (10) is bounded in a similar way by permuting the Brownian motions $B_1, \ldots, B_4$. This completes the proof. •

4 Compact Support Property

We shall need control of $\sup_x u_t(x)$. Given the Green's function representation

$$u_t(x) = P_t f(x) + \int_0^t \int p_{t-s}(x-y)|u_s(y)(1-u_s(y))|^{1/2} dW_{y,s}$$

this is equivalent to controling $\sup_x N_t(x)$ where

$$N_t(x) = \int_0^t \int p_{t-s}(x-y)|u_s(y)(1-u_s(y))|^{1/2} dW_{y,s}.$$

This is done by controling all increments $|N_t(x) - N_t(y)|$ for x, y in dyadic grids (as in say the proof of the modulus of continuity of Brownian motion). Estimates of the sort in the following lemma occur in several papers ([4],[5],[11]) but since none are quite suited to our needs we prove another.

Lemma 4.1 . *For* $\epsilon, t > 0, A \in \mathrm{R}$

$$P(|N_s(x)| \geq \epsilon, \exists s \leq t, \; x \geq A) \leq c_1 \epsilon^{-20}(t \vee t^{22})(f, P_t \mathrm{I}(A,\infty)).$$

Proof. We use the estimates, for $0 < s < t$

$$\begin{aligned}
\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 dz\, ds &\leq C|x-y| \wedge t^{1/2}, \\
\int_0^s \int (p_{t-r}(x-z) - p_{s-r}(x-z))^2 dz\, dr &\leq C|t-s|^{1/2} \wedge s^{1/2}.
\end{aligned}$$

Applying Burkholder's and then Holder's inequalities we have, taking $p \geq 2$,

$$\begin{aligned}
& E(|N_t(x) - N_t(y)|^{2p}) \\
\leq\; & C(p) E((\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 u_s(z) dz\, ds)^p) \\
\leq\; & C(p)(|x-y| \wedge t^{1/2})^{p-1} E(\int_0^t \int (p_{t-s}(x-z) - p_{t-s}(y-z))^2 u_s^p(z) dz\, ds) \\
\leq\; & C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
& \qquad E(\int_0^t (t-s)^{-1/2} \int (p_{t-s}(x-z) + p_{t-s}(y-z)) u_s(z) dz\, ds) \\
\leq\; & C(p)(|x-y| \wedge t^{1/2})^{p-1} t^{1/2} (f, p_t(x-\cdot) + p_t(y-\cdot)).
\end{aligned}$$

Similarly, for $0 \leq s < t$

$$\begin{aligned}
& E(|N_t(x) - N_s(x)|^{2p}) \\
\leq\; & C(p) E((\int_s^t \int p_{t-r}^2(x-z) u_r(z) dz\, dr)^p)
\end{aligned}$$

$$+C(p)E((\int_0^s\int(p_{t-r}(x-z)-p_{s-r}(x-z))^2u_r(z)dz\,dr)^p)$$

$$\leq C(p)(\int_s^t\int p_{t-r}^2(x-z)dz\,dr)^{p-1}E(\int_s^t\int p_{t-r}^2(x-z)u_r(z)dz\,dr)$$

$$+C(p)(|t-s|\wedge s)^{(p-1)/2}E(\int_0^s\int(p_{t-r}(x-z)-p_{s-r}(x-z))^2u_r(z)dz\,dr)$$

$$\leq C(p)|t-s|^{(p-1)/2}t^{1/2}(f,p_t(x-\cdot)+p_s(x-\cdot)).$$

Define $x_n^j=t_n^j=j2^{-n}$ for $j\in\mathbb{Z},n\in\mathbb{N}$. Define the events

$$A_{j,k,n}^1(\epsilon)=\{|N_{t_n^j}(x_n^{k+1})-N_{t_n^j}(x_n^k)|\geq\epsilon 2^{-n/10}\}$$
$$A_{j,k,n}^2(\epsilon)=\{|N_{t_n^j}(x_n^k)-N_{t_n^{j-1}}(x_n^k)|\geq\epsilon 2^{-n/10}\}.$$

Set $n_0=\inf(n\geq 1:2^{-n}\leq t^{1/2})$. Then if $j\geq 1$

$$\sum_{n\geq n_0}\sum_{1\leq j\leq 2^nt}\sum_{k\geq 2^nA}P(A_{j,k,n}^1(\epsilon))$$

$$\leq \sum_{n\geq n_0}\sum_{1\leq j\leq 2^nt}\sum_{k\geq 2^nA}\epsilon^{-2p}2^{np/5}E(|N_{t_n^j}(x_n^{k+1})-N_{t_n^j}(x_n^k)|^{2p})$$

$$\leq C(p)\epsilon^{-2p}t^{1/2}\sum_{n\geq n_0}2^{n(2-(4p/5))}$$
$$\sum_{1\leq j\leq 2^nt}\sum_{k\geq 2^nA}2^{-n}(f,p_{t_n^j}(x_n^k-\cdot)+p_{t_n^j}(x_n^{k+1}-\cdot))$$

$$\leq C(p)\epsilon^{-2p}t^{1/2}\sum_{n\geq n_0}2^{n(2-(4p/5))}$$
$$\sum_{1\leq j\leq 2^nt}\left(\int_A^\infty f(x)(\int p_{t_n^j}(x-y)dy+2^{-n}(t_n^j)^{-1/2}\mathrm{I}(x\geq A-2^{-n}))dx\right)$$

$$\leq C(p)\epsilon^{-2p}t^{1/2}\sum_{n\geq n_0}2^{n(2-(4p/5))}$$
$$\sum_{1\leq j\leq 2^nt}\left((f,P_t\mathrm{I}(A,\infty))(t_n^j/t)^{-1/2}+(t_n^j)^{-1/2}(f,(A-2^{-n_0},\infty))\right)$$

$$\leq C(p)\epsilon^{-2p}(t\vee t^{1/2})\sum_{n\geq n_0}2^{n(3-(4p/5))}(f,P_t\mathrm{I}(A,\infty))\sum_{1\leq j\leq 2^nt}2^{-n}(t_n^j)^{-1/2}$$

$$\leq C(p)\epsilon^{-2p}(t^{3/2}\vee t)(f,P_t\mathrm{I}(A,\infty))\quad\text{if } p>15/4.$$

The same bound (with a different constant) holds when A^1 is replaced by A^2 provided that we take $p>25/3$. Define

$$A(\epsilon)=\bigcup_{n\geq n_0}\bigcup_{1\leq j\leq 2^nA}\bigcup_{k\geq 2^nA}A_{j,k,n}^1(\epsilon)\cup A_{j,k,n}^2(\epsilon).$$

Then $P(A(\epsilon)) \leq C(t \vee t^{3/2})\epsilon^{-20}(f, P_t I(A, \infty))$. On the set $A^c(\epsilon)$ we may estimate $|N_s(x)|$ for $s \leq t, x \geq A$ by an infinite sum of increments over neighbouring dyadics in the usual manner. Moreover we need at most $2t$ increments over step length 2^{-n_0} and two steps (one in space and one in time) of length 2^{-n} for $n \geq 2$. So

$$|N_s(x)| \leq 2t\epsilon 2^{-1/10} + 2\sum_{n\geq 2} \epsilon 2^{-n/10} \leq c_6\epsilon(1+t).$$

The set $A(c_6^{-1}\epsilon(1+t)^{-1})$ then leads to the desired result. •

We now establish a compact support property by considering the Laplace functional of a solution, adapting the method used for super-Brownian motion in Dawson, Iscoe and Perkins [1].

Proposition 4.2 *Let u be a solution to (1) such that $R(u_0) \leq 0$. Then for all $t \geq 0$, $b \geq 4t^{1/2}$*

$$P(\sup_{s\leq t} R(u_s) \geq b) \leq c_3(t^{-1/2} \vee t^{23})e^{-b^2/16t}.$$

Remark. By considering $1-u$ we obtain a corresponding result about L_t.

Proof. Fix $\psi : \mathrm{R} \to [0,1]$ continuous, integrable and with $(x : \psi(x) > 0) = (0, \infty)$. Let $\psi_b(x) = \psi(x-b)$. For $0 < a < b$ define stopping times

$$\tau_a = \inf(t \geq 0 : u_t(x) \geq 1/2, \exists x \geq a), \quad \rho_b = \inf(t \geq 0 : (u_t, \psi_b) > 0).$$

Fix t and let $(\phi_s^\lambda(x) : s \in [0,t], x \in \mathrm{R})$ be the unique non-negative bounded solution to

$$\begin{cases} -\dot{\phi}^\lambda &= (1/2)\Delta\phi^\lambda - (1/4)(\phi^\lambda)^2 + \lambda\psi_b \\ \phi_t^\lambda &\equiv 0 \end{cases}$$

The existence and uniqueness for this equation is discussed in [3]. Comparing with the solution to the same equation without the $-(1/4)(\phi^\lambda)^2$ term shows that $\phi_s^\lambda(x) \leq \lambda\int_0^{t-s} P_{t-s-r}\psi_b(x)dr$. The function $h(x) = 12(b-x)^{-2}$ solves $h'' = (1/2)h^2$ on $(-\infty, b)$. Arguing as in the proof of the maximum principle shows that $\phi_s^\lambda(x) \leq 12(b-x)^{-2}$ for all $x < b, s \leq t, \lambda > 0$. Using the Feynman Kac representation for ϕ^λ as in [1] lemma 3.5 we have for any $r \in (x, b)$

$$\begin{aligned} \phi_s^\lambda(x) &\leq 12(b-r)^{-2}P_x(\inf(t : B_1(t) = r) \leq t-s) \\ &\leq 24(b-r)^{-2}P_0(B_1(t) \geq r-x). \end{aligned}$$

Supposing that $b \geq 4t^{1/2}, x \leq b - 2t^{1/2}$ we choose $r = b - t^{1/2}$ to find

$$\phi_s^\lambda(x) \leq 24t^{-1}e^{(b-x)^2/8t} \quad \forall s \leq t. \tag{24}$$

Ito's formula gives

$$\begin{aligned} & d(e^{-(u_s,\phi_s^\lambda)-\lambda\int_0^s (u_r,\psi_b)dr}) \\ = \; & e^{-(u_s,\phi_s^\lambda)-\lambda\int_0^s (u_r,\psi_b)dr}\Big(|u_s(x)(1-u_s(x))|^{1/2}\phi_s^\lambda(x)dW_{x,s} \\ & +(u_s, -\dot{\phi}_s^\lambda - (1/2)\Delta\phi_s^\lambda - \lambda\psi_b) + (1/2)(u_s(1-u_s), (\phi_s^\lambda)^2)ds\Big). \end{aligned}$$

So, using the integrability of ϕ_s^λ to show the stochastic integral is a martingale,

$$\begin{aligned}
& E(1-e^{-(u_{t\wedge\tau_a},\phi^\lambda_{t\wedge\tau_a})-\lambda\int_0^{t\wedge\tau_a}(u_r,\psi_b)dr}) \\
= \;& E(1-e^{-(u_0,\phi_0^\lambda)}) \\
& +E(\int_0^{t\wedge\tau_a} e^{-(u_s,\phi_s^\lambda)-\lambda\int_0^s(u_r,\psi_b)dr}((1/4)u_s-(1/2)u_s(1-u_s),(\phi_s^\lambda)^2)ds \\
\leq \;& E(1-e^{-(u_0,\phi_0^\lambda)})+E(\int_0^t((1/4)u_s\mathrm{I}(-\infty,a),(\phi_s^\lambda)^2)ds).
\end{aligned} \tag{25}$$

