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Self-regularization of projection methods with a posteriori discretization level choice for severely ill-posed problems

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Abstract

It is well known that projection schemes for certain linear ill–posed problems $Ax = y$ can be regularized by a proper choice of the discretization level only, where no additional regularization is needed. The previous study of this self–regularization phenomenon was restricted to the case of so–called moderately ill–posed problems, i.e., when the singular values $\sigma_k(A)$, $k = 1, 2, ...$, of the operator A tend to zero with polynomial rate. The main accomplishment of the present paper is a new strategy for a discretization level choice that provides optimal order accuracy also for severely ill–posed problems, i.e., when $\sigma_k(A)$ tend to zero exponentially. The proposed strategy does not require a priori information regarding the solution smoothness and the exact rate of $\sigma_k(A)$.

1 Introduction

In this paper, we wish to recover an element x of some Hilbert space $\mathbb{X}$ from observations near

$$y = Ax, \tag{1}$$

where A is some injective linear compact and infinitely smoothing operator acting from $\mathbb{X}$ into another Hilbert space $\mathbb{Y}$, while the solution $x = A^{-1}y$ has only a finite smoothness in some sense. The inner product and corresponding norm on each of the Hilbert spaces $\mathbb{X}$ and $\mathbb{Y}$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. (It will be always clear from the context which space is concerned.)
Inverse problems involving infinitely smoothing operators often arise in scientific context, ranging from tomography [17], non-destructive detection [6], to satellite geodetic explorations [7]. These problems are severely ill–posed in the sense that noisy data y_δ with arbitrarily small noise level δ,

$$\|y - y_\delta\| \le \delta,$$

can lead to disproportionally large deviations in the solution.

More precisely, if $x = A^{-1}y$ belongs to some subspace U continuously embedded in $\mathbb{X}$, and the singular values τ_k of the canonical embedding operator $J_U : U \to \mathbb{X}$ tend to zero with polynomial rate, say $O(k^{-\mu})$, (solution has a finite smoothness), while the singular values $\sigma_k(A)$ of the operator A tend to zero exponentially (A is an infinitely smoothing operator), then, as it was shown by Mair [12], one can expect to recover the solution $x = A^{-1}y$ in the space $\mathbb{X}$ with the accuracy $O\left(\ln^{-\mu} \frac{1}{\delta}\right)$ only, where δ is the noise level and $\mu > 0$ can be taken as the smoothness index of x. Thus, in order to obtain stable approximations to $x = A^{-1}y$, regularization methods have to be applied. Order–optimal regularization methods of the worst–case error for severely ill–posed problems were constructed in [12], [19], [3],[4]. These methods require an exact knowledge of the smoothness index μ and/or the exponential rate of $\sigma_k(A)$. However, such a priori information is rarely available in practice. This drawback was overcome recently in [9], [16], where it was shown that the combination of a well-known discrepancy principle and the simplest version of Tikhonov's regularization method is always order–optimal for severely ill–posed problems. It should be noted that this combination does not require the a priori information mentioned above.

Note, that regularized problems are usually defined in an infinite–dimensional setting and have to be discretized for an implementation.

At the same time, in practice even noisy observations

$$y_\delta = Ax + \delta\xi, \quad \|\xi\| \le 1, \tag{2}$$

are possible only in a discretized form. To be more precise, we have only a vector $\{y_{\delta,i}\}_{i=1}^n \in \mathbb{R}^n$ defined by $y_{\delta,i} = \langle y_\delta, \varphi_i\rangle = \langle Ax, \varphi_i\rangle + \delta\langle \xi, \varphi_i\rangle$, $i = 1, 2, ..., n$, where

$$V_n = \operatorname{span}\{\varphi_1, ..., \varphi_n\} \subset \mathbb{Y}$$

is some finite–dimensional subspace.

It was shown in [14] that an effective projection scheme with properly choosen discretization level n allows to obtain a regularization effect; no additional regularization of the problem is needed. This phenomenon is sometimes called self–regularization or regularization by projection. Self–regularization with a priori chosen discretization level was analyzed in [1], [2], [15], [13]. In these papers it is assumed that the smoothness index μ of $x = A^{-1}y$ is known a priori. An a posteriori regularization strategy, that yields optimal order of accuracy without using knowledge of μ, was proposed recently in [10], [18].

