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Asymptotic behaviour for a phase-field model with hysteresis in one-dimensional thermo-visco-plasticity

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Abstract

The asymptotic behaviour for $t \to \infty$ of the solutions to a one-dimensional model for thermo-visco-plastic behaviour is investigated in this paper. The model consists of a coupled system of nonlinear partial differential equations, representing the equation of motion, the balance of the internal energy, and a phase evolution equation, determining the evolution of a phase variable. The phase evolution equation can be used to deal with relaxation processes. Rate-independent hysteresis effects in the strain-stress law and also in the phase evolution equation are described by using the mathematical theory of hysteresis operators.

1 Introduction

In this paper, an initial-boundary value problem for a system of partial differential equations involving hysteresis operators is considered, and the asymptotic behaviour of the solutions to this system is investigated. The system has been derived in [KSS01b] to model one-dimensional thermo-visco-plastic developments connected with solid-solid phase transitions taking also into account the hysteresis effects appearing on the macroscopic scale as a consequence of effects on the micro- and/or mesoscale.

To describe such developments, one is considering the evolution of the displacement u, of the absolute temperature θ, and of a phase variable w, which is usually a so-called *generalized freezing index*, see [KS00c]. For a wire of unit length, the evolution of these fields is determined by the following system:

$$\rho u_{tt} - \mu u_{xxt} = \sigma_x + f(x,t), \quad \text{a.e. in} \quad \Omega_\infty, \tag{1.1}$$

$$\sigma = \mathcal{H}_1[u_x, w] + \theta \mathcal{H}_2[u_x, w], \quad \text{a.e. in} \quad \Omega_\infty, \tag{1.2}$$

$$\left(C_V\theta + \mathcal{F}_1[u_x, w]\right)_t - \kappa\theta_{xx} = \mu u_{xt}^2 + \sigma u_{xt} + g(x,t,\theta), \quad \text{a.e. in} \quad \Omega_\infty, \tag{1.3}$$

$$\nu w_t = -\psi, \quad \text{a.e. in} \quad \Omega_\infty, \tag{1.4}$$

$$\psi = \mathcal{H}_3[u_x, w] + \theta \mathcal{H}_4[u_x, w], \quad \text{a.e. in} \quad \Omega_\infty, \tag{1.5}$$

$$u(0,t) = 0, \quad \mu u_{xt}(1,t) + \sigma(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \quad \text{a.e. in} \quad (0,\infty), \tag{1.6}$$

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0, \quad w(\cdot,0) = w_0, \quad \text{a.e. in} \quad \Omega, \tag{1.7}$$

with $\Omega_\infty := \Omega \times (0,\infty)$ and $\Omega := [0,1]$.

The equation (1.1) is the equation of motion, (1.3) is the balance of internal energy, and (1.4) is the phase evolution equation. By the constitutive law (1.2), the elastoplastic stress σ is determined, and the constitutive law (1.4) defines the thermodynamic force ψ. The boundary condition (1.6) means that the wire is fixed at $x = 0$, stress-free at $x = 1$, and thermally insulated at both ends. Here, x denotes the space variable, t denotes the time, and the indices x and t denote the differentiation with respect to space and time, respectively.

The mass density ρ, the viscosity μ, the specific heat C_V, the heat conductivity κ, and the kinetic relaxation coefficient ν are supposed to be positive constants. The initial data for the displacement, the velocity, the temperature, and the phase variable considered in (1.7) are denoted by u_0, u_1, θ_0, and w_0, respectively. Finally, the nonlinearities $\mathcal{H}_i$, $1 \le i \le 4$, and $\mathcal{F}_1$ are hysteresis operators (see below), where one needs to take into account $u_x(x,\cdot)|_{[0,t]}$ and $w(x,\cdot)|_{[0,t]}$ to compute $\mathcal{H}_i[u_x, w](x,t)$ and $\mathcal{F}_1[u_x, w](x,t)$.

These operators are supposed to reflect some *memory* in the material on the macroscale, resulting from effects in the micro/mesoscale. Such effects can lead to *hysteresis loops*, as they are for example observed in the macroscopic strain-stress relation (ε - σ, where $\varepsilon = u_x$ is the linearized strain) determined from measurements in uniaxial load-deformation of materials like *shape memory alloys*. The curves show a strong dependence on the temperature, but many of them are *rate-independent*, i.e., they are independent of the speed with which they are traversed.

There are other approaches to model hysteretic behaviour by considering systems similar to parts of (1.1)–(1.5), where the operators $\mathcal{F}_1$ and $\mathcal{H}_i$, for $1 \le i \le 4$, are superposition operators. These models are derived by considering a free energy, which is a superposition operator, involving a potential which has (one or more) concave parts. The concave parts of the potential correspond to instable physical states, and these instabilities are supposed to produce the observed hysteresis effects. Such approaches have successfully been used and investigated in a number of papers, see, e.g., [BS96, DH82, RZ97, Vis96] and the references therein, but the modelling by non-convex free energies has its limits, since a non-convex part of the potential alone does not ensure that hysteresis loops are present, see, e.g., [Mül01]. Moreover, the simple superposition operator cannot represent all the complicated hysteresis curves that are observed in experiments.

Hence, to describe such structures, the more general *hysteresis operators* have been introduced and used in a number of papers, see, e.g., the monographs [BS96, Kre96, KP89, Vis94] to this subject and the references therein. For a final time $T > 0$, an operator $\mathcal{H} : C[0,T] \to \mathrm{Map}[0,T] := \{v : [0,T] \to \mathbb{R}\}$ is a *hysteresis operator* if it is rate-independent and causal according to the following definitions. The operator $\mathcal{H}$ is called *rate-independent*, if for every $v \in C[0,T]$ and every continuous increasing (not necessary strictly increasing) function $\alpha : [0,T] \to [0,T]$ with $\alpha(0) = 0$ and $\alpha(T) = T$ it holds that

$$\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t)), \quad \forall\, t \in [0,T]. \tag{1.8}$$

An operator $\mathcal{H} : D(\mathcal{H}) (\subseteq \mathrm{Map}[0,T]) \to \mathrm{Map}[0,T]$ is said to be *causal*, if for every

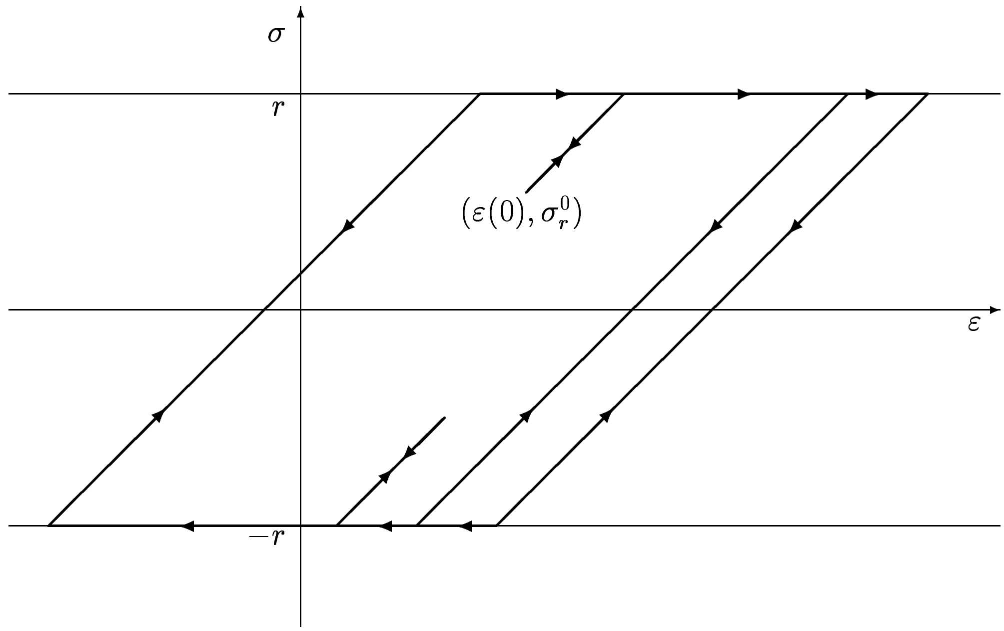

Figure 1: An example for the evolution of $(\varepsilon(t), \mathcal{S}_r[\sigma_r^0, \varepsilon](t))$, starting in $(\varepsilon(0), \sigma_r^0)$.