As $\lambda\to\infty$ so $\phi^\lambda\uparrow\phi^\infty\in[0,\infty]$. Letting $\lambda\to\infty$ in (25) gives

$$\begin{aligned}
& P(\rho_b<\tau_a\wedge t) \\
\leq \;& \lim_{\lambda\to\infty} E(1-e^{-(u_{t\wedge\tau_a},\phi^\lambda_{t\wedge\tau_a})-\lambda\int_0^{t\wedge\tau_a}(u_r,\psi_b)dr}) \\
\leq \;& E(1-e^{-(u_0,\phi_0^\infty)})+E(\int_0^t((1/4)u_s\mathrm{I}(-\infty,a),(\phi_s^\infty)^2)ds) \\
\leq \;& \int_{-\infty}^0\phi_0^\infty(x)dx+\int_0^t\int_{-\infty}^a\int_{-\infty}^0(1/4)p_s(x-y)(\phi_s^\infty(x))^2dy\,dx\,ds.
\end{aligned}$$

Choosing $a=b/2$ and using the bounds in (24) we have

$$\begin{aligned}
& P(\rho_b<\tau_a\wedge t) \\
\leq \;& \int_{-\infty}^0 24t^{-1}e^{(b-x)^2/8t}dx+\int_0^t\int_{-\infty}^{b/2}\int_{-\infty}^0 p_s(x-y)144t^{-2}e^{(b-x)^2/4t}dy\,dx\,ds \\
\leq \;& 96b^{-1}e^{-b^2/8t}+144t^{-1}\int_{-\infty}^{b/2}e^{(b-x)^2/4t}dx \\
\leq \;& Cb^{-1}e^{-b^2/16t} \\
\leq \;& Ct^{-1/2}e^{-b^2/16t}.
\end{aligned} \tag{26}$$

But from lemma 4.1 a) we have for $b\geq 4t^{1/2}$

$$\begin{aligned}
P(\tau_{b/2}\leq t) &= P(P_sf(x)+N_s(x)\geq 1/2,\ \exists x\geq b/2, s\leq t) \\
&\leq P(P_t\mathrm{I}(-\infty,0)(b/2)+N_s(x)\geq 1/2,\ \exists x\geq b/2, s\leq t) \\
&\leq P(N_s(x)\geq 1/2-P_0(B_1(1)\geq 2),\ \exists x\geq b/2, s\leq t) \\
&\leq C(t\vee t^{22})(\mathrm{I}(-\infty,0),P_t\mathrm{I}(b/2,\infty)) \\
&\leq C(t\vee t^{22})t^{1/2}e^{-b^2/8t}
\end{aligned}$$

which combined with (26) completes the proof. •

Corollary 4.3 *Let u be a solution to (1) with $-\infty<L(u_0)<R(u_0)<\infty$. Then the path $t\to R(u_t)$ is, almost surely, right continuous with left limits. At the at most countably many jumps $R(u_t)\leq\lim_{s\uparrow t}R(u_s)$.*

Proof. We prove the desired regularity on a fixed (but arbitrary) time interval $[0, M]$. We have $P(-\infty < \inf_{s\in[0,M]} L(u_s) < \sup_{s\in[0,M]} R(u_s) < \infty) = 1$ from proposition 4.2. Let $s_n^j = j2^{-n}$, $\Delta_n = \{s_n^0, s_n^1, \ldots\}$ and $\Delta = \bigcup_{n\geq 0} \Delta_n$. For $j, n \geq 0$ define $A_{j,n} = (\sup_{s_n^j \leq s \leq s_n^{j+1}} R(u_s) - R(u_{s_n^j}) \geq 2^{-n/4})$. Then, also from proposition 4.2, $P(A_{j,n}) \leq C\exp(-2^{n/2}/16)$. By Borel Cantelli there exist $A(\omega), N(\omega)$ with $P(A, N < \infty) = 1$, $R(u_t)(\omega) \in [-A(\omega), A(\omega)]$, $\forall t \in [0, M]$ and $\omega \in \bigcap_{n\geq N(\omega)} \bigcap_{j\leq M2^n} A_{j,n}^c$.

We now fix a sample path for which $A(\omega), N(\omega) < \infty$. Choose $0 \leq s \leq t \leq M$ with $s \in \Delta, t - s \leq 2^{-N}$. We may choose a sequence $s = s_0 < s_1 < \ldots < s_{k-1} < t \leq s_k$ with $s_i \in \Delta_{n_i}, s_{i+1} - s_i = 2^{-n_i}, n_{i+1} > n_i$ for $i = 0, \ldots, k-1$. Note that $s_k - s_{k-1} \leq |t - s|$. Then

$$\begin{aligned}
& R(u_t) - R(u_s) \\
= \; & (R(u_t) - R(u_{s_{k-1}})) + (R(u_{s_{k-1}}) - R(u_{s_{k-2}})) + \ldots + (R(u_{s_1}) - R(u_{s_0})) \\
\leq \; & 2^{-n_{k-1}/4} + \ldots + 2^{-n_0/4} \\
\leq \; & C(t-s)^{1/4}. \qquad (27)
\end{aligned}$$

Suppose that $\limsup_{s\uparrow t, s\in\Delta, s\leq t} R(u_s) > \liminf_{s\uparrow t, s\in\Delta, s\leq t} R(u_s)$ for some $t \in (0, M]$. Then we may obtain a contradiction to (27). Hence we may define $S_t = \lim_{s\uparrow t, s\in\Delta, s\leq t} R(u_s)$. From it's definition $t \to S_t$ is left continuous. If for some $t \in [0, M)$ $\limsup_{s\downarrow t, s>t} S_s > \liminf_{s\downarrow t, s>t} S_s$ then we can again obtain a contradiction to (27). Hence $t \to S_t$ has right limits on $[0, M)$. Also (27) implies that $R(u_t) \leq S_t$ for all $t \in [0, M]$ and that S_t can only jump downwards. Since $R(u_t) = S_t$ on $\mathbb{Q}\cap[0, M]$ we have $R(u_t) = S_t$ at continuity points of S_t in $[0, M]$. We may exhaust the jumps of S_t by a sequence of stopping times $T_k, k = 1, 2, \ldots$ Using proposition 4.2 and the strong Markov property at time T_k

$$P(\sup_{T_k \leq s \leq T_k + 2^{-n}} R(u_s) - R(u_{T_k}) \geq 2^{-n/2}) \leq C\exp(-2^{n/4}/16).$$

Then $P(\limsup_{s\downarrow T_k} R(u_s) \leq R(u_{T_k})\ \forall k = 1, \ldots) = 1$. From the continuity of $u_t(x)$ we have that $\liminf_{s\downarrow t, s>t} R(u_s) \geq R(u_t)$. Combining these observations shows that $R(u_t) = S_{t+}$ at jump points completing the proof. •

5 Death

Proposition 5.1 *a) For $m \geq 1$, $\sup_{f \leq \mathrm{I}(0,m)} P((u_{m^{100}}, 1) > 0) \leq Cm^{-1}$.*

b) For $\beta \in [0, \infty)$ there exists $C(\beta) < \infty$ such that whenever $m \geq 1$ and $(f, (-5m^{51} - m^{\beta}, m^{\beta} + 5m^{51})) \leq m, m \geq 1$ then $P((u_{m^{100}}, (-m^{\beta}, m^{\beta})) > 0) \leq c_6 m^{-1}$.

Remarks. We sketch the argument of the proof in words. One can prove that super Brownian motion dies out by calculating the Laplace transform $E(\exp(-\lambda(u_t, 1)))$ and letting $\lambda \to \infty$. An exact calculation of the Laplace transform is impossible for our eqaution but the method will provide a suitable bound provided we can show that for long periods of time

$$||u(t, \cdot)||_{\infty} < 1 - \epsilon \quad \text{for some} \quad \epsilon > 0. \tag{28}$$

To do this we use the fact that the total mass is a non-negative martingale and so its brackets process converges. This implies that the amount of noise decreases and eventually cannot counteract the effect of the heat kernel which by itself would lead to (28).

In part b) we allow mass at a distance $5m^{51}$ from the area of interest. From proposition 4.2 this mass should only travel $O(m^{50})$ in time m^{100} thus should not affect the mass inside the area of interest. We do not have the additive property of super Brownian motion (or a subadditive property) so that this requires several small changes in the above sketched argument. Since part a) follows from part b) and proposition 4.2 we give only the details for b).

Part b) suggests that a solution with initial mass m is likely to die out by time m^{100}. In section 7 we shall show that from initial condition $f \leq \mathrm{I}(0, m)$, a solution dies in time $O(m^2)$.

Finally a slight change in the argument would show that $(f, 1) < \infty$ is sufficient to ensure a solution will die in finite time.

Proof. β will be fixed throughout the proof and we suppress notation for its dependence in all variables. Write $t_j = jm^5$ for $j = 0, 1, \ldots$ and let $J_j = [-m^{\beta} - jm^{51}, m^{\beta} + jm^{51}]$ for $j = 0, 1, \ldots, 5$. Define the events for $j = 0, 1, \ldots$

$$A_j = \{ \int_{t_j}^{t_{j+1}} \int_{J_3} u_s(z)(1 - u_s(z)) dz\, ds \leq m^{-80}, \sup_{s \leq t_{j+1}} (u_s, \mathrm{I}_{J_3}) \leq m^2 \}.$$

On the event A_j there is little noise in J_3 during $[t_j, t_{j+1}]$ and we shall see that there is enough time for the heat kernel to drag the solution uniformly below 1/2 on J_2. The first step is to show that the events A_j occur frequently. Define $\psi_1(s, x) = P_{m^{100}-s} \mathrm{I}_{J_4}(x)$ and note that for $x \in J_3, s \leq m^{100}, m \geq 1$

$$\begin{aligned} \psi_1(s, x) &= P_x(B_1(s) \in J_4) \\ &\geq P_0(|B_1(m^{100})| \leq m^{51}) \\ &= P_0(|B_1(1)| \leq m) \geq 1/2. \end{aligned}$$

The process $(u_s, \psi_1(s))$ is a martingale on $[0, m^{100}]$ so that

$$\begin{aligned}
& E((\int_0^{m^{100}} \int_{J_3} u_s(z)(1-u_s(z))dz\, ds)^{1/4}) \\
\leq\ & E((\int_0^{m^{100}} \int 4\psi_1^2(s,z)u_s(z)(1-u_s(z))dz\, ds)^{1/4}) \\
=\ & CE([(u_\cdot, \psi_1(\cdot))]_{m^{100}}^{1/4}) \\
\leq\ & CE(\sup_{s \leq m^{100}} (u_s, \psi_1(s))^{1/2}) \quad \text{(Burkhölder's inequality)} \\
\leq\ & CE((u_0, \psi_1(0))^{1/2}) \\
\leq\ & C(E((u_0, \psi_1(0))))^{1/2} \\
\leq\ & C(m + \int_{J_5^c} \psi_1(0,x)dx)^{1/2} \\
\leq\ & Cm^{1/2} \qquad (29)
\end{aligned}$$

where in the last step we use the assumption that $(u_0, \mathrm{I}_{J_5}) \leq m$ and an easy upper bound on ψ_1. Also

$$\begin{aligned}
& P(\sup_{s \leq m^{100}} (u_s, \mathrm{I}_{J_3}) \geq m^2) \\
\leq\ & P(\sup_{s \leq m^{100}} 2(u_s, \psi_1(s)) \geq m^2) \\
\leq\ & Cm^{-2}E((u_0, \psi_1(0))) \\
\leq\ & Cm^{-1}. \qquad (30)
\end{aligned}$$

Define $N_n = \sum_{j=1}^n \mathrm{I}_{A_j}$.

$$\begin{aligned}
& P(N_{m^{95}} < m^{95} - m^{86}) \\
\leq\ & P(\int_0^{m^{100}} \int_{J_3} u_s(z)(1-u_s(z))dz\, ds \geq m^6) + P(\sup_{s \leq m^{100}} (u_s, \mathrm{I}_{J_3}) \geq m^2) \\
\leq\ & Cm^{-1} \qquad (31)
\end{aligned}$$

using (29,30) and Chebychev's inequality. This completes the first step.

