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On the bifurcation of the Biot slow wave in a porous medium

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Abstract

Propagation of the slow Biot wave is investigated within the low-frequency range. For the first time it is proven theoretically that longitudinal wave of the second kind is not propagatory if its wave number is lower than some critical value. This critical wave number is a bifurcation point, above which longitudinal wave of the second kind becomes to be propagatory. Asymptotical formulae for phase velocity and attenuation of P2 wave are derived.

Introduction

A fundamental theory for the propagation of elastic waves through an isotropic and macroscopically homogeneous fluid-saturated porous medium was developed by M.Biot [1-2]. Biot showed that one shear wave and two longitudinal waves, namely the fast (P1) and slow (P2) modes, can propagate in a fluid-saturated porous medium. The shear wave and the longitudinal wave of the first kind (P1) predicted by the theory are similar to the corresponding waves in an ordinary isotropic elastic medium. The most interesting phenomenon, predicted by Biot, is the existance of a longitudinal waves of the second kind (P2), the so-called Biot slow wave. The main feature of this waves is that its phase velocity is always lower than both compressional wave vlocities in a fluid and in a solid. The slow wave was observed experimentally at ultarsonic frequencies in *artificial rocks* made of sintered glass beads [3] and confimed theoretically [4]. Later it was registered in *natural granular soils* (Monterey sand) also at ultarsonic frequencies [5].

In the low frequency range the Biot theory assumes that slow wave is highly dispersive and strongly attenuated below some critical frequency, which depends on the pores size in the skeleton and the viscosity of the fluid. This critical frequency is typically around $1-10$ kHz for water saturated porous materials of around 1 $Darcy$ permeability [6]. The Biot slow wave is characterized by the out-of-phase motions of the solid skeleton and pore fluid. This relative motion is very sensitive to the viscosity of the fluid and the dynamic permeability of the porous medium.

Although Biot's theory has been thoroughly studied during last 40 years, the question of why slow wave cannot be detected in low-permeability materials such as natural rocks is still open. First attempt to answer this question was made in [7]. The authors suggested that this lack of perceivable slow wave propagation is probably because of clay particles which are deposited both within the pores and on the surfaces of the rock grains. They reduce porosity and permeability of the rock and greatly increase the viscous losses. The latter results in increasing attenuation of the slow wave and its complete disapparence.

Propagation of the slow wave in air-filled porous materials (porous ceramics of $2 - 70$ *Darcy* and natural rocks of $200 - 700$ *mDarcy* permeability) was investigated experimentally [8] between 10 and 500 kHz. The velocity and attenuation coefficient were measured as functions of frequency. It was shown that in the low-frequency limit the phase velocity and attenuation of the Biot slow wave are essentially determined by the permeability of the porous material.

The purpose of this papers is to prove analytically that the Biot slow wave is not propagatory below some critical wave number which is proportional to permeability of the media. In should be emphasized here that because of the fact that we consider the propagation of elastic waves through an infinite space in the absence of external forces, one must set the wave number k to be real and define frequency $\omega = \omega(k)$, which can be complex, as a solution of corresponding dispersion equation. In contrast to the widely used Biot's model, various phenomenological parameters of which it is difficult or impossible to measure, we rely on the more simple mathematical model of saturated poroelastic materials, proposed by K. Wilmanski [9-12]. This model leads to similar results as the classical Biot model. Detailed comparison of the models is presented in [13].

1. Problem Statement

1.1. Mathematical Model

Consider propagation of the bulk waves through an infinite space Ω occupied by a saturated porous medium. The set of balance equations describing the porous two-component medium has the following general form ($x \in \Omega,\ t \in [0,T]$) [9-12]:

Mass conservation equations

$$\frac{\partial \rho^F}{\partial t} + \mathrm{div}\big(\rho^F \mathbf{v}^F\big) = 0,$$

$$\frac{\partial \rho^S}{\partial t} + \mathrm{div}(\rho^S \mathbf{v}^S) = 0. \tag{1.1}$$

Here, ρ is the mass density, $\mathbf{v}$ is the velocity vector and indices F and S indicate fluid or solid phases, respectively.