$v_1, v_2 \in D(\mathcal{H})$ and every $t \in [0,T]$ we have the implication

$$v_1(\tau) = v_2(\tau),\ \forall\, \tau \in [0,T] \quad \Rightarrow \mathcal{H}[v_1](t) = \mathcal{H}[v_2](t). \tag{1.9}$$

An example for a hysteresis operator is the so-called *stop operator*, which is also know as *Prandtl's normalized elastic-perfectly plastic element.* To define this operator, we consider some yield limit $r > 0$, an initial stress $\sigma_r^0 \in [-r, r]$, and a finial time $T > 0$. For any input function $\varepsilon \in W^{1,1}(0,T)$, we have (see, e.g., [BS96, KP89, Kre96, Vis94]) a unique solution $\sigma_r \in W^{1,1}(0,T)$ to the variational inequality

$$\sigma_r(t) \in [-r, r], \quad \forall\, t \in [0,T], \quad \sigma_r(0) = \sigma_r^0, \tag{1.10}$$

$$(\varepsilon_t(t) - \sigma_{r,t}(t))\,(\sigma_r(t) - \eta) \geq 0, \quad \forall\, \eta \in [-r, r], \quad \text{a.e. in} \quad (0,T). \tag{1.11}$$

This defines the stop operator

$$\mathcal{S}_r : [-r, r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) \ :\ (\sigma_r^0, \varepsilon) \mapsto \sigma_r. \tag{1.12}$$

An example for the evolution of the input and the output for the stop operator is presented in Figure 1. Connected to the stop operator $\mathcal{S}_r$ is another important hysteresis operator, the so-called *play operator* $\mathcal{P}_r$ defined by

$$\mathcal{P}_r : [-r, r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) \ :\ (\sigma_r^0, \varepsilon) \mapsto \varepsilon - \mathcal{S}_r[\sigma_r^0, \varepsilon]. \tag{1.13}$$

For all $\sigma_r^0, \sigma_r^{0,1}, \sigma_r^{0,2} \in [-r, r]$ and all $\varepsilon, \varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, these operators satisfy (see, e.g., [BS96, KP89, Kre96])

$$\left\| \mathcal{S}_r[\sigma_r^0, \varepsilon] \right\|_{C[0,T]} \leq r, \qquad \left| \mathcal{S}_r[\sigma_r^0, \varepsilon] \right|^2 = \big(\mathcal{S}_r[\sigma_r^0, \varepsilon]\big)_t\, \varepsilon_t, \quad \text{a.e. in} \quad (0,T), \tag{1.14}$$

$$
\begin{aligned}
&\left|\mathcal{S}_r[\sigma_r^{0,1},\varepsilon_1](t)-\mathcal{S}_r[\sigma_r^{0,2},\varepsilon_2]\right| \\
&\le |\varepsilon_1(t)-\varepsilon_2(t)|+\max\left\{\max_{0\le\tau\le t}|\varepsilon_1(\tau)-\varepsilon_2(\tau)|\,,\,\left|\sigma_r^{0,1}-\sigma_r^{0,1}\right|\right\},\quad \forall\, t\in[0,T],
\end{aligned} \tag{1.15}
$$

$$
\left(\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0,\varepsilon]\right)_t+\left|\left(r\mathcal{P}_r[\sigma_r^0,\varepsilon]\right)_t\right|=\mathcal{S}_r[\sigma_r^0,\varepsilon]\varepsilon_t,\quad \text{a.e. in}\quad (0,T). \tag{1.16}
$$

The inequality (1.15) allows to extend the stop and the play operator to Lipschitz continuous operators on $[-r,r]\times C[0,T]$. These operators are not differentiable, which is quite typical for hysteresis operators, since nontrivial hysteresis operators are at best Lipschitz continuous or only locally Lipschitz continuous in suitable functions spaces, but they are not differentiable. This leads to problems for the mathematical investigation of equations involving hysteresis operators. To overcome this difficulties, one is applying inequalities and equalities similar to (1.16). Using the notation of [BS96, Chapter 2.5], this equation means that $\frac{1}{2}\mathcal{S}_r^2[\sigma_r^0,\cdot]$ is the *clockwise admissible potential* and $r\mathcal{P}_r[\sigma_r^0,\cdot]$ is the corresponding *dissipation operator* for the operator $\mathcal{S}_r[\sigma_r^0,\cdot]$.

Let $\mathrm{Map}[0,\infty) := \{v : [0,\infty)\to\mathbb{R}\}$. An operator $\mathcal{H} : D(\mathcal{H})(\subset \mathrm{Map}[0,\infty)\times \mathrm{Map}[0,\infty))\to \mathrm{Map}[0,\infty)$ is said to be *causal*, if for every $(\varepsilon_1,w_1),(\varepsilon_2,w_2)\in D(\mathcal{H})$ and every $t\ge 0$ we have the implication

$$
\varepsilon_1(\tau)=\varepsilon_2(\tau),\, w_1(\tau)=w_2(\tau),\quad \forall\,\tau\in[0,t]\quad \Rightarrow \mathcal{H}[\varepsilon_1,w_1](t)=\mathcal{H}[\varepsilon_2,w_2](t). \tag{1.17}
$$

Moreover, the operator $\mathcal{H}$ generates an operator $\overline{\mathcal{H}}$ mapping (ε,w) with $\varepsilon,w:\Omega\times[0,T]\to\mathbb{R}$ such that $(\varepsilon(x,\cdot),w(x,\cdot))\in D(\mathcal{H})$ for a.e. $x\in\Omega$ to the function on $\Omega\times[0,T]$ defined by

$$
\overline{\mathcal{H}}[\varepsilon,w](x,t)=\mathcal{H}[\varepsilon(x,\cdot),w(x,\cdot)](t),\quad \forall\, t\in[0,T],\quad \text{for a.e. } x\in\Omega. \tag{1.18}
$$

In the sequel, we will no longer distinguish between $\mathcal{H}$ and the generated operator $\overline{\mathcal{H}}$. This holds especially for $\mathcal{H}_1,\dots,\mathcal{H}_4$, and $\mathcal{F}_1$, since for these operators the same notation will be used for the causal operators discussed in the assumptions in the next section and the operators generated from these operators, which are the operators considered in the system (1.1)–(1.7).

The hysteresis phenomena described by hysteresis operators are often related to changes between different configurations within the wire. In the system above, these configurations are described by the phase parameter w, and the evolution of these configurations is described by the phase evolution equation (1.4). Such an equation allows to take also into account relaxation processes that appear in addition to the rate independent hysteresis loops, which are modeled by the hysteresis operators.

Let recall some results for systems with hysteresis operators similar to the one above. In [GKS00, KS98a, KS00b, KS00c, KS02, KSZ00], a multi-dimensional phase transition is considered without taking mechanical effects into account. This corresponds to investigate (1.3)–(1.5) without a dependence on u or σ. The one-dimensional thermoelastoplastic hysteresis without considering relaxation processes in the phase transition, i.e., (1.1)–(1.3) with no dependence on w, has been studied in [KS97, KS98b].

For the complete system (1.1)–(1.7) above with an additional Ginzburg term u_{xxxx} on the left-hand side of (1.1) and boundary condition $u = u_{xx} = 0$ on $\partial\Omega$ for u, the global existence and uniqueness of a solution has been shown in [KSS01a].

The system (1.1)–(1.7) has been derived and investigated in [KSS01b]. Therein, the existence, uniqueness, and regularity of a strong solution has been proved (see Theorem 2 in Section 2.3), and it has also been shown that the Clausius-Duhem inequality and therefore the second principle of thermodynamics is satisfied for the solution.

In present work, we are dealing with the asymptotic behaviour for $t \to \infty$ for the system under consideration. After discussing the assumptions in Section 2.1, the results are presented in Theorem 1 in Section 2.2. The a-priori estimates derived in Section 3 are used in Section 4 to prove this theorem.

2 Asymptotic behaviour of solutions

2.1 Assumptions

The assumptions used in the investigation of the asymptotic behaviour of the solution to (1.1)–(1.7) are now presented and discussed. Let $C_{\text{loc}}[0,\infty)$ denote the set of all functions from $[0,\infty)$ to $\mathbb{R}$ that are in $C[0,T]$ for all $T > 0$. For $t \geq 0$, the seminorm $|\cdot|_{[0,t]}$ on $C_{\text{loc}}[0,\infty)$ and on $C[0,T]$ for $T \geq t$ is defined by

$$|f|_{[0,t]} = \max_{0 \leq s \leq t} |f(s)| \,. \tag{2.1}$$

We will use the following assumptions:

(H1) We have $u_0 \in H^2(\Omega)$, $u_1 \in W^{1,\infty}(\Omega)$, $\theta_0 \in H^1(\Omega)$, $w_0 \in H^1(\Omega)$, and there is some $\delta > 0$ such that $\theta_0(x) \geq \delta$ for all $x \in \overline{\Omega}$. Moreover, the compatibility condition $u_0(0) = u_1(0) = 0$ is satisfied.

