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Convergence of the stochastic weighted particle method for the Boltzmann equation

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Abstract

This paper studies convergence of the stochastic weighted particle method for the Boltzmann equation. First the method is extended by introducing new stochastic reduction procedures, in order to control the number of simulation particles. Then, under rather general conditions, convergence to the solution of the Boltzmann equation is proved. Finally, numerical experiments are performed illustrating both convergence and considerable variance reduction, for the specific problem of calculating tails of the velocity distribution.

Contents

1. Introduction

Direct Simulation Monte Carlo (DSMC) is presently the most widely used numerical algorithm in kinetic theory [2]. In this method, a system of simulation particles

$$\Big(x_i(t), v_i(t)\Big), \qquad i = 1, \ldots, n, \qquad t \geq 0,$$

is used to approximate the behaviour of the real gas. Independent motion (free flow) of the particles and their pairwise interactions (collisions) are separated using a splitting procedure with a time increment Δt. During the free flow step, particles are moved according to their velocities,

$$x_i(t + \Delta t) = x_i(t) + \int_t^{t+\Delta t} v_i(s)\, ds, \qquad i = 1, \ldots, n,$$

and boundary conditions are taken into account. During the collision step, particle pairs $(x, v), (y, w)$ are randomly chosen in small cells of the position space, according to the collision probability for the interparticle potential. The post-collision velocities

$$v^* = v^*(v, w, e) = v + e\,(e, w - v), \qquad w^* = w^*(v, w, e) = w - e\,(e, w - v) \tag{1.1}$$

are determined by randomly selecting a direction vector e from the unit sphere $\mathcal{S}^2$ in the Euclidean space $\mathcal{R}^3$. Here $(\,.\,,.\,)$ denotes the scalar product in $\mathcal{R}^3$. The number of collisions is computed from the local collision frequency. The limiting behaviour (as $n \to \infty$) of this algorithm has been studied in [9]. In particular, during the collision step, the particle system approximates the solution of the spatially homogeneous **Boltzmann equation** (cf. [3], [4])

$$\frac{\partial}{\partial t} f(t, v) \;=\; \int_{\mathcal{R}^3} \int_{\mathcal{S}^2} B(v, w, e) \Big[f(t, v^*)\, f(t, w^*) - f(t, v)\, f(t, w)\Big] de\, dw, \tag{1.2}$$

where B is the collision kernel.

A basic problem in many applications of DSMC (e.g., flows with high density gradients, or low Mach number flows) are large statistical fluctuations, so that variance reduction is a challenging task. To this end, a modification of DSMC called **stochastic weighted particle method** (SWPM) was proposed in [6]. Convergence of this method has been studied in [7] (see also [10]). In SWPM a system of weighted particles is used, which allows one to resolve low density regions with a moderate number of simulation particles (cf. [8]). SWPM is based on a partial random weight transfer during collisions. This leads to an increase in the number of particles, so that appropriate reduction procedures are needed to control this quantity. Various deterministic procedures with different conservation properties were proposed in [5], and some error estimates were found. However, so far there was no convergence proof for SWPM with reduction.

The **purpose of this paper** is to fill this gap. The basic idea is the introduction of new stochastic reduction procedures that, on the one hand, do not possess all conservation properties of the deterministic procedures, but, on the other hand, have the correct expectation for a much larger class of functionals. This idea is rather natural in the context of stochastic particle methods. Under very general assumptions on the reduction procedure, we prove convergence to the solution of the Boltzmann equation (1.2). Numerical experiments are performed illustrating both convergence and considerable variance reduction, for the specific problem of calculating tails of the velocity distribution.

The paper is organized as follows. In Section 2.1 a family of Markov jump processes is introduced, and its relationship to the Boltzmann equation is discussed on a heuristic level. In Section 2.2 conditions on the various components of the processes are given, and the convergence theorem is formulated. Section 3 is concerned with the proof of the main result. The proof is based on an auxiliary theorem, where the assumptions concerning the reduction procedure are less restrictive, but also less explicit. Previously known results concerning convergence of SWPM without reduction are obtained as a corollary. In Section 4 we provide several examples of stochastic reduction procedures and show that they satisfy the assumptions of the main theorem. Section 5 contains results of some numerical experiments.

2. Stochastic reduction

2.1. Markov process

We introduce a family of Markov processes $Z^{(n)}$, $n = 1, 2, \dots$, and study its asymptotic behaviour as $n \to \infty$. The parameter n can be considered as the number of particles at time zero or as the inverse of an average particle weight. It will be sometimes omitted in order not to overload the formulas.

The process

$$Z(t) = \Big((g_i(t), v_i(t)) \,, \quad i = 1, \dots, m(t) \Big) , \qquad t \geq 0 \,,$$

is determined by the generator

$$\mathcal{A}\Phi(z) = \int_{\mathcal{Z}} [\Phi(\tilde{z}) - \Phi(z)] \, Q(z, d\tilde{z}) \,, \tag{2.1}$$

where Φ is an appropriate test function,

$$z \in \mathcal{Z} = \Big\{ \Big(m; (g_1, v_1), \dots, (g_m, v_m) \Big) \; : \; m = 0, 1, 2, \dots \,, \; g_i > 0 \,, \; v_i \in \mathcal{R}^3 \Big\} \,, \tag{2.2}$$

and

$$Q(z, d\tilde{z}) = \begin{cases} Q_{\mathrm{coll}}(z; d\tilde{z}) \,, & \text{if} \quad m \leq m_{\max}(n) \,, \\ Q_{\mathrm{red}}(z; d\tilde{z}) \,, & \text{otherwise.} \end{cases} \tag{2.3}$$

In case of **collision**, the transition measure is

$$Q_{\mathrm{coll}}(z; d\tilde{z}) = \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \int_{S^2} \delta_{J_{\mathrm{coll}}(z; i, j, e)}(d\tilde{z}) \, q_{\mathrm{coll}}(z; i, j, e) \, de \,, \tag{2.4}$$

where δ denotes the Dirac measure. The jump transformation is (cf. (1.1))

$$[J_{\mathrm{coll}}(z; i, j, e)]_k = \begin{cases} (v_k, g_k) & , \text{if} \quad k \leq m \,, \; k \neq i, j \,, \\ (v_i, g_i - G(z; i, j, e)) & , \text{if} \quad k = i \,, \\ (v_j, g_j - G(z; i, j, e)) & , \text{if} \quad k = j \,, \\ (v_i^*, G(z; i, j, e)) & , \text{if} \quad k = m + 1 \,, \\ (v_j^*, G(z; i, j, e)) & , \text{if} \quad k = m + 2 \,, \end{cases} \tag{2.5}$$

where

$$G(z;i,j,e) = \frac{1}{1+\kappa(z;i,j,e)}\,\min(g_i,g_j)\,, \qquad \kappa(z;i,j,e) \ge 0\,, \tag{2.6}$$

and (cf. (1.1))

$$v_i^* = v^*(v_i,v_j,e)\,, \qquad v_j^* = w^*(v_i,v_j,e)\,. \tag{2.7}$$

The intensity function is

$$q_{\mathrm{coll}}(z;i,j,e) = (1+\kappa(z;i,j,e))\,\max(g_i,g_j)\,B(v_i,v_j,e) \tag{2.8}$$

so that

$$G(z;i,j,e)\,q_{\mathrm{coll}}(z;i,j,e) = g_i\,g_j\,B(v_i,v_j,e)\,. \tag{2.9}$$

In case of **reduction**, the transition measure is

$$Q_{\mathrm{red}}(z;d\tilde{z}) = \int_{\Theta_{\mathrm{red}}(z)} \delta_{J_{\mathrm{red}}(z;\theta)}(d\tilde{z})\,q_{\mathrm{red}}(z;d\theta)\,, \tag{2.10}$$

where

$$[J_{\mathrm{red}}(z;\theta)]_k = (\tilde{v}_k(z;\theta),\tilde{g}_k(z;\theta))\,, \qquad k=1,\ldots,\tilde{m}(z;\theta)\,, \tag{2.11}$$

and θ belongs to some parameter set $\Theta_{\mathrm{red}}(z)$. Let

$$Q_{\mathrm{red}}(z;d\tilde{z}) = \pi_{\mathrm{red}}(n)\,P_{\mathrm{red}}(z;d\tilde{z})\,, \qquad q_{\mathrm{red}}(z;d\theta) = \pi_{\mathrm{red}}(n)\,p_{\mathrm{red}}(z;d\theta)\,, \tag{2.12}$$

where P_{red} and p_{red} are probability measures. Note that (2.4) can be written in a form analogous to (2.10), with

$$\Theta_{\mathrm{coll}}(z) = \{(i,j,e)\,:\,1 \le i \ne j \le m\,, \quad e \in \mathcal{S}^2\}\,.$$

The starting point for the study of the convergence behaviour is the representation

$$\Phi(Z(t)) = \Phi(Z(0)) + \int_0^t \mathcal{A}(\Phi)(Z(s))\,ds + M(t)\,, \tag{2.13}$$

where $M(t)$ is a martingale satisfying

$$E\,M(t)^2 = E\int_0^t [\mathcal{A}\Phi^2 - 2\,\Phi\,\mathcal{A}\Phi](Z(s))\,ds\,. \tag{2.14}$$

For

$$\Phi(z) = \sum_{i=1}^m g_i\,\varphi(v_i) \tag{2.15}$$

one obtains

$$\Phi(Z(t)) \;=\; \int_{\mathcal{R}^3} \varphi(v)\,\mu(t,dv)\,, \tag{2.16}$$

where

$$\mu(t,dv) \;=\; \sum_{i=1}^{m(t)} g_i(t)\,\delta_{v_i(t)}(dv) \tag{2.17}$$

is called the **empirical measure** of the process. For $k=1,2$, it follows from (2.4), (2.9) that

$$\begin{aligned}
&\int_{\mathcal{Z}} [\Phi(\tilde z)-\Phi(z)]^k\, Q_{\mathrm{coll}}(z,d\tilde z) = \qquad\qquad (2.18)\\
&\quad \frac{1}{2}\sum_{1\le i\ne j\le m}\int_{\mathcal{S}^2} [\Phi(J_{\mathrm{coll}}(z;i,j,e))-\Phi(z)]^k\, q_{\mathrm{coll}}(z;i,j,e)\,de\\
&= \frac{1}{2}\sum_{1\le i\ne j\le m}\int_{\mathcal{S}^2} [\varphi(v_i^*)+\varphi(v_j^*)-\varphi(v_i)-\varphi(v_j)]^k\, G(z;i,j,e)^k\, q_{\mathrm{coll}}(z;i,j,e)\,de\\
&= \frac{1}{2}\sum_{1\le i\ne j\le m} g_i\,g_j\int_{\mathcal{S}^2} [\varphi(v_i^*)+\varphi(v_j^*)-\varphi(v_i)-\varphi(v_j)]^k\, G(z;i,j,e)^{k-1}\, B(v_i,v_j,e)\,de\,,
\end{aligned}$$

and from (2.10), (2.11) that

$$\begin{aligned}
\int_{\mathcal{Z}} [\Phi(\tilde z)-\Phi(z)]^k\, Q_{\mathrm{red}}(z,d\tilde z) \;&=\; \int_{\Theta_{\mathrm{red}}(z)} [\Phi(J_{\mathrm{red}}(z;\theta))-\Phi(z)]^k\, q_{\mathrm{red}}(z;d\theta) \qquad (2.19)\\
&=\; \int_{\Theta_{\mathrm{red}}(z)} \left[\sum_{i=1}^{\tilde m}\tilde g_i\,\varphi(\tilde v_i)-\sum_{i=1}^{m} g_i\,\varphi(v_i)\right]^k q_{\mathrm{red}}(z;d\theta)\,.
\end{aligned}$$

Using the property (cf. (1.1))

$$v^*(v,v,e)=w^*(v,v,e)=v\,, \tag{2.20}$$

we conclude that (cf. (2.1))

$$\mathcal{A}(\Phi)(z)=\frac{1}{2}\sum_{i,j=1}^{m} g_i\,g_j\int_{\mathcal{S}^2}[\varphi(v_i^*)+\varphi(v_j^*)-\varphi(v_i)-\varphi(v_j)]\,B(v_i,v_j,e)\,de+R(\varphi,z)\,,$$

where $R(\varphi,z)=R_1(\varphi,z)-R_2(\varphi,z)$,

$$R_1(\varphi,z)=\chi_{\{m>m_{\max}(n)\}}\int_{\Theta_{\mathrm{red}}(z)}\left[\sum_{i=1}^{\tilde n}\tilde g_i\,\varphi(\tilde v_i)-\sum_{i=1}^{m} g_i\,\varphi(v_i)\right] q_{\mathrm{red}}(z;d\theta) \tag{2.21}$$

and

$$\begin{aligned}
&R_2(\varphi,z)= \qquad\qquad (2.22)\\
&\quad \chi_{\{m>m_{\max}(n)\}}\,\frac{1}{2}\sum_{1\le i\ne j\le m} g_i\,g_j\int_{\mathcal{S}^2}[\varphi(v_i^*)+\varphi(v_j^*)-\varphi(v_i)-\varphi(v_j)]\,B(v_i,v_j,e)\,de\,.
\end{aligned}$$

Consequently, one obtains (cf. (2.17), (2.7))

$$\begin{aligned}
\mathcal{A}(\Phi)(Z(s)) \;=\; &R(\varphi,Z(s))+\\
&\frac{1}{2}\int_{\mathcal{R}^3}\int_{\mathcal{R}^3}\int_{\mathcal{S}^2}[\varphi(v^*)+\varphi(w^*)-\varphi(v)-\varphi(w)]\,B(v,w,e)\,de\,\mu(s,dv)\,\mu(s,dw)\,,
\end{aligned}$$

and (2.13) takes the form (cf. (2.16))

$$\int_{\mathcal{R}^3} \varphi(v)\, \mu(t,dv) = \qquad (2.23)$$
$$\int_{\mathcal{R}^3} \varphi(v)\, \mu(0,dv) + \int_0^t \mathcal{B}(\varphi, \mu(s))\, ds + \int_0^t R(\varphi, Z(s))\, ds + M(\varphi, t)\,,$$

with the notation

$$\mathcal{B}(\varphi,\nu) = \frac{1}{2} \int_{\mathcal{R}^3} \int_{\mathcal{R}^3} \int_{\mathcal{S}^2} [\varphi(v^*) + \varphi(w^*) - \varphi(v) - \varphi(w)]\, B(v,w,e)\, de\, \nu(dv)\, \nu(dw)\,. \qquad (2.24)$$

The expected **limiting equation** (as $n \to \infty$) is therefore

$$\int_{\mathcal{R}^3} \varphi(v)\, \lambda(t,dv) = \int_{\mathcal{R}^3} \varphi(v)\, \lambda(0,dv) + \int_0^t \mathcal{B}(\varphi, \lambda(s))\, ds\,. \qquad (2.25)$$

Note that, under appropriate assumptions on the collision kernel, (2.25) is a weak form of the Boltzmann equation (1.2).