Momentum conservation equations

$$\rho^F \left[\frac{\partial}{\partial t} + (v_j^F, \frac{\partial}{\partial x_j})\right] v_i^F - \frac{\partial}{\partial x_j} T_{ij}^F + \pi(v_i^F - v_i^S) = 0,$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v_j^S, \frac{\partial}{\partial x_j}) \right] v_i^S - \frac{\partial}{\partial x_j} T_{ij}^S - \pi (v_i^F - v_i^S) = 0, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the inner product.

Balance equation for the change of porosity

$$\frac{\partial \Delta_n}{\partial t} + \left(v_i^S, \frac{\partial}{\partial x_i}\right) \Delta_n + n_E \mathrm{div}(\mathbf{v}^F - \mathbf{v}^S) = -\frac{\Delta_n}{\tau}, \tag{1.3}$$

where τ is the relaxation time of porosity, assumed to be constant. $\mathbf{T}^F$ and $\mathbf{T}^S$ are the partial stress tensors, π is a positive parameter which is constant in the model used in this paper.

Constitutive relations for linear poroelastic materials

$$\mathbf{T}^F = -p^F \mathbf{1} - \beta \Delta_n \mathbf{1}, \quad p^F = p_0^F + \kappa(\rho^F - \rho_0^F), \tag{1.4}$$

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S \mathrm{div} \mathbf{u}^S \mathbf{1} + 2\mu^S \mathrm{symgrad} \mathbf{u}^S + \beta \Delta_n \mathbf{1}, \tag{1.5}$$

where p^F is the pore pressure, p_0^F and ρ_0^F are the initial values of pore pressure and fluid mass density, respectively, κ is the constant compressibility coefficient of the fluid depending only on the equilibrium value of porosity n_E. $\Delta_n = n - n_E$ is the change of the porosity, and β denotes the coupling coefficient of the components. $\mathbf{T}_0^S$ denotes a constant reference value of the partial stress tensor in the skeleton, λ^S and μ^S are the Lamé constants of the skeleton, which depend only on n_E, and $\mathbf{u}^S$ is the displacement vector for the solid phase with

$$\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}. \tag{1.6}$$

1.2. Dimensionless variables and parameters

Let us rewrite the system of equation (1.1)-(1.6) in a dimensionless form. For this purpose we introduce the following dimensionless variables and parameters [13]:

$$\hat{\rho}^F = \frac{\rho^F}{\rho_0^S}, \qquad \hat{\rho}^S = \frac{\rho^S}{\rho_0^S}, \qquad \hat{\mathbf{v}}^F = \frac{\mathbf{v}^F}{U_{||}^S}, \qquad \hat{\mathbf{v}}^S = \frac{\mathbf{v}^S}{U_{||}^S},$$

where ρ_0^S is the initial value of the skeleton mass density and $U_{||}^S = \sqrt{(\lambda^S + 2\mu^S)/\rho_0^S}$ is a velocity of a longitudinal wave in an unbounded elastic medium. Also one has

$$\hat{x} = \frac{x}{U_{||}^S \tau}, \qquad \hat{t} = \frac{t}{\tau}, \qquad \hat{\mathbf{u}} = \frac{\mathbf{u}}{U_{||}^S \tau}, \qquad \hat{p}^F = \frac{p^F}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\kappa} = \frac{\kappa}{(U_{||}^S)^2},$$

$$\hat{\pi} = \frac{\pi \tau}{\rho_0^S}, \qquad \hat{\beta} = \frac{\beta}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\lambda}^S = \frac{\lambda^S}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\mu}^S = \frac{\mu^S}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\alpha} = \alpha U_{||}^S.$$

After the change of variables and parameters the original system (1.1)-(1.6) keeps its form except of the right-hand side in the equation for the change of porosity. One gets there $-\Delta_n$. For typographical reasons we omit below the symbol $\hat{}$ characterising dimensionless quantities.

1.3. Dispersion equation for the bulk waves

Let us investigate propagation of bulk waves through the porous medium. We confine ourselves to the consideration of a 1D problem, i.e. we study the propagation of longitudinal waves only. In 1D case the system (1.1)-(1.6) takes the following form (for convenience strain tensor e^S has been introduced and we have assumed that $\beta = 0$):

$$\frac{\partial \rho^F}{\partial t} + \frac{\partial}{\partial x}\left(\rho^F v^F\right) = 0, \tag{1.7}$$

$$\frac{\partial \rho^S}{\partial t} + \frac{\partial}{\partial x}(\rho^S v^S) = 0. \tag{1.8}$$