But it is worth to note that the previous study of self–regularization was restricted to the case of so–called moderately ill–posed problems, i.e., when the singular values of the operator A tend to zero with polynomial rate. To the best of our knowledge there are no papers devoted to the analysis of the self–regularization phenomenon for severely ill–posed problems.

The main accomplishment of this paper is to propose a new strategy for a discretization level choice that provides the accuracy of optimal order $O\left(\ln^{-\mu}\frac{1}{\delta}\right)$ for severely ill–posed problems and does not require an exact knowledge of μ and $\{\sigma_k(A)\}$. For fixed data one has to compute the discretized solution for a number of subsequent discretization levels, then, from this series, our algorithm will select one with optimal order of accuracy.

In Section 2 our assumptions will be explained and motivated. The algorithm will be developed in Section 3, and in Section 4 some easy numercal experiments will be performed.

2 Pojection methods

In order to define regularization by projection for the linear ill–posed operator equation (1) we consider a sequence of finite–dimensional subspaces V_n, $n = 1, 2, \dots$, $\dim V_n = n$, whose union is dense in $\mathbb{Y}$, and the corresponding sequence of projected equations

$$Q_n A x = Q_n y_\delta \tag{3}$$

where Q_n denotes the orthogonal projection onto V_n. Let $A_n = Q_n A$. A regularized approximate solution x_n^δ is determined from (3) as the unique element of $\mathbb{X}$ that has minimal norm among all minimizers of the residual $\|A_n x - Q_n y_\delta\|$. It is well known that the unique element minimizing the residual is given by $x_n^\delta = A_n^\dagger y_\delta$, where $A_n^\dagger$ denotes the Moore–Penrose generalized inverse of A_n.

Now we are going to establish the main assumptions of this paper which will be motivated by means of a special projection method: the so–called method of least error (also called dual least squares method). In the case

$$\ker(A) = 0, \ \ker(A^*) = 0, \tag{4}$$

this method has the property

$$\|x - x_n\| = \min_{u \in A^* V_n} \|x - u\|,$$

where x_n is the solution of (3) for exact data.

Moreover, it is well known that $x_n^\delta = A_n^\dagger y_\delta \in A^* V_n$ holds. The approximate solution x_n^δ can be represented in the form

$$x_n^\delta = \sum_{k=1}^{n} d_k A^* \varphi_k,$$

where the vector $(d_1,\, d_2,\, ...,\, d_n)$ is found from the following system of linear equations

$$\sum_{k=1}^{n} d_k \langle A^*\varphi_k, A^*\varphi_l\rangle = \langle \varphi_l, y_\delta\rangle, \ l = 1, 2, ..., n.$$

Keeping in mind that $x = A^{-1}y$ has a finite smoothness, it is natural to assume that for some $c_\mu \geq 1$ and $\mu > 0$

$$\|A^{-1}y - A_n^\dagger y\| = \inf_{u \in A^* V_n} \|A^{-1}y - u\| \leq c_\mu n^{-\mu}, \ \mu \in [\mu_0, \mu_1], \ n = 1, 2, ..., \tag{5}$$

and one knows only a finite interval $[\mu_0, \mu_1]$ containing the unknown μ which can be considered as the smoothness index of $x = A^{-1}y$.

Following [5], the influence of non-vanishing data noise can be estimated as

$$\|A_n^\dagger y - A_n^\dagger y_\delta\| \leq \lambda_n^{-1/2} \|Q_n(y - y_\delta)\| \leq \lambda_n^{-1/2}\delta, \tag{6}$$

where λ_n is the smallest positive eigenvalue of $A_n A_n^* = Q_n A A^* Q_n$, and $\lambda_n^{\frac{1}{2}} = O(\sigma_n(A))$. For operators involved in severely ill–posed problems, $\sigma_n(A),\ n = 1, 2, ...,$ tend to zero exponentially. Therefore, in view of (6) it is natural to assume that for some $a > 0,\ q > 1$

$$e^{an}\delta \leq \|A_n^\dagger y - A_n^\dagger y_\delta\| \leq e^{anq}\delta. \tag{7}$$

This assumption seems to be realistic, when A, e.g., is a Fredholm integral operator with analytic kernel. If the operator A is not well studied then the exponent a is rarely known exactly. In this case q reflects the magnitude of a gap in our knowledge of A and is assumed to be known.