(H2) We assume that $g : \Omega \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathèodry function such that there are functions $g_1, g_2 : \Omega_\infty \to [0,\infty)$, with

$$\begin{gathered} g_1 \in L^1(\Omega_\infty) \cap L^2(\Omega_\infty), \quad g_2 \in L^1(0,T; L^\infty(\Omega)) \cap L^2(0,T; L^\infty(\Omega)), \\ -g_2(x,t)s \leq g(x,t,s) \leq g_1(x,t) + g_2(x,t)s, \quad \forall\,(x,t) \in \Omega_\infty, s \geq 0. \end{gathered}$$

(H3) The operators $\mathcal{H}_1, \ldots, \mathcal{H}_4\,, \mathcal{F}_1 : C_{\text{loc}}[0,\infty) \times C_{\text{loc}}[0,\infty) \to C_{\text{loc}}[0,\infty)$ are causal and map $W^{1,1}_{\text{loc}}(0,\infty) \times W^{1,1}_{\text{loc}}(0,\infty)$ into $W^{1,1}_{\text{loc}}(0,\infty)$. The operators map $C[0,T] \times C[0,T]$ continuously into $C[0,T]$ for all $T > 0$, and for all $\varepsilon, w \in C_{\text{loc}}[0,\infty)$ it holds

$$\mathcal{F}_1[\varepsilon, w](t) \geq 0, \quad \forall\, t \geq 0.$$

(H4) There exist causal operators $\mathcal{F}_2 : W^{1,1}_{\text{loc}}(0,\infty) \times W^{1,1}_{\text{loc}}(0,\infty) \to W^{1,1}_{\text{loc}}(0,\infty)$, $\mathcal{D}_1, \mathcal{D}_2 : W^{1,1}_{\text{loc}}(0,\infty) \times W^{1,1}_{\text{loc}}(0,\infty) \to L^1_{\text{loc}}(0,\infty)$, $\mathcal{G} : W^{1,1}_{\text{loc}}(0,\infty) \to W^{1,1}_{\text{loc}}(0,\infty)$, and a non-decreasing function k_1 such that for all $\varepsilon, w \in W^{1,1}_{\text{loc}}(0,\infty)$ it holds

i)

$$|\mathcal{D}_1[\varepsilon, w]| = \varepsilon_t \mathcal{H}_1[\varepsilon, w] + (\mathcal{G}[w])_t\, \mathcal{H}_3[\varepsilon, w] - (\mathcal{F}_1[\varepsilon, w])_t\,, \quad \text{a.e. in} \quad (0,\infty),$$
$$|\mathcal{D}_2[\varepsilon, w]| = \varepsilon_t \mathcal{H}_2[\varepsilon, w] + (\mathcal{G}[w])_t\, \mathcal{H}_4[\varepsilon, w] - (\mathcal{F}_2[\varepsilon, w])_t\,, \quad \text{a.e. in} \quad (0,\infty).$$

ii)

$$|(\mathcal{G}[w])_t\,(t)|^2 \le k_1\left(|w|_{[0,t]}\right) w_t(t)\,(\mathcal{G}[w])_t\,(t), \quad \text{for a.e.}\, t \in (0,\infty).$$

(H5) We have $\mathcal{F}_{1,0}, \mathcal{F}_{2,0} \in L^1(\Omega)$ such that for all $\varepsilon, w \in W^{1,1}_{\text{loc}}(0,\infty; L^2(\Omega))$ with $\varepsilon(\cdot,0) = u_{0,x}$ and $w(\cdot,0) = w_0$ a.e. on Ω it holds that

$$\mathcal{F}_1[\varepsilon, w](\cdot,0) = \mathcal{F}_{1,0}, \quad \mathcal{F}_2[\varepsilon, w](\cdot,0) = \mathcal{F}_{2,0}, \quad \text{a.e. in} \quad \Omega.$$

(H6) There are non-decreasing functions $k_2, k_3, k_4 : [0,\infty) \to [0,\infty)$ such that for all $\varepsilon, w \in C_{\text{loc}}[0,\infty)$ it holds:

i)

$$\max_{1 \le i \le 4} |\mathcal{H}_i[\varepsilon, w](t)| \le k_2\left(|\varepsilon|_{[0,t]} + |w|_{[0,t]}\right), \quad \forall\, t \ge 0.$$

ii)

$$-\mathcal{F}_2[\varepsilon, w](t) \;\le\; k_3\left(|\varepsilon|_{[0,t]} + |w|_{[0,t]}\right)(1 + \mathcal{F}_1[\varepsilon, w](t))\,, \quad \forall\, t \ge 0.$$

iii) If $\varepsilon, w \in W^{1,1}_{\text{loc}}(0,\infty)$ then

$$\max_{1 \le i \le 4} |(\mathcal{H}_i[\varepsilon, w])_t\,(t)| + |(\mathcal{F}_1[\varepsilon, w])_t\,(t)|$$
$$\le k_4\left(|\varepsilon|_{[0,t]} + |w|_{[0,t]}\right)\left(|\varepsilon_t(t)| + \sqrt{w_t(t)\,(\mathcal{G}[w])_t\,(t)}\right), \qquad \text{for a.e.}\, t \in (0,\infty).$$

(H7) We have $f \in L^\infty(0,\infty; L^2(\Omega))$ and there exists functions $f_\infty \in L^2(\Omega)$, $F \in L^2(0,\infty; H^1(\Omega)) \cap H^1(0,\infty; L^2(\Omega)) \cap L^\infty(\Omega_\infty)$, and positive constants K_0, K_1 such that

$$f - f_\infty \in L^1(0,\infty; L^2(\Omega)), \qquad F(x,t) = \int_1^x f(\xi,t)\,\mathrm{d}\xi\,, \quad \text{for a.e.}\,(x,t) \in \Omega_\infty,$$
$$\|f_\infty\|_{L^1(\Omega)}\,|\varepsilon(t)| \le (1 - K_0)\,|\mathcal{F}_1[\varepsilon, w](t)| + K_1, \quad \forall\, \varepsilon, w \in C_{\text{loc}}[0,\infty), t \ge 0. \quad (2.2)$$

For the formulation of the remaining assumptions, we use the following notations, which are well defined by **(H1)**:

$$\varepsilon_{0,\min} := \min\{u_{0,x}(x) \,:\, x \in \overline{\Omega}\}, \quad \varepsilon_{0,\max} := \max\{u_{0,x}(x) \,:\, x \in \overline{\Omega}\}, \tag{2.3}$$

$$w_{0,\min} := \min\{w_0(x) \,:\, x \in \overline{\Omega}\}, \quad w_{0,\max} := \max\{w_0(x) \,:\, x \in \overline{\Omega}\}. \tag{2.4}$$

(H8) For each $\varepsilon_\Delta > 0$, there exists $\varepsilon_- \leq \varepsilon_{0,\min}$, $\varepsilon_+ \geq \varepsilon_{0,\max}$, $w_\Delta > 0$, $w_- \leq w_{0,\min}$, and $w_+ \geq w_{0,\max}$ such that for all $\varepsilon, w \in C_{\text{loc}}[0,\infty)$ and all $t \geq 0$ holds:

i) If $\varepsilon(t) \geq \varepsilon_+$,

$$\varepsilon_{0,\min} \leq \varepsilon(0) \leq \varepsilon_{0,\max}, \quad \varepsilon_- - \varepsilon_\Delta \leq \varepsilon(\tau) \leq \varepsilon_+ + \varepsilon_\Delta, \quad \forall \tau \in [0,t], \tag{2.5}$$

$$w_{0,\min} \leq w(0) \leq w_{0,\max}, \quad w_- - w_\Delta \leq w(\tau) \leq w_+ + w_\Delta, \quad \forall \tau \in [0,t], \tag{2.6}$$

hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \geq \|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \geq 0. \tag{2.7}$$

ii) If $\varepsilon(t) \leq \varepsilon_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_1[\varepsilon, w](t) \leq -\|F\|_{L^\infty(\Omega_\infty)}, \quad \mathcal{H}_2[\varepsilon, w](t) \leq 0. \tag{2.8}$$

iii) If $w(t) \geq w_+$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \geq 0, \quad \mathcal{H}_4[\varepsilon, w](t) \geq 0. \tag{2.9}$$

iv) If $w(t) \leq w_-$, (2.5), and (2.6) hold then we have

$$\mathcal{H}_3[\varepsilon, w](t) \leq 0, \quad \mathcal{H}_4[\varepsilon, w](t) \leq 0. \tag{2.10}$$

(H9) For every $\varepsilon, w \in W^{1,1}_{\text{loc}}(0,\infty)$ with ε and w bounded and

$$\int_0^\infty \left(|\mathcal{D}_1[\varepsilon, w](t)| + |\mathcal{D}_2[\varepsilon, w](t)|\right) \,\mathrm{d}t < \infty,$$

there exists $\varepsilon_\infty \in \mathbb{R}$ such that $\lim_{t\to\infty} \varepsilon(t) = \varepsilon_\infty$.