$$\rho^F \left[\frac{\partial}{\partial t} + (v^F, \frac{\partial}{\partial x})\right] v^F + \kappa \frac{\partial \rho^F}{\partial x} + \pi(v^F - v^S) = 0, \tag{1.9}$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v^S, \frac{\partial}{\partial x})\right] v^S - (\lambda^S + 2\mu^S)\frac{\partial e^S}{\partial x} - \pi(v^F - v^S) = 0, \tag{1.10}$$

$$\frac{\partial e^S}{\partial t} = \frac{\partial v^S}{\partial x}, \tag{1.11}$$

$$\frac{\partial \Delta_n}{\partial t} + \left(v^S, \frac{\partial}{\partial x}\right)\Delta_n + n_E \frac{\partial}{\partial x}(v^F - v^S) = -\Delta_n, \tag{1.12}$$

Consider the propagation of the harmonic waves whose frequency is ω and wave number is k. Below we use the following dimensionless parameters: $\hat{\omega} = \omega\tau$ and $\hat{k} = kU_{||}^S \tau$ (the upper symbol $\hat{}$ is again omitted in further consideration). Substituting solutions in the form

$$\rho^F - \rho_0^F = R^F \exp\left(i\left(kx - \omega t\right)\right), \qquad \rho^S - \rho_0^S = R^S \exp\left(i\left(kx - \omega t\right)\right),$$

$$v^F = V^F \exp\left(i\left(kx - \omega t\right)\right), \qquad v^S = V^S \exp\left(i\left(kx - \omega t\right)\right), \tag{1.13}$$

$$e^S = E \exp\left(i\left(kx - \omega t\right)\right), \qquad \Delta_n = D \exp\left(i\left(kx - \omega t\right)\right)$$

into equation system (1.7)-(1.12) one gets the system of algebraic equations for the unknown amplitudes, namely:

$$\omega R^F - krV^F = 0,$$
$$\omega R^S - kV^S = 0,$$

$$\omega r V^F - k c_f^2 R^F + i\pi(V^F - V^S) = 0,$$
$$\omega V^S - kE - i\pi(V^F - V^S) = 0,$$
$$\omega E + kV^S = 0,$$
$$(\omega + i)D - kn_E(V^F - V^S) = 0. \tag{1.14}$$

Here $r = \rho_0^F/\rho_0^S$, $c_f = U^F/U_{||}^S$, and sound velocity in a fluid $U^F = \sqrt{\kappa}$. Requesting that the determinant of this system must vanish yields the dispersion equation for longitudinal waves:

$$\mathcal{F}(k,\omega) = 0, \tag{1.15}$$

where

$$\mathcal{F}(k,\omega) = r(\omega^2 - c_f^2 k^2)(\omega^2 - k^2) + i\omega\pi\left((1+r)\omega^2 - k^2(1+rc_f^2)\right). \tag{1.16}$$

It should be reminded that similar to our previous research on surface waves [14-17], we consider the solutions of (1.7)-(1.12) in the absence of external forces. In this case it is necessary to derive ω as a function of the real wave number $k \in R^1$. Thus, $\mathrm{Re}\omega/k$ defines the phase velocity of a wave and $\mathrm{Im}\omega$ gives its attenuation.

Our goal is to prove that solution $\omega_{P2}(k)$ of dispersion equation (1.15), corresponding to the Biot slow wave, possess a bifurcation. It takes place in some critical point k_{cr} (bifurcation point), in small neighbourhood of which solution of equation (1.15) splits into several branches.

2. Bifurcation of the Biot slow wave

Let us rewrite equation (1.15) as

$$r(\tilde{\omega}^2 - c_f^2)(\tilde{\omega}^2 - 1) + i\tilde{\omega}\frac{1}{\tilde{k}}\left((1+r)\tilde{\omega}^2 - (1+rc_f^2)\right) = 0, \tag{2.1}$$

where $\tilde{\omega} = \omega/k$ and $\tilde{k} = k/\pi$. Obviously, for the case $k \gg 1$ (high-frequency range) equation (2.1) has the roots (note that here $1/\tilde{k} \ll 1$ is assumed to be a small parameter)

$$\tilde{\omega}_{P1} = \pm 1 - \frac{i}{2}\frac{1}{\tilde{k}} \mp \frac{4 + r - rc_f^2}{8r(1-c_f^2)}\frac{1}{\tilde{k}^2} + O\Big(\frac{1}{\tilde{k}^3}\Big) \tag{2.2}$$