Define $B_j = \{u_t(x) \leq P_{t-t_j}u_{t_j}(x) + Cm^{-1},\ \forall x \in J_2, t \in [t_j, t_{j+1}]\}$. The second step is to show that $A_j \cap B_j^c$ is unlikely. Fix j, m and define

$$\begin{aligned}
M_t(x) &= u_{t_j+t}(x) - P_{t-t_j}u_{t_j}(x) \\
&= \int_0^t \int p_{t-r}(x-z)|u_{t_j+r}(z)(1-u_{t_j+r}(z))|^{1/2} dW_{z,r}.
\end{aligned}$$

We shall control $M_s(x)$ again by controlling it's increments over dyadics. We consider $M_t(x) - M_t(y)$ is the terminal value of a martingale with (in a slight abuse of notation)

$$[M_t(x)-M_t(y)] = \int_0^t \int (p_{t-r}(x-z)-p_{t-r}(y-z))^2 u_{t_j+r}(z)(1-u_{t_j+r}(z))dz\,dr. \tag{32}$$

We now consider $|x-y| \leq 1$ and $t \leq m^5$. We use also the bounds $|p_t(x)-p_t(y)| \leq Ct^{-1}|x-y|$ and $|p_t(x)| \leq Ct^{-1/2}$. We split the space integral in (32) into two halves, over J_3 and J_3^c. The first half is bounded on A_j by (applying Hölder's inequality)

$$\begin{aligned}
&(\int_0^t \int (p_{t-r}(x-z)-p_{t-r}(y-z))^{5/2} dz\,dr)^{4/5} \\
&\quad .(\int_0^t \int_{J_3} u_r^5(z)(1-u_r(z))^5 dz\,dr)^{1/5} \\
\leq\ & C(\int_0^t \|p_{t-r}(x-\cdot)-p_{t-r}(y-\cdot)\|_\infty^{3/2})^{4/5} m^{-20} \\
\leq\ & C(\int_0^t (t-s)^{-3/4} \wedge |x-y|^{3/2}(t-s)^{-3/2} ds)^{4/5} m^{-20} \\
\leq\ & C|x-y|^{2/5} m^{-20}.
\end{aligned}$$

When $x, y \in J_2$ the second half is bounded by

$$\begin{aligned}
&\int_0^t \|p_{t-r}(x-\cdot)-p_{t-r}(y-\cdot)\|_\infty \int_{J_3^c} p_{t-r}(x-z)+p_{t-r}(y-z)dz\,dr \\
\leq\ & C\int_0^t (t-r)^{-1/2} \wedge |x-y|(t-r)^{-1} dr P_0(|B_1(m^5)| \geq m^{51}) \\
\leq\ & C|x-y|^{2/5} m^{-20}.
\end{aligned}$$

Now write $x_n^k = t_n^k = k2^{-n}$. Then if $n \geq 0, x_n^k, x_n^{k+1} \in J_2, 2^{-n} \leq t_n^l \leq m^5$

$$\begin{aligned}
&P(|M_{t_n^l}(x_n^{k+1}) - M_{t_n^l}(x_n^k)| \geq m^{-6}2^{-n/20}, A_j | \mathcal{F}_{t_n^l}) \\
\leq\ & P(|M_{t_n^l}(x_n^{k+1}) - M_{t_n^l}(x_n^k)| \geq m^{-6}2^{-n/20}, \\
&\qquad [M_{t_n^l}(x_n^{k+1}) - M_{t_n^l}(x_n^k)] \leq C2^{-2n/5}m^{-20} | \mathcal{F}_{t_n^l}) \\
\leq\ & 2\exp(-Cm^8 2^{3n/10})
\end{aligned}$$

using [9] corollory IV.37.12. In an entirely similar way we have

$$P(|M_{t_n^l}(x_n^k) - M_{t_n^{l-1}}(x_n^k)| \geq m^{-6}2^{-n/20}, A_j | \mathcal{F}_{t_n^l}) \leq 2\exp(-Cm^8 2^{n/10}).$$

Now define

$$\begin{aligned}
\tilde{B}_{k,l,n} &= \{|M_{t_n^l}(x_n^{k+1}) - M_{t_n^l}(x_n^k)| \vee |M_{t_n^l}(x_n^k) - M_{t_n^{l-1}}(x_n^k)| \geq m^{-6}2^{-n/20}\} \\
\tilde{B}_j^c &= \bigcup_{n\geq 0} \bigcup_{1\leq l \leq m^5 2^n} \bigcup_{k: x_n^k, x_n^{k+1} \in J_2} \tilde{B}_{k,l,n}.
\end{aligned}$$

Then defining $q(m) = ||P(\tilde{B}_j^c \cap A_j | \mathcal{F}_{t_n^j})||_\infty$ we have

$$q(m) \le C \sum_{n \ge 0} 2^{2n} m^5 (m^\beta + m^{51}) \exp(-Cm^8 2^{n/10}) < \infty.$$

On the set $\tilde{B}_j$ we may estimate $|M_t(x)|$ for $t \le m^5, x \in J_2$ in the usual manner by an infinite sum over dyadic increments. Indeed we need at most m^5 steps over intervals of length 1 and two steps over intervals of length 2^{-n} for each $n \ge 1$. Thus for $t \le m^5, x \in J_2$

$$|M_t(x)| \le m^{-1} + m^{-6} \sum_{n \ge 1} 2^{-n/20} \le Cm^{-1}$$

when $C = 1 + 2^{-1/20}(1 - 2^{-1/20})^{-1}$. Hence

$$P(B_j^c \cap A_j | \mathcal{F}_{t_n^j}) \le P(\tilde{B}_j^c \cap A_j | \mathcal{F}_{t_n^j}) \le q_m \to 0 \quad \text{as} \quad m \to \infty. \tag{33}$$

Define $\tilde{t}_j = t_j + m^{9/2}$. Then on $A_j \cap B_j$, for $t \in [\tilde{t}_j, t_{j+1}], x \in J_2$

$$\begin{aligned} u_t(x) &\le P_{t-t_j} u_{t_j}(x) + Cm^{-1} \\ &\le (u_{t_j}, \mathrm{I}_{J_3}) ||p_{t-t_j}(\cdot)||_\infty + \int_{J_3^c} p_{t-t_j}(z-n)dz + Cm^{-1} \\ &\le Cm^2 (t - t_j)^{-1/2} + Cm^{-1} \\ &\le Cm^{-1/4}. \end{aligned} \tag{34}$$

This completes the second step.

Define $C_j = \{(u_{t_{j+1}}, \mathrm{I}_{J_1}) = 0\}$. The third step is to show that $A_j \cap C_j^c$ is unlikely. Again fix j, m. As in section 2 we may assume that we have coupled to u a process $\tilde{u}$ with $u_{\tilde{t}_j} = \tilde{u}_{\tilde{t}_j}$ and solving on $[\tilde{t}_j, t_{j+1}]$

$$\dot{\tilde{u}} = (1/2)\Delta \tilde{u} + \sigma(\tilde{u}) dW$$

where

$$\sigma(\tilde{u}, x) = \begin{cases} |\tilde{u}(1-\tilde{u})|^{1/2} & \text{if } |\tilde{u}| \le 1/2 \text{ or if } x \in J_2^c \\ |\tilde{u}/2|^{1/2} & \text{if } |\tilde{u}| \ge 1/2 \text{ and } x \in J_2. \end{cases}$$

We can construct $\tilde{u}$ so that $\tilde{u} \ge 0$ and so that $u_t = \tilde{u}_t$ for all $t \le \inf(s \ge \tilde{t}_j : ||u||_\infty \ge 1/2)$. Thus by (34), for sufficiently large m, we see that $u_t = \tilde{u}_t$ for all $t \in [\tilde{t}_j, t_{j+1}]$ on the set $A_j \cap B_j$. Then if $D_j = \{(\tilde{u}_{\tilde{t}_j}, \mathrm{I}_{J_3}) \le m^2\}$

$$\begin{aligned} & P(A_j \cap B_j \cap C_j^c | \mathcal{F}_{t_j}) \\ = \; & P(A_j \cap B_j \cap \{(\tilde{u}_{t_{j+1}}, \mathrm{I}_{J_1}) > 0\} | \mathcal{F}_{t_j}) \\ \le \; & P(\{(\tilde{u}_{t_{j+1}}, \mathrm{I}_{J_1}) > 0\} \cap D_j | \mathcal{F}_{t_j}). \end{aligned}$$

For the process $\tilde{u}$ we may argue again as in [1]. Let $0 \leq \phi_0(x) \leq 1$ satisfy $(\phi_0(x) > 0) = J_1$. Let $\psi_2^\lambda(s,x)$ solve

$$\begin{aligned} -\dot{\psi}_2^\lambda &= (1/2)\Delta\psi_2^\lambda - (1/4)(\psi_2^\lambda)^2 \quad \text{on } [\tilde{t}_j, t_{j+1}] \\ \psi_2^\lambda(t_{j+1}) &= \lambda\phi_0. \end{aligned}$$

We may now estimate $E(\exp(-(\tilde{u}_s, \psi_2^\lambda(s)))$ by expanding it according to Ito's formula:

$$\begin{aligned} & P(\{(\tilde{u}_{t_{j+1}}, \mathrm{I}_{J_1}) > 0\} \cap D_j | \mathcal{F}_{\tilde{t}_j}) \\ = & \lim_{\lambda\to\infty} E((1 - \exp(-(\tilde{u}_{t_{j+1}}, \psi_2^\lambda(t_{j+1}))))\mathrm{I}_{D_j} | \mathcal{F}_{\tilde{t}_j}) \\ = & \lim_{\lambda\to\infty} (1 - \exp(-(\tilde{u}_{\tilde{t}_j}, \psi_2^\lambda(\tilde{t}_j))))\mathrm{I}_{D_j} + \lim_{\lambda\to\infty} E(\int_{\tilde{t}_j}^{t_{j+1}} \exp(-(\tilde{u}_s, \psi_2^\lambda(s))) \\ & \cdot \Big((\tilde{u}_s, (1/2)\Delta\psi_2^\lambda(s) + \dot{\psi}_2^\lambda(s)) - (1/2)(\sigma(\tilde{u}_s), (\psi_2^\lambda(s))^2)\Big)\, ds \mathrm{I}_{D_j} | \mathcal{F}_{\tilde{t}_j}) \\ \leq & \lim_{\lambda\to\infty} (\tilde{u}_{\tilde{t}_j}, \psi_2^\lambda(\tilde{t}_j))\mathrm{I}_{D_j} + \lim_{\lambda\to\infty} E(\int_{\tilde{t}_j}^{t_{j+1}} \int_{J_2^c} \tilde{u}_s(x)(\psi_2^\lambda(s,x))^2 dx\, ds | \mathcal{F}_{\tilde{t}_j}) \\ \leq & \lim_{\lambda\to\infty} \left(m^2 \|\psi_2^\lambda(\tilde{t}_j)\|_\infty + \int_{J_2^c} \psi_2^\lambda(\tilde{t}_j, x) dx + \int_{\tilde{t}_j}^{t_{j+1}} \int_{J_2^c} (\psi_2^\lambda(s,x))^2 dx\, ds \right) \quad (35) \end{aligned}$$

From [1] lemma 3.1 we have $|\psi_s^\lambda(x)| \leq 4/(t_{j+1} - s)$. Using this we see that the first term of (35) is bounded by Cm^{-3}. Also from [1] lemma 3.1 we have also the bound, for $x \notin J_1$

$$\psi_2^\lambda(s,x) \leq C(t_{j+1} - s)^{-1} \exp(-\tilde{C} d(x, J_1)(t_{j+1} - s)^{-1/2})$$

where $d(x, J_1) = \inf(|y - x| : y \in J_1)$. Using this one may show that the second and third terms of (35) are also bounded by Cm^{-3}. This proves that $P(A_j \cap B_j \cap C_j^c | \mathcal{F}_{t_j}) \leq Cm^{-3}$ and combining with (33) that

$$P(A_j \cap C_j^c | \mathcal{F}_{t_j}) \leq 1/2 \quad (36)$$

for sufficiently large m. This completes the third part of the proof.