2.2. Convergence theorem

We assume that the initial condition of equation (2.25) satisfies

$$\lambda(0,\mathcal{R}^3) \;<\; \infty\,, \qquad (2.26)$$

$$\int_{\mathcal{R}^3} \|v\|^2\, \lambda(0,dv) \;<\; \infty\,, \qquad (2.27)$$

and that the initial state of the process is such that (cf. (2.17))

$$\mu(0,\mathcal{R}^3) = \sum_{i=1}^{m(0)} g_i(0) \;\le\; C_0\,, \qquad (2.28)$$

$$\limsup_{n\to\infty} E \int_{\mathcal{R}^3} \|v\|^2\, \mu(0,dv) < \infty \qquad (2.29)$$

and

$$\lim_{n\to\infty} E\varrho(\mu(0), \lambda(0)) = 0\,, \qquad (2.30)$$

where

$$\varrho(\nu_1,\nu_2) = \sup_{\|\varphi\|_L \le 1} \left| \int_{\mathcal{R}^3} \varphi(v)\, \nu_1(dv) - \int_{\mathcal{R}^3} \varphi(v)\, \nu_2(dv) \right|, \qquad (2.31)$$

$$\|\varphi\|_L = \max\left\{ \|\varphi\|_\infty\,;\ \sup_{v,w\in\mathcal{R}^3} \frac{|\varphi(v)-\varphi(w)|}{\|v-w\|} \right\}, \qquad \|\varphi\|_\infty = \sup_{v\in\mathcal{R}^3} |\varphi(v)|\,. \qquad (2.32)$$

The assumptions concerning the collision kernel are

$$\int_{\mathcal{S}^2} B(v,w,e)\,de \;\; \le \;\; C_B\,, \tag{2.33}$$

$$\int_{\mathcal{S}^2} |B(v,w,e)-B(v_1,w_1,e)|\,de \;\; \le \;\; C_L\Big[\|v-v_1\|+\|w-w_1\|\Big], \tag{2.34}$$

and the assumption concerning the weight transfer parameter (cf. (2.6)) is

$$\kappa(z;i,j,e) \;\; \le \;\; C_\kappa\,. \tag{2.35}$$

The individual particle weights are assumed to satisfy

$$g_i(t) \;\; \le \;\; g_{\max}(n)\,, \qquad \forall\, t\ge 0\,, \quad i=1,\ldots,m(t)\,, \tag{2.36}$$

where

$$\lim_{n\to\infty} g_{\max}(n) \;\; = \;\; 0\,. \tag{2.37}$$

The mass of the system (cf. (2.17)) is assumed to be uniformly bounded,

$$\mu(t,\mathcal{R}^3)=\sum_{i=1}^{m(t)} g_i(t) \;\; \le \;\; C_\mu\,, \qquad \forall\, t\ge 0\,. \tag{2.38}$$

Furthermore, we assume that the particle number bound indicating reduction satisfies

$$\lim_{n\to\infty} m_{\max}(n) \;\; = \;\; \infty \tag{2.39}$$

and that the parameter of the waiting time before reduction satisfies

$$\lim_{n\to\infty} \pi_{\mathrm{red}}(n) \;\; = \;\; \infty\,. \tag{2.40}$$

Finally, we need some assumptions concerning the reduction procedure. They are formulated using the sets (cf. (2.2))

$$\mathcal{Z}(\varepsilon) \;\; = \;\; \Big\{z\in\mathcal{Z}\,:\, m\le(1-\varepsilon)\,m_{\max}(n)\Big\}, \qquad \varepsilon\in[0,1]\,, \tag{2.41}$$

and

$$\tilde{\mathcal{Z}}(\varepsilon)=\left\{z\in\mathcal{Z}\,:\,\sum_{i=1}^{m} g_i\le C_0\,(1+\varepsilon)\right\}, \qquad \varepsilon\ge 0\,. \tag{2.42}$$

Two assumptions are related to **conservation properties** (for bounded continuous φ and $\varphi(v)=\|v\|^2$), namely

$$\int_{\mathcal{Z}} \Phi(\tilde z)\,P_{\mathrm{red}}(z;d\tilde z) \;\; = \;\; \Phi(z)\,, \qquad \forall z\in\mathcal{Z}\setminus\mathcal{Z}(0)\,, \tag{2.43}$$

and

$$\limsup_{n\to\infty} \pi_{\mathrm{red}}(n) \sup_{z\in\tilde{\mathcal{Z}}(\varepsilon)\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} [\Phi(\tilde{z})-\Phi(z)]^2\, P_{\mathrm{red}}(z;d\tilde{z}) \;<\; \infty\,, \qquad \forall\,\varepsilon\geq 0\,, \tag{2.44}$$

where Φ is defined in (2.15). Two other assumptions are related to the **reduction property** itself - there exists some $\delta>0$ such that

$$\sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z;d\tilde{z}) \leq (1-\delta)\, m_{\max}(n) \tag{2.45}$$

and

$$\lim_{n\to\infty} \frac{1}{m_{\max}(n)^2} \sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z;d\tilde{z})\right]^2 P_{\mathrm{red}}(z;dz_1) = 0\,, \tag{2.46}$$

where (cf. (2.2))

$$\Phi_0(z) = m\,, \qquad z\in\mathcal{Z}\,. \tag{2.47}$$

Theorem 2.1 *Under the above assumptions,*

$$\lim_{n\to\infty} E \sup_{t\in[0,T]} \varrho(\mu(t),\lambda(t)) = 0\,, \qquad \forall\, T>0\,. \tag{2.48}$$

Remark 2.2 *In case of mass conservation*

$$\mu(t,\mathcal{R}^3) \;=\; \mu(0,\mathcal{R}^3)\,, \qquad \forall\, t\geq 0\,, \tag{2.49}$$

condition (2.38) is fulfilled with $C_\mu = C_0$, *according to (2.28). Otherwise, condition*

$$\limsup_{n\to\infty} g_{\max}(n)\, m_{\max}(n) \;<\; \infty \tag{2.50}$$

implies (2.38).

Remark 2.3 *Assumption (2.43) represents the fact that the reduced system has the correct expectation. Note that (2.43) is not fulfilled (for sufficiently general test functions) in the case of deterministic reduction.*

Remark 2.4 *Assumption (2.43) implies* **mass conservation on average** *so that*

$$\sum_{i=1}^m g_i = \int_{\mathcal{Z}} \left(\sum_{i=1}^{\tilde{m}} \tilde{g}_i\right) P_{\mathrm{red}}(z;d\tilde{z}) \;\leq\; g_{\max}(n) \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z;d\tilde{z})$$

and

$$\int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z;d\tilde{z}) \;\geq\; \frac{1}{g_{\max}(n)} \sum_{i=1}^m g_i\,.$$

Thus, for z such that

$$\sum_{i=1}^{m} g_i \;=\; (1-\varepsilon)\, m_{\max}(n)\, g_{\max}(n)\,, \tag{2.51}$$

one obtains

$$\int_{\mathcal{Z}} \Phi_0(\tilde z)\, P_{\mathrm{red}}(z; d\tilde z) \;\geq\; (1-\varepsilon)\, m_{\max}(n)\,. \tag{2.52}$$

Consider $z \in \tilde{\mathcal{Z}}(\delta) \setminus \mathcal{Z}(0)$ such that $\sum_{i=1}^{m} g_i = C_0\,(1+\delta)$ and choose ε so that (2.51) is fulfilled. Then (2.52) and (2.45) imply

$$\frac{C_0\,(1+\delta)}{g_{\max}(n)} \;\leq\; (1-\delta)\, m_{\max}(n)\,.$$

Thus, a **necessary condition** *for the consistency of the assumptions is*

$$C_0 \;<\; \inf_n\, m_{\max}(n)\, g_{\max}(n)\,. \tag{2.53}$$

Remark 2.5 *It is well-known (cf. [1]) that, under the assumptions (2.33), (2.34), there exists a unique solution in $L^1(\mathcal{R}^3)$ of equation (1.2) satisfying*

$$\int_{\mathcal{R}^3} f(t,v)\, dv \;=\; \int_{\mathcal{R}^3} f(0,v)\, dv\,, \qquad t \geq 0\,,$$

and

$$\int_{\mathcal{R}^3} \|v\|^2 f(t,v)\, dv \;=\; \int_{\mathcal{R}^3} \|v\|^2 f(0,v)\, dv\,, \qquad t \geq 0\,.$$

The corresponding measure-valued function $\lambda(t,dv) = f(t,v)\,dv$ solves the weak equation (2.25). Existence for that equation can also be established using Lemma 3.6 below and a fixed point argument.

Remark 2.6 *The results hold for*

$$v^*(v,w,e) = \frac{v+w}{2} + e\,\frac{\|v-w\|}{2}\,, \qquad w^*(v,w,e) = \frac{v+w}{2} - e\,\frac{\|v-w\|}{2}$$

instead of (1.1) (cf. (2.20) and (3.10) below).

3. Proof of the main result

We start with a theorem, where the assumptions concerning the reduction procedure are less restrictive, but also less explicit.

Theorem 3.1 *Assume (2.33), (2.34), (2.26), (2.36), (2.37), (2.38), (2.43), (2.44), (2.30), and*

$$\lim_{r\to\infty} \sup_{t\in[0,T]} \lambda(t,\{\|v\|\geq r\}) = 0\,, \tag{3.1}$$

$$\lim_{r\to\infty} \limsup_{n\to\infty} E \sup_{t\in[0,T]} \mu(t,\{\|v\|\geq r\}) = 0\,, \tag{3.2}$$

$$\lim_{n\to\infty} E \int_0^T \chi_{\{m(s)>m_{\max}(n)\}}\, ds = 0\,. \tag{3.3}$$

Then (2.48) holds.

Corollary 3.2 *Suppose $m_{\max}(n)=\infty$ so that no reduction procedure is needed. Assumptions (3.3), (2.43) and (2.44) are obviously fulfilled. Moreover, since*

$$\max_i g_i(t) \;\leq\; \max_i g_i(0)\,,$$

(2.49) and

$$\int_{\mathcal{R}^3} \|v\|^2\,\mu(t,dv) \;=\; \int_{\mathcal{R}^3} \|v\|^2\,\mu(0,dv)\,,$$

assumptions (2.37), (2.38), (3.2) are fulfilled under appropriate assumptions concerning the initial state. Thus, the conclusion of Theorem 3.1 holds. This is the previous result of [10], [7].

Remark 3.3 *Condition $C_\kappa<\infty$ (cf. (2.35)) is not explicitly used in the proof of Theorem 3.1. However, it is needed to avoid explosion in the case $m_{\max}(n)=\infty$. It was used in the old papers. Letting $C_\kappa\to\infty$ with $m\to\infty$, explosion can be reached. Condition*

$$m_{\max}(n)<\infty \tag{3.4}$$

is used in (3.48), in the proof of Theorem 2.1.

The proof of Theorem 3.1 follows [10]. For any $r>0$, we consider the function χ_r on $\mathcal{R}^3$,

$$\chi_r(v) = \begin{cases} 1 &, \text{ if } \|v\|\leq r\,,\\ r+1-\|v\|\,, & \text{ if } \|v\|\in[r,r+1]\,,\\ 0 &, \text{ if } \|v\|\geq r+1\,,\end{cases}$$

and denote

$$\varphi_r(v) = \varphi(v)\,\chi_r(v)\,,\quad v\in\mathcal{R}^3\,.$$

Note that

$$\|\varphi_r\|_L \leq 2\,\|\varphi\|_L\,, \tag{3.5}$$

We prepare the proof by several lemmas.