(H10) For every ε, w as in **(H9)** there exists $w_\infty \in \mathbb{R}$ such that $\lim_{t\to\infty} w(t) = w_\infty$.

Remark 2.1. There are important cases where the operators $\mathcal{H}_i$ are decoupled and may include some contribution from a superposition operator. Considering causal operators $\tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_4 : C_{\text{loc}}[0,\infty)) \to C_{\text{loc}}[0,\infty)$ and non-negative functions $h_1, \ldots, h_4 \in C^2_{\text{loc}}(\mathbb{R})$, we can define the operators $\mathcal{H}_1, \ldots, \mathcal{H}_4$ by setting for all $\varepsilon, w \in C_{\text{loc}}[0,\infty)$ and all $t \geq 0$

$$\mathcal{H}_i[\varepsilon, w](t) := \begin{cases} h_i'(\varepsilon(t)) + \tilde{\mathcal{H}}_i[\varepsilon](t), & \text{for} \quad i = 1,2, \\ h_i'(w(t)) + \tilde{\mathcal{H}}_i[w](t), & \text{for} \quad i = 3,4. \end{cases} \tag{2.11}$$

If we have clockwise admissible potentials for $\tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_4$, i.e., if we have causal operators $\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_4 : C_{\text{loc}}[0,\infty) \to C_{\text{loc}}[0,\infty)$ which are mapping $W^{1,1}_{\text{loc}}(0,\infty)$ in $W^{1,1}_{\text{loc}}(0,\infty)$ and causal operators $\tilde{\mathcal{D}}_1, \ldots, \tilde{\mathcal{D}}_4 : W^{1,1}_{\text{loc}}(0,\infty) \to L^1_{\text{loc}}(0,\infty)$ with

$$\left|\tilde{\mathcal{D}}_i[v]\right| = v_t \tilde{\mathcal{H}}_i[v], - \left(\tilde{\mathcal{F}}_i[v]\right)_t \quad \text{a.e. in} \quad (0,\infty), \quad \forall\, v \in W^{1,1}_{\text{loc}}[0,\infty), i = 1, \ldots, 4, \tag{2.12}$$

then **(H4)** holds with $\mathcal{G}$ being the identity and $\mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1, \mathcal{D}_2$ defined by

$$\mathcal{F}_1[\varepsilon, w](t) := h_1(\varepsilon(t)) + \tilde{\mathcal{F}}_1[\varepsilon](t) + h_3(w(t)) + \tilde{\mathcal{F}}_3[w](t), \tag{2.13}$$

$$\mathcal{F}_2[\varepsilon, w](t) := h_2(\varepsilon(t)) + \tilde{\mathcal{F}}_2[\varepsilon](t) + h_4(w(t)) + \tilde{\mathcal{F}}_4[w](t), \tag{2.14}$$

$$\mathcal{D}_1[\varepsilon, w](t) := \left|\tilde{\mathcal{D}}_1[\varepsilon](t)\right| + \left|\tilde{\mathcal{D}}_3[w](t)\right|, \quad \mathcal{D}_2[\varepsilon, w](t) := \left|\tilde{\mathcal{D}}_2[\varepsilon](t)\right| + \left|\tilde{\mathcal{D}}_4[w](t)\right|, \tag{2.15}$$

for all $\varepsilon, w \in C_{\text{loc}}[0,\infty)$ and $t \geq 0$.

If $h_1(r) = h_1^* r^2$ with some positive constant h_1^* then the corresponding operator $\mathcal{H}_1$ models a linear elasticity with a hysteretic modification.

Remark 2.2. A sufficient condition for **(H8)** to be satisfied is that the two following assumptions **(H11)** and **(H12)** hold. These assumptions are especially useful, if the operators $\mathcal{H}_1, \ldots, \mathcal{H}_4$ are decoupled as in the Remarks 2.1, 2.4–2.6. The notation of an *outward pointing* operator used in these assumptions is introduced and discussed in the forthcoming paper [KK].

The more general formulation in **(H8)** is helpful, if the operators are coupled, e.g., if they are derived from multi-dimensional stop or Prandtl-Ishlinskii operators (see, e.g., [Kre96, KS00c, KS01, KS02]).

(H11) For each $\varepsilon_\Delta > 0$, there exists $\varepsilon_- \leq \varepsilon_{0,\min}$ and $\varepsilon_+ \geq \varepsilon_{0,\max}$ such that for all $w \in C_{\text{loc}}[0,\infty)$ with $w_{0,\min} \leq w(0) \leq w_{0,\max}$ the operator mapping $\varepsilon \in C_{\text{loc}}[0,\infty)$ to $\mathcal{H}_1[\varepsilon, w] \in C_{\text{loc}}[0,\infty)$ is *pointing outwards with bound* $\|F\|_{L^\infty(\Omega_\infty)}$ *in the* ε_Δ*-neighbourhood of* $[\varepsilon_-, \varepsilon_+]$ *for initial values in* $[\varepsilon_{0,\min}, \varepsilon_{0,\max}]$ and that the same holds for $\mathcal{H}_2$ just with bound 0, that is to say for all $\varepsilon \in C_{\text{loc}}[0,\infty)$ and all $t \geq 0$ holds:

i) If $\varepsilon(t) \geq \varepsilon_+$ and (2.5) hold then we have (2.7).

ii) If $\varepsilon(t) \leq \varepsilon_-$ and (2.5) hold then we have (2.8).

(H12) There are $w_\Delta > 0$, $w_- \leq w_{0,\min}$, and $w_+ \geq w_{0,\max}$ such that for all $\varepsilon \in C_{\text{loc}}[0,\infty)$ with $\varepsilon_{0,\min} \leq \varepsilon(0) \leq \varepsilon_{0,\max}$ the operators $C_{\text{loc}}[0,\infty) \ni w \mapsto \mathcal{H}_3[\varepsilon, w]$ and $C_{\text{loc}}[0,\infty) \ni w \mapsto \mathcal{H}_4[\varepsilon, w]$ are pointing outwards with bound 0 in the w_Δ-neighbourhood of $[w_-, w_+]$ for initial values in $[w_{0,\min}, w_{0,\max}]$, that is to say for all $w \in C_{\text{loc}}[0,\infty)$ and $t \geq 0$ holds

i) If $w(t) \geq w_+$ and (2.6) hold then we have (2.9).

ii) If $w(t) \leq w_-$ and (2.6) hold then we have (2.10).

Remark 2.3. If we use $\tilde{\mathcal{H}}_3 = \tilde{\mathcal{H}}_4 \equiv 0$ in Remark 2.1 then $\mathcal{H}_3$ and $\mathcal{H}_4$ are superposition operators and the assumption **(H12)** holds if and only if there are $w_\Delta > 0$, $w_- \le w_{0,\min}$, and $w_+ \ge w_{0,\max}$ such that

- For all $s \in [w_+, w_+ w_\Delta]$ holds $h_3'(s) \ge 0 \quad h_4'(s) \ge 0$.
- For all $s \le [w_- - w_\Delta, w_-]$ holds $h_3'(s) \le 0, \quad h_4'(s) \le 0$.

Similar assumption has been used in [And80, Peg87, RZ97]. A direct translation of this assumption leads to an assumption similar to **(H12)**, but with (2.6) replaced by $w_- - w_\Delta \le w(t) \le w_+ + w_\Delta$. This is a stronger assumption then **(H12)** and will be denoted by **(H12+)**. There are important hysteresis operators satisfying **(H12)**, but not **(H12+)**.

In a similar way, one can consider a stronger version **(H11+)** of **(H11)**, where $\varepsilon_- - \varepsilon_\Delta \le \varepsilon(t) \le \varepsilon_+ + \varepsilon_\Delta$ is used instead of (2.5).

Remark 2.4. If for the functions and operators in Remark 2.1 there are positive constants $K_{2,1}, \ldots, K_{2,4}$ such that

$$\left|\tilde{\mathcal{H}}_i[v](t)\right| \le K_{2,i}, \quad \forall\, t \ge 0,\, v \in C_{\mathrm{loc}}[0,\infty),\, 1 \le i \le 4, \tag{2.16}$$

$$\pm \lim_{r \to \pm\infty} h_1'(r) > K_{2,1} + \|F\|_{L^\infty(\Omega_\infty)}, \quad \pm \lim_{r \to \pm\infty} h_i'(r) > K_{2,i}, \quad \forall\, 2 \le i \le 4, \tag{2.17}$$

then the assumptions **(H11+)** and **(H12+)** are satisfied, and **(H11)**, **(H12)**, and **(H8)** hold therefore. Moreover, the condition (2.2) in **(H7)** is satisfied if the other assumptions in **(H7)** hold.