Define $\bar{N}_n = \sum_{i=1}^n \mathrm{I}(A_j^c \cup C_j)$. Then

$$\begin{aligned} P(\bar{N}_n \leq n/4) &= P(\sum_{i=1}^n \mathrm{I}(\mathrm{I}(A_j \cap C_j^c) - \mathrm{I}(A_j^c \cup C_j)) \geq n/2) \\ &\leq (4/n^2) E((\sum_{i=1}^n \mathrm{I}(\mathrm{I}(A_j \cap C_j^c) - \mathrm{I}(A_j^c \cup C_j)))^2) \\ &\leq 4/n \end{aligned}$$

using (36). Recall that $N_n = \sum_{i=1}^n \mathrm{I}(A_j)$. Then using (31) we have

$$\begin{aligned} P(\bigcup_{j=1}^{m^{95}} C_j) &\geq P(\bar{N}_{m^{95}} > (1/4)m^{95}, N_n \geq (4/5)m^{95}) \\ &\geq 1 - P(\bar{N}_n \leq (1/4)m^{95}) - P(N_m^{95} < m^{95} - m^{86}) \qquad (37) \\ &\geq 1 - Cm^{-1}. \qquad (38) \end{aligned}$$

From proposition 4.2 we have

$$\begin{aligned} & P(\bigcup_{j=1}^{m^{95}} C_j \cap \{(u_{m^{100}}, J_0) > 0\}) \\ \leq & \sum_{j=1}^{m^{95}} P((u_{t_j}, J_1) = 0, (u_{m^{100}}, J_0) > 0) \\ \leq & \; Cm^{95} m^{2300} e^{-m^2/16} \leq Cm^{-1}. \end{aligned}$$

Combining this with (38) completes the proof. •

Lemma 5.2 *Suppose that* $-\infty < L(f) < R(f) < \infty$. *Then*

$$P(\sup_{s \leq t} |R_s - L_s| \geq 14t^{(1/2)-(1/1000)}) \leq C(f)t^{-1/2000}.$$

Proof. We fix $t \geq e$. Write $\delta = 1/1000$ and $s_j = jt^{1-3\delta}$. We suppress dependence on that initial data f.

Define

$$\begin{aligned} \tilde{R}_j &= \sup(x : \int_x^\infty u_{s_j}(z)dz \geq t^{(1/2)-\delta}), \\ \tilde{L}_j &= \inf(x : \int_{-\infty}^x (1 - u_{s_j}(z))dz \geq t^{(1/2)-\delta}). \end{aligned}$$

From the compact support property we have $-\infty < L_{s_j} < \tilde{L}_j, \tilde{R}_j < R_{s_j} < \infty$. For $s_j \leq t$, by Chebychev's inequality and lemma 3.2 b)

$$\begin{aligned} P(\int_{\tilde{L}_j}^\infty u_{s_j}(z)dz \geq t^{5\delta}) &\leq P(\int_{-\infty}^\infty u_{s_j}(z) \int_{-\infty}^z (1 - u_{s_j}(y))dy\, dz \geq t^{(1/2)+4\delta}) \\ &\leq Ct^{-4\delta} \log^{1/2}(t). \end{aligned}$$

Now apply proposition 5.1 with $m = t^{5\delta}, \beta = 200$ to see

$$P((u_{s_j+t^{500\delta}}, (\tilde{L}_j + 5t^{205\delta}, \tilde{L}_j + 5t^{205\delta} + t)) > 0, \int_{\tilde{L}_j}^\infty u_{s_j}(z)dz \leq t^{5\delta}) \leq Ct^{-5\delta}.$$

The mass inside the interval $(\tilde{R}_j - 5t^{205\delta} - t, \tilde{R}_j - 5t^{205\delta})$ is controled in a similar way. Then we have

$$\begin{aligned}
& P(\exists s_j \leq t \text{ such that } (u_{s_j+t^{500\delta}}, (\tilde{L}_j + 5t^{205\delta}, \tilde{L}_j + 5t^{205\delta} + t)) \\
& \qquad +(u_{s_j+t^{500\delta}}, (\tilde{R}_j - 5t^{205\delta} - t, \tilde{R}_j - 5t^{205\delta})) > 0) \\
\leq \; & Ct^{3\delta}(t^{-4\delta} \log^{1/2}(t) + t^{-5\delta}) \\
\leq \; & Ct^{-\delta/2}. \qquad (39)
\end{aligned}$$

From the compact support property proposition 4.2 we have also

$$P(\sup_{s \leq 2t} |R_t| \vee |L_t| \geq t^{3/4}) \leq Ct^{23} e^{-t^{6/4}/32t} \leq Ct^{-\delta/2}. \qquad (40)$$

Combining (39, 40) with the fact that $\tilde{L}_j - \tilde{R}_j \leq 2t^{(1/2)-\delta}$ we have

$$P(|R_{s_j+t^{500\delta}} - L_{s_j+t^{205\delta}}| \geq 2t^{(1/2)-\delta} + 10t^{205\delta}, \exists s_j \leq t) \leq Ct^{-\delta/2}.$$

We now need to interpolate between the grid points. But from lemma 4.2

$$\begin{aligned}
& P(\sup_{s \in [s_j + t^{500\delta}, s_{j+2}]} (R_s - R_{s_j+t^{500\delta}}) \vee (L_{s_j+t^{500\delta}} - L_s) \geq t^{(1/2)-\delta}) \\
\leq \; & Ct^{23} e^{-t^{1-2\delta}/32t^{1-3\delta}}.
\end{aligned}$$

A similar estimate allows interpolation over the interval $[0, s_2]$ which completes the proof. •

6 A single wavefront

Throughout this section u is a solution to (1) started at f with $-\infty < L(f) < R(f) < \infty$. Define

$$v_t^{(n)}(x) = u_{n^2t}(nx). \tag{41}$$

Lemma 6.1 *For continuous integrable ϕ the processes $((v_t^{(n)}, \phi) : t \geq 0)_{n=1,2,\ldots}$ are tight.*

Proof. Rescaling the Green's function representation gives, for integrable ϕ,

$$(v_t^{(n)}, \phi) = (v_0^{(n)}, P_t\phi) + n^{1/2} \int_0^t \int |v_s^{(n)}(x)(1 - v_s^{(n)}(x))|^{1/2} P_{t-s}\phi(x) d\bar{W}_{x,s} \tag{42}$$

for some new white noise $\bar{W}$. Note that all terms in (42) are continuous in t. The first term on the right hand side of (42) converges to $(\mathrm{I}(-\infty, 0], P_t\phi)$. We shall check the Kolmogorov tightness criterion for the stochastic integral in (42). For $0 < s < t$

$$\begin{aligned}
& n^{1/2} \int_0^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x) d\bar{W}_{x,r} \\
& -n^{1/2} \int_0^s \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{s-r}\phi(x) d\bar{W}_{x,r} \\
= \; & n^{1/2} \int_s^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x) d\bar{W}_{x,r} \\
& +n^{1/2} \int_0^s \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} (P_{t-r}\phi(x) - P_{s-r}\phi) d\bar{W}_{x,r}.
\end{aligned}$$

Using the moment bounds from lemma 3.2

$$\begin{aligned}
& E((n^{1/2} \int_s^t \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} P_{t-r}\phi(x) d\bar{W}_{x,r})^4) \\
\leq \; & C\|\phi\|_\infty^4 E((\int_s^t \int n v_r^{(n)}(x)(1 - v_r^{(n)}(x)) dr\, dx)^2) \\
\leq \; & C(\phi)(t-s) E \int_s^t (\int u_{n^2r}(x)(1 - u_{n^2r}(x)) dx)^2 dr \\
\leq \; & C(\phi, f)(t-s)^2.
\end{aligned}$$

Using the bound $\|P_t\phi - P_s\phi\|_\infty \leq \|\phi\|_\infty(|t-s|s^{-1} \wedge 1)$ we also have

$$\begin{aligned}
& E((n^{1/2} \int_0^s \int |v_r^{(n)}(x)(1 - v_r^{(n)}(x))|^{1/2} (P_{t-r}\phi(x) - P_{s-r}\phi) d\bar{W}_{x,r})^4) \\
\leq \; & C(\phi) E((\int_0^s \int u_{n^2r}(nx)(1 - u_{n^2r}(x) dx(|t-s|^2(s-r)^{-2} \wedge 1) dr)^2)
\end{aligned}$$

$$\begin{aligned}
&\leq \; C(\phi)(t-s)\int_0^s E((\int u_{n^2r}(nx)(1-u_{n^2r}(x)dx)^2)(|t-s|^2(s-r)^{-2}\wedge 1)dr \\
&\leq \; C(\phi,f)(t-s)^2.
\end{aligned}$$

This checks Kolmogorov's criterion and completes the proof. •

Theorem 6.2 *a) Given $\epsilon>0, T<\infty$ then for all sufficiently large n there is a coupling of processes $(\bar{u}_t, B_t : t\geq 0)$ with B a Brownian motion started at 0, $\bar{u}$ a solution to (1) started at f and*

$$P(\sup_{t\leq T}|(R(u_{n^2t})/n)-B_t|\vee|(L(u_{n^2t})/n)-B_t|\geq\epsilon)\leq\epsilon.$$

b) If u is a solution to (1) started at f then the processes $(R(u_{n^2t})/n : t\geq 0)_{n=1,2,...}$ converge in distribution to a Brownian motion started at 0.

Proof. Part b) follows directly from part a). The key step in proving part a) is to show that the measure valued processes $(v_t^{(n)}(x)dx : t\geq 0)_{n=1,2,\ldots}$, as defined in (41), converge in distribution and that the limit has the law of the process

$$\mu_t(dx) = \mathrm{I}(x\leq B_t)dx \tag{43}$$

where B_t is a standard Brownian motion started at 0.