Proposition 1 *Let the assumptions (4), (5), (7) hold. Then for*

$$n_* = \min\{n : \ c_\mu n^{-\mu} \leq \delta e^{aqn}\}$$

we obtain

$$\|A^{-1}y - x_{n_*}^\delta\| \leq 2^{\mu+1} e c_\mu (aq)^\mu \ln^{-\mu} \frac{1}{\delta}.$$

Proof. Consider

$$\bar{n} := \left\lfloor (aq)^{-1} \left[\ln \frac{1}{\delta} - \mu \ln \ln \frac{1}{\delta} + \ln c_\mu (2aq)^\mu \right] \right\rfloor$$

where $\lfloor \tau \rfloor$ means the smallest integer that is larger than τ.

Without loss of generality we may assume that for δ small enough

$$\mu \ln \ln \frac{1}{\delta} - \ln c_\mu (2aq)^\mu \le \frac{1}{2} \ln \frac{1}{\delta}.$$

Then

$$c_\mu \bar{n}^{-\mu} \le c_\mu (2aq)^\mu \ln^{-\mu} \frac{1}{\delta} \le \delta e^{aq\bar{n}},$$

and from the definition of n_* we conclude that

$$n_* \le \bar{n}.$$

It is now easy to derive from assumptions (5), (7) the desired bound:

$$\begin{aligned} \|A^{-1}y - x_{n_*}^\delta\| &\le \|A^{-1}y - A_{n_*}^\dagger y\| + \|A_{n_*}^\dagger y - A_{n_*}^\dagger y_\delta\| \le \\ &\le c_\mu n_*^{-\mu} + e^{aqn_*}\delta \le 2\delta e^{aqn_*} \le 2\delta e^{aq\bar{n}} \le \\ &\le 2^{\mu+1} e c_\mu (aq)^\mu \ln^{-\mu} \frac{1}{\delta}. \qquad \blacksquare \end{aligned}$$

Note that $n = n_*$ is an optimal choice for the discretization level, because under the assumptions (5), (7) it balances the discretization error with the data noise. But this optimal choice requires a priori information on the parameters μ, c_μ, a, q and for this reason it is not practicable. In the next section we will introduce our adaptive a posteriori discretization level choice that yields the optimal rate $O(\ln^{-\mu} \frac{1}{\delta})$ without using knowledge of μ, c_μ, a.

3 Adaptive discretization level choice

In this section we define a new principle for an a posteriori choice of the discretization level. It distinguishes from the residual principle discussed in [20], [10] for moderately ill–posed problems, where the discretization level is chosen minimal with the property $\|Ax_n^\delta - y_\delta\| \le c\delta$. Since such a choice needs the function y_δ, it is possible only under accessing to infinitely many discrete data.

The idea of our principle has its origin in the paper [11] devoted to statistical estimation from direct white noise observations that corresponds to (1) with identity operator A, but with random noise data. In the context of ill–posed problems of the form (1) with compact operators acting along some Hilbert scale (the case of moderately ill–posed problems), but still with random noise, this idea has been realized in [8] for adaptive estimating the value of a linear functional on the solution of (1). If, as it is usual for statisticians, we will treat the discretization error term (5) and the data noise term (6) as bias and variance, respectively, then the idea is to choose the minimal n for which the bias is still dominated by the variance. For pseudodifferential equations of negative order with deterministic noise (moderately ill–posed problem) the same idea was used in the paper [18] also devoted to adaptive regularization by projection. But in all papers just listed it has been essentially used that the order of the data noise term (variance) is a priori known. In view of (7) such an assumption is rather restrictive. Therefore we will combine the above–mentioned idea with successive testing the hypothesis that the exponent aq from (7) is less than some term of the progression

$$B_q = \left\{ b_j : \ b_j = b_0 q^j, \ j = 0, 1, ... \right\}.$$

If μ_0 and b_0 are the minimal expected smoothness index from (5) and the minimal expected exponent aq from (7), respectively, then in view of Proposition 1 it is natural to choose the discretization level from the finite set

$$\mathbb{N}_\delta = \{n : \ n = 1, 2, ...; \ n \le N\},$$

where

$$N = b_0^{-1}(\ln \frac{1}{\delta} - \mu_0 \ln \ln \frac{1}{\delta}).$$

To shorten notation we assume that N is an integer. Let

$$M_j = \left\{ m \in \mathbb{N}_\delta : \ \forall k, n \ge m; \ k, n \in \mathbb{N}_\delta \ \ \|x_n^\delta - x_k^\delta\| \le 2\delta[e^{b_j n} + e^{b_j k}] \right\} \tag{8}$$