Remark 2.5. Consider yield limits $r_{i,j} \in \mathbb{R}$, initial values $\sigma^0_{i,j} \in [-r_{i,j}, r_{i,j}]$, and weights $\phi_{i,j} > 0$. Defining $\tilde{\mathcal{H}}_i[\cdot]$ as the sum $\sum_j \phi_{i,j} \mathcal{S}_{r_{i,j}}[\sigma^0_{i,j}, \cdot]$, one has by (1.16) that (2.12) holds with $\tilde{\mathcal{F}}_i$ being the sum $\sum_j \phi_{i,j} \mathcal{S}^2_{r_{i,j}}[\sigma^0_{i,j}, \cdot]/2$ and $\tilde{\mathcal{D}}_i$ being the sum $\sum_j \phi_{i,j} \left|(r\mathcal{P}_r[\sigma^0_r, \cdot])_t\right|$. For $\mathcal{H}_i$ as in Remark 2.1, one can use (1.14), (1.15), and **(H1)** to show that **(H3)**–**(H5)** are satisfied.

Moreover, we have (2.16) and the inequalities in (2.17) hold for appropriate functions h_i . The last remark then yields that even the strong formulations **(H11+)** and **(H12+)** of **(H11)** and **(H12)** are satisfied. For $h_3 \equiv h_4 \equiv$, i.e., $\mathcal{H}_3$ and $\mathcal{H}_4$ being the weighted sum of stop operators depending on w, this would not work, and one can easily see that **(H12+)** will not hold in this case. But, by investigating the behaviour of the stop operator one can show that **(H12)** holds, see also [KK]. But, if $h_1 \equiv 0$ or $h_2 \equiv 0$, i.e., if $\mathcal{H}_1$ and $\mathcal{H}_2$ are the weighted sum of stop operators depending on ε, one can consider **(H11)** for some ε_Δ which is bigger then the double of all the involved yield limits $r_{i,j}$ and observes that **(H11)** is not satisfied.

For all functions h_i, the assumptions **(H9)** and **(H10)** are not satisfied for the corresponding operators $\mathcal{H}_1, \ldots, \mathcal{H}_4$, since for oscillations that are smaller then all involved yield limits $r_{i,j}$, the play operators stay constant after the first oscillation.

Remark 2.6. For $i = 1, \dots, 4$, we consider a non-negative weight function $\phi_i \in L^1(0,\infty)$ and a function $\sigma_i^0 \in W^{1,\infty}(0,\infty)$ such that $\sigma_i^0(r) \in [-r, r]$ for all $r \geq 0$, $|(\sigma_i^0)_r| \leq 1$ a.e. on $(0,\infty)$, and $\sigma_r^0(r') = 0$ for all $r' \geq R_i$ for some $R_i > 0$. Now, we define $\tilde{\mathcal{H}}_i : C_{\text{loc}}[0,\infty) \to C_{\text{loc}}[0,\infty)$ as the *Prandtl-Ishlinskii operator*

$$\tilde{\mathcal{H}}_i[v] := \int_0^\infty \phi_i(r)\mathcal{S}_r[\sigma_i^0(r), v]\, \mathrm{d}r\,, \quad \forall\, v \in C_{\text{loc}}[0,\infty). \tag{2.18}$$

A clockwise admissible potential for this operator is defined by $\tilde{\mathcal{F}}_i : C_{\text{loc}}[0,\infty) \to C_{\text{loc}}[0,\infty)$ with

$$\tilde{\mathcal{F}}_i[v] := \frac{1}{2}\int_0^\infty \phi_i(r)\mathcal{S}_r^2[\sigma_i^0(r), v]\, \mathrm{d}r\,, \quad \forall\, v \in C_{\text{loc}}[0,\infty), \tag{2.19}$$

since Proposition 2.5.5. in [BS96] yields that (2.12) holds for

$$\tilde{\mathcal{D}}_i[v] := \left| \frac{\partial}{\partial t} \int_0^\infty r\phi_i(r)\mathcal{P}_r[\sigma_r^0, v]\, \mathrm{d}r \right|, \quad \forall\, v \in W^{1,1}_{\text{loc}}[0,\infty). \tag{2.20}$$

Defining now $\mathcal{H}_i$ and $\mathcal{F}_i$ as in Remark 2.1, and using well know properties of the stop operator one can show that **(H3)**–**(H6)** hold.

Applying (2.15), (2.20), and properties of the play operator, we see that **(H9)** holds, if and only if

$$\int_0^s r\left(\phi_1(r) + \phi_2(r)\right)\, \mathrm{d}r\ > 0, \quad \forall\, s > 0. \tag{2.21}$$

For **(H10)**, we get a analogous condition, just with $\phi_1 + \phi_2$ replaced by $\phi_3 + \phi_4$. If one wants to ensure as in Remark 2.1 that **(H11)** and **(H12)** are satisfied, one has to require that (2.16) holds, which is equivalent to the condition

$$\int_0^\infty r\phi_i(r)\, \mathrm{d}r\ < K_{2,i} < +\infty, \quad \forall\, 1 \leq i \leq 4. \tag{2.22}$$

If this condition is satisfied, we see that **(H11)** and **(H12)** holds for appropriate functions h_i, but this argumentation can not be applied if $\mathcal{H}_i = \tilde{\mathcal{H}}_i$ for some $i \in 1, \dots, 4$.

In [KK], it is proved that **(H12)** holds for $\mathcal{H}_3 := \tilde{\mathcal{H}}_3$ and $\mathcal{H}_4 := \tilde{\mathcal{H}}_4$, independently of (2.22). Moreover, there it is shown that for $\mathcal{H}_1 := \tilde{\mathcal{H}}_1$ the condition in **(H11)** holds if and only if $\int_0^\infty r\phi_1(r)\, \mathrm{d}r\ = \infty$, and that an analogous equivalence holds for $\mathcal{H}_2 := \tilde{\mathcal{H}}_2$.

2.2 The asymptotic result

The following theorem is the main result of this paper:

Theorem 1. *Assume that **(H1)–(H8)** are satisfied and that a solution* (u, θ, w) *to* (1.1)–(1.7) *is given such that*

$$u \in H^2_{\rm loc}(0,\infty; L^2(\Omega)) \cap H^1_{\rm loc}(0,\infty; H^2(\Omega)), \tag{2.23}$$
$$\theta \in H^1_{\rm loc}(0,\infty; L^2(\Omega)) \cap L^2_{\rm loc}(0,\infty; H^2(\Omega)), \tag{2.24}$$
$$w \in H^2_{\rm loc}(0,\infty; L^2(\Omega)) \cap H^1_{\rm loc}(0,\infty; H^1(\Omega)), \tag{2.25}$$
$$\theta(x,t) > 0, \quad \forall\, x \in \overline{\Omega}, t \geq 0. \tag{2.26}$$

a) We have a constant $\theta_* > 0$ *such that*

$$\lim_{t\to\infty} \|u_{xt}(\cdot,t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|u_t(\cdot,t)\|_{C(\overline{\Omega})} = 0, \tag{2.27}$$
$$\sigma(\cdot,t) \underset{t\to\infty}{\longrightarrow} -F_\infty, \quad in \quad L^2(\Omega), \tag{2.28}$$
$$\lim_{t\to\infty} \|\theta_x(\cdot,t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \left\|\theta_x(\cdot,t) - \overline{\theta}(t)\right\|_{C(\overline{\Omega})} = 0, \tag{2.29}$$
$$\theta(x,t) \geq \theta_*, \quad \forall\, x \in \overline{\Omega},\, t \geq 0, \tag{2.30}$$

with

$$F_\infty(x) := \int_1^x f_\infty(\xi)\,\mathrm{d}\xi\,, \quad \overline{\theta}(t) := \int_\Omega \theta(x,t)\,\mathrm{d}x\,, \quad \forall\, x \in \overline{\Omega},\, t \geq 0. \tag{2.31}$$

b) If $\mathcal{G}$ *is the identity operator, then we have*

$$\lim_{t\to\infty} \|w_t(\cdot,t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|\psi(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{2.32}$$
$$\lim_{t\to\infty} \left\|\left(\mathcal{F}_1[u_x,w]\right)_t(\cdot,t)\right\|_{L^2(\Omega)} = \sum_{i=1}^{4} \lim_{t\to\infty} \left\|\left(\mathcal{H}_i[u_x,w]\right)_t(\cdot,t)\right\|_{L^2(\Omega)} = 0. \tag{2.33}$$

c) If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, *and* $f \equiv 0$, *then we have*

$$\theta(\cdot,t) \underset{t\to\infty}{\longrightarrow} \|\theta_0\|_{L^1(\Omega)} + \frac{\rho}{2C_V}\|u_1\|^2_{L^2(\Omega)}\,, \quad in \quad L^\infty(\Omega), \tag{2.34}$$
$$\lim_{t\to\infty} \|\mathcal{H}_2[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.35}$$

d) If $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, $f \equiv 0$, *and* $\mathcal{G}$ *is the identity operator, then we have*

$$\lim_{t\to\infty} \|\mathcal{H}_4[u_x,w](\cdot,t)\|_{L^2(\Omega)} = 0. \tag{2.36}$$

*e) If **(H9)** holds then there exists a* $u_\infty \in W^{1,\infty}(\Omega)$ *such that*

$$u(\cdot,t) \underset{t\to\infty}{\longrightarrow} u_\infty \quad weakly\text{-}star\ in \quad W^{1,\infty}(\Omega), \tag{2.37}$$
$$u_x(\cdot,t) \underset{t\to\infty}{\longrightarrow} u_{\infty,x}, \quad a.e.\ in \quad \Omega. \tag{2.38}$$

*f) If **(H10)** holds then there exists a $w_\infty \in L^\infty(\Omega)$ such that*

$$w(\cdot, t) \underset{t\to\infty}{\longrightarrow} w_\infty \quad \textit{weakly-star in} \quad L^\infty(\Omega) \quad \textit{and a.e. in } \Omega. \tag{2.39}$$

Remark 2.7. We see that (2.27) yields that for $t \to \infty$ the viscous part of the stress tends to zero, and by (2.28) the stress tends to $-F_\infty$, which is the potential corresponding to the limit f_∞ for $t \to \infty$ of the applied force f. Moreover, by (2.29), we see that the temperature becomes more and more uniform in space.