Lemma 6.1 implies that the processes $(v_t^{(n)}(x)dx : t\geq 0)_{n=1,2,...}$ are tight (see [8]). We extract a convergent subsequence, which we continue to label $v^{(n)}$. By changing probability space we may take measure valued processes $\mu^{(n)},\mu$ with $\mu^{(n)}_{\cdot} \stackrel{\mathcal{D}}{=} v_{\cdot}^{(n_r)}(x)dx$ and $\mu^{(n)}\stackrel{a.s.}{\to}\mu$. Thus for any $t\geq 0$ and ϕ continuous with compact support we have

$$\sup_{s\leq t}|(\mu_s^{(n)},\phi)-(\mu_s,\phi)|\to 0. \tag{44}$$

Note that $\sup_{t\geq 0}\mu_t^{(n)}(\phi)\leq\|\phi\|_1$ a.s. Using this we may extend the convergence in (44) to continuous integrable ϕ. Note also that $\mu_0 = \mathrm{I}(x\leq 0)dx$ a.s.

Suppose $\phi_s(x)$ is smooth, bounded and $\sup_{s\in[0,t]}|\phi_s|\vee|\Delta\phi_s|\vee|\dot{\phi}_s|$ is integrable. Then

$$\begin{aligned}
(\mu_s^{(n)},\phi_s) &= (\mu_0^{(n)},\phi_0)+\int_0^s(\mu_r^{(n)},(1/2)\Delta\phi_r+\dot{\phi}_r)dr+m_s^{(n)}(\phi) \quad \text{for } s\leq t \\
m_s^{(n)}(\phi) & \quad \text{is a continuous martingale} \\
[m^{(n)}(\phi)]_s &\stackrel{\mathcal{D}}{=} \int_0^s\int n\phi_r^2(x)u_{n^2r}(nx)(1-u_{n^2r}(nx))dx\,dr.
\end{aligned}$$

Note that $|m_s^{(n)}(\phi)|\leq(1+s)\|\sup_{s\in[0,t]}|\phi_s|\vee|\Delta\phi_s|\vee|\dot{\phi}_s|\|_1$. We may pass to the limit to see that

$$\begin{aligned}
(\mu_s,\phi_s) &= (\mu_0,\phi_0)+\int_0^s(\mu_r,(1/2)\Delta\phi_r+\dot{\phi}_r)dr+m_s(\phi) \quad \text{for } s\leq t \\
m_s(\phi) & \quad \text{is a continuous martingale.}
\end{aligned}$$

We shall now identify the brackets process of m_t for certain ϕ. The crucial lemma is as follows.

Lemma 6.3 *Fix $\psi \geq 0$, smooth, of compact support and with $\int \psi(x)dx = 1$. Let $\phi_s = P_{t-s}\psi$. Then as $n \to \infty$*

$$E\left(\left(\int_0^t \int n\phi_s^2(x) u_{n^2 s}(nx)(1 - u_{n^2 s}(nx)) dx\, ds \right.\right.$$
$$\left.\left. - \int_0^t \int 2\phi_s(x)\phi_s'(x) u_{n^2 s}(nx) dx\, ds \right)^2 \right) \to 0.$$

We delay the proof of this lemma to the end of this section.

Choose $\psi_1, \ldots, \psi_n \in C_c$. For $\phi_s(x)$ as in lemma 6.3, $0 \leq r_1 < \ldots < r_n \leq r \leq s$

$$\begin{aligned}
& E((m_s(\phi) - m_r(\phi))^2 (\mu_{r_1}, \psi_1) \ldots (\mu_{r_n}, \psi_n)) \\
= \; & \lim_{n\to\infty} E((m_s^{(n)}(\phi) - m_r^{(n)}(\phi))^2 (\mu_{r_1}^{(n)}, \psi_1) \ldots (\mu_{r_n}^{(n)}, \psi_n)) \\
= \; & \lim_{n\to\infty} E(([m^{(n)}(\phi)]_s - [m^{(n)}(\phi)]_r)(\mu_{r_1}^{(n)}, \psi_1) \ldots (\mu_{r_n}^{(n)}, \psi_n)) \\
= \; & \lim_{n\to\infty} E(\int_r^s \int n|u_{n^2 q}(nx)(1 - u_{n^2 p}(nx))|\phi_q^2(x) dx\, dq \\
& \qquad\qquad (v_{r_1}^{(n)}, \psi_1) \ldots (v_{r_n}^{(n)}, \psi_n)) \\
= \; & \lim_{n\to\infty} E(\int_r^s \int 2\phi_q(x)\phi_q'(x) u_{n^2 q}(nx) dx\, dq (v_{r_1}^{(n)}, \psi_1) \ldots (v_{r_n}^{(n)}, \psi_n)) \\
= \; & E(\int_r^s (\mu_s, 2\phi_q\phi_q') dq (\mu_{r_1}, \psi_1) \ldots (\mu_{r_n}, \psi_n))
\end{aligned}$$

using lemma 6.3 in the penultimate step. This calculates the brackets process and shows that the limit point μ_t satisfies the following martingale problem: For all $\psi \geq 0$, smooth, of compact support and with $\int \psi(x)dx = 1$

$$\begin{aligned}
(\mu_s, P_{t-s}\psi) &= (\mathrm{I}(-\infty, 0), P_t\psi) + m_s(\psi) \quad \text{for } s \leq t, \\
m_s(\psi) & \quad \text{is a continuous martingale with} \\
[m_s(\psi)] &= \int_0^s (\mu_r, 2P_{t-r}\psi\, P_{t-r}\psi') dr \text{ for } s \leq t.
\end{aligned}$$

Applying Ito's formula shows that μ_t given by (43) also satisfies this martingale problem. It remains to show that uniqueness of solutions holds. By polarisation show that

$$[m_s(\psi), m_s(\tilde{\psi})] = \int_0^s (\mu_r, P_{t-r}\psi P_{t-r}\tilde{\psi}' + P_{t-r}\tilde{\psi} P_{t-r}\psi') dr.$$

Applying Ito's formula gives

$$
\begin{aligned}
&E((\mu_t,\psi_1)\dots(\mu_t,\psi_k)) \\
= \;& \prod_{i=1}^{k}(\mathrm{I}(-\infty,0),P_t\psi_i) \qquad (45)\\
&+\sum_{i,j=1;i\neq j}^{k} E\Big(\int_0^t (\mu_s, P_{t-s}\psi_i P_{t-r}\psi_j' + P_{t-r}\psi_j P_{t-r}\psi_i')\prod_{k\neq i,j}(\mu_s, P_{t-s}\psi_k)ds\Big).
\end{aligned}
$$

The identity $E((\mu_t,\psi)) = (\mathrm{I}(-\infty,0),P_t\psi)$ can be extended by a monotone class argument to hold for all non-negative ψ. Similarly using induction and (45) the moments $E((\mu_t,\psi_1)\dots(\mu_t,\psi_k))$ are determined. Since $\mu_t(\psi)\leq \|\psi\|_1$ these moments determine the distribution of μ_t and, as usual, uniqueness of the one dimensional distributions implies uniqueness for the martingale problem. This completes the proof of convergence as measure valued processes.

To obtain the coupling stated in part a) we fix $\epsilon\in(0,1), T<\infty$ and choose $k\geq 1$ such that $P_0(\sup_{t\leq T}B_t\geq k)\leq\epsilon$. We also choose $0\leq\phi\in C_c$ with $(\phi>0)=(0,1)$ and define $\psi_k(x) = ((x+k+1)\wedge(k+1-x)\wedge 1)_+$, an approximation to $\mathrm{I}(-k,k)$.

Since $\mu^{(n)}\stackrel{\mathcal{D}}{=}v^{(n)}$ it has a jointly continuous density $\bar{v}_t^{(n)}(x)$ and defining $\bar{u}_t(x)=\bar{v}_{t/n^2}^{(n)}(x/n)$ then, by extending the probability space to define a suitable white noise, $\bar{u}$ is a solution to (1) started at f. Defining $\bar{B}_t=\sup(x:\mu_t((x,\infty))>0)$ gives a processes whose finite dimensional distributions are those of a Brownian motion. Letting $B_t=\limsup(\bar{B}_s: s\leq t, s\in\mathbb{Q}, s\to t)$ produces a Brownian motion started at 0 and so that $\mu_t(x)dx=\mathrm{I}(x\leq B_t)dx,\ \forall t\geq 0$.

We consider sufficiently large n so that $R(\bar{v}_0^{(n)}), L(\bar{v}_0^{(n)})\in[-1,1]$. Then

$$
\begin{aligned}
P(\sup_{t\leq T}R(\bar{v}_t^{(n)})\geq k) &= P(\sup_{t\leq T}(\mu^{(n)},\phi(\cdot-k))>0)\\
&\to P(\sup_{t\leq T}(\mu,\phi(\cdot-k))>0)\\
&= P_0(\sup_{t\leq T}B_t\geq k)\leq\epsilon.
\end{aligned}
$$

(The convergence holds since the set is a continuity set for the limit law.) So for sufficiently large n we have

$$
P(\sup_{t\leq T}|R(\bar{u}_{n^2t})/n|\vee|L(\bar{u}_{n^2t})/n|\vee|B_t|\geq k)\leq 5\epsilon. \qquad (46)
$$

From lemma 5.2, for sufficiently large n,

$$
P(\sup_{t\leq T}|R(\bar{u}_{n^2t})-L(\bar{u}_{n^2t})|/n\geq\epsilon)\leq\epsilon. \qquad (47)
$$

From (44), for sufficiently large n,

$$P(\sup_{t\leq T} |(\bar{v}_t^{(n)}, \psi_k) - (\mathrm{I}(x \leq B_t), \psi_k)| \geq \epsilon) \leq \epsilon. \tag{48}$$

On the intersection of the three sets in (46,47,48) we have

$$\begin{aligned}
R(\bar{u}_{n^2t})/n &= R(\bar{v}_t^{(n)}) \\
&\leq L(\bar{v}_t^{(n)}) + \epsilon \\
&= (\mathrm{I}(x \leq L(\bar{v}_t^{(n)}), \psi_k) - (k + (1/2)) + \epsilon \\
&\leq (\bar{v}_t^{(n)}, \psi_k) - (k + (1/2)) + \epsilon \\
&\leq (\mathrm{I}(x \leq B_t), \psi_k) - (k + (1/2)) + 2\epsilon \\
&= B_t + 2\epsilon.
\end{aligned}$$

Similarly $L(\bar{u}_{n^2t})/n \geq B_t - 2\epsilon$ which gives the desired coupling. •

Proof of lemma 6.3.

Fix ψ as in the statement of the lemma. Let X_1, X_2 be independent variables with density $\psi(x)$ and independent of the Brownian motion $\underline{B}$. Define $f^{(n)}(x) = f(nx)$.

Lemma 6.4 *Let ψ, ϕ_t as in lemma 6.3. Let $u_0 = f$ satisfy $-\infty < L(f) < R(f) < \infty$. Then*

$$\begin{aligned}
&\Big| E(\int_0^t \int n\phi_s^2(x) u_{n^2s}(nx)(1 - u_{n^2s}(nx)) dx\, ds \\
&\qquad\qquad -2 \int_0^t \int P_s f(x) \phi_s(x) \phi_s'(x) dx\, ds \Big| \\
&\leq 2 \int \int P_t\psi(x) P_t\psi(y) f^{(n)}(x)(1 - f^{(n)}(y)) \mathrm{I}(y \leq x) dy\, dx + |e(n,t)|
\end{aligned}$$

where $e(n,t)$ is independent of f and $1 \geq |e(n,t)| \to 0$ as $n \to \infty$.