Let us study the properties of the sequence

$$m_j = \min\{m : \ m \in M_j\}, \ j = 1, 2, ...$$

Lemma 1 *The sequence $\{m_j\}$ is monotone non–increasing:*

$$N \ge m_1 \ge m_2 \ge ... \ge m_j \ge ... \ge 1.$$

Proof. It follows immediately from the fact that the set M_j becomes larger if j grows, i.e. $M_j \subset M_{j+1}$ and

$$m_j = \min\{m : \ m \in M_j\} \ge \min\{m : \ m \in M_{j+1}\} = m_{j+1}. \qquad \blacksquare$$

Define the integer ν such that

$$b_\nu = \max\{b_j : \ b_j \in B_q, \ b_j \le a\}.$$

Without loss of generality we may assume that $\nu > 1$.

Lemma 2 *Let the assumtions (4), (5), (7) hold. Assume that δ is small enough such that*

$$\delta^{q-1} \ln^{\mu_0(q-1)} \frac{1}{\delta} < \frac{1}{8}. \tag{9}$$

Then for any $j = 1, 2, ..., \nu - 1$

$$m_j \ge q^{-1}(N - b_0^{-1} \ln 6).$$

Proof. We prove the lemma considering the cases $n_* < m_j$ and $n_* \geq m_j$ separately, where n_* is defined in Proposition 1.

Assume that $m_j > n_*$. Then for $k = N$, $n = m_j$ from the definition (8) of m_j and assumptions (5), (7) we obtain

$$\begin{aligned}
4\delta e^{b_j N} &\geq 2\delta[e^{b_j N} + e^{b_j m_j}] \geq \|x_N^\delta - x_{m_j}^\delta\| \geq \|A^{-1}y - x_N^\delta\| \\
&- \|A^{-1}y - x_{m_j}^\delta\| \geq \|A_N^\dagger y - A_N^\dagger y_\delta\| - \|A^{-1}y - A_N^\dagger y\| \\
&- c_\mu m_j^{-\mu} - \delta e^{aqm_j} \geq e^{aN}\delta - c_\mu N^{-\mu} - c_\mu m_j^{-\mu} - \delta e^{aqm_j} \\
&\geq e^{aN}\delta - 2c_\mu m_j^{-\mu} - \delta e^{aqm_j} \geq e^{aN}\delta - 3e^{aqm_j}\delta.
\end{aligned}$$

Using (9) one can rewrite it as

$$\begin{aligned}
3e^{aqm_j} &\geq e^{aN}\left(1 - 4e^{(b_j - a)N}\right) \geq e^{aN}\left(1 - 4e^{(b_{\nu-1} - b_\nu)N}\right) \\
&= e^{aN}\left(1 - 4e^{q^{\nu-1}b_0(1-q)N}\right) \geq e^{aN}\left(1 - 4\delta^{q-1}\ln^{\mu_0(q-1)}\frac{1}{\delta}\right) \\
&\geq \frac{1}{2}e^{aN}.
\end{aligned}$$

Thus,

$$m_j \geq q^{-1}(N - a^{-1}\ln 6) \geq q^{-1}(N - b_0^{-1}\ln 6),$$

and we obtain the statement of the lemma under the assumption that $m_j > n_*$.