Under the additional conditions in part c) of Theorem 1, the convergence of the temperature for $t \to \infty$ is shown, and if $\mathcal{H}_2$ and $\mathcal{H}_4$ are special operators, like, e.g. stop operators, one could also show some convergence for u and w, by adapting the argument in [RZ97, Lemma 4.5]. In the general case it is still an open questions, if one can show convergence, or if up to $t \to \infty$ oscillations can appear. This is similar to [RZ97], where the system (1.1)–(1.3) with $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{F}_1$ being nonlinear superposition operators of u_x has been considered. Also in this paper there is no convergence result for θ or u_x in the general case.

Remark 2.8. If **(H8)** does not hold then one can still get some of the results in Theorem 1, if some additional assumptions are satisfied.

i) If **(H4)**ii) and **(H6)** with $k_1, \ldots, k_4$ replaced by positive constants hold then one can still show the results a)–d).

ii) If **(H11)**, **(H4)**ii) with k_1 replaced by a positive constant, and **(H6)** without the $|w|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove that the results a)–e) are satisfied.

iii) If **(H12)** and **(H6)** without the $|\varepsilon|_{[0,t]}$-term in the evaluation of k_2, k_3, k_4 hold then one can prove the results a)–d) and f).

Remark 2.9. In many applications, the operator $\mathcal{G}$ in **(H4)** is the identity, see, e.g., [Kre00, KS97, KS98b, KS00c, KS01, KS02, KSZ01], such that the results b) and d) in Theorem 1 can be applied.

If $\mathcal{G}$ is not the identity operator, one could get still some informations about the limiting behaviour of $\mathcal{G}[w]_t$ and therefore about the behaviour of the time derivatives of $\mathcal{F}_1[u_x, w]$ and $\mathcal{H}_i[u_x, w]$, if for $w \in W^{2,1}_{\text{loc}}(0, T)$ the "second order energy inequality" (see [Kre96, Section II. 4])

$$\frac{\partial}{\partial t}\left(w_t \left(\mathcal{G}[w]\right)_t\right) \le w_{tt} \left(\mathcal{G}[w]\right)_t$$

holds a.e. on $(0, \infty)$. In this case, only minor changes in the proof would be necessary. But, to the knowledge of the author, in all cases where an operator $\mathcal{G}$ as in **(H4)** is derived, which is not the identity, the operator is a stop operator, see [KS98a, KS00b, KS00c, KS00a, KSZ01]. In this case, $\mathcal{G}[w]$ is only of bounded variation, and the second order inequality holds only in the sense of distribution. To be able to deal with $\mathcal{G}$ of this kind, one would have to use methods similar to [KSZ00].

2.3 Existence of solutions

Before proving the asymptotic result, it will be recalled that there is a solution to the problem under considerations satisfying the regularity and positivity demands presented in Theorem 1, at least if some additional assumptions are satisfied. This assumption will be

(H13) It holds $f \in H^1_{\text{loc}}(0,\infty; L^2(\Omega))$.

(H14) There is a function $g_0 \in L^\infty_{\text{loc}}(\Omega_\infty)$ such that for every $T > 0$ there is a positive constant $K_{3,T}$

$$\left|\frac{\partial g}{\partial \theta}\right| \leq K_{3,T} \quad \text{a.e. in} \quad \Omega \times (0,T) \times \mathbb{R}, \qquad g_0(x,t) \geq 0, \quad \text{a.e. in} \quad \Omega_\infty,$$
$$g(x,t,\theta) = g_0(x,t), \quad \forall\,(x,t,\theta) \in \Omega \times (0,\infty) \times (-\infty, 0].$$

(H15) For every $T > 0$ there are positive constants $K_{4,T}, \dots, K_{9,T}$ such that for all $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in C_{\text{loc}}[0,\infty)$ it holds:

i) We have for all $t \in [0,T]$:

$$|\mathcal{H}_2[\varepsilon,w](t)| + |\mathcal{H}_4[\varepsilon,w](t)| \leq K_{4,T},$$
$$\max_{1\leq i\leq 4} |\mathcal{H}_i[\varepsilon_1,w_1](t) - \mathcal{H}_i[\varepsilon_2,w_2](t)| \leq K_{5,T}\left(|\varepsilon_1 - \varepsilon_2|_{[0,t]} + |w_1 - w_2|_{[0,t]}\right).$$

ii) If $\varepsilon, \varepsilon_1, \varepsilon_2, w, w_1, w_2 \in W^{1,1}_{\text{loc}}(0,\infty)$ then the inequality in **(H4)**ii) with $k_1\left(|w|_{[0,t]}\right)$ replaced by $K_{6,T}$ holds for a.e. $t \in (0,T)$ and

$$\max_{1\leq i\leq 4} |(\mathcal{H}_i[\varepsilon,w])_t\,(t)| \leq K_{7,T}\left(|\varepsilon_t(t)| + |w_t(t)|\right), \quad \text{for a.e. } t \in (0,T),$$
$$|(\mathcal{F}_1[\varepsilon,w])_t\,(t)| \leq K_{8,T}\left(|\varepsilon_t(t)| + |w_t(t)|\right), \quad \text{for a.e. } t \in (0,T), \tag{2.40}$$
$$|\mathcal{F}_1[\varepsilon_1,w_1](t) - \mathcal{F}_1[\varepsilon_2,w_2](t)| \leq K_{9,T}\Bigg(|\varepsilon_1(0) - \varepsilon_2(0)| + |w_1(0) - w_2(0)| + \int_0^t \left(|\varepsilon_{1,t}(\tau) - \varepsilon_{2,t}(\tau)| + |w_{1,t}(\tau) - w_{2,t}(\tau)|\right) \mathrm{d}\tau\Bigg), \quad \forall\, t \in [0,T]. \tag{2.41}$$

Thanks to Theorem 2.1 in [KSS01b], we have

Theorem 2. *Assume that* ***(H1)–(H3)***, ***(H4)****i), and* ***(H13)–(H15)*** *hold. Then the system* (1.1)–(1.7) *has a unique strong solution* (u,θ,w) *such that* (2.23)–(2.25) *hold. This solution satisfies also* (2.26).

Remark 2.10. If $\mathcal{H}_1$ as in (2.11) is modelling a linear elasticity with a bounded hysteretic modification as in the Remark 2.1, then one has $\mathcal{F}_1[\varepsilon,w](t) = h_1^*\varepsilon^2(t) + \dots$. Hence, in general the estimates (2.40) and (2.41) in **(H15)**ii) are not satisfied, and the existence result in [KSS01b] can therefore not be applied. To be able to use this result, one has to approximate the linear elastic term $2h_1^2\varepsilon$ for big ε by a bounded function. This is a somehow unexpected feature of combining these assumptions, since the authors of [KSS01b] want that their assumptions also include the case of linear elasticity, and avoid to use the assumption that $\mathcal{H}_1$ is bounded. They also do not assume explicitly that $\mathcal{H}_3$ is bounded, but combining the estimate (2.40) with **(H4)**i) and the continuity of $\mathcal{F}_1$ on $C[0,T]\times C[0,T]$ (see **(H3)**), one can show that for all $\varepsilon, w \in W^{1,1}_{\mathrm{loc}}[0,\infty)$ holds

$$\max\left(|\mathcal{H}_1[\varepsilon,w](t)|\,,\,|\mathcal{H}_3[\varepsilon,w](t)|\right) \le K_{8,T}, \quad \forall\, t\in[0,T].$$

Hence, at least formally, the existence result in [KSS01b] can only be applied if the operators $\mathcal{H}_1$ and $\mathcal{H}_3$ are bounded. But, if we examine the proof of the global existence result in [KSS01b], then we see that in the a-priori estimates therein the assumptions corresponding to **(H15)**ii) are used after the uniform estimates for u_x and w have been derived. Hence, this a-priori estimates can also be used, if one is considering a weakened version of **(H15)**, where $K_{8,T}$ and $K_{9,T}$ are replaced by $k_{5,T}\left(|\varepsilon|_{[0,t]}+|w|_{[0,t]}\right)$ and $k_{6,T}\left(|\varepsilon_1|_{[0,t]}+|\varepsilon_2|_{[0,t]}+|w_1|_{[0,t]}+|w_2|_{[0,t]}\right)$, respectively, with non-decreasing functions $k_{5,T}, k_{6,T} : [0,\infty)\to[0,\infty)$. A careful examination of the local existence proof in [KSS01b, Section 3] should allow to find a way do deal also with this weakened assumption, such that one can show also the existence of solutions to (1.1)–(1.7) for unbounded $\mathcal{H}_1$ and $\mathcal{H}_3$. In [KSS01a] the authors of [KSS01b] consider such an assumption for a modified version of the system (1.1)–(1.7).