Proof of lemma 6.4. Using lemma 3.1 a) we have

$$\begin{aligned}
&E(\int_0^t \int n u_{n^2s}(nx)(1 - u_{n^2s}(nx)) \phi_s^2(x) dx\, ds) \\
= \; & E(\int_0^t \int n u_{n^2(t-s)}(nx)(1 - u_{n^2(t-s)}(nx)) \phi_{(t-s)}^2(x) dx\, ds) \\
= \; & \int_0^t \int \phi_{t-s}^2(x) E_{\underline{0}}(f(nx + B_1(n^2(t-s))) \\
&\qquad (1 - f(nx + B_2(n^2(t-s)))) n e^{-L_{1,2}(n^2(t-s))/2}) dx\, ds \\
= \; & \int_0^t \int (P_s\psi(x))^2 E_{\underline{0}}(f^{(n)}(x + B_1(t-s))
\end{aligned}$$

$$(1 - f^{(n)}(x + B_2(t-s))ne^{-nL_{1,2}(t-s)/2})dx\,ds$$

$$\begin{aligned}
&= E_{\underline{0}}\Big(\int_0^t f^{(n)}(X_1 + B_1(t))(1 - f^{(n)}(X_2 + B_2(t))) \\
&\qquad ne^{-n(L_t^0 - L_s^0)(X_1+B_1-X_2-B_2)/2}(1/2)dL_s^0(X_1 + B_1 - X_2 - B_2)\Big) \\
&= E_{\underline{0}}(f^{(n)}(X_1 + B_1(t))(1 - f^{(n)}(X_2 + B_2(t)))(1 - e^{-nL_t^0(X_1+B_1-X_2-B_2)/2})) \\
&= E_{\underline{0}}(f^{(n)}(X_1 + B_1(t))(1 - f^{(n)}(X_2 + B_2(t)))\mathrm{I}(\tau \le t)) + e(n,t) \qquad (49)
\end{aligned}$$

where $\tau = \inf(t : X_1 + B_1(t) = X_2 + B_2(t))$ and

$$|e(n,t)| \le E(|1 - e^{-nL_t^0(X_1+B_1-X_2-B_2)/2} - \mathrm{I}(\tau \le t)|) \to 0 \quad \text{as } n \to \infty.$$

Let $f_0(x) = \mathrm{I}(x \le 0)$. Note that $f^{(n)} \to f_0$ as $n \to \infty$.

$$\begin{aligned}
&E_{\underline{0}}(f_0(X_1 + B_1(t))(1 - f_0(X_2 + B_2(t)))\mathrm{I}(\tau \le t)) \\
&= P_{\underline{0}}(X_1 + B_1(t) \le 0, X_2 + B_2(t) \ge 0, \tau \le t) \\
&= 2P_{\underline{0}}(X_1 + B_1(t) \le 0, X_2 + B_2(t) \ge 0, X_2 \le X_1) \quad \text{(reflection principle)} \\
&= 2P_{\underline{0}}(B_2(t) \le X_2 \le X_1 \le B_1(t)) \\
&= 2E_{\underline{0}}\Big(\int_{B_2(t)}^{B_1(t)} \int_{\bar z}^{B_1(t)} \psi(z)\psi(\bar z)dz\,d\bar z\ \mathrm{I}(B_1(t) \ge B_2(t))\Big) \\
&= E_{\underline{0}}\Big(\Big(\int_{B_2(t)}^{B_1(t)} \psi(z)dz\Big)^2 \mathrm{I}(B_1(t) \ge B_2(t))\Big) \\
&= (1/2)E_{\underline{0}}\Big(\Big(\int_{B_2(t)}^{B_1(t)} \phi_t(z)dz\Big)^2\Big).
\end{aligned}$$

This is a smooth function of $(t, B_1(t), B_2(t))$ and may be expanded by Ito's formula. This leads, after some calculation, to

$$\begin{aligned}
&= E_{\underline{0}}\Big(\int_0^t (1/2)\phi_s^2(B_1(s)) + (1/2)\phi_s^2(B_2(s))ds\Big) \\
&= \int_0^t \int p_s(x)\phi_s^2(x)dx\,ds \\
&= 2\int_0^t \int P_s f_0(x)\phi_s(x)\phi_s'(x)dx\,ds.
\end{aligned}$$

By replacing $\psi(y)$ by $\psi(y + a)$ we see that the same equality still holds if f_0 is replaced by $f_a(x) = \mathrm{I}(x \le a)$. For f as in the lemma but also smooth we have

$$\begin{aligned}
&2\int_0^t \int P_s f(x)\phi_s(x)\phi_s'(x)dx\,ds \\
&= -2\int_{-\infty}^{\infty} f'(a) \int_0^t \int P_s f_a(x)\phi_s(x)\phi_s'(x)dx\,ds\,da
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{\infty} f'(a) E_{\underline{0}}(f_a(X_1+B_1(t))(1-f_a(X_2+B_2(t)))\mathrm{I}(\tau \le t))da \\
&= -E_{\underline{0}}(\int_{-\infty}^{\infty} f'(a)\mathrm{I}(X_1+B_1(t) \le a \le X_2+B_2(t), \tau \le t)da) \\
&= E_{\underline{0}}((f(X_1+B_1(t)) - f(X_2+B_2(t)))\mathrm{I}(\tau \le t, X_2+B_2(t) \ge X_1+B_1(t))).
\end{aligned}
$$

The same equality must then also hold without the smoothness assumption on f. Then

$$
\begin{aligned}
&|E_{\underline{0}}(f(X_1+B_1(t))(1-f(X_2+B_2(t)))\mathrm{I}(\tau \le t)) \\
&\qquad -2\int_0^t \int P_s f(x)\phi_s(x)\phi_s'(x)dx\,ds| \\
&\le E_{\underline{0}}(f(X_2+B_2(t))(1-f(X_1+B_1(t)))\mathrm{I}(\tau \le t, X_2+B_2(t) \ge X_1+B_1(t))) \\
&\quad +E_{\underline{0}}(f(X_1+B_1(t))(1-f(X_2+B_2(t))\mathrm{I}(\tau \le t, X_2+B_2(t) \le X_1+B_1(t))) \\
&\le 2E_{\underline{0}}(f(X_1+B_1(t))(1-f(X_2+B_2(t)))\mathrm{I}(X_2+B_2(t) \le X_1+B_1(t))) \\
&\le 2\int\int P_t\psi(x)P_t\psi(y)f(x)(1-f(y))\mathrm{I}(y \le x)dy\,dx \qquad (50)
\end{aligned}
$$

Combining (49) and (50) completes the proof of lemma 6.4. •

We can now finish the proof of lemma 6.3. Let $\mathcal{F}_t = \sigma(u_s : s \le t)$.

$$
\begin{aligned}
&E((\int_0^t \int \phi_s^2(x) n u_{n^2 s}(nx)(1-u_{n^2 s}(nx))dx\,ds \\
&\qquad -2\phi_s(x)\phi_s'(x)u_{n^2 s}(nx)dx\,ds)^2) \\
&= 2E\int_0^t \left(\int n\phi_s^2(x)u_{n^2 s}(nx)(1-u_{n^2 s}(nx)) - 2\phi_s(x)\phi_s'(x)u_{n^2 s}(nx)dx\right) \\
&\quad \left(\int_s^t \int n\phi_r^2(y)u_{n^2 r}(ny)(1-u_{n^2 r}(ny)) - 2\phi_r(y)\phi_r'(y)u_{n^2 r}(ny)dy\,dr\right) ds \\
&\le 2E\int_0^t Z_s E((\int_s^t \int n\phi_r^2(y)u_{n^2 r}(ny)(1-u_{n^2 r}(ny)) \\
&\qquad -2\phi_r(y)\phi_r'(y)u_{n^2 r}(ny)dy\,dr|\mathcal{F}_{n^2 s})ds \qquad (51)
\end{aligned}
$$

where $0 \le Z_s \le C(\phi)(1+\int u_{n^2 s}(x)(1-u_{n^2 s}(x))dx)$ is square integrable by lemma 3.2. We use the Markov property and lemma 6.4 to bound the conditional expectation in (51) by (noting that the compact support property implies that the hypotheses of this lemma are satisfied)

$$
2\int\int P_{t-s}\psi(x)P_{t-s}\psi(y)u_{n^2 s}(nx)(1-u_{n^2 s}(ny))\mathrm{I}(y \ge x)dx\,dy + e(n,t-s). \qquad (52)
$$

Note that both terms in (52) are bounded. Substituting the bound (52) into (51) will produce a term that vanishes as $n \to \infty$ provided that the terms in

(52) converge to zero in probability as $n \to \infty$. This is immediate for $e(n, t-s)$ by lemma 6.4. For the first term in (52) we have

$$
\begin{aligned}
& E\Big(\int\int P_{t-s}\psi(x)P_{t-s}\psi(y)u_{n^2 s}(nx)(1-u_{n^2 s}(ny))\mathrm{I}(y \le x)dx\,dy\,ds\Big) \\
= & \int\int P_{t-s}\psi(x)P_{t-s}\psi(y)\mathrm{I}(y \le x) \\
& E_{\underline{0}}\big(f(nx + B_1(n^2 s))(1 - f(ny + B_2(n^2 s)))e^{-L^{n(y-x)}_{n^2 s}(B_1 - B_2)/2}\big)dx\,dy \\
= & \int\int P_{t-s}\psi(x)P_{t-s}\psi(y)\mathrm{I}(y \le x) \\
& E_{\underline{0}}\big(f^{(n)}(x + B_1(s))(1 - f^{(n)}(y + B_2(s)))e^{-nL^{y-x}_s(B_1 - B_2)/2}\big)dx\,dy \\
\le & \int\int P_{t-s}\psi(x)P_{t-s}\psi(y)\mathrm{I}(y \le x) \\
& E\big(\mathrm{I}(x + B_1(s) \le 0, y + B_2(s) \ge 0)e^{-nL^{y-x}_s(B_1 - B_2)/2}\big)dx\,dy \\
+ & \int\int P_{t-s}\psi(x)P_{t-s}\psi(y)\mathrm{I}(y \le x) \\
& \big(P(|x + B_1(s)| \in [0, |R(f)|/n]) + P(|y + B_2(s)| \in [0, |L(f)|/n])\big)dx\,dy.
\end{aligned}
$$

Note that $e^{-nL^{y-x}_s(B_1-B_2)/2} \to 0$ on the set $\{y \le x, x + B_1(s) \ge 0, y + B_2(s) \le 0\}$ and both integrals on the right hand side converge to zero by the dominated convergence theorem. This finishes the proof of lemma 6.3. •

7 Multiple wavefronts

In this section we fix f_i, $i = 1, \ldots, 2l$ with $-\infty < L(f_i) < R(f_i) < \infty$ and also $a_1, \ldots, a_{2l} \in \mathrm{R}$. We consider the initial conditions

$$f^n(x) = \sum_{i=1}^{2l} (-1)^i f_i(x - na_i).$$

Theorem 7.1 *Let u^n be a solution to (1) started at f^n. Set $v_t^n(x) = u^n_{n^2 t}(nx)$. Then $v^n_\cdot(x)dx$ converge in distribution as $n \to \infty$ as continuous M valued processes. The limit has the law of*

$$v_t^\infty(x)dx = \sum_{i=1}^{2l} (-1)^i \mathrm{I}(x \leq X_i(t))dx \tag{53}$$

where $(X_1, \ldots, X_{2l})$ is a system of annihilating Brownian motions started at $(a_1, \ldots, a_{2l})$ and where we have the convention that $X_i(t) = -\infty$ for values of t after the annihilation of particle i.