Now let us consider the remaining case. Under the assumption $n_* \geq m_j$ we can repeat the previous argument with m_j replaced by n_*. It gives the inequality

$$n_* \geq q^{-1}(N - b_0^{-1}\ln 6) \geq b_1^{-1}\ln\frac{1}{\delta} - b_1^{-1}\mu_0\ln\ln\frac{1}{\delta} + O(1).$$

On the other hand, from the proof of Proposition 1 one knows that

$$n_* \leq \bar{n} \leq (aq)^{-1}\ln\frac{1}{\delta} - \mu(aq)^{-1}\ln\ln\frac{1}{\delta} + O(1),$$

where $aq > b_1$ by definition. Therefore the hypothesis, that for some $j \in \{1, 2, ..., \nu - 1\}$ $n_* \geq m_j$, leads to the relation $\ln\frac{1}{\delta} = O\left(\ln\ln\frac{1}{\delta}\right)$ being in contradiction with the assumption that δ is small enough. Thus for such δ the case $n_* \geq m_j$ is impossible. This completes the proof of the lemma. ■

As a consequence it has been established that

$$n_* < q^{-1}(N - b_0^{-1}\ln 6). \tag{10}$$

Lemma 3 *Assume that $b_j \in B_q$ is such that $b_j \geq aq$. Then under the assumptions (4), (5), (7)*

$$\|A^{-1}y - x_{m_j}^\delta\| \leq 6e2^{\mu+1}c_\mu b_j^\mu \ln^{-\mu}\frac{1}{\delta},$$

and

$$m_j < q^{-1}(N - b_0^{-1}\ln 6).$$

Proof. Consider

$$n_j = \min\left\{n:\ c_\mu n^{-\mu} \le \delta e^{b_j n}\right\}.$$

With an argument like that in the proof of Proposition 1 we get the estimate

$$\|A^{-1}y - x_{n_j}^\delta\| \le 2\delta e^{b_j n_j} \le 2^{\mu+1} e c_\mu b_j^\mu \ln^{-\mu}\frac{1}{\delta}. \tag{11}$$

Note also that for any $k, n \in \mathbb{N}_\delta$ such that $k, n \ge n_j$

$$\begin{aligned}
\|x_n^\delta - x_k^\delta\| &\le \|A^{-1}y - x_n^\delta\| + \|A^{-1}y - x_k^\delta\| \le c_\mu n^{-\mu} + e^{aqn}\delta \\
&+ c_\mu k^{-\mu} + e^{aqk}\delta \le c_\mu n^{-\mu} + e^{b_j n}\delta + c_\mu k^{-\mu} \\
&+ e^{b_j k}\delta \le 2\delta\left[e^{b_j n} + e^{b_j k}\right].
\end{aligned}$$

It means that n_j belongs to the set M_j defined by (8), and

$$n_j \ge m_j := \min\{m:\ m \in M_j\}. \tag{12}$$

Then from (11) one has

$$\begin{aligned}
\|A^{-1}y - x_{m_j}^\delta\| &\le \|A^{-1}y - x_{n_j}^\delta\| + \|x_{n_j}^\delta - x_{m_j}^\delta\| \le 2\delta e^{b_j n_j} \\
&+ 2\delta\left[e^{b_j n_j} + e^{b_j m_j}\right] \le 6\delta e^{b_j n_j} \\
&\le 6e2^{\mu+1} c_\mu b_j^\mu \ln^{-\mu}\frac{1}{\delta},
\end{aligned}$$

as claimed.

It is now easy to derive the remaining assertion concerning m_j. Namely, from (10) and (12) it follows that

$$m_j \le n_j \le \min\left\{n:\ c_\mu n^{-\mu} \le \delta e^{aqn}\right\} = n_* < q^{-1}(N - b_0^{-1}\ln 6).$$

The lemma is proved. ∎

Now we are in a position to describe a new strategy for an adaptive discretization level choice.

First, we obtain a family of regularized approximate solutions $\{x_n^\delta\}$ associated with $n \in \mathbb{N}_\delta$. Second, for every $b_j \in B_q,\ j = 1, 2, ...,$ we choose adaptively the discretization level $m_j \in \mathbb{N}_\delta$ as minimal m from the set (8) until

$$m_j < q^{-1}(N - b_0^{-1}\ln 6).$$

Let m_l denote the maximal (or the first) m_j satisfying this condition. By the construction, such m_l corresponds to some $b_l = b_0 q^l \in B_q$.

The regularized approximate solution we are interested in is defined now as $x_{m_{l+2}}^\delta$, where m_{l+2} is the minimal m from the set (8) corresponding to $b_{l+2} = b_l q^2 \in B_q$.