and

$$\tilde{\omega}_{P2} = \pm c_f - \frac{i}{2r}\frac{1}{\tilde{k}} - \frac{1 - c_f^2(1+4r)}{8r^2(1-c_f^2)(\pm c_f)}\frac{1}{\tilde{k}^2} + O\Big(\frac{1}{\tilde{k}^3}\Big), \tag{2.3}$$

which define the velocities and attenuations of forward and backward directed longitudinal waves of the first and second kinds, respectively. It is evident, that in the high frequency limit, phase velocities of P1 and P2 waves do not depend on frequency ω.

Next let us consider low-frequency range, when $k \ll 1$ and, consequently, $\tilde{k} \ll 1$. Solutions of equation (2.1) are sought in the following form:

$$\tilde{\omega} = \tilde{\omega}_0 + \tilde{k}\tilde{\omega}_1 + \tilde{k}^2\tilde{\omega}_2 + \dots \tag{2.4}$$

For the longitudinal P1 wave of forward and backward directions one obtains:

$$\tilde{\omega}_{P1} = \pm\sqrt{\frac{1+rc_f^2}{1+r}} - \tilde{k}\frac{ir(1-c_f^2)^2}{2(1+rc_f^2)(1+r)^2}$$
$$\pm\tilde{k}^2\sqrt{\frac{1+r}{1+rc_f^2}}\frac{r^2(1-c_f^2)^3(2(1-r)(1+rc_f^2)+1-c_f^2)}{8(1+r)^4(1+rc_f^2)^2} + O(\tilde{k}^3). \tag{2.5}$$

Nowever for the P2 wave construction of asymptotic solution for the corresponding root of (2.1) is much more complicated. We prove later on that there exists some critical value of wave number k_{cr}, below which longitudinal wave of the second kind is not propagatory. Thus, asymptotic expansion of corresponding root of (2.1) has a different structure depending on whether wave number of P2 wave is lower or higher then its critical value k_{cr}.

Substitution of (2.4) into (2.1) yields for the forward directed P2 wave:

$$\tilde{\omega}^f_{P2} = -i\frac{rc_f^2}{1+rc_f^2}\tilde{k} - i\frac{r^3c_f^4(1+rc_f^4)}{(1+rc_f^2)^4}\tilde{k}^3 + O(\tilde{k}^4). \tag{2.6}$$

Solution for the backward directed P2 wave is sought in the form

$$\tilde{\omega} = \frac{1}{\tilde{k}}\tilde{\omega}_0 + \tilde{\omega}_1 + \tilde{k}\tilde{\omega}_2 + \dots \tag{2.7}$$

and it leads to the expansion

$$\tilde{\omega}^b_{P2} = -i\frac{1+r}{r}\frac{1}{\tilde{k}} + i\frac{r(r+c_f)}{(1+r)^2}\tilde{k} + O(\tilde{k}^2). \tag{2.8}$$

Obviously, expansions (2.6) and (2.8) consist of the imaginary terms only. The latter means that phase velocity of P2 wave is equal to zero, i.e. the wave is not propagatory. However, these expansions are valid only if wave number k is less then some critical value k_{cr}. In other words there exists a bifurcation point k_{cr} in small neighbourhood of which solution of equation (1.15) splits into several branches. Let us prove this statement. Consider dispersion equation (1.15). It is easy to see that exact solution for P2 wave is given by:

$$k^2 = \frac{1}{2rc_f^2}\Big(r\omega^2(1+c_f^2) + i\pi\omega(1+rc_f^2)$$
$$+\sqrt{r^2\omega^4(1-c_f^2)^2 - \pi^2\omega^2(1+rc_f^2)^2 + 2ir\pi\omega^3(1-c_f^2)(1-rc_f^2)}\ \Big) \tag{2.9}$$

Proposition. There exists some critical value of wave number $k_{cr} \in R^+$ such that:

a) if $0 < k < k_{cr}$ then equation (2.9) has two pure imaginary roots $\omega_1(k)$ and $\omega_2(k)$, $\mathrm{Re}\omega_j(k) = 0,\ j = 1, 2$;

b) if $k = k_{cr}$ then equation (2.9) has one multiple pure imaginary root, i.e. $\omega_1(k) = \omega_2(k)$, $\mathrm{Re}\omega_j(k) = 0,\ j = 1, 2$;

c) if $k > k_{cr}$ then equation (2.9) has no pure imaginary roots.