Lemma 3.4 *Assume (2.36), (2.37), (2.38), (2.33), (2.43), (2.44) and (3.3). Then (cf. (2.23))*

$$\limsup_{n\to\infty} E \sup_{t\in[0,T]} \sup_{\|\varphi\|_L\le 1} |M(\varphi_r,t)| = 0\,.$$

Proof. The set $D_r := \{\varphi_r \,:\, \|\varphi\|_L \le 1\}$ is compact in $\mathcal{C}(\{\|v\|\le r+1\})$. Consequently, for any $\varepsilon>0$, there exists a finite subset $\{\psi_i\,;\ i=1,\ldots,I(\varepsilon)\}$ of D_r such that

$$\min_i \|\psi-\psi_i\|_\infty \le \varepsilon\,, \qquad \forall\,\psi\in D_r\,.$$

This implies the estimate

$$|M(\varphi_r,t)| \;\le\; \sup_{\|\psi\|_\infty\le\varepsilon} |M(\psi,t)| + \sum_{i=1}^{I(\varepsilon)} |M(\psi_i,t)|\,. \tag{3.6}$$

According to (2.18), (2.19), **(2.33)** and **(2.43)**, it follows that

$$|\mathcal{A}(\Phi)(Z(s))| \;\le\; 2\,\|\varphi\|_\infty\, C_B\, \mu(s,\mathcal{R}^3)^2\,.$$

Using **(2.38)**, we obtain (cf. (2.13), (2.15))

$$\begin{aligned}|M(\varphi,t)| \;&\le\; 2\,\|\varphi\|_\infty \sup_{s\in[0,T]}\mu(s,\mathcal{R}^3)\left[1+C_B\,t\sup_{s\in[0,T]}\mu(s,\mathcal{R}^3)\right]\\ &\le\; 2\,\|\varphi\|_\infty\, C_\mu\,[1+C_B\,t\,C_\mu]\,,\end{aligned}$$

so that (3.6) implies

$$\sup_{t\in[0,T]}\sup_{\|\varphi\|_L\le 1}|M(\varphi_r,t)| \le 2\,\varepsilon\, C_\mu\,[1+C_B\,T\,C_\mu] + \sum_{i=1}^{I(\varepsilon)}\sup_{t\in[0,T]}|M(\psi_i,t)|\,. \tag{3.7}$$

The martingale inequality gives

$$E\sup_{t\in[0,T]}|M(\varphi,t)| \le 2\left(E\,M(\varphi,T)^2\right)^{\frac12}\,. \tag{3.8}$$

Using the elementary identity $a^2-b^2=2(a-b)b+(a-b)^2$, one obtains

$$\mathcal{A}\Phi^2(z) = 2\,\Phi(z)\,\mathcal{A}\Phi(z) + \int_{\mathcal{Z}}[\Phi(\tilde z)-\Phi(z)]^2\,Q(z,d\tilde z)\,,$$

so that, according to (2.18), (2.19),

$$\begin{aligned}\mathcal{A}\Phi^2(z)-2\,\Phi(z)\,\mathcal{A}\Phi(z) &= \chi_{\{m(z)\le m_{\max}(n)\}}\times\\ &\frac12\sum_{1\le i\ne j\le m} g_i\,g_j\int_{\mathcal{S}^2}[\varphi(v_i^*)+\varphi(v_j^*)-\varphi(v_i)-\varphi(v_j)]^2\,G(z;i,j,e)\,B(v_i,v_j,e)\,de\\ &+\;\chi_{\{m(z)>m_{\max}(n)\}}\int_{\Theta_{\mathrm{red}}(z)}\left[\sum_{i=1}^{\tilde n}\tilde g_i\,\varphi(\tilde v_i)-\sum_{i=1}^{m} g_i\,\varphi(v_i)\right]^2 q_{\mathrm{red}}(z;d\theta)\,.\end{aligned}$$

Using (2.6) and **(2.33)**, we conclude that (cf. (2.10), (2.11))

$$\mathcal{A}\Phi^2(z) - 2\,\Phi(z)\,\mathcal{A}\Phi(z) \le$$
$$8\,\|\varphi\|_\infty^2\,C_B \sum_{1\le i\ne j\le m} g_i\,g_j\,\min(g_i,g_j) + \chi_{\{m(z)>m_{\max}(n)\}} \int_{\mathcal{Z}} [\Phi(\tilde z)-\Phi(z)]^2\,Q_{\rm red}(z;d\tilde z)\,.$$

Now (2.14), **(2.36)** and **(2.38)** imply

$$E\,M(\varphi,T)^2 \le 8\,\|\varphi\|_\infty^2\,C_B\,T\,g_{\max}(n)\,C_\mu^2 + \left[\sup_{z\in\tilde{\mathcal{Z}}(\varepsilon')\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} [\Phi(\tilde z)-\Phi(z)]^2\,Q_{\rm red}(z;d\tilde z)\right] E\int_0^T \chi_{\{m(s)>m_{\max}(n)\}}\,ds\,, \tag{3.9}$$

where ε' is such that $C_\mu \le C_0\,(1+\varepsilon')$. Using (3.7), (3.8), (3.9) and $\sqrt{a^2+b^2}\le|a|+|b|$, we conclude that

$$E \sup_{t\in[0,T]}\sup_{\|\varphi\|_L\le 1} |M(\varphi_r,t)| \le 2\,\varepsilon\,C_\mu\,[1+C_B\,T\,C_\mu] + 2\,I(\varepsilon)\,C_\mu\,\sqrt{8\,C_B\,T\,g_{\max}(n)}$$
$$+2\left(E\int_0^T \chi_{\{m(s)>m_{\max}(n)\}}\,ds\right)^{\frac12} \sum_{i=1}^{I(\varepsilon)} \left[\sup_{z\in\tilde{\mathcal{Z}}(\varepsilon')\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} [\Phi_i(\tilde z)-\Phi_i(z)]^2\,Q_{\rm red}(z;d\tilde z)\right]^{\frac12}.$$

Using **(2.37)**, **(2.44)** (for $\varphi=\psi_i$) and **(3.3)**, we conclude that

$$\limsup_{n\to\infty} E \sup_{t\in[0,T]}\sup_{\|\varphi\|_L\le 1} |M(\varphi_r,t)| \le 2\,\varepsilon\,C_\mu\,[1+C_B\,T\,C_\mu]\,, \qquad \forall\,\varepsilon>0\,,$$

and the assertion follows. ∎

Lemma 3.5 *Assume (2.38), (2.33) and (3.3). Then (cf. (2.22))*

$$\lim_{n\to\infty} E\int_0^T \sup_{\|\varphi\|_L\le 1} |R_2(\varphi_r,Z(s))|\,ds = 0\,.$$

Proof. It follows from **(2.33)** that

$$|R_2(\varphi,z)| \le \chi_{\{m(z)>m_{\max}(n)\}}\,2\,\|\varphi\|_\infty\,C_B \sum_{1\le i\ne j\le m} g_i\,g_j\,.$$

Thus, using **(2.38)**, we obtain

$$\int_0^T \sup_{\|\varphi\|_L\le 1} |R_2(\varphi_r,Z(s))|\,ds \le 2\,C_B\,C_\mu^2 \int_0^T \chi_{\{m(s)>m_{\max}(n)\}}\,ds\,,$$

and the assertion follows from **(3.3)**. ∎

Lemma 3.6 *Assume (2.33) and (2.34). Then (cf. (2.24), (2.31), (2.32))*

$$|\mathcal{B}(\varphi,\nu)-\mathcal{B}(\varphi,\nu_1)| \le 2\,\|\varphi\|_L\,(C_B+C_L)\,\varrho(\nu,\nu_1)\,[\nu(\mathcal{R}^3)+\nu_1(\mathcal{R}^3)]\,.$$

Proof. Introduce

$$b(\varphi)(v,w) = \frac{1}{2}\int_{\mathcal{S}^2} [\varphi(v^*(v,w,e)) + \varphi(w^*(v,w,e)) - \varphi(v) - \varphi(w)]\, B(v,w,e)\, de$$

and

$$b_1(\varphi,\nu)(v) = \int_{\mathcal{R}^3} b(\varphi)(v,w)\,\nu(dw)\,, \quad b_2(\varphi,\nu)(v) = \int_{\mathcal{R}^3} b(\varphi)(v,w)\,\nu(dv)\,.$$

According to **(2.33)**, **(2.34)**, and since (cf. (1.1))

$$\begin{aligned} \|v^*(v,w,e) - v^*(v_1,w_1,e)\| &\leq 2\,\|v-v_1\| + \|w-w_1\|\,, \\ \|w^*(v,w,e) - w^*(v_1,w_1,e)\| &\leq \|v-v_1\| + 2\,\|w-w_1\|\,, \end{aligned}$$

one obtains

$$\begin{aligned} &|b(\varphi)(v,w) - b(\varphi)(v_1,w_1)| \leq \\ &\qquad 2\,\|\varphi\|_\infty \int_{\mathcal{S}^2} |B(v,w,e) - B(v_1,w_1,e)|\, de + 2\,\|\varphi\|_L \Big[\|v-v_1\| + \|w-w_1\|\Big] C_B \\ &\leq\ 2\,\|\varphi\|_L\,(C_L + C_B)\Big[\|v-v_1\| + \|w-w_1\|\Big] \end{aligned} \tag{3.10}$$

and

$$|b_i(\varphi,\nu)(v) - b_i(\varphi,\nu)(v_1)| \leq 2\,\|\varphi\|_L\,(C_B + C_L)\,\nu(\mathcal{R}^3)\,\|v-v_1\|\,, \qquad i=1,2\,. \tag{3.11}$$

It follows from (3.11) and

$$|b_i(\varphi,\nu)(v)| \leq 2\,\|\varphi\|_\infty\, C_B\,\nu(\mathcal{R}^3)\,, \qquad i=1,2\,,$$

that

$$\|b_i(\varphi,\nu)\|_L \leq 2\,\|\varphi\|_L\,(C_B + C_L)\,\nu(\mathcal{R}^3)\,, \qquad i=1,2\,. \tag{3.12}$$

Finally, using (3.12) and the notation

$$\langle \varphi, \nu \rangle = \int_{\mathcal{R}^3} \varphi(v)\,\nu(dv)\,, \tag{3.13}$$

one obtains

$$\begin{aligned} |\mathcal{B}(\varphi,\nu) - \mathcal{B}(\varphi,\nu_1)| &\leq |\langle b_2(\varphi,\nu),\nu\rangle - \langle b_2(\varphi,\nu),\nu_1\rangle| + |\langle b_1(\varphi,\nu_1),\nu\rangle - \langle b_1(\varphi,\nu_1),\nu_1\rangle| \\ &\leq \Big[\|b_2(\varphi,\nu)\|_L + \|b_1(\varphi,\nu_1)\|_L\Big]\,\varrho(\nu,\nu_1) \\ &\leq 2\,\|\varphi\|_L\,(C_B + C_L)\,[\nu(\mathcal{R}^3) + \nu_1(\mathcal{R}^3)]\,\varrho(\nu,\nu_1)\,, \end{aligned}$$

and the assertion follows. ■

Proof of Theorem 3.1. Note that **(2.43)** implies $R_1(\varphi,z) = 0$ (cf. (2.21), (2.10), (2.11), (2.12)). Thus, according to (2.23), (2.25), we obtain (cf. (3.13))

$$\begin{aligned} &|\langle \varphi, \mu(t)\rangle - \langle \varphi, \lambda(t)\rangle| \leq \\ &\qquad |\langle \varphi_r, \mu(t)\rangle - \langle \varphi_r, \lambda(t)\rangle| + |\langle \varphi - \varphi_r, \mu(t)\rangle| + |\langle \varphi - \varphi_r, \lambda(t)\rangle| \\ &\leq\ |\langle \varphi_r, \mu(0)\rangle - \langle \varphi_r, \lambda(0)\rangle| + \int_0^t |\mathcal{B}(\varphi_r, \mu(s)) - \mathcal{B}(\varphi_r, \lambda(s))|\, ds + \\ &\qquad |M(\varphi_r,t)| + \int_0^t |R_2(\varphi_r, Z(s))|\, ds + \|\varphi\|_\infty \Big[\mu(t,\{\|v\| \geq r\}) + \lambda(t,\{\|v\| \geq r\})\Big]. \end{aligned} \tag{3.14}$$

Using (3.5), **(2.26)**, **(2.38)** and **Lemma 3.6**, we conclude from (3.14) that (cf. (2.31))

$$\begin{aligned}
&\varrho(\mu(t),\lambda(t)) \le \\
&\quad 2\,\varrho(\mu(0),\lambda(0)) + 4\,(C_B+C_L)\int_0^t \varrho(\mu(s),\lambda(s))\Big[\mu(s,\mathcal{R}^3)+\lambda(s,\mathcal{R}^3)\Big]\,ds + \\
&\quad \sup_{\|\varphi\|_L\le 1}|M(\varphi_r,t)| + \int_0^t \sup_{\|\varphi\|_L\le 1}|R_2(\varphi_r,Z(s))|\,ds + \mu(t,\{\|v\|\ge r\}) + \lambda(t,\{\|v\|\ge r\}) \\
&\le\; 4\,(C_B+C_L)\,[C_\mu+\lambda(0,\mathcal{R}^3)]\int_0^t \varrho(\mu(s),\lambda(s))\,ds + \\
&\quad \sup_{s\in[0,T]}\sup_{\|\varphi\|_L\le 1}|M(\varphi_r,s)| + \int_0^T \sup_{\|\varphi\|_L\le 1}|R_2(\varphi_r,Z(s))|\,ds + \\
&\quad \sup_{s\in[0,T]}\mu(s,\{\|v\|\ge r\}) + \sup_{s\in[0,T]}\lambda(s,\{\|v\|\ge r\}) + 2\,\varrho(\mu(0),\lambda(0))\,.
\end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned}
&\sup_{t\in[0,T]}\varrho(\mu(t),\lambda(t)) \le \exp\big(4\,(C_B+C_L)\,T\,[C_\mu+\lambda(0,\mathcal{R}^3)]\big)\times \\
&\quad\Bigg[\sup_{t\in[0,T]}\sup_{\|\varphi\|_L\le 1}|M(\varphi_r,t)| + \int_0^T \sup_{\|\varphi\|_L\le 1}|R_2(\varphi_r,Z(s))|\,ds + \\
&\quad \sup_{t\in[0,T]}\mu(t,\{\|v\|\ge r\}) + \sup_{t\in[0,T]}\lambda(t,\{\|v\|\ge r\}) + 2\,\varrho(\mu(0),\lambda(0))\Bigg]\,.
\end{aligned}$$