We stress that the exact values of the parameters μ, c_μ and a from (5), (7) are not involved in the construction of $x_{m_{l+2}}^\delta$. It depends only on the three design parameters μ_0, b_0 and q reflecting our a priori knowledge of the problem.

We turn to the main result of this paper.

Theorem 1. *Assume that the conditions of Lemma 2 hold. Then*

$$\|A^{-1}y - x^{\delta}_{m_{l+2}}\| \le c \ln^{-\mu}\frac{1}{\delta},$$

where $c \le 6e2^{\mu+1}c_{\mu}(aq^4)^{\mu}$.

Proof. From the very definition it follows that

$$b_{\nu} \le a \le b_{\nu+1} \le aq \le b_{\nu+2} \le aq^2 \le b_{\nu+3} \le aq^3 \le b_{\nu+4} \le aq^4.$$

Then from Lemma 1 and Lemma 3 one has

$$m_{\nu+3} \le m_{\nu+2} < q^{-1}(N - b_0^{-1}\ln 6).$$

It means that $m_l \ge m_{\nu+2}$. On the other hand, Lemma 2 gives $m_l < m_{\nu-1}$. Therefore, m_l can take only the values $m_l = m_{\nu}$, $m_l = m_{\nu+1}$ or $m_l = m_{\nu+2}$, and as a consequence, $b_{l+2} \in \{b_{\nu+2}, b_{\nu+3}, b_{\nu+4}\}$. Thus, in any case $b_{l+2} \ge aq$, and the statement of the Theorem follows from Lemma 3. ■

The theorem just proven shows that the regularized approximate solution $x^{\delta}_{m_{l+2}}$ with adaptively chosen discretization level m_{l+2} yields the optimal rate of accuracy $O\left(\ln^{-\mu}\frac{1}{\delta}\right)$ without using knowledge of μ, c_{μ}, a.

4 Numerical results

As in [3], in order to demonstrate the performance of our method, we consider the integral equation with logarithmic kernel

$$Ax(t) := \int_0^1 \ln(t-\tau)x(\tau)d\tau = y(t), \quad t \in [2,3]. \tag{13}$$

Since $[0,1]\cap[2,3] = \emptyset$, the kernel is analytic with respect to t, τ, and the integral equation (13) is severely ill-posed in the above mentioned sense.

For testing the algorithm we consider three cases for which the solutions of (13) are known explicitly. We choose $y(t) = y_k(t),\ k = 1,2,3$, given by

$$\begin{aligned}
y_1(t) &= t\ln t - (t-\frac{1}{2})\ln(t-\frac{1}{2}) - \frac{1}{2},\\
y_2(t) &= \frac{t^2}{2}\ln t + \frac{(t-1)^2}{2}\ln(t-1) - (t^2 - t + \frac{1}{4})\ln(t-\frac{1}{2}) - \frac{3}{8},\\
y_3(t) &= \frac{t^2}{2}\ln t - \frac{t^2-1}{2}\ln(t-1) - \frac{t}{2} - \frac{1}{4}.
\end{aligned}$$

One easily checks that the functions

$$\begin{aligned}
x_1(\tau) &= \begin{cases} 1, & \tau \in [0,\frac{1}{2}],\\ 0, & \tau \in (\frac{1}{2},1],\end{cases}\\
x_2(\tau) &= \begin{cases} \tau, & \tau \in [0,\frac{1}{2}],\\ 1-\tau, & \tau \in (\frac{1}{2},1],\end{cases}\\
x_3(\tau) &= \tau.
\end{aligned}$$

are such that $Ax_k(t) = y_k(t),\ k = 1,2,3$, i.e. x_k is the solution of (13) for $y = y_k$.

Now we generate noisy data $y_\delta(t) = y_{\delta,k}(t),\ k = 1,2,3$, in the form of piecewise linear functions interpolating the values $y_{\delta,k}(t_i) = y_k(t_i) + \delta z_i$ at the points $t_i = 2 + \frac{i}{M},\ i = 0,1,2,\ldots,M$, where $M = 5000$, z_i are random numbers such that $|z_i| \le 1$, and δ characterizes the level of noise in the data taking the values $\delta = 10^{-7}$ or $\delta = 10^{-8}$.