Remark 2.11. For non-negative functions $h_1,\dots,h_4 \in C^2_{\mathrm{loc}}(\mathbb{R})$ with $h_1',\dots,h_4' \in W^{1,\infty}(\mathbb{R})$ and operators $\tilde{\mathcal{H}}_1,\dots,\tilde{\mathcal{H}}_4$ as in Remark 2.5 or as in Remark 2.6 with non-negative weight functions $\phi_1,\dots,\phi_4 \in L^1(0,\infty)$ satisfying (2.22) one can use (1.14) and (1.15) to show that **(H15)** holds. If one is using the weakened version of **(H15)** (see Remark 2.10), one needs only $h_2', h_4' \in W^{1,\infty}(\mathbb{R})$, $h_1'', h_3'' \in L^\infty(\mathbb{R})$, and (2.22) for $i=2$ and $i=4$.

3 Uniform a-priori estimates

In this section, it will be assumed that **(H1)**–**(H8)** are satisfied and that a solution (u,θ,w) to (1.1)–(1.7) is given, such that (2.23)–(2.26) hold. To prepare the proof of the asymptotic results in the next section, some a-priori estimates are derived that are uniform with respect to time.

Before this is done, we consider the balance law for the energy and a immediate consequence:

Remark 3.1. Multiplying (1.1) by u_t and adding the result to balance law (1.3) for the internal energy, we get the balance law for the energy

$$\left(C_V\theta + \frac{\rho}{2}u_t^2 + \mathcal{F}_1[u_x,w]\right)_t - \kappa\theta_{xx} = \left(u_t(\mu u_{tx}+\sigma)\right)_x + g + u_t f, \quad \text{a.e. in} \quad \Omega_\infty. \tag{3.1}$$

For $t > 0$, we integrate this equation over $\Omega \times (0,t)$, and use Green's formula, (1.6), (1.7), **(H1)**, and **(H5)**, to show that

$$C_V \overline{\theta}(t) + \frac{\rho}{2} \|u_t(\cdot,t)\|^2_{L^2(\Omega)} = I_0 + I_1(t), \quad \forall\, t \geq 0 \tag{3.2}$$

holds for the $\overline{\theta}$ defined in (2.31),

$$I_0 := C_V \|\theta_0\|_{L^1(\Omega)} + \frac{\rho}{2} \|u_1\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{1,0}(x)\,\mathrm{d}x \; > 0, \tag{3.3}$$

$$I_1(t) := \int_0^t \int_\Omega \big(g(x,\tau,\theta(x,\tau)) + u_t(x,\tau) f(x,\tau)\big)\,\mathrm{d}x\,\mathrm{d}\tau - \int_\Omega \big(\mathcal{F}_1[u_x,w](x,t)\big)\,\mathrm{d}x\,, \quad \forall\, t \geq 0. \tag{3.4}$$

In the sequel, for $1 \leq p < \infty$, the notation $\|\cdot\|_p$ will be used as abbreviation for the $L^p(\Omega)$-norm, and $\|\cdot\|_\infty$ will denote the $C(\overline{\Omega})$-norm, i.e., the maximum norm on $\overline{\Omega}$. Moreover, C_i, for $i \in \mathbb{N}$, will always denote generic positive constants, independent of time, space, and the considered solution.

Thanks to (2.23)–(2.26) and **(H3)**, we can assume without losing generality that σ and ϕ are continuous (maybe unbounded) functions on $\overline{\Omega_\infty} = \overline{\Omega} \times [0,\infty)$, such that (1.2) and (1.5) hold for all $(x,t) \in \overline{\Omega_\infty}$. Because of (1.7), (2.3), (2.4), we can apply the assumption **(H8)** for $\varepsilon(\cdot) := u_x(x,\cdot)$ and $w(\cdot) := w(x,\cdot)$. For the sake of notational convenience, we assume in the remaining part of this section without losing generality that $\rho = \mu = C_V = \kappa = \nu = 1$.

In the following estimates, some ideas from [KSS01b, RZ97, SZZ98] are used.

Lemma 3.2. *There are two positive constants C_1, C_2 such that*

$$\sup_{0 \leq t} \big(\|\theta(\cdot,t)\|_1 + \|u_t(\cdot,t)\|_2 + \|\mathcal{F}_1[u_x,w](\cdot,t)\|_1\big) \;\leq\; C_1, \tag{3.5}$$

$$\int_0^\infty \big(\|g(\cdot,t,\theta(\cdot,t))\|_1 + \|g(\cdot,t,\theta(\cdot,t))\|_1^2\big)\,\mathrm{d}t \;\leq C_2. \tag{3.6}$$

Proof. Let

$$\Psi(t) := \int_\Omega \big(\mathcal{F}_1[u_x,w](x,t) - f_\infty(x) u(x,t) + K_1\big)\,\mathrm{d}x\,, \quad \forall\, t \geq 0. \tag{3.7}$$

Now, we get from (3.2) by using (2.31), (2.26), (3.3), (3.4), Hölder's inequality, Young's inequality, **(H1)**, **(H2)**, **(H5)**, and **(H7)** that for all $t \geq 0$

$$\left(\|\theta(\cdot,t)\|_1 + \frac{1}{2}\|u_t(\cdot,t)\|_2^2 + \Psi(t)\right) \;<\; C_3 + \int_0^t \Big(\|g_1(\cdot,\tau)\|_1 + \|g_2(\cdot,t)\|_\infty \|\theta(\cdot,\tau)\|_1 + \frac{1}{2}\|f(\cdot,\tau) - f_\infty\|_2 + \frac{1}{2}\|f(\cdot,\tau) - f_\infty\|_2 \|u_t(\cdot,\tau)\|_2^2\Big)\,\mathrm{d}\tau\,. \tag{3.8}$$

By (3.7), Hölder's inequality, (1.6), **(H3)**, and **(H7)**, we have

$$\Psi(t) \geq K_0 \left\| \mathcal{F}_1[u_x, w](\cdot, t) \right\|_1 , \quad \forall\, t \geq 0.$$

Hence, because of (3.8), we can apply Gronwall's Lemma (see below), **(H2)**, and **(H7)** to show that that (3.5) and (3.6) are satisfied. □

The following version of Gronwall's Lemma can be derived from Proposition 1.4.2 in [BS96].

Lemma 3.3 (Gronwall's Lemma). *Let $a \in L^1_{\mathrm{loc}}(0,\infty)$ and $c \in L^\infty(0,\infty)$ denote non-negative functions. If a function $v \in C_{\mathrm{loc}}[0,\infty)$ satisfies*

$$0 \leq v(t) \leq c(t) + \int_0^t a(\tau) v(\tau)\, \mathrm{d}\tau\,, \quad \textit{for a.e. } t \in (0,\infty),$$

then

$$0 \leq v(t) \leq \|c\|_{L^\infty(0,\infty)} \exp\left(\int_0^t a(\tau)\, \mathrm{d}\tau \right), \quad \forall\, t \geq 0.$$