According to **Lemma 3.4** and **Lemma 3.5**, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} E\sup_{t\in[0,T]}\varrho(\mu(t),\lambda(t)) \le \exp\big(4\,(C_B+C_L)\,T\,[C_\mu+\lambda(0,\mathcal{R}^3)]\big)\times \\
&\quad\Bigg[\limsup_{n\to\infty} E\sup_{t\in[0,T]}\mu(t,\{\|v\|\ge r\}) + \sup_{t\in[0,T]}\lambda(t,\{\|v\|\ge r\}) + 2\,\limsup_{n\to\infty} E\varrho(\mu(0),\lambda(0))\Bigg]\,,
\end{aligned}$$

for arbitrary $r>0$. Using **(3.1), (3.2)**, we finally obtain

$$\begin{aligned}
&\limsup_{n\to\infty} E\sup_{t\in[0,T]}\varrho(\mu(t),\lambda(t)) \le \\
&\quad 2\,\exp\big(4\,(C_B+C_L)\,T\,[C_\mu+\lambda(0,\mathcal{R}^3)]\big)\limsup_{n\to\infty} E\varrho(\mu(0),\lambda(0))\,,
\end{aligned}$$

so that (2.48) follows from **(2.30)**. ∎

In the remaining part of this section we show that, under the additional assumptions of Theorem 2.1, conditions (3.1), (3.2) and (3.3) are satisfied.

Remark 3.7 *Since*

$$\lambda(t,\{\|v\|\ge r\}) \le \frac{1}{r^2}\int_{\mathcal{R}^3}\|v\|^2\,\lambda(t,dv)\,,$$

condition (3.1) follows from assumption (2.27).

Consider

$$Z_{t,z}(s)\,, \quad s \ge t \ge 0\,, \quad z \in \mathcal{Z}\,, \qquad Z_{t,z}(t) = z\,,$$

and let (cf. (2.41))

$$\tau_{t,z} = \inf\{u > t \;:\; Z_{t,z}(u) \in \mathcal{Z} \setminus \mathcal{Z}(0)\}\,, \qquad t \ge 0\,, \quad z \in \mathcal{Z}\,, \tag{3.15}$$

be the first moment of reaching $\mathcal{Z} \setminus \mathcal{Z}(0)$. The joint distribution function of $(\tau_{t,z}, Z_{t,z}(\tau_{t,z}))$ is denoted by $P_{t,z}$. Note that

$$P_{t,z}(ds, d\tilde{z}) = \delta_{t,z}(ds, d\tilde{z})\,, \qquad z \in \mathcal{Z} \setminus \mathcal{Z}(0)\,. \tag{3.16}$$

Let $\sigma_{t,z}$ be the moment of the first jump of the process starting in z at time t. The joint distribution function of $(\sigma_{t,z}, Z_{t,z}(\sigma_{t,z}))$ is denoted by $Q_{t,z}$. Note that (cf. (2.3), (2.12))

$$Q_{t,z}(ds, d\tilde{z}) = \mathrm{Prob}(t + \xi \in ds)\, P_{\mathrm{red}}(z; d\tilde{z})\,, \qquad z \in \mathcal{Z} \setminus \mathcal{Z}(0)\,, \tag{3.17}$$

where ξ has exponential distribution with parameter $\pi_{\mathrm{red}}(n)$. Let $\tau'_{t,z}$ denote the moment of the first reduction jump, when starting in z at time t. Introduce the kernel

$$K(t, z; dt_1, dz_1) = \int_t^\infty \int_{\mathcal{Z} \setminus \mathcal{Z}(0)} P_{t,z}(ds, d\tilde{z})\, Q_{s,\tilde{z}}(dt_1, dz_1)\,, \tag{3.18}$$

which represents the **joint distribution of time and state after the first reduction jump** of the process starting in z at time t, and the iterated kernels

$$K^{(l+1)}(t, z; dt_2, dz_2) \;=\; \int_t^\infty \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, K^{(l)}(t_1, z_1; dt_2, dz_2)\,, \qquad l \ge 1\,, \tag{3.19}$$

where $K^{(1)} = K$.

Lemma 3.8 *Assume (2.43), (2.38), (2.28), (2.29). Then (cf. (2.41))*

$$\lim_{r\to\infty} \limsup_{n\to\infty} E \sup_{t\in[0,T]} \mu(t, \{\|v\| \ge r\}) \;\le\; C_\mu \limsup_{n\to\infty} \sup_{z \in \tilde{\mathcal{Z}}(0)} K^{(l)}(0, z; [0,T], \mathcal{Z})\,, \tag{3.20}$$

for any $l \ge 1$.

Proof. Introduce the function

$$F(t, z) = E_{t,z} \sup_{u\in[t,T]} \mu(u, \{\|v\| \ge r\})\,, \qquad t \in [0,T]\,, \quad z \in \mathcal{Z}\,. \tag{3.21}$$

Using the strong Markov property, one obtains

$$\begin{aligned} F(t,z) \;&=\; \int_t^\infty \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, E_{t,z}\Big\{ \sup_{u\in[t,T]} \mu(u, \{\|v\| \ge r\}) \,\Big|\, \tau'_{t,z} = t_1, Z(\tau'_{t,z}) = z_1 \Big\} \\ &\le\; \int_T^\infty \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, E_{t,z}\Big\{ \sup_{u\in[t,T]} \mu(u, \{\|v\| \ge r\}) \,\Big|\, \tau'_{t,z} = t_1, Z(\tau'_{t,z}) = z_1 \Big\} \\ &\quad + \int_t^T \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, E_{t,z}\Big\{ \sup_{u\in[t,t_1]} \mu(u, \{\|v\| \ge r\}) \,\Big|\, \tau'_{t,z} = t_1, Z(\tau'_{t,z}) = z_1 \Big\} \\ &\quad + \int_t^T \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, E_{t,z}\Big\{ \sup_{u\in[t_1,T]} \mu(u, \{\|v\| \ge r\}) \,\Big|\, \tau'_{t,z} = t_1, Z(\tau'_{t,z}) = z_1 \Big\} \\ &\le\; E_{t,z} \sup_{u\in[t,\min(T,\tau'_{t,z})]} \mu(u, \{\|v\| \ge r\}) + \int_t^T \int_{\mathcal{Z}} K(t, z; dt_1, dz_1)\, F(t_1, z_1)\,. \end{aligned} \tag{3.22}$$

Iterating (3.22), one obtains

$$F(t,z) \;\leq\; \int_t^T \int_{\mathcal{Z}} K^{(l)}(t,z;dt_1,dz_1)\,F(t_1,z_1) + \sum_{k=1}^{l-1} \int_t^T \int_{\mathcal{Z}} K^{(k)}(t,z;dt_1,dz_1)\,f(t_1,z_1) + f(t,z)\,, \qquad l \geq 1\,, \tag{3.23}$$

where

$$f(t,z) \;=\; E_{t,z} \sup_{u\in[t,\min(T,\tau'_{t,z})]} \mu(u,\{\|v\|\geq r\})\,.$$

Using the inequality

$$\mu(t,\{\|v\|\geq r\}) \;\leq\; \frac{1}{r^2}\int_{\mathcal{R}^3} \|v\|^2\,\mu(t,dv)$$

and the fact that the function $\int_{\mathcal{R}^3} \|v\|^2\,\mu(u,dv)$ takes at most two different values for $u \in [t,\min(T,\tau'_{t,z})]$, one obtains

$$\begin{aligned} f(t,z) \;&\leq\; \frac{1}{r^2}\,E_{t,z} \sup_{u\in[t,\min(T,\tau'_{t,z})]} \int_{\mathcal{R}^3} \|v\|^2\,\mu(u,dv) \\ &\leq\; \frac{1}{r^2}\left[E_{t,z}\int_{\mathcal{R}^3}\|v\|^2\,\mu(t,dv) + E_{t,z}\int_{\mathcal{R}^3}\|v\|^2\,\mu(\tau'_{t,z},dv)\,\chi_{\{\tau'_{t,z}\leq T\}}\right] \\ &=\; \frac{1}{r^2}\left[\Phi(z) + \int_t^T\int_{\mathcal{Z}} \Phi(z_1)\,K(t,z;dt_1,dz_1)\right], \end{aligned} \tag{3.24}$$

where (cf. (2.15))

$$\Phi(z) = \sum_{i=1}^m g_i\,\|v_i\|^2\,.$$

Note that (cf. (2.5))

$$\int_t^\infty \int_{\mathcal{Z}} \Phi(\tilde{z})\,P_{t,z}(ds,d\tilde{z}) = \Phi(z)\,. \tag{3.25}$$

Using **(2.43)** with $\varphi(v)=\|v\|^2$, we obtain (cf. (3.17))

$$\int_s^T \int_{\mathcal{Z}} \Phi(z_1)\,Q_{s,\tilde{z}}(dt_1,dz_1) \;\leq\; \int_{\mathcal{Z}} \Phi(z_1)\,P_{\text{red}}(\tilde{z};dz_1) = \Phi(\tilde{z})\,. \tag{3.26}$$

It follows from (3.25) and (3.26) that (cf. (3.18))

$$\begin{aligned} &\int_t^T \int_{\mathcal{Z}} \Phi(z_1)\,K(t,z;dt_1,dz_1) = \int_t^T \int_{\mathcal{Z}} \Phi(z_1) \int_t^\infty \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} P_{t,z}(ds,d\tilde{z})\,Q_{s,\tilde{z}}(dt_1,dz_1) \\ &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} \int_s^T \int_{\mathcal{Z}} \Phi(z_1)\,Q_{s,\tilde{z}}(dt_1,dz_1)\,P_{t,z}(ds,d\tilde{z}) \leq \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} \Phi(\tilde{z})\,P_{t,z}(ds,d\tilde{z}) \\ &\leq\; \Phi(z)\,, \end{aligned}$$

and (cf. (3.19))

$$\int_t^T \int_{\mathcal{Z}} K^{(k)}(t,z;dt_1,dz_1)\,\Phi(z_1) \le \Phi(z)\,, \qquad k \ge 1\,. \tag{3.27}$$

Note that $F(t,z) \le C_\mu$, according to **(2.38)** and (3.21). Using (3.24), (3.27), (3.23) one obtains

$$\begin{aligned} F(t,z) &\le \int_t^T \int_{\mathcal{Z}} K^{(l)}(t,z;dt_1,dz_1)\,F(t_1,z_1) + \frac{2\,l}{r^2}\,\Phi(z) \\ &\le C_\mu\,K^{(l)}(t,z;[t,T],\mathcal{Z}) + \frac{2\,l}{r^2}\,\Phi(z)\,. \end{aligned} \tag{3.28}$$

Finally, (3.20) follows from (3.28), **(2.28)** and **(2.29)**. ■

Lemma 3.9 *Assume (2.28) and (2.40). Then*

$$\limsup_{n\to\infty} E\,\frac{1}{T}\int_0^T \chi_{\{m(s)>m_{\max}(n)\}}\,ds \le \tag{3.29}$$

$$\limsup_{n\to\infty}\ \sup_{z\in\tilde{\mathcal{Z}}(0)} K^{(l)}(0,z;[0,T],\mathcal{Z}) + \sum_{k=1}^{l} \limsup_{n\to\infty}\ \sup_{z\in\tilde{\mathcal{Z}}(0)} K^{(k)}(0,z;[0,\infty),\mathcal{Z}\setminus\mathcal{Z}(0))\,,$$

for any $l \ge 1$.