In order to define regularization by projection for noisy equations $Ax = y_{\delta,k},\ k = 1,2,3$, with the operator A as in (13) and noisy data $y_{\delta,k}$, let us use the method of least error mentioned in Section 2. As test spaces let us take the finite–dimensional subspaces of piecewise constant functions

$$V_n = \operatorname{span}\{\varphi_1, \varphi_2, ..., \varphi_n\}, n = 1,2,\ldots,N,$$

where

$$\varphi_i(t) = \begin{cases} 1, & 2 + \frac{i-1}{n} \le t \le 2 + \frac{i}{n}, \\ 0, & \text{else.} \end{cases}$$

In this case the regularized solutions $x^\delta_{n,k} = A^\dagger_n y_{\delta,k},\ k = 1,2,3$, are defined as a linear combinations of the trial functions

$$A^*\varphi_i(\tau) = (2 + \frac{i}{n} - \tau)\ln(2 + \frac{i}{n} - \tau) - (2 + \frac{i-1}{n} - \tau)\ln(2 + \frac{i-1}{n} - \tau) - \frac{1}{n},$$
$$i = 1,2,\ldots,n; \quad \tau \in [0,1],$$

and the computations mentioned in Section 2 can be performed explicitly.

To demonstrate our algorithm we should indicate the values of the parameters μ_0, b_0 and q. For fixed δ the parameters μ_0 and b_0 can be chosen depending on the maximal discretization level N. In our numerical experiments N takes the values between 20 and 24, and we choose $\mu_0 = 0,\ b_0 = 0.8$.

The choice of q is of particular importance for our algorithm, because the error estimate presented in Theorem 1 crucially depends on this parameter. From (6) and (7) it follows that q depends mainly on the operator A. Therefore, one can choose this parameter using some test problem with known solution for the same operator A. As such a test problem we use here the equation (13) with noisy data $y_{\delta,1}(t),\ \delta = 10^{-8}$. The exact solution of this problem is $x_1(\tau)$. Applying our algorithm with different values of q we obtain the results presented in Table 1. Keeping in mind that the smoothness index μ for the solution $x_1(\tau)$ is relatively small (it can be estimated as $\mu = \frac{1}{2}$) one can not expect to reach high accuracy for the problem (13) with such a solution. Nevertheless, the results presented in Table 1 show that a reasonable choice for q would be $q = 1.5$.

Numerical results for problem (13) with noisy data $y_{\delta,k},\ k = 2,3$, are presented in Tables 2 and 3, respectively. They show that within the framework of our algorithm the same value $q = 1.5$ allows to reach a good level of accuracy for both problems. At the same time, it should be noted that the error of the projection scheme has a very unstable behavior. For example, for problem (13) with noisy data $y_{\delta,1}$ the discretization level $m = 6$ gives the error $0.16996\ldots$, and the discretization level $m = 11$ gives the error $2.7665 \cdot 10^{-2}$ for the problem with noisy data $y_{\delta,2}$. These values are slightly superior to the error obtained with $q = 1.5$. However, our algorithm automatically finds the discretization level that gives the accuracy of the same order.

q	m_{l+2}	$\|x_1 - x^{\delta}_{m_{l+2}}\|$
1.3	8	0.17532...
1.5	5	0.17112...
1.7	3	0.24133...
1.9	1	0.40128...

Table 1: Numerical results for $k = 1,\ \delta = 10^{-8},\ N = 24$.

q	m_{l+2}	$\|x_2 - x^{\delta}_{m_{l+2}}\|$
1.3	8	0.75615...
1.5	3	5.1825×10^{-2}...
1.7	1	0.14965...
1.9	1	0.14965...

Table 2: Numerical results for $k = 2,\ \delta = 10^{-7},\ N = 20$.

q	m_{l+2}	$\|x_3 - x^{\delta}_{m_{l+2}}\|$
1.3	8	3.3175×10^{-2}...
1.5	4	4.8860×10^{-4}...
1.7	3	3.7980×10^{-3}...
1.9	1	0.39058...

Table 3: Numerical results for $k = 3,\ \delta = 10^{-8},\ N = 21$.

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