We have been unable to follow the method of proof used in Theorem 6.2 since we cannot find a suitably simple martingale problem for the system of annihilating Brownian motions. Instead we use the following argument: the $2l$ wavefronts move independently until they begin to overlap. For large n the positions of the wavefronts move approximately as Brownian motions so with large probabilty the first collision will be between exactly two wavefronts and the others will still be seperated by a distance $O(n)$. By lemma 5.2 the colliding wavefronts will have total width $O(n^{1-(1/500)})$. By lemma 7.2 below these two colliding wavefronts will die in time $O(n^{2-(1/250)})$ which is too quick for the other wavefronts to have interfered or for another collision to have occured. After the first annihilation we are left with $2l - 2$ wavefronts and the argument can be repeated.

Lemma 7.2

$$\lim_{\theta \to \infty} \limsup_{m \to \infty} \sup_{f \leq \mathrm{I}(0,m)} P((u_{\theta m^2}, 1) > 0) = 0.$$

Proof. Let $\psi_a(x) = ((a - x) \wedge 1)_+$. Take independent solutions u^l, u^r to (1), driven by independent white noises W^l, W^r and with initial conditions $u_0^l = \psi_0, u_0^r = \psi_1$. Define for $m \geq 1$

$$T^m = \inf(t \geq 0 : R_t(u^l) \geq L_t(u^r) + m).$$

Using the method of proposition 2.2 we may construct a further solution $\bar{u}^m$ to (1) with respect to another independent white noise $\bar{W}$ but with initial condition $\bar{u}_0^m(x) = u^r_{T^m}(x - m) - u^l_{T^m}(x)$. Define

$$u_t^m(x) = (u_t^r(x - m) - u_t^l(x))\mathrm{I}(t \leq T^m) + \bar{u}^m_{t-T^m}(x)\mathrm{I}(t > T^m).$$

Note that $u_0^m(x) = \psi_1(x-m) - \psi_0(x) \geq \mathrm{I}(x \in (0,m))$. Then for nice ϕ

$$\begin{aligned}
&(u_t^m, \phi) - (u_0^m, \phi) \\
= &\int_0^{t\wedge T^m} (u_s^r(\cdot - m) - u_s^l, (1/2)\Delta\phi)ds + \int_{t\wedge T^m}^t (\bar{u}_{s-T^m}^m, (1/2)\Delta\phi)ds \\
&+ \int_0^{t\wedge T^m} \int |u_s^r(x-m)(1-u_s^r(x-m))|^{1/2}\phi(x)dW_{s,x-m}^r \\
&- \int_0^{t\wedge T^m} \int |u_s^l(x)(1-u_s^l(x))|^{1/2}\phi(x)dW_{s,x}^l \\
&+ \int_{t\wedge T^m}^t \int |\bar{u}_s^m(x)(1-\bar{u}_s^m(x))|^{1/2}\phi(x)dW_{s,x}^m \\
= &\int_0^t (u_s^m, (1/2)\Delta\phi)ds + \int_0^t \int |u_t^m(x)(1-u_t^m(x))|^{1/2}\phi(x)dW_{s,x}
\end{aligned}$$

provided we define a martingale measure W by (for bounded measurable ψ)

$$\begin{aligned}
W_t(\psi) \;=\; &\int_0^{t\wedge T^m} \int \psi(x)\mathrm{I}(u^r(x-m) < 1)dW_{s,x-m}^r \\
&- \int_0^{t\wedge T^m} \int \psi(x)\mathrm{I}(u^r(x-m) = 1)dW_{s,x}^l \\
&+ \int_{t\wedge T^m}^t \int \psi(x)dW_{s,x}^m.
\end{aligned}$$

To check that u^m is a solution to (1) we need to check that W is a white noise. But this is true if $[W(\psi)]_t = \int_0^t (\psi^2, 1)ds$ (see [11]) and this follows immediately from the independence of W^r, W^l, W^m.

We sketch the remaining argument in words. Since the two wavefronts move independently and their positions move approximately as Brownian motions, the collision time should occur in time $O(m^2)$. At the collision time the combined width of the two wavefronts should by lemma 5.2 be $O(m^{1-(1/500)})$. The argument can then be repeated a finite number of times until the wavefront is of size $O(m^{1/50})$. Then proposition 5.1 a) ensures that that with high probability it dies in a further time $O(m^2)$.

Let $v_t^{m,l}(x) = 1 - u_{m^2t}^l(mx)$, $v_t^{m,r}(x) = u_{m^2t}^r(m(x-1))$. Then from theorem 6.2 for $\phi \in C_c$

$$((v_\cdot^{m,l}, \phi), (v_\cdot^{m,r}, \phi)) \xrightarrow{\mathcal{D}} \left(\int_{B_1(\cdot)}^\infty \phi(x)dx, \int_{-\infty}^{B_2(\cdot)} \phi(x)dx \right) \tag{54}$$

where (B_1, B_2) is a two-dimensional Brownian motion started at $(0,1)$. Using the fact that $|v_t^{m,l}(x)| \vee |v_t^{m,r}(x)| \leq 1$ we may extend (54) to continuous integrable ϕ. Fix continuous $\phi_0 > 0$ with $\int \phi_0 = 1$. Then

$$
\begin{aligned}
& \limsup_{m\to\infty} P(T^m > \theta m^2) \\
\leq\ & \limsup_{m\to\infty} P((v_t^{m,l},\phi_0) + (v_t^{m,r},\phi_0) \geq 1, \forall t \leq \theta) \\
\leq\ & P_{0,1}\Big(\int_{B_1(t)}^{\infty} \phi_0(x)dx + \int_{-\infty}^{B_2(t)} \phi_0(x)dx \geq 1, \forall t \leq \theta\Big) \\
=\ & P_{0,1}(B_2(t) \geq B_1(t), \forall t \leq \theta) \\
=\ & \int_{-(2\theta)^{-1/2}}^{(2\theta)^{-1/2}} p_1(x)dx \\
\leq\ & (\pi\theta)^{-1/2}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
& P(T^m \leq \theta m^2, R_{T^m}(u^r(\cdot - m)) - L_{T^m}(u^l) \geq 2(\theta m^2)^{(1/2)-(1/1000)}) \\
\leq\ & P(R_t(u^l) - L_t(u^l) \geq (\theta m^2)^{(1/2)-(1/1000)}, \exists t \leq \theta m^2) \\
& + P(R_t(u^r) - L_t(u^r) \geq (\theta m^2)^{(1/2)-(1/1000)}, \exists t \leq \theta m^2) \\
\leq\ & C(\theta m^2)^{-1/2000}
\end{aligned}
$$

by lemma 5.2. So using the comparison of solutions in proposition 2.2 we have

$$
\begin{aligned}
& \limsup_{m\to\infty} \sup_{f: f \leq \mathrm{I}(0,m)} Q^f(R_t(w) - L_t(1-w) \geq 2(\theta m^2)^{(1/2)-(1/1000)}, \forall t \leq \theta m^2) \\
& \leq (\pi\theta)^{-1/2}.
\end{aligned} \tag{55}
$$

Now define $m_0 = m, t_0 = 0, t_n = \theta m_{n-1}^2, m_n = 2t_n^{(1/2)-(1/1000)}$. Define $\tau_D = \inf(t : (w_t, 1) = 0)$, $\tau_0 = 0$ and $\tau_n = \inf(t \geq \tau_{n-1} : R_t(w) - L_t(1-w) \leq m_n)$. Fix $f \leq \mathrm{I}(0,m)$. From (55) and the strong Markov property we have for fixed n and sufficiently large m

$$Q^f(\tau_n - \tau_{n-1} \geq t_n | \tau_{n-1} < \infty) \leq 2(\pi\theta)^{-1/2}.$$

So for sufficiently large m

$$Q^f(\tau_{3000} \geq t_1 + \ldots t_{3000}) \leq 6000(\pi\theta)^{-1/2}.$$

We now consider $\theta \geq 1$ so that $m_{3000} \leq (2\theta)^{3000} m^{1/50}$. From lemma 5.1 a) we have

$$Q^f(\tau_D - \tau_{3000} \geq (2\theta)^{300000} m^2 | \tau_{3000} < \infty) \leq C(2\theta)^{-3000} m^{-1/50}.$$

So for all sufficiently large m

$$
\begin{aligned}
& Q^f(\tau_D \geq (3000(2\theta)^{3000} + (2\theta)^{300000})m^2) \\
\leq\ & Q^f(\tau_D \geq t_1 + \ldots t_{3000} + (2\theta)^{300000} m^2) \\
\leq\ & 6001(\pi\theta)^{-1/2}
\end{aligned}
$$

completing the proof. •

Proof of theorem 7.1.

Given $\epsilon_0 > 0, T < \infty, m \geq 1, \phi_1, \ldots \phi_m \in C_c$ with $\|\phi_i\|_\infty \leq 1$ we shall show, for sufficiently large n, there exists a solution u to (1) started at f^n and a process v^∞ arising from annihilating Brownian motions $(X_1, \ldots, X_{2l})$ as in (53) such that

$$\begin{aligned} P(\sup_{t \leq T} |(v_t^n, \phi_j) - (v_t^\infty, \phi_j)| \geq \epsilon_0, \exists j = 1, \ldots, m) &\leq \epsilon_0 & (56) \\ P(|X_i - a_i| \geq \epsilon_0, \exists i = 1, \ldots, 2l) \leq \epsilon_0. & & (57) \end{aligned}$$

This implies the desired convergence in distributions.

Given $\epsilon > 0$, for sufficiently large n we may, by Theorem 6.2, find a probability space with the following variables: independent solutions $(\bar{u}^k : k = 1, \ldots, 2l)$ to (1), started at f_k, driven by independent white noises W^k and Brownian motions $\bar{B}_k$ such that, if we set $\bar{v}_t^{k,n}(x) = \bar{u}_{n^2t}^k(nx)$, then $P(A^c) \leq \epsilon$ where

$$A^c = \{\sup_{t \leq T} |n^{-1} R(\bar{u}_{n^2t}^k) - \bar{B}_t^k| \vee |n^{-1} L(\bar{u}_{n^2t}^k) - \bar{B}_t^k| \geq \epsilon, \ \exists k = 1, \ldots, 2k\}. \quad (58)$$

We fix such an n and solutions $(\bar{u}^k : k = 1, \ldots, 2l)$ (suppressing the dependence on n). Define

$$u_t^k(x) = \bar{u}_t^k(x - na_k), \quad B_t^k = \bar{B}_t^k + na_i \quad \text{for} i = 1, \ldots, 2l.$$

We now construct our solution u to (1) started at f^n by using the processes $(u^k : k = 1, \ldots, 2l)$ as the basic ingredients. We seem to need a fair amount of notation alas. We shall define stopping times $T_1, \ldots, T_l$ that mark the times of successive collisions and subsets $(1, \ldots, 2l) = S_0 \supset S_1 \supset \ldots \supset S_l = \emptyset$ that list the labels of the remaining wavefronts after each collision. S_k will have $2l - 2k$ elements that we list in increasing order as $s(k, 1) < \ldots < s(k, 2l - 2k)$. Define $T_0 = \bar{T}_0 = 0$ and for $k = 1, \ldots, 2l$

$$\begin{aligned} T^{k,i} &= \inf(t \geq T^{k-1} : R_t(u^{s(k-1,i)}) = L_t(u^{s(k-1,i+1)})) \quad \text{for } i \leq |S_{k-1}| - 1 \\ T^k &= \inf(T^{k,i} : i = 1, \ldots, 2l - 2k + 1) \\ J^k &= \inf(i : T^k = T^{k,i}) \\ a(k) &= s(k-1, J^k) \\ b(k) &= s(k-1, J^k + 1) \\ S_k &= S_{k-1} - \{a(k), b(k)\}. \end{aligned}$$