Proof. Applying the change $\omega = -i\pi\Omega,\ \Omega \geq 0$, equation (2.9) can be rewritten as

$$F_1(\Omega) = F_2(\Omega), \tag{2.10}$$

where

$$F_1(\Omega) = \Omega\sqrt{\Omega^2r^2(1-c_f^2)^2 - 2r\Omega(1-c_f^2)(1-rc_f^2) + (1+rc_f^2)^2}, \tag{2.11}$$

$$F_2(\Omega) = 2rc_f^2\tilde{k}^2 + \Omega^2 r(1+c_f^2) - \Omega(1+rc_f^2), \tag{2.12}$$

and, as above, $\tilde{k} = k/\pi$. It should be noted here that function under the square root $g(\Omega) = \Omega^2r^2(1-c_f^2)^2 - 2r\Omega(1-c_f^2)(1-rc_f^2) + (1+rc_f^2)^2$ is always positive. Consider equation (2.10). First let us investigate behaviour of functions $F_1(\Omega)$, $F_2(\Omega)$ as $\Omega \to \infty$. Obviously,

$$\frac{F_1(\Omega)}{\Omega^2} \sim r(1-c_f^2) \text{ and } \frac{F_2(\Omega)}{\Omega^2} \sim r(1+c_f^2) \tag{2.13}$$

i.e. $F_2(\Omega)$ is steeper than $F_1(\Omega)$. Consequently, if $\tilde{k} = 0$ then function $(F_2 - F_1)(\Omega)$ has two real roots: $\Omega = 0$ and some Ω_*, so that $(F_2 - F_1)(\Omega) < 0$ in $(0,\ \Omega_*)$.

Next we calculate stationary points for $F_1(\Omega)$ and $F_2(\Omega)$ and inflation points for $F_1(\Omega)$. One can easily check, that function $F_1(\Omega)$ has two stationary points, namely $\Omega_1^{(1)} \approx (1 + (1+7r)c_f^2)/(2r)$ and $\Omega_1^{(2)} \approx (1 + (1-5r)c_f^2)/r$ and function $F_2(\Omega)$ has one stationary point $\Omega_2 = (1+rc_f^2)/(2r(1+c_f^2))$ such that:

$$\Omega_2 < \Omega_1^{(1)} < \Omega_1^{(2)}. \tag{2.14}$$

Function $F_1(\Omega)$ has unique inflation point

$$\Omega_{inf} \approx \frac{1 - rc_f^2 - \sqrt[3]{2}\sqrt[3]{r}\sqrt[3]{c_f^2}(1 - \sqrt[3]{2}\sqrt[3]{r}\sqrt[3]{c_f^2})}{r(1-c_f^2)} \tag{2.15}$$

and it being known that $\Omega_1^{(1)} < \Omega_{inf} < \Omega_1^{(2)}$ as well as that $F_1(\Omega)$ is concave if $\Omega < \Omega_i$ and $F_1(\Omega)$ is convex if $\Omega > \Omega_i$. Preceding analysis allows us to conclude that there exists unique point of tangency of functions $F_1(\Omega)$ and $F_2(\Omega)$ (see Fig.1).

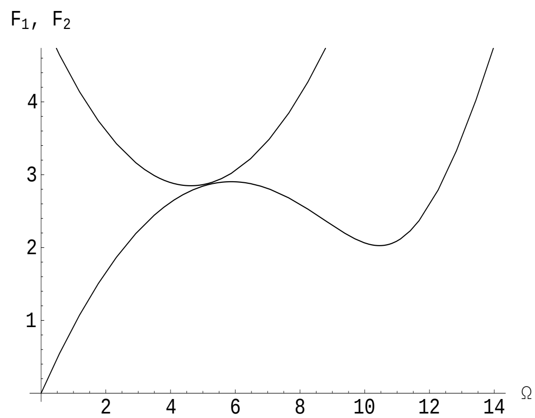

Figure 1: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} = \ \tilde{k}_{cr} = 16.97$