Proof. Introduce the function

$$H(t,z) = E_{t,z}\int_t^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,, \qquad t\in[0,T]\,, \quad z\in\mathcal{Z}\,. \tag{3.30}$$

For $z \in \mathcal{Z}(0)$, the strong Markov property implies (cf. (3.15))

$$\begin{aligned} H(t,z) &= \int_t^\infty \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} E_{t,z}\left(\int_t^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du \,\middle|\, \tau_{t,z}=s\,,\ Z(\tau_{t,z})=\tilde{z}\right) P_{t,z}(ds,d\tilde{z}) \\ &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} E_{t,z}\left(\int_t^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du \,\middle|\, \tau_{t,z}=s\,,\ Z(\tau_{t,z})=\tilde{z}\right) P_{t,z}(ds,d\tilde{z}) \\ &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} E_{t,z}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du \,\middle|\, \tau_{t,z}=s\,,\ Z(\tau_{t,z})=\tilde{z}\right) P_{t,z}(ds,d\tilde{z}) \\ &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} E_{s,\tilde{z}}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\right) P_{t,z}(ds,d\tilde{z}) \\ &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} H(s,\tilde{z})\,P_{t,z}(ds,d\tilde{z})\,. \end{aligned} \tag{3.31}$$

For $\tilde{z} \in \mathcal{Z}\setminus\mathcal{Z}(0)$ and $s\in[0,T]$, one obtains

$$H(s,\tilde{z}) = \int_s^\infty \int_{\mathcal{Z}} E_{s,\tilde{z}}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du \,\middle|\, \sigma_{s,\tilde{z}}=t\,,\ Z(\sigma_{s,\tilde{z}})=z\right) Q_{s,\tilde{z}}(dt,dz)$$

$$
\begin{aligned}
&= \int_s^\infty \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} E_{s,\tilde z}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,\Big|\,\sigma_{s,\tilde z}=t\,,\ Z(\sigma_{s,\tilde z})=z\right) Q_{s,\tilde z}(dt,dz)\\
&\quad+\int_T^\infty \int_{\mathcal{Z}(0)} E_{s,\tilde z}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,\Big|\,\sigma_{s,\tilde z}=t\,,\ Z(\sigma_{s,\tilde z})=z\right) Q_{s,\tilde z}(dt,dz)\\
&\quad+\int_s^T \int_{\mathcal{Z}(0)} E_{s,\tilde z}\left(\int_s^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,\Big|\,\sigma_{s,\tilde z}=t\,,\ Z(\sigma_{s,\tilde z})=z\right) Q_{s,\tilde z}(dt,dz)\\
&\le (T-s)\,Q_{s,\tilde z}([s,\infty),\mathcal{Z}\setminus\mathcal{Z}(0)) + (T-s)\,Q_{s,\tilde z}([T,\infty),\mathcal{Z}(0))\\
&\quad+\int_s^T \int_{\mathcal{Z}(0)} E_{s,\tilde z}\left(\int_t^T \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,\Big|\,\sigma_{s,\tilde z}=t\,,\ Z(\sigma_{s,\tilde z})=z\right) Q_{s,\tilde z}(dt,dz)\\
&\quad+\int_s^T \int_{\mathcal{Z}(0)} E_{s,\tilde z}\left(\int_s^t \chi_{(m_{\max}(n),\infty)}(m(u))\,du\,\Big|\,\sigma_{s,\tilde z}=t\,,\ Z(\sigma_{s,\tilde z})=z\right) Q_{s,\tilde z}(dt,dz)\\
&= (T-s)\Big[Q_{s,\tilde z}([s,\infty),\mathcal{Z}\setminus\mathcal{Z}(0)) + Q_{s,\tilde z}([T,\infty),\mathcal{Z}(0))\Big] \qquad (3.32)\\
&\quad+\int_s^T \int_{\mathcal{Z}(0)} H(t,z)\,Q_{s,\tilde z}(dt,dz) + \int_s^T (t-s)\,Q_{s,\tilde z}(dt,\mathcal{Z}(0))\,.
\end{aligned}
$$

Thus, (3.31) and (3.32) imply, for $z\in\mathcal{Z}(0)$,

$$
H(t,z) \le \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} P_{t,z}(ds,d\tilde z) \int_s^T \int_{\mathcal{Z}(0)} Q_{s,\tilde z}(dt_1,dz_1)\,H(t_1,z_1) + h(t,z)\,, \qquad (3.33)
$$

where

$$
\begin{aligned}
h(t,z) &= \int_t^T \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} \Big\{ \int_s^T (u-s)\,Q_{s,\tilde z}(du,\mathcal{Z}(0)) + \qquad (3.34)\\
&\qquad (T-s)\Big[Q_{s,\tilde z}([T,\infty),\mathcal{Z}(0)) + Q_{s,\tilde z}([s,\infty),\mathcal{Z}\setminus\mathcal{Z}(0))\Big]\Big\} P_{t,z}(ds,d\tilde z)\,.
\end{aligned}
$$

Note that inequality (3.33) holds also for $z\in\mathcal{Z}\setminus\mathcal{Z}(0)$, according to (3.16) and (3.32). Thus, using the kernel (3.18), inequality (3.33) implies

$$
H(t,z) \le \int_t^T \int_{\mathcal{Z}(0)} K(t,z;dt_1,dz_1)\,H(t_1,z_1) + h(t,z)\,, \qquad t\in[0,T]\,, \quad z\in\mathcal{Z}\,. \qquad (3.35)
$$

Iterating (3.35) one obtains

$$
\begin{aligned}
H(t,z) &\le \int_t^T \int_{\mathcal{Z}} K^{(l)}(t,z;dt_1,dz_1)\,H(t_1,z_1)\\
&\qquad + \sum_{k=1}^{l-1} \int_t^T \int_{\mathcal{Z}} K^{(k)}(t,z;dt_1,dz_1)\,h(t_1,z_1) + h(t,z)\,, \qquad l\ge 1\,. \qquad (3.36)
\end{aligned}
$$

Next we estimate the function (3.34), which consists of three terms. According to (3.17), for $s\in[0,T]$ and $\tilde z\in\mathcal{Z}\setminus\mathcal{Z}(0)$, the distribution function $Q_{s,\tilde z}(du,\mathcal{Z})$ corresponds to a random variable $s+\xi$, where ξ has exponential distribution with parameter $\pi_{\mathrm{red}}(n)$. Thus, we obtain

$$
\int_s^T (u-s)\,Q_{s,\tilde z}(du,\mathcal{Z}(0)) \le \frac{1}{\pi_{\mathrm{red}}(n)}
$$

and

$$(T-s)\,Q_{s,\tilde z}([T,\infty),\mathcal{Z}(0)) \le (T-s)\,\mathrm{Prob}(s+\xi\ge T) \le \frac{1}{\pi_{\mathrm{red}}(n)}\,,$$

so that the first two terms are estimated by $\frac{2}{\pi_{\mathrm{red}}(n)}$. The third term is estimated by

$$T\int_t^T\int_{\mathcal{Z}\setminus\mathcal{Z}(0)} Q_{s,\tilde z}([s,\infty),\mathcal{Z}\setminus\mathcal{Z}(0))P_{t,z}(ds,d\tilde z) \;\;\le\;\; T\,K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}(0))\,.$$

Since $H(t,z)\le T$ (by definition (3.30)), it follows from (3.36) that

$$H(t,z) \;\;\le\;\; T\,K^{(l)}(t,z;[t,T],\mathcal{Z}) + \frac{2\,l}{\pi_{\mathrm{red}}(n)} + T\sum_{k=1}^{l} K^{(k)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}(0))\,,$$

so that (3.29) is a consequence of **(2.28)** and **(2.40)**. ■

Note that, to complete the proof of Theorem 2.1, it remains to check that the right-hand sides of (3.20) and (3.29) vanish, for some $l\ge 1$. Indeed, conditions (3.1), (3.2) and (3.3) are then satisfied, according to Remark 3.7, Lemma 3.8 and Lemma 3.9.

Lemma 3.10 *Consider subsets* $\mathcal{Z}_1,\ldots,\mathcal{Z}_{l+1}\subset\mathcal{Z}$, *where* $l\ge 1$. *Then, for any* $\Delta t>0$,

$$\sup_{t\ge 0\,,\,z\in\mathcal{Z}_{l+1}} K^{(l+1)}(t,z;[t,t+l\,\Delta t],\mathcal{Z}) \;\;\le\;\; \sum_{k=1}^{l}\alpha(\mathcal{Z}_k,\Delta t) + \sum_{k=2}^{l+1}\beta(\mathcal{Z}_k,\mathcal{Z}_{k-1})\,, \qquad (3.37)$$

where

$$\alpha(\mathcal{Z}',\Delta t) \;\;=\;\; \sup_{t\ge 0\,,\,z\in\mathcal{Z}'} K(t,z;[t,t+\Delta t],\mathcal{Z}) \qquad (3.38)$$

and

$$\beta(\mathcal{Z}',\mathcal{Z}'') \;\;=\;\; \sup_{t\ge 0\,,\,z\in\mathcal{Z}'} K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}'')\,, \qquad \mathcal{Z}',\mathcal{Z}''\subset\mathcal{Z}\,. \qquad (3.39)$$

Proof. We first prove

$$\sup_{t\ge 0\,,\,z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,t+l\,\Delta t],\mathcal{Z}) \;\;\le\;\; \sum_{k=1}^{l}\alpha(\mathcal{Z}_k,\Delta t) + \sum_{k=2}^{l}\beta(\mathcal{Z}_k,\mathcal{Z}_{k-1})\,. \qquad (3.40)$$

For $l=1$, the assertion follows from definition (3.38). For $l\ge 1$, we obtain from (3.19) that

$$\begin{aligned}
&K^{(l+1)}(t,z;[t,t+(l+1)\Delta t],\mathcal{Z}) = \\
&\qquad \int_t^\infty\int_{\mathcal{Z}} K(t,z;dt_1,dz_1)\,K^{(l)}(t_1,z_1;[t,t+(l+1)\Delta t],\mathcal{Z}) \\
&\quad = \int_t^{t+\Delta t}\int_{\mathcal{Z}} K(t,z;dt_1,dz_1)\,K^{(l)}(t_1,z_1;[t,t+(l+1)\Delta t],\mathcal{Z})
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{t+\Delta t}^{t+(l+1)\Delta t} \int_{\mathcal{Z}} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,t+(l+1)\Delta t],\mathcal{Z}) \\
\le\ & K(t,z;[t,t+\Delta t],\mathcal{Z}) \\
&+ \int_{t+\Delta t}^{t+(l+1)\Delta t} \int_{\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t_1,t_1+l\Delta t],\mathcal{Z}) \\
&+ \int_{t+\Delta t}^{t+(l+1)\Delta t} \int_{\mathcal{Z}\setminus\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,t+(l+1)\Delta t],\mathcal{Z}) \\
\le\ & K(t,z;[t,t+\Delta t],\mathcal{Z}) \\
&+ \sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,t+l\Delta t],\mathcal{Z}) + K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_l)\,,
\end{aligned}
$$

and, using the definitions (3.38), (3.39),

$$
\begin{aligned}
\sup_{t\ge 0\,,\, z\in\mathcal{Z}_{l+1}} K^{(l+1)}(t,z;[t,t+(l+1)\Delta t],\mathcal{Z}) &\le \sup_{t\ge 0\,,\, z\in\mathcal{Z}_{l+1}} K(t,z;[t,t+\Delta t],\mathcal{Z}) \\
&+ \sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,t+l\Delta t],\mathcal{Z}) + \sup_{t\ge 0\,,\, z\in\mathcal{Z}_{l+1}} K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_l) \\
&= \alpha(\mathcal{Z}_{l+1},\Delta t) + \sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,t+l\Delta t],\mathcal{Z}) + \beta(\mathcal{Z}_{l+1},\mathcal{Z}_l)\,.
\end{aligned}
$$

Thus, (3.40) follows by induction. Note that (3.19) implies

$$
\begin{aligned}
&K^{(l+1)}(t,z;[t,t+l\Delta t],\mathcal{Z}) = \\
&\quad \int_t^{t+l\Delta t} \int_{\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,t+l\Delta t],\mathcal{Z}) \\
&\quad + \int_t^{t+l\Delta t} \int_{\mathcal{Z}\setminus\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,t+l\Delta t],\mathcal{Z}) \\
&\le \sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,t+l\Delta t],\mathcal{Z}) + K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_l)\,, \qquad l\ge 1\,.
\end{aligned}
\tag{3.41}
$$

Using (3.41), (3.40) and (3.39), one obtains (3.37). ■

Lemma 3.11 *Consider subsets $\mathcal{Z}_0,\mathcal{Z}_1,\ldots,\mathcal{Z}_l\subset\mathcal{Z}$, where $l\ge 1$. Then (cf. (3.39))*

$$
\sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) \ \le\ \sum_{k=1}^{l} \beta(\mathcal{Z}_k,\mathcal{Z}_{k-1})\,. \tag{3.42}
$$

Proof. For $l=1$, the assertion follows from (3.39). For $l\ge 1$, we obtain from (3.19) that

$$
\begin{aligned}
K^{(l+1)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) &= \int_t^{\infty}\int_{\mathcal{Z}} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) \\
&= \int_t^{\infty}\int_{\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) \\
&\quad + \int_t^{\infty}\int_{\mathcal{Z}\setminus\mathcal{Z}_l} K(t,z;dt_1,dz_1)\, K^{(l)}(t_1,z_1;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) \\
&\le \sup_{t\ge 0\,,\, z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) + K(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_l)
\end{aligned}
$$

and

$$\sup_{t\geq 0\,,\,z\in\mathcal{Z}_{l+1}} K^{(l+1)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) \leq \sup_{t\geq 0\,,\,z\in\mathcal{Z}_l} K^{(l)}(t,z;[t,\infty),\mathcal{Z}\setminus\mathcal{Z}_0) + \beta(\mathcal{Z}_{l+1},\mathcal{Z}_l)\,.$$

Thus, (3.42) follows by induction. ■

Lemma 3.12 *Assume (2.38), (2.35), (2.33). Then (cf. (3.38))*

$$\lim_{n\to\infty} \alpha(\mathcal{Z}(\varepsilon),\Delta t) = 0\,, \qquad \forall\,\Delta t < \frac{\varepsilon}{2\,(1+C_\kappa)\,C_B\,C_\mu}\,, \qquad \varepsilon\in(0,1)\,. \tag{3.43}$$