When a collision occurs we shall use a new independent solution of (1) to follow the annihilation of the two colliding wavefronts. For $k = 1, \ldots, l$ let $\bar{w}^k$ be solutions to (1) driven by an independent white noises $\bar{W}^k$ and with initial conditions

$$\bar{w}_0^k = u_{T^k}^{b(k)} - u_{T^k}^{a(k)}.$$

Let $w^k_{t-T^k} = \bar{w}^k_t$ for $t \geq T^k$ and let $w^0 \equiv 0$. Fixing $\epsilon \in (0,1]$ we plan that the colliding pairs of wave fronts die out in time $n^2\epsilon$. So we define a process $\bar{u}_t$ taking the values

$$\begin{cases} \sum_{j=1}^{2l-2k}(-1)^{s(k,j)}u_t^{s(k,j)} + w_t^k & \text{on } [T^k, (T^k+n^2\epsilon)\wedge T^{k+1}) \text{ for } k \leq l \\ \sum_{j=1}^{2l-2k}(-1)^{s(k,j)}u_t^{s(k,j)} & \text{on } [(T^k+n^2\epsilon)\wedge T^{k+1}, T^{k+1}),\ k \leq l-1 \\ 0 & \text{on } [T^l+n^2\epsilon, \infty). \end{cases}$$

$\bar{u}$ will be the desired solution on the set where nothing goes wrong. We shall modify the definition whenever the annihilating pairs of wavefronts live too long, collide with another wavefront or when another collision occurs during their annihilation. Define

$$\begin{aligned} S_1^k &= \inf(t \geq T^k : R(w_t^k) \geq L(u_t^{s(k-1,J^k+2)}) \text{ or } L(1-w_t^k) \leq R(u_t^{s(k-1,J^k-1)})) \\ S_2^k &= \inf(t : (w_t^k, 1) = 0) \\ S &= \inf(S_1^k \wedge T^{k+1} : S_1^k \wedge T^{k+1} < T^k + n^2\epsilon) \\ &\qquad \wedge \inf(T^k + n^2\epsilon : T^k + n^2\epsilon < S_2^k). \end{aligned}$$

(We need to define $R(u_t^{s(k-1,-1)}) = -\infty, L_t(u_t^{s(k-1,2l-2k+3)}) = \infty$ to ensure S_1^k is well defined).

The stopping time S is the first time something goes wrong. Let $\tilde{u}$ be a solution to (1) driven by yet another independent white noise $\tilde{W}$ and with starting condition $\tilde{u}_0 = \bar{u}_S$. Then we define

$$u_t = \bar{u}_t \mathrm{I}(t < S) + \tilde{u}_{t-S}\mathrm{I}(t \geq S).$$

It is possible to check (using the same argument as in lemma 7.2) that u is a solution to (1) started at f^n.

Let $(X_t^k : k = 1, \ldots, 2l)$ be the system of annihilating Brownian motions induced by $(B^k : k = 1, \ldots, 2l)$ and let v^∞ be the induced measure valued process by the recipe (53). Also let $v_t^n(x) = u_{n^2t}(nx)$. We shall now check that (56) is satisfied which will finish the proof.

We define various good sets:

$$\begin{aligned} B_1 &= \{R(\bar{u}_t^j) - L(\bar{u}_t^j) \leq 14(n^2T)^{(1/2)-(1/1000)},\ \forall t \leq n^2T, j = 1, \ldots, 2l\} \\ B_2 &= \{(w_{T^k+n^2\epsilon}, 1) = 0,\ \forall k = 1, \ldots, l\} \\ B_3 &= \{\sup_{t\in[T^k, T^k+n^2\epsilon]} (R(w_t^k) - R(w_{T^k}^k)) \vee (L(1-w_{T^k}^k) - L(1-w_t^k)) \\ &\qquad \leq n\epsilon^{1/3},\ \forall k = 1, \ldots, l\} \\ B &= B_1 \cup B_2 \cup B_3 \\ S^{i,j} &= \inf(t \geq 0 : B_t^j - B_t^i \leq 2\epsilon) \\ \bar{S}^{i,j} &= \inf(t \geq 0 : B_t^j - B_t^i \leq -2\epsilon) \end{aligned}$$

$$
\begin{aligned}
C_1 &= \{[S^{i,j}, \bar{S}^{i,j} + n^2\epsilon] \text{ are disjoint intervals for } i < j\} \\
C_2 &= \{|B^k_t - B^i_s| \vee |B^k_t - B^j_s| > \epsilon^{1/3} + 2\epsilon, \\
&\qquad \forall k \neq i, j,\ \forall s, t \in [S^{i,j}, \bar{S}^{i,j} + \epsilon],\ \forall i, j : 1 \leq i < j \leq l\} \\
C_3 &= \{|B^i_t - B^j_t| \leq \epsilon^{1/3}, \forall t \in [S^{i,j}, \bar{S}^{i,j} + \epsilon],\ \forall i, j : 1 \leq i < j \leq l\} \\
C &= C_1 \cup C_2 \cup C_3.
\end{aligned}
$$

To prove (56) it is enough to prove the following two claims.

Claim I: On the set $A \cap B \cap C$ we have $|(v^n_t, \phi) - (v^\infty_t, \phi)| \leq C\epsilon^{1/3}$ for $t \leq T$ and $\phi \in (\phi_1, \ldots, \phi_m)$.

Claim II: $\lim_{\epsilon \to 0} \limsup_{n \to \infty} P(A \cap B \cap C) = 0$.

We shall assume that n is chosen large enough so that

$$
R(u_0^{k-1}) + \epsilon < L(u_0^k) - \epsilon < R(u_0^k) + \epsilon < L(u_0^{k+1}) + \epsilon
$$

for all suitable k. Thus at time zero the wavefronts are seperated and in the correct order and on A the Brownian motions also start in the correct order. Then on A we have $n^2 T^{i,j} \in [S^{i,j}, \bar{S}^{i,j}]$. So on $A \cap C_1$ we may match the sequence of collisions $T^1, \ldots, T^l$ exactly with the sucessive collisions in the annihlating Brownian motions. In particular the k'th annihilation is between the Brownian particles $B^{a(k)}, B^{b(k)}$ and occurs during the interval $[S^{a(k),b(k)}, \bar{S}^{a(k),b(k)}]$.

We now check claim I. On $A \cap C_1$ we have $T^{k+1} \geq n^2 \bar{S}^{a(k),b(k)} + n^2\epsilon \geq T^k + n^2\epsilon$. On $A \cap C_2 \cap B_3$ we have for $t \in [T^k, T^k + n^2\epsilon]$, $t \leq T$

$$
\begin{aligned}
n^{-1} R(w^k_t) &\leq n^{-1} R(w^k_{T^k}) + \epsilon^{1/3} \\
&= n^{-1} R(u^{b(k)}_{T^k}) + \epsilon^{1/3} \\
&\leq B^{b(k)}_{n^{-2}T^k} + \epsilon^{1/3} + \epsilon \\
&< B^{s(k-1,J^k+2)}_{n^{-2}t} - \epsilon \\
&\leq n^{-1} L(u_t^{s(k-1,J^k+2)})
\end{aligned}
$$

and similarly $L(1 - w^k_t) > R(u_t^{s(k-1,J^k-1)})$. On B_2 we have $\inf(T^k + n^2\epsilon : T^k + n^2\epsilon < S^k_2) = \infty$. This paragraph has then checked that on $A \cap B \cap C$ we have $S = \infty$ and the solution u agrees with the process $\bar{u}$ constructed from the independent parts $(u^j : j = 1, \ldots, 2l)$.

We work now on the set $A \cap B \cap C$. Fix ϕ with $||\phi||_\infty \leq 1$. For $t \in [\bar{S}^{a(k),b(k)} + \epsilon, S^{a(k+1),b(k+1)}]$, $k = 0, \ldots, l-1$ we have

$$
\begin{aligned}
(v^n_t, \phi) &= (u_{n^2 t}(n\cdot), \phi) \\
&= (\bar{u}_{n^2 t}(n\cdot), \phi) \\
&= \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (u^{s(k,j)}_{n^2 t}(n\cdot), \phi)
\end{aligned}
$$

$$= \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (\bar{u}^{s(k,j)}_{n^2 t}(n\cdot), \phi(\cdot + na_{s(k,j)})).$$

Also for such t

$$(v^\infty_t, \phi) = \sum_{j=1}^{2l-2k} (-1)^{s(k,j)} (\mathrm{I}(x \le \bar{B}^{s(k,j)}_t), \phi(\cdot + na_{s(k,j)})).$$

So $|(v^\infty_t, \phi) - (v^n_t, \phi)| \le 2l\epsilon$ by (58).

For $t \in [S^{a(k),b(k)}, \bar{S}^{a(k),b(k)} + \epsilon]$, $k = 0, \dots, l$ there are two possible extra errors:

$$\begin{aligned} n^{-1}|(w^k_t, \phi)| &\le n^{-1}(R(w^k_t) - L(1 - w^k_t)) \\ &\le 2\epsilon^{1/3} + n^{-1}(R(w^k_{T^k}) - L(1 - w^k_{T^k})) \\ &= 2\epsilon^{1/3} + n^{-1}(R(u^{b(k)}_{T^k}) - L(u^{a(k)}_{T^k})) \\ &\le 2\epsilon^{1/3} + 2\epsilon + (B^{b(k)}_{n^{-2}T^k} - B^{a(k)}_{n^{-2}T^k}) \\ &\le 3\epsilon^{1/3} + 2\epsilon \end{aligned}$$

(using the definition of C_3) and

$$\left|\int_{X^{a(k)}_t}^{X^{b(k)}_t} \phi(x)dx\right| \le |X^{b(k)}_t - X^{a(k)}_t| \le \epsilon^{1/3}.$$

This proves the first claim.

By lemma 5.2 $P(B_1^c) \le Cl(n^2T)^{-1/2000}$. By lemma 7.2

$$\begin{aligned} &P(B_2^c \cap B_1) \\ \le\ & 2l \sup(Q^f((w_{n^2\epsilon}, 1) > 0) : R(f) - L(1-f) \le 28(n^2T)^{(1/2)-(1/1000)}) \le \epsilon \end{aligned}$$

for sufficiently large n. We have

$$P(B_3^c) \le 2l \sup(Q^f(R(w_t) \ge n\epsilon^{1/3}, \exists t \le n^2\epsilon) : f \le \mathrm{I}(-\infty, 0))$$

which, by proposition 4.2, is bounded by ϵ for sufficiently large n.

Finally $p(\epsilon, y_1, \dots, y_{2l}) := P(C^c | B^1_0 = y_1, \dots, B^{2l}_0 = y_{2l})$ is independent of n and (by elementary properties fo Brownian motions) converges to zero as $\epsilon \to 0$ uniformly on $(y : |y_i - a_i| \le \delta)$ for small δ. But on A we have $|B^i_0 - a_i| \le 2\epsilon + (R(f_i) - L(f_i))/n$ (which also implies (57)). We have shown that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} P(B^c) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} P(A \cap C^c) = 0.$$

Combining this with (58) proves claim II. •

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