Thus, using that $F_1^{'}(\Omega) = F_2^{'}(\Omega)$ one can define point of tangency, i.e. critical value Ω_{cr}:

$$\Omega_{cr} \approx \frac{1}{2r} + 2c_f^2(1 + 3rc_f^2 - 2c_f^2), \tag{2.16}$$

which is positive by virtue of physical sense. Corresponding critical value of wave number is defined from equation (2.10) and is given by:

$$\tilde{k}_{cr} \approx c_f\Big(1 + \frac{1}{2rc_f^2}\Big), \quad \text{i.e.} \quad k_{cr} \approx c_f\Big(1 + \frac{1}{2rc_f^2}\Big)\pi. \tag{2.17}$$

Therefore, it was proven that there exist some critical real value k_{cr} for which equation (2.9) has one multiple imaginary root $\omega_{cr} = -i\pi\Omega_{cr}$.

Next we prove that if $k < k_{cr}$ then equation (2.9) has two pure imaginary roots and if $k > k_{cr}$ then equation (2.9) has no imaginary roots. Consider following multiplicative expansions

$$k = k_{cr}\Big(1 \pm \epsilon k_1 \pm \epsilon^2 k_2 + \dots\Big)$$

and

$$\omega = \Omega_{cr}\Big(\frac{\omega_{cr}}{\Omega_{cr}} + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots\Big), \tag{2.18}$$

where ϵ is a small parameter. Note that here any ϵ can be chosen. For example one may set $\epsilon = c_f^\delta,\ 0 < \delta \leq 1$. Subsitution of (2.18) into (2.9) yields the bifurcation equation. >From its $O(\epsilon)$ approximation it follows that $k_1 = 0$. From the next $O(\epsilon^2)$ approximation one has:

$$\pm k_2 = \frac{1}{4}\frac{\omega_1^2\Omega_{cr}^2}{k_{cr}^2}\mathcal{A} \tag{2.19}$$

with

$$\mathcal{A} = \frac{1+c_f^2}{c_f^2} + \frac{1-c_f^2}{c_f^2 g(\Omega_{cr})\sqrt{g(\Omega_{cr})}}\Big(- r^3(1-c_f^2)^3\Omega_{cr}^3 + 3r^2(1-c_f^2)^2(1-rc_f^2)\Omega_{cr}^2$$
$$-3r(1-c_f^2)(1+r^2c_f^4)\Omega_{cr} + (1-rc_f^2)(1+rc_f^2)^2\Big) > 0. \tag{2.20}$$

It is obvious, that for given $k_2 > 0$ equation (2.19) has two real solutions for ω_1 if plus sign is chosen in its left-hand side. The letter means that we consider expansion $k = k_{cr}\Big(1 + \epsilon^2 k_2 + \dots\Big)$ and $k > k_{cr}$. Consequently, equation (2.10) has no any solution (see Fig.2).

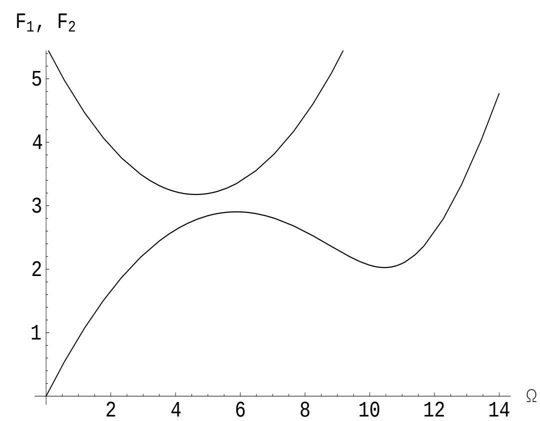

Figure 2: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} = 17.5 >\ \tilde{k}_{cr}$

Vice versa, if $k = k_{cr}\Big(1 - \epsilon^2 k_2 + \dots\Big) < k_{cr}$ then for given $k_2 > 0$ equation (2.19), as well as equation (2.9), has two imaginary roots (see Fig.3). Thus, **Proposition** was proven.