Proof. It follows from (3.18) that, for $u\geq t$,

$$\begin{aligned} K(t,z;[t,u],\mathcal{Z}) &= \int_t^\infty \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} P_{t,z}(ds,d\tilde{z})\,Q_{s,\tilde{z}}([t,u],\mathcal{Z}) \\ &= \int_t^u \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} P_{t,z}(ds,d\tilde{z})\,Q_{s,\tilde{z}}([s,u],\mathcal{Z}) \leq P_{t,z}([t,u],\mathcal{Z}) = 1-\mathrm{Prob}(\tau_{t,z}\geq u)\,. \end{aligned} \tag{3.44}$$

Each collision increases the number of particles by at most 2, so that at least

$$k(z) = \frac{m_{\max}(n)-m(z)}{2} \tag{3.45}$$

jumps take place before the particle number bound $m_{\max}(n)$ is crossed for the first time. Therefore, we obtain $\tau_{t,z}\geq\sigma_{t,z}(k(z))$, and

$$\mathrm{Prob}(\tau_{t,z}\geq u) \geq \mathrm{Prob}(\sigma_{t,z}(k(z))\geq u)\,, \tag{3.46}$$

where $\sigma_{t,z}(k)$, $k=1,2,\ldots$, denotes the moment of the k-th jump of the process starting in z at time t. The waiting times between the jumps of the process have the parameter $\pi_{\mathrm{coll}}(z)$. Using **(2.38), (2.35), (2.33)**, we obtain (cf. (2.4), (2.8))

$$\begin{aligned} \pi_{\mathrm{coll}}(z) &= Q_{\mathrm{coll}}(z,\mathcal{Z}) = \frac{1}{2}\sum_{1\leq i\neq j\leq m}\int_{\mathcal{S}^2}[1+\kappa(z;i,j,e)]\,\max(g_i,g_j)\,B(v_i,v_j,e)\,de \\ &\leq (1+C_\kappa)\,C_B\,m\sum_{i=1}^m g_i \leq (1+C_\kappa)\,C_B\,C_\mu\,m_{\max}(n)\,. \end{aligned}$$

We conclude that

$$\mathrm{Prob}(\sigma_{t,z}(k(z))\geq u) \;\geq\; \mathrm{Prob}(\sigma'_{t,z}(k(z))\geq u)\,, \tag{3.47}$$

where $\sigma'_{t,z}(k)$ denotes the k-th jump time of a process with waiting time parameter (cf. (3.4))

$$(1+C_\kappa)\,C_B\,C_\mu\,m_{\max}(n)\,. \tag{3.48}$$

According to (3.45), (2.41), one obtains

$$k(z) \;\geq\; k_{\min}(n,\varepsilon)\,, \qquad \forall\,z\in\mathcal{Z}(\varepsilon)\,,$$

where

$$k_{\min}(n,\varepsilon) \quad = \quad \left[\frac{\varepsilon\, m_{\max}(n)}{2}\right], \tag{3.49}$$

and $[x]$ denotes the integer part of a real number x. Consequently, for all $z \in \mathcal{Z}(\varepsilon)$,

$$\mathrm{Prob}(\sigma'_{t,z}(k(z)) \geq u) \geq \mathrm{Prob}(\sigma'_{t,z}(k_{\min}(n,\varepsilon)) \geq u) = \mathrm{Prob}\left(t + \sum_{i=1}^{k_{\min}(n,\varepsilon)} \xi_i \geq u\right), \tag{3.50}$$

where (ξ_i) are independent random variables exponentially distributed with parameter (3.48). Using (3.44), (3.46), (3.47) and (3.50), one obtains

$$\sup_{t\geq 0\,,\, z\in\mathcal{Z}(\varepsilon)} K(t,z;[t,t+\Delta t],\mathcal{Z}) \quad \leq \quad 1 - \mathrm{Prob}\left(\sum_{i=1}^{k_{\min}(n,\varepsilon)} \xi_i \geq \Delta t\right).$$

Since (cf. (3.48), (3.49))

$$E \sum_{i=1}^{k_{\min}(n,\varepsilon)} \xi_i = \frac{k_{\min}(n,\varepsilon)}{(1+C_\kappa)\, C_B\, C_\mu\, m_{\max}(n)} \quad \to \quad \frac{\varepsilon}{2\,(1+C_\kappa)\, C_B\, C_\mu}, \qquad \text{as} \quad n\to\infty,$$

and

$$\mathrm{Var}\left(\sum_{i=1}^{k_{\min}(n,\varepsilon)} \xi_i\right) = \frac{k_{\min}(n,\varepsilon)}{[(1+C_\kappa)\, C_B\, C_\mu\, m_{\max}(n)]^2} \quad \to \quad 0, \qquad \text{as} \quad n\to\infty,$$

one obtains, using **(2.39)**,

$$\sum_{i=1}^{k_{\min}(n,\varepsilon)} \xi_i \quad \to \quad \frac{\varepsilon}{2\,(1+C_\kappa)\, C_B\, C_\mu} \qquad \text{in probability, as} \quad n\to\infty.$$

Thus, (3.43) follows. ■

Lemma 3.13 *Assume (2.45) and (2.46). Then*

$$\lim_{n\to\infty} \beta(\tilde{\mathcal{Z}}(\delta),\mathcal{Z}(\varepsilon)) \quad = \quad 0\,, \qquad \forall\, \varepsilon\in[0,\delta)\,. \tag{3.51}$$

Proof. It follows from (3.18), (3.17) that

$$K(t,z;[t,\infty),\mathcal{Z}'') = \int_t^\infty \int_{\mathcal{Z}\setminus\mathcal{Z}(0)} P_{t,z}(ds,dz_1)\, P_{\mathrm{red}}(z_1;\mathcal{Z}'') = E\, P_{\mathrm{red}}(Z_{t,z}(\tau_{t,z});\mathcal{Z}'')$$

and therefore

$$\beta(\mathcal{Z}',\mathcal{Z}'') \quad = \quad \sup_{t\geq 0\,,\, z\in\mathcal{Z}'} E\, P_{\mathrm{red}}(Z_{t,z}(\tau_{t,z});\mathcal{Z}\setminus\mathcal{Z}'')\,.$$

Consider $z \in \tilde{\mathcal{Z}}(\delta)$ and denote $z' = Z_{t,z}(\tau_{t,z})$. Using **(2.45)** and mass conservation during collision jumps, one obtains

$$\begin{aligned}
&P_{\mathrm{red}}(z'; \mathcal{Z} \setminus \mathcal{Z}(\varepsilon)) = P_{\mathrm{red}}(z'; \{z_1 : \Phi_0(z_1) > (1-\varepsilon)\, m_{\max}(n)\}) \\
&\quad \leq \; P_{\mathrm{red}}\left(z'; \left\{z_1 : \Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z'; d\tilde{z}) > (\delta-\varepsilon)\, m_{\max}(n)\right\}\right) \\
&\quad \leq \; \frac{1}{[(\delta-\varepsilon)\, m_{\max}(n)]^2} \int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z'; d\tilde{z})\right]^2 P_{\mathrm{red}}(z'; dz_1) \\
&\quad \leq \; \frac{1}{[(\delta-\varepsilon)\, m_{\max}(n)]^2} \sup_{z \in \tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red}}(z; d\tilde{z})\right]^2 P_{\mathrm{red}}(z; dz_1).
\end{aligned}$$

Consequently, (3.51) follows from **(2.46)**. ■

Lemma 3.14 *Assume (2.40), (2.43) and (2.44). Then*

$$\lim_{n\to\infty} \beta(\tilde{\mathcal{Z}}(\delta'), \tilde{\mathcal{Z}}(\delta'')) \;=\; 0\,, \qquad \forall\, 0 \leq \delta' < \delta'' < \infty\,. \tag{3.52}$$

Proof. Consider $z \in \tilde{\mathcal{Z}}(\delta')$ and denote $z' = Z_{t,z}(\tau_{t,z})$. Let $\Phi(z) = \sum_{i=1}^m g_i$. Using **(2.43)** and mass conservation during collision jumps, one obtains

$$\begin{aligned}
&P_{\mathrm{red}}(z'; \mathcal{Z} \setminus \tilde{\mathcal{Z}}(\delta'')) = P_{\mathrm{red}}(z'; \{z_1 : \Phi(z_1) > C_0\,(1+\delta'')\}) \\
&\quad \leq \; P_{\mathrm{red}}\,(z'; \{z_1 : \Phi(z_1) - \Phi(z') > C_0\,(\delta''-\delta')\}) \\
&\quad \leq \; \frac{1}{[C_0\,(\delta''-\delta')]^2} \int_{\mathcal{Z}} \left[\Phi(z_1) - \int_{\mathcal{Z}} \Phi(\tilde{z})\, P_{\mathrm{red}}(z'; d\tilde{z})\right]^2 P_{\mathrm{red}}(z'; dz_1) \\
&\quad \leq \; \frac{1}{[C_0\,(\delta''-\delta')]^2} \sup_{z \in \tilde{\mathcal{Z}}(\delta')\setminus\mathcal{Z}(0)} \int_{\mathcal{Z}} \left[\Phi(z_1) - \int_{\mathcal{Z}} \Phi(\tilde{z})\, P_{\mathrm{red}}(z; d\tilde{z})\right]^2 P_{\mathrm{red}}(z; dz_1).
\end{aligned}$$

Consequently, (3.52) follows from **(2.40)** and **(2.44)**. ■

Lemma 3.15 *Assume (2.38), (2.35), (2.33), (2.45), (2.46) (2.40), (2.43) and (2.44). Then*

$$\limsup_{n\to\infty} \sup_{z \in \tilde{\mathcal{Z}}(0)} K^{(l+1)}(0, z; [0,T], \mathcal{Z}) \;=\; 0 \tag{3.53}$$

for sufficiently large l.

Proof. According to (3.37), property (3.53) is fulfilled provided that

$$\tilde{\mathcal{Z}}(0) \subset \mathcal{Z}_{l+1}\,, \tag{3.54}$$

$$\lim_{n\to\infty} \beta(\mathcal{Z}_k, \mathcal{Z}_{k-1}) \;=\; 0\,, \qquad k = 2, \ldots, l+1\,, \tag{3.55}$$

and

$$\lim_{n\to\infty} \alpha(\mathcal{Z}_k, \Delta t) \;=\; 0\,, \qquad k = 1, \ldots, l\,, \tag{3.56}$$

for some $\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_{l+1}$ and Δt such that

$$T \leq l\,\Delta t\,. \tag{3.57}$$

Consider the choice

$$\mathcal{Z}_k = \mathcal{Z}(\varepsilon) \cap \tilde{\mathcal{Z}}(\delta_k)\,, \qquad k = 1, \ldots, l\,, \qquad \mathcal{Z}_{l+1} = \tilde{\mathcal{Z}}(\delta_{l+1})\,,$$

where $\varepsilon \in [0, \delta)$ and

$$0 \leq \delta_{l+1} < \ldots < \delta_1 \leq \delta\,.$$

Note that (3.54) is fulfilled. Choose Δt according to (3.43) and l such that (3.57) holds. Then (3.56) is fulfilled according to **Lemma 3.12**. Moreover, one obtains

$$\begin{aligned} \beta(\mathcal{Z}_k, \mathcal{Z}_{k-1}) &\leq \beta(\mathcal{Z}_k, \mathcal{Z}(\varepsilon)) + \beta(\mathcal{Z}_k, \tilde{\mathcal{Z}}(\delta_{k-1})) \\ &\leq \beta(\tilde{\mathcal{Z}}(\delta), \mathcal{Z}(\varepsilon)) + \beta(\tilde{\mathcal{Z}}(\delta_k), \tilde{\mathcal{Z}}(\delta_{k-1})) \end{aligned}$$

so that (3.55) is fulfilled according to **Lemma 3.13** and **Lemma 3.14**. ■

Lemma 3.16 *Assume (2.45), (2.46) (2.40), (2.43) and (2.44). Then*

$$\limsup_{n\to\infty} \sup_{z\in\tilde{\mathcal{Z}}(0)} K^{(l)}(0, z; [0,\infty), \mathcal{Z} \setminus \mathcal{Z}(0)) = 0\,, \tag{3.58}$$

for any $l = 1, 2, \ldots$.

Proof. According to (3.42), property (3.58) is fulfilled provided that

$$\mathcal{Z}_0 \subset \mathcal{Z}(0)\,, \qquad \tilde{\mathcal{Z}}(0) \subset \mathcal{Z}_l$$

and

$$\lim_{n\to\infty} \beta(\mathcal{Z}_k, \mathcal{Z}_{k-1}) = 0\,, \qquad k = 1, \ldots, l\,,$$

for some $\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_l$. According to **Lemma 3.13** and **Lemma 3.14**, it is sufficient to choose

$$\mathcal{Z}_0 = \mathcal{Z}(\varepsilon)\,, \qquad \mathcal{Z}_k = \tilde{\mathcal{Z}}(\delta_k)\,, \qquad k = 1, \ldots, l\,,$$

where $\varepsilon \in [0, \delta)$ and $0 \leq \delta_l < \ldots < \delta_1 \leq \delta$. ■

According to Lemma 3.15 and Lemma 3.16, the right-hand sides of (3.20) and (3.29) vanish, if l is sufficiently large. This completes the proof of Theorem 2.1.

4. Examples of reduction procedures

Here we introduce a class of reduction procedures and check the assumptions of Theorem 2.1.

Given the state (2.2), we form γ groups

$$z_i = \Gamma_i(z) = \Big((g_{i,j}, v_{i,j})\,, \quad j = 1,\ldots,m_i\Big)\,, \quad i = 1,\ldots,\gamma(z)\,, \qquad \sum_{i=1}^{\gamma(z)} m_i = m\,.$$

In each group, a reduction procedure $P_{\mathrm{red},i}$ is applied independently, and the new state is formed of the group results, i.e.

$$P_{\mathrm{red}}(z; d\tilde{z}) \;=\; \int_{\mathcal{Z}} \ldots \int_{\mathcal{Z}} \delta_{(\tilde{z}_1,\ldots,\tilde{z}_\gamma)}(d\tilde{z}) \prod_{i=1}^{\gamma} P_{\mathrm{red},i}(z_i; d\tilde{z}_i)\,.$$

Note that, for Φ defined in (2.15) and Φ_0 defined in (2.47),

$$\Phi(\tilde{z}) = \Phi(\tilde{z}_1) + \ldots + \Phi(\tilde{z}_\gamma)\,, \qquad \Phi_0(\tilde{z}) = \Phi_0(\tilde{z}_1) + \ldots + \Phi_0(\tilde{z}_\gamma)\,.$$

Assumption (2.43) is fulfilled if

$$\int_{\mathcal{Z}} \Phi(\tilde{z})\, P_{\mathrm{red},i}(z; d\tilde{z}) \;=\; \Phi(z)\,, \qquad i = 1,\ldots,\gamma(z)\,, \tag{4.1}$$

since

$$\begin{aligned}\int_{\mathcal{Z}} \Phi(\tilde{z})\, P_{\mathrm{red}}(z; d\tilde{z}) \;&=\; \int_{\mathcal{Z}} \Phi(\tilde{z}_1)\, P_{\mathrm{red},1}(z_1; d\tilde{z}_1) + \ldots + \int_{\mathcal{Z}} \Phi(\tilde{z}_\gamma)\, P_{\mathrm{red},\gamma}(z_\gamma; d\tilde{z}_\gamma) \\ &=\; \Phi(z_1) + \ldots + \Phi(z_\gamma) = \Phi(z)\,.\end{aligned}$$

Assumption (2.44) takes the form

$$\limsup_{n\to\infty} \pi_{\mathrm{red}}(n) \sup_{z\in\tilde{\mathcal{Z}}(\varepsilon)\setminus\mathcal{Z}(0)} \sum_{i=1}^{\gamma} \int_{\mathcal{Z}} \left[\Phi(\tilde{z}) - \Phi(z_i)\right]^2\, P_{\mathrm{red},i}(z_i; d\tilde{z}) < \infty\,, \qquad \forall\,\varepsilon \ge 0\,. \tag{4.2}$$

Assumption (2.45) is fulfilled if

$$\sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \sum_{i=1}^{\gamma} \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red},i}(z_i; d\tilde{z}) \;\le\; (1-\delta)\, m_{\max}(n)\,, \tag{4.3}$$

and **assumption (2.46)** takes the form

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{m_{\max}(n)^2}\times \\ &\quad \sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \sum_{i=1}^{\gamma} \int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red},i}(z_i; d\tilde{z})\right]^2 P_{\mathrm{red},i}(z_i; dz_1) = 0\,.\end{aligned} \tag{4.4}$$

Example 4.1 *Given a state (2.2), we define*

$$\Theta_{\mathrm{red}}^{(1)}(z) = \{1,\ldots,m\}\,, \qquad p_{\mathrm{red}}^{(1)}(z; \{j\}) = \frac{g_j}{\sum_{k=1}^{m} g_k}$$

and

$$J^{(1)}_{\mathrm{red}}(z;j) = \left(\sum_{k=1}^m g_k\,, v_j\right), \qquad j=1,\ldots,m\,,$$

i.e. a random index is chosen, and the weight of the system is given to the particle with that index. Thus, mass is conserved. Moreover, one obtains (cf. (4.1))

$$\int_{\mathcal{Z}} \Phi(\tilde z)\, P^{(1)}_{\mathrm{red}}(z;d\tilde z) = \sum_{j=1}^m \varphi(v_j)\, g_j = \Phi(z)\,, \qquad \forall z\,, \tag{4.5}$$

and (cf. (4.2))

$$\int_{\mathcal{Z}} [\Phi(\tilde z)-\Phi(z)]^2\, P^{(1)}_{\mathrm{red}}(z;d\tilde z) \le \tag{4.6}$$

$$\int_{\mathcal{Z}} \Phi(\tilde z)^2\, P^{(1)}_{\mathrm{red}}(z;d\tilde z) = \sum_{j=1}^m \left(\sum_{k=1}^m g_k\right) \varphi(v_j)^2 g_j \le \|\varphi\|^2 \left(\sum_{k=1}^m g_k\right)^2 .$$

Note that (cf. (4.3), (4.4))

$$P^{(1)}_{\mathrm{red}}(z;\{\tilde z:\ \Phi_0(\tilde z)=1\}) \;=\; 1\,. \tag{4.7}$$

Example 4.2 *Given a state (2.2), we consider m independent uniform random numbers, i.e.*

$$\Theta^{(2)}_{\mathrm{red}}(z) = [0,1]^m\,, \qquad p^{(2)}_{\mathrm{red}}(z;d\theta) = d\theta_1\ldots d\theta_m\,,$$

and some

$$\bar g \;\ge\; \max_k g_k\,.$$

We define

$$g'_k(\theta_k) = \begin{cases} \bar g\,, & \textit{if}\quad \bar g\,\theta_k \le g_k\,,\\ 0\,, & \textit{otherwise}\,, \end{cases} \qquad k=1,\ldots,m\,,$$

and

$$J^{(2)}_{\mathrm{red}}(z;\theta) = \{(g'_k, v_k)\,,\ k=1,\ldots,m\ :\ g'_k=\bar g\}\,.$$

During this the procedure of "individual reduction", each particle either gets weight $\bar g$ or gets weight zero being removed from the system.

Note that

$$\int_0^1 g'_i\, d\theta_i = g_i\,, \qquad \int_0^1 (g'_i)^2\, d\theta_i = g_i\,\bar g\,. \tag{4.8}$$

For Φ defined in (2.15), one obtains (cf. (4.1))

$$\int_{\mathcal{Z}} \Phi(\tilde{z})\, P^{(2)}_{\text{red}}(z;d\tilde{z}) = \int_{\Theta^{(2)}_{\text{red}}(z)} \Phi(J^{(2)}_{\text{red}}(z;\theta))\, p^{(2)}_{\text{red}}(z;d\theta)$$
$$= \int_0^1 \ldots \int_0^1 \left(\sum_{i=1}^m g_i' \,\varphi(v_i)\right) d\theta_1 \ldots d\theta_m = \sum_{i=1}^m g_i\, \varphi(v_i) = \Phi(z) \tag{4.9}$$

and (cf. (4.2))

$$\int_{\mathcal{Z}} [\Phi(\tilde{z}) - \Phi(z)]^2\, P^{(2)}_{\text{red}}(z;d\tilde{z}) = \int_{\Theta^{(2)}_{\text{red}}(z)} \left[\sum_{i=1}^m g_i'\,\varphi(v_i) - \sum_{i=1}^m g_i\,\varphi(v_i)\right]^2 p^{(2)}_{\text{red}}(z;d\theta)$$
$$= \int_0^1 \ldots \int_0^1 \left[\sum_{i=1}^m (g_i' - g_i)\,\varphi(v_i)\right]^2 d\theta_1 \ldots d\theta_m = \sum_{i=1}^m \int_0^1 \left[(g_i'-g_i)\,\varphi(v_i)\right]^2 d\theta_i$$
$$= \sum_{i=1}^m \varphi(v_i)^2\, g_i\,(\bar{g} - g_i) \le \|\varphi\|^2\, \bar{g} \sum_{i=1}^m g_i\,. \tag{4.10}$$

Note that, for Φ_0 defined in (2.47),

$$\Phi_0(\tilde{z}) \;=\; \frac{1}{\bar{g}} \sum_{i=1}^m g_i'\,.$$

Thus, according to (4.8), one obtains (cf. (4.3))

$$\int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P^{(2)}_{\text{red}}(z;d\tilde{z}) \;=\; \frac{1}{\bar{g}} \sum_{i=1}^m g_i \tag{4.11}$$

and (cf. (4.4))

$$\int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P^{(2)}_{\text{red}}(z;d\tilde{z})\right]^2 P^{(2)}_{\text{red}}(z;dz_1) =$$
$$\int_{\Theta^{(2)}_{\text{red}}(z)} \left[\frac{1}{\bar{g}} \sum_{i=1}^m (g_i' - g_i)\right]^2 p^{(2)}_{\text{red}}(z;d\theta) = \frac{1}{\bar{g}^2} \sum_{i=1}^m \int_0^1 (g_i' - g_i)^2\, d\theta_i$$
$$= \frac{1}{\bar{g}^2} \sum_{i=1}^m g_i\,(\bar{g} - g_i) \le \frac{1}{\bar{g}} \sum_{i=1}^m g_i\,. \tag{4.12}$$

Now we consider a mixture of $0 \le \gamma_1 \le \gamma$ groups with the mass conserving reduction procedure $P^{(1)}_{\text{red}}$ and of $\gamma - \gamma_1$ groups with the “individual” reduction procedure $P^{(2)}_{\text{red}}$. **Necessary conditions** are a restriction on the choice of the groups,

$$\sum_{j=1}^{m_i} g_{i,j} \;\le\; g_{\max}(n)\,, \qquad \forall\, i = 1, \ldots, \gamma_1\,, \tag{4.13}$$

and a restriction on the choice of the weight bounds,

$$\bar{g}_i \;\le\; g_{\max}(n)\,, \qquad \forall\, i = \gamma_1 + 1, \ldots, \gamma\,. \tag{4.14}$$

Note that **condition (4.1)** is satisfied, according to (4.5) and (4.9).

Using (4.6) and (4.10), one obtains

$$\sum_{i=1}^{\gamma} \int_{\mathcal{Z}} [\Phi(\tilde{z}) - \Phi(z_i)]^2 \; P_{\mathrm{red},i}(z_i; d\tilde{z}) \leq \|\varphi\|^2 \left[\sum_{i=1}^{\gamma_1} \left(\sum_{j=1}^{m_i} g_{i,j} \right)^2 + \sum_{i=\gamma_1+1}^{\gamma} \bar{g}_i \sum_{j=1}^{m_i} g_{i,j} \right].$$

Thus, **condition (4.2)** takes the form

$$\limsup_{n\to\infty} \pi_{\mathrm{red}}(n) \sup_{z\in\tilde{\mathcal{Z}}(\varepsilon)\setminus\mathcal{Z}(0)} \left[\sum_{i=1}^{\gamma_1} \left(\sum_{j=1}^{m_i} g_{i,j} \right)^2 + \sum_{i=\gamma_1+1}^{\gamma} \bar{g}_i \sum_{j=1}^{m_i} g_{i,j} \right] < \infty , \qquad \forall \varepsilon \geq 0 . \quad (4.15)$$

Since (cf. (4.13))

$$\sum_{i=1}^{\gamma_1} \left(\sum_{j=1}^{m_i} g_{i,j} \right)^2 \leq g_{\max}(n) \sum_{i=1}^{\gamma_1} \sum_{j=1}^{m_i} g_{i,j}$$

and (cf. (4.14))

$$\sum_{i=\gamma_1+1}^{\gamma} \bar{g}_i \sum_{j=1}^{m_i} g_{i,j} \leq g_{\max}(n) \sum_{i=\gamma_1+1}^{\gamma} \sum_{j=1}^{m_i} g_{i,j} ,$$

condition (4.15) reduces to (cf. (2.38))

$$\limsup_{n\to\infty} \pi_{\mathrm{red}}(n)\, g_{\max}(n) < \infty .$$

Using (4.7) and (4.11), one obtains

$$\sum_{i=1}^{\gamma} \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red},i}(z_i; d\tilde{z}) = \gamma_1 + \sum_{i=\gamma_1+1}^{\gamma} \frac{1}{\bar{g}_i} \sum_{j=1}^{m_i} g_{i,j} .$$

Thus, **condition (4.3)** takes the form

$$\sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \left[\gamma_1 + \sum_{i=\gamma_1+1}^{\gamma} \frac{1}{\bar{g}_i} \sum_{j=1}^{m_i} g_{i,j} \right] \leq (1-\delta)\, m_{\max}(n) . \qquad (4.16)$$

Using (4.7) and (4.12), one obtains

$$\sum_{i=1}^{\gamma} \int_{\mathcal{Z}} \left[\Phi_0(z_1) - \int_{\mathcal{Z}} \Phi_0(\tilde{z})\, P_{\mathrm{red},i}(z_i; d\tilde{z}) \right]^2 P_{\mathrm{red},i}(z_i; dz_1) \leq \sum_{i=\gamma_1+1}^{\gamma} \frac{1}{\bar{g}_i} \sum_{j=1}^{m_i} g_{i,j} .$$

Thus, **condition (4.4)** takes the form

$$\lim_{n\to\infty} \frac{1}{m_{\max}(n)^2} \sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \left[\sum_{i=\gamma_1+1}^{\gamma} \frac{1}{\bar{g}_i} \sum_{j=1}^{m_i} g_{i,j} \right] = 0$$

and follows from (4.16) and **(2.39)**.

For the following examples, it remains to check (4.13), (4.14) and (4.16).