Remark. One can also prove **Proposition** applying the same procedure to the dispersion equation (1.15). Taking into account that $\mathcal{F}(k_{cr}, \omega_{cr}) = 0$ and $\mathcal{F}_\omega^{'}(k_{cr}, \omega_{cr}) = 0$, one can define critical values k_{cr} and ω_{cr}. Next one has to substitute expansions (2.18) into (1.15). As above one obtains at $O(\epsilon)$ approximation that $k_1 = 0$. From the next $O(\epsilon^2)$ approximation one gets:

$$\pm k_2 = \frac{1}{2}\frac{\omega_1^2\Omega_{cr}^2}{k_{cr}^2}\mathcal{A}_1 \tag{2.21}$$

with

$$\mathcal{A}_1 = \frac{-6r\Omega_{cr}^2 + 3(1+r)\Omega_{cr} - r(1+c_f^2)\tilde{k}_{cr}^2}{-r(1+c_f^2)\Omega_{cr}^2 + (1+rc_f^2)\Omega_{cr} - 2rc_f^2\tilde{k}_{cr}^2} > 0. \tag{2.22}$$

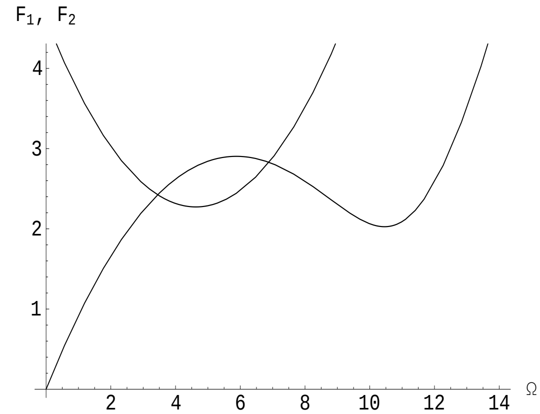

Figure 3: Numerical example: $r = 0.1,\ c_f = 0.3,\ \tilde{k} = 16.0 <\ \tilde{k}_{cr}$

Analogously to (2.19), equation (2.21) has two real solutions for ω_1 if for given $k_2 > 0$ plus sign is chosen in its left-hand side.

Therefore we conclude that P2 wave is not propagatory if its wave number is less than critical value k_{cr}. Otherwise, the frequency of P2 wave is given by

$$\omega_{P2} = \Omega_{cr}\left(\frac{\omega_{cr}}{\Omega_{cr}} + \epsilon\omega_1\right) + O(\epsilon^2) \tag{2.23}$$

with

$$\omega_1 = 2\frac{k_{cr}}{\Omega_{cr}}\sqrt{\frac{k_2}{\mathcal{A}}}. \tag{2.24}$$

Consequently, phase velocity of forward and backward directed P2 wave is defined by $\pm\mathrm{Re}(\omega_{P2})/k$, where $k = k_{cr}\left(1 + \epsilon^2 k_2\right) + O(\epsilon^3)$.

Numerical example. Second formula in (2.17) shows clearly that critical wave number is directly proportional to the permeability π. Thus, corresponding critical wave length's dependence on π is through an inverse proportionality. To obtain estimates of critical wave length, we take the following typical values of parameters [18]: $\rho_0^F = 0.2 \cdot 10^3\ \frac{kg}{m^3},\ \ \rho_0^S = 2.0 \cdot 10^3\ \frac{kg}{m^3},\ \ U^F = 0.9 \cdot 10^3\ \frac{m}{s},\ \ U^S = 3.0 \cdot 10^3\ \frac{m}{s}$ i.e. $r = 0.1$ and $c_f = 0.3$ as in given above figures. Also $\tau = 4 \cdot 10^{-6}\ \ s$. For $\pi = 10^9\ \frac{kg}{m^3 \cdot s}$, $\pi = 10^8\ \frac{kg}{m^3 \cdot s}$, and $\pi = 10^7\ \frac{kg}{m^3 \cdot s}$ critical wave length is equal to 0.22 cm, 2.22 cm, and 22.21 cm, respectively. Corresponding critical frequency for the lowest value of π is about 20 kHz. We conclude that the Biot slow wave becomes to be propagatory with rather short wave length and, consequently, it cannot be detected in the low frequency range of interest in seismology $(1 - 100\ Hz)$.

3. Conclusions

The results presented in the paper concern propagation of the Biot slow wave at low frequencies. For the first time it was proven analytically that longitudinal waves of the second kind is not propagatory if its wave number is lower than some critical value. This critical wave number is a bifurcation point, above which longitudinal wave of the second kind becomes to be propagatory. Asymptotical formulae (2.17) derived here show clearly that critical wave number is directly proportional to permeability π. However even for low permeable materials critical wave number is rather big. Thus we conclude that propagating slow modes have rather short wave length and, consequently, they cannot be observable at very low frequencies of interest in seismology.

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