Example 4.3 *The simplest procedure is individual reduction in one group. Consider the case*

$$\gamma = 1\,, \qquad \gamma_1 = 0\,, \qquad \bar{g}_1 = g_{\max}(n)\,.$$

Then condition (4.16) takes the form

$$\sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \frac{1}{g_{\max}(n)} \sum_{i=1}^{m} g_i = \frac{1}{g_{\max}(n)}\, C_0\,(1+\delta) \;\le\; (1-\delta)\, m_{\max}(n)\,,$$

which is fulfilled for sufficiently small δ provided that (2.53) holds.

However, this procedure just compensates the effect of blow-up. It is of little practical use, but may serve as a test example for illustrating convergence properties.

In order to construct reduction procedures relevant for the calculation of tail functionals, we introduce a shell structure of the velocity space,

$$\begin{aligned} V_i &= \left\{ v\in\mathcal{R}^3 \,:\, r_i \le \frac{\|v-V\|}{\sqrt{T}} < r_{i+1} \right\}, \qquad i=1,\ldots,S-1\,,\\ V_S &= \left\{ v\in\mathcal{R}^3 \,:\, \frac{\|v-V\|}{\sqrt{T}} \ge r_S \right\} \qquad S\ge 1\,, \end{aligned}$$

where $0 = r_1 < \ldots < r_S < \infty$ and V, T are the mean velocity and the temperature of the system. For each shell, a certain weight $\bar{g}_i$ satisfying

$$\bar{g}_i \;\le\; g_{\max}(n)\,, \qquad \forall\, i=1,\ldots,S\,. \tag{4.17}$$

is chosen. Then the corresponding subsystem is either divided into groups with weight $\bar{g}_i$ (and possibly one with lower weight) to which mass preserving reduction from Example 4.1 is applied, or kept as one group, subject to individual reduction from Example 4.2 with parameter $\bar{g}_i$. Note that conditions (4.13) and (4.14) are satisfied. A sufficient condition for (4.16) is then

$$\sup_{z\in\tilde{\mathcal{Z}}(\delta)\setminus\mathcal{Z}(0)} \sum_{i=1}^{S} \left\lceil \frac{1}{\bar{g}_i} \sum_{j\,:\,v_j\in V_i} g_j \right\rceil \;\le\; (1-\delta)\, m_{\max}(n)\,, \tag{4.18}$$

where the bracket means rounding up to the next integer.

We gives two examples for the choice of the quantities $\bar{g}_i$, corresponding to different reduction strategies. In the first case one would like to have precisely n particles after the reduction. In the second case one would like to have at least n/S particles in each shell after the reduction.

Example 4.4 *Define*

$$n'_i \;=\; \left\lceil \frac{1}{g_{\max}(n)} \sum_{j\,:\,v_j\in V_i} g_j \right\rceil, \qquad i=1,\ldots,S\,,$$

where the bracket means rounding up to the next integer. Then the parameters

$$\bar{g}_i \quad = \quad \frac{1}{n_i'} \sum_{j\,:\,v_j \in V_i} g_j\,, \qquad i = 1, \ldots, S\,,$$

satisfy (4.17). The left-hand side of condition (4.18) is estimated as

$$\sup_{z \in \tilde{\mathcal{Z}}(\delta) \setminus \mathcal{Z}(0)} \sum_{i=1}^{S} n_i' \le \sup_{z \in \tilde{\mathcal{Z}}(\delta) \setminus \mathcal{Z}(0)} \frac{1}{g_{\max}(n)} \sum_{i=1}^{m} g_i + S \quad \le \quad \frac{C_0\,(1+\delta)}{g_{\max}(n)} + S\,.$$

If

$$g_{\max}(n) \quad = \quad C_g\, \frac{C_0}{n} \qquad \text{for some} \quad C_g > 1\,, \tag{4.19}$$

and

$$m_{\max}(n) \quad = \quad C_m\, n \qquad \text{for some} \quad C_m > 1\,, \tag{4.20}$$

then

$$\frac{C_0\,(1+\delta)}{g_{\max}(n)} + S \quad \le \quad n \tag{4.21}$$

and (4.18) is satisfied, for sufficiently small δ *and sufficiently large* n *. According to (4.21), all conditions are satisfied when replacing* n_i' *by arbitrary* $n_i \ge n_i'$ *such that*

$$\sum_{i=1}^{S} n_i \quad = \quad n\,. \tag{4.22}$$

Example 4.5 *Define*

$$\bar{g}_i \quad = \quad \min\left(g_{\max}(n)\,, \frac{1}{n_i} \sum_{j\,:\,v_j \in V_i} g_j \right), \qquad i = 1, \ldots, S\,,$$

where n_i *denotes the desired minimum number of particles in the shell* V_i *after reduction. Note that condition (4.17) is satisfied. The left-hand side of (4.18) is estimated as*

$$\sum_{i=1}^{S} n_i + S + \sup_{z \in \tilde{\mathcal{Z}}(\delta) \setminus \mathcal{Z}(0)} \frac{1}{g_{\max}(n)} \sum_{i=1}^{m} g_i \quad \le \quad \sum_{i=1}^{S} n_i + S + \frac{C_0\,(1+\delta)}{g_{\max}(n)}$$

so that condition (4.18) is satisfied, for sufficiently small δ *and sufficiently large* n *, provided that (4.19), (4.20) and (4.22) hold. Note that (4.19), (4.20) imply (2.50).*

5. Numerical experiments

We consider the relaxation of a mixture of two Maxwellians

$$f_0(v) \quad = \quad \alpha\, M_1(v) + (1-\alpha)\, M_2(v)\,,$$

where

$$M_i(v) \quad = \quad \frac{1}{(2\pi\, T_i)^{3/2}} \exp\left(-\frac{\|v-V_i\|}{2\,T_i}\right)$$

and

$$\alpha = 0.1\,, \qquad V_1 = (94.82, 0, 0)\,, \qquad V_2 = (-10.54, 0, 0)\,, \qquad T_1 = T_2 = 1\,,$$

to a Maxwellian $M_\infty(v)$ with

$$V_\infty = (0,0,0)\,, \qquad T_\infty = 334\,. \tag{5.1}$$

Our main interest is the calculation of a tail functional,

$$\int_{\{\|v\|>105\}} f(t,v)\, dv\,, \qquad t \in [0,50]\,, \tag{5.2}$$

i.e. of the mass outside some ball with large radius. SWPM with the reduction procedure from Example 4.4 is applied. The relevant parameters are

$$n = 10^5\,, \qquad m_{\max}(n) = 2\,n\,, \qquad g_{\max}(n) = \frac{2}{n}$$

and

$$S = 51\,, \qquad r_i = 2.3\,(i-1)\,, \quad i = 1,\ldots,S\,. \tag{5.3}$$

Confidence bands for the functional (5.2), calculated with DSMC and SWPM, are displayed in **Figure 1**. The left graph shows that both algorithms model the same time evolution. The right graph shows how the algorithms approximate the very small value at equilibrium,

$$\int_{\{\|v\|>105\}} M_\infty(v)\, dv \quad = \quad 3.2\cdot 10^{-7}\,.$$

Beside convergence, which is the main objective of the present paper, this example illustrates the variance reduction effect, which was the motivation for introducing SWPM. The number of repetitions is chosen in such a way that DSMC and SWPM need almost the same CPU time. In this particular example, SWPM is about 16 times more time consuming (per trajectory), so that the gain factor in efficiency of about 50 results from a tremendous variance reduction.

An interesting feature of SWPM is the variable number of simulation particles displayed in **Figure 2**. First we note that SWPM produces a correct result despite the

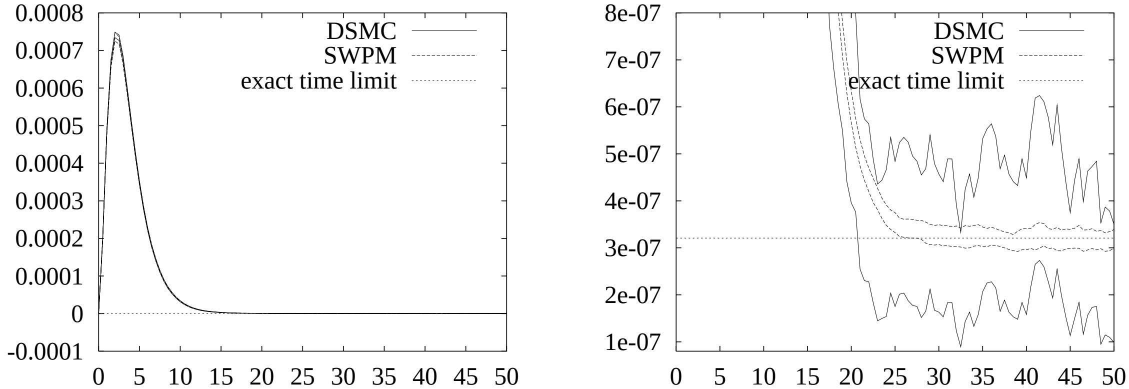

Figure 1: *Tail functional (5.2) and zoom (right).*

strong fluctuation in the overall number of particles (left graph) and the correspondingly large number of reduction steps. This illustrates the main theoretical result of this paper. The variance reduction effect is explained best by looking at the curves for the number of simulation particles in the tail (right graph). At the beginning, both algorithms do not have particles inside the tail. Then SWPM (with $\kappa = 1$ in (2.6)) produces such particles rather quickly and keeps them. At the end, in SWPM 10% of the particles are in the tail, while in DSMC the relative amount corresponds to the value of the functional. However, the number of particles is just an illustrative quantity, the important point are their correct weights. This is achieved by using a relatively large number of shells (cf. (5.3)).

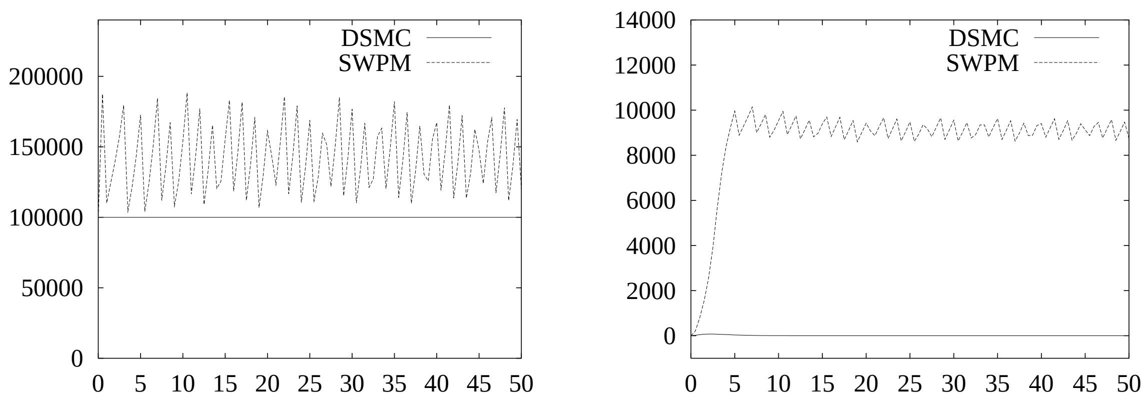

Figure 2: *Number of particles in the system (left) and in the tail (right).*

It should be noted that in SWPM momentum and energy are conserved in the mean, while DSMC conserves these quantities pathwise. However, convergence for higher moments is observed. Results for the second energy component

$$\int_{\mathcal{R}^3} v_2^2 \, f(t,v) \, dv \,, \qquad t \in [0,50] \,, \tag{5.4}$$

are displayed in **Figure 3**. The corresponding asymptotic value is given in (5.1). Again, the left graph shows that both algorithms model the same time evolution. The right graph shows how the algorithms approximate the equilibrium value. For this functional the variances of both methods are roughly the same, and the gain factor in efficiency of DSMC just corresponds to the lower effort per trajectory mentioned above.

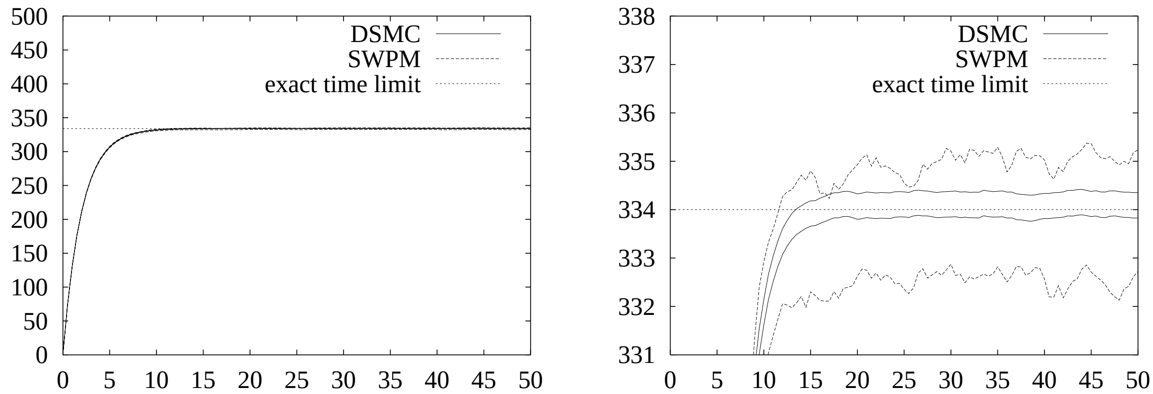

Figure 3: *Functional (5.4) and zoom (right).*

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