To prepare the following estimates, we now consider the transformation due to Andrews [And80], which is also used, e.g., in [Peg87, RZ97, KSS01b], and introduce functions $p, q, \tilde{\sigma} : \overline{\Omega_\infty} \to \mathbb{R}$ that are defined by

$$p(x,t) := \int_1^x u_t(\xi, t)\, \mathrm{d}\xi\,, \quad q(x,t) := u_x(x,t) - p(x,t), \quad \forall\, (x,t) \in \overline{\Omega_\infty}, \tag{3.9}$$

$$\tilde{\sigma}(x,t) := \sigma(x,t) + F(x,t), \quad \forall\, (x,t) \in \overline{\Omega_\infty}, \tag{3.10}$$

with F as in **(H7)**. Recalling (1.1)–(1.7) and **(H7)**, we see that

$$p_t - p_{xx} = \tilde{\sigma}, \quad \text{a.e. in} \quad \Omega_\infty, \tag{3.11}$$

$$p(1,t) = p_x(0,t) = 0, \quad \text{a.e. in} \quad (0,T), \quad p(x,0) = \int_1^x u_1(\xi)\, \mathrm{d}\xi\,, \quad \text{a.e. in} \quad \Omega, \tag{3.12}$$

$$q_t = -\tilde{\sigma}, \quad \text{a.e. in} \quad \Omega, \tag{3.13}$$

$$q(x,0) = u_{0,x}(x) - \int_1^x u_1(\xi)\, \mathrm{d}\xi\,, \quad \text{a.e. in} \quad \Omega. \tag{3.14}$$

Lemma 3.4. *There are positive constant C_4, C_5 such that*

$$\sup_{0\leq t} \left(\|p_x(\cdot,t)\|_2 + \|p(\cdot,t)\|_\infty \right) \leq C_4, \tag{3.15}$$

$$\sup_{0\leq t} \left(\|u_x(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty + \|u(\cdot,t)\|_\infty + \|q(\cdot,t)\|_\infty \right) \leq C_5. \tag{3.16}$$

Proof. In the light of the estimate for u_t in (3.5) and the definition of p in (3.9), we see that (3.15) holds. Considering **(H8)** for $\varepsilon_\Delta := 2C_4 + 1$, we get $\varepsilon_- < \varepsilon_{0,\min}$, $\varepsilon_{0,\max} < \varepsilon_+$, $w_- < w_{0,\min}$, and $w_+ > w_{0,\max}$ such that the remaining conditions in **(H8)** are satisfied. Now,

$$u_x(x,t) \in [\varepsilon_- - 2C_4, \varepsilon_+ + 2C_4], \quad w(x,t) \in [w_-, w_+], \quad \forall\, (x,t) \in \overline{\Omega_\infty}. \tag{3.17}$$

is proved by contradiction. Suppose that (3.17) does not hold. Then there is some $\delta \in (0, \min\{w_\Delta, 1\})$ such that $u_x \leq \varepsilon_- - 2C_4 - \delta$ and/or $u_x \geq \varepsilon_+ + 2C_4 + \delta$ and/or $w \leq w_- - \delta$ and/or $w \geq w_+ + \delta$ somewhere in $\overline{\Omega_\infty}$. We have $u_x(x,0) = u_{0,x}(x) \in [\varepsilon_-, \varepsilon_+]$ and $w(x,0) = w_0(x) \in [w_-, w_+]$ for all $x \in \overline{\Omega}$ because of (2.3) and (2.4). Since (2.23) and (2.25) yield that w and u_x are continuous on $\overline{\Omega_\infty}$, we get $x_1 \in \overline{\Omega}$, $t_1 > 0$ such that

$$\left(u_x(x_1,t_1) \in \{\varepsilon_- - 2C_4 - \delta, \varepsilon_+ + 2C_4 + \delta\} \text{ and/or } w(x_1,t_1) \in \{w_+ + \delta,\, w_- - \delta\}\right), \tag{3.18}$$

$$\varepsilon_- - 2C_4 - \delta < u_x(x,t) < \varepsilon_+ + 2C_4 + \delta, \qquad \forall\, t \in [0,t_1), x \in \overline{\Omega}, \tag{3.19}$$

$$\varepsilon_- - 2C_4 - \delta \leq u_x(x,t_1) \leq \varepsilon_+ + 2C_4 + \delta, \quad \forall\, x \in \overline{\Omega}, \tag{3.20}$$

$$w_- - \delta < w(x,t) < w_+ + \delta, \qquad \forall\, t \in [0,t_1), x \in \overline{\Omega}, \tag{3.21}$$

$$w_- - \delta \leq w(x,t_1) \leq w_+ + \delta, \quad \forall\, x \in \overline{\Omega}. \tag{3.22}$$

Hence, we see that (2.5) with $\varepsilon := u_x(x,\cdot)$ and (2.6) with $w := w(x,\cdot)$ hold for all $x \in \overline{\Omega}$ and $t \leq t_1$, and it remains only to check the first condition in **(H8)**i)–iv) if one wants to apply one the corresponding inequalities (2.7)–(2.10). Since u_x and w are uniformly continuous on $\overline{\Omega} \times [0,t_1]$, there is some open neighborhood $U \subset \overline{\Omega}$ of x_1 such that

$$|u_x(x,t) - u_x(x_1,t)| + |w(x,t) - w(x_1,t)| \leq \frac{\delta}{8}, \quad \forall\, x \in U,\, t' \in [0,t_1]. \tag{3.23}$$

Now, we consider the case $u_x(x_1,t_0) = \varepsilon_+ + 2C_4 + \delta$. Since u_x is continuous on $\overline{\Omega} \times [0,t_1]$ and $u_x(x_1,0) \leq \varepsilon_+$, we get some $t_0 \in (0,t_1)$ such that

$$\varepsilon_+ + \frac{\delta}{2} = u_x(x_1,t_0), \qquad \varepsilon_+ + \frac{\delta}{2} < u_x(x_1,t) < \varepsilon_+ + 2C_4 + \delta, \quad \forall\, t \in (t_0,t_1). \tag{3.24}$$

Combining this with (3.23), we conclude that $u_x(x,t) \geq \varepsilon_+$ for all $x \in U, t \in (t_0,t_1)$. In the light of (2.7) in **(H8)**i), we see that

$$\|F\|_{L^\infty(\Omega_\infty)} \leq \mathcal{H}_1[u_x,w](x,t), \quad 0 \leq \mathcal{H}_2[u_x,w](x,t), \quad \forall\, x \in U,\, t \in (t_0,t_1). \tag{3.25}$$

Applying (1.2) and that $\theta > 0$ on Ω_∞ by (2.26), we observe that $\sigma \geq -F$, a.e. in $U \times (t_0,t_1)$. Thanks to (3.13) and (3.10), we deduce that $q_t \leq 0$ a.e. in $U \times (t_0,t_1)$. This leads to

$$\int_U (q(x,t_1) - q(x,t_0))\, \mathrm{d}x\, \mathrm{d}\tau = \int_U \int_{t_0}^{t_1} q_t(x,t)\, \mathrm{d}t\, \mathrm{d}x \leq 0.$$

On the other hand, using (3.9), (3.15), (3.23), (3.24), and $u_x(x_1,t_0) = \varepsilon_+ + \frac{\delta}{2}$, we conclude that

$$\int_U (q(x,t_1) - q(x,t_0))\,\mathrm{d}x \ \geq\ \int_U \big(u_x(x,t_1) - C_4 - (u_x(x,t_0) + C_4)\big)\,\mathrm{d}x$$
$$\geq \int_U \left(u_x(x_1,t_1) - \frac{\delta}{8}\ -\ \left(u_x(x_1,t_0) + \frac{\delta}{8}\right) - 2C_4\right)\,\mathrm{d}x \ \geq\ \int_U \frac{\delta}{4}\,\mathrm{d}x\ > 0.$$

Hence, we have derived a contradiction. By an analogous argumentation, we get a contradiction, if $u_x(x_1,t_1) = \varepsilon_- - 2C_4 - \delta$.

Now, we will deal with the case $w(x_1,t_1) = w_+ + \delta$. Applying the continuity of w, we get some $t_0 \in (0,t_1)$ such that

$$w(x_1,t_0) = w_+ + \frac{\delta}{2}, \qquad w_+ + \frac{\delta}{2} < w(x_1,t) < w_+ + \delta, \quad \forall\, t \in (t_0,t_1). \tag{3.26}$$

Combining this with (3.23) we see that $w(x,t) \geq w_+$ for all $x \in U, t \in (t_0,t_1)$. Therefore, we conclude from (2.9) in **(H8)**iii) that

$$\mathcal{H}_3[u_x,w](x,t) \geq 0, \quad \mathcal{H}_4[u_x,w](x,t) \geq 0, \quad \forall\, x \in U,\ t \in (t_0,t_1). \tag{3.27}$$

Since $\theta > 0$ a.e. on Ω_∞ by (2.26), we deduce now from (1.5) and (1.4) that $w_t \leq 0$ a.e. in $U \times (t_0,t_1)$. This leads to

$$\int_U (w(x,t_1) - w(x,t_0))\,\mathrm{d}x\ =\ \int_U \int_{t_0}^{t_1} w_t(x,t)\,\mathrm{d}t\ \mathrm{d}x\ \mathrm{d}x\ \leq 0.$$

Since $w(x_1,t_1) = w_+ + \delta$, (3.26), and (3.23) yield that the integral on the left-hand side has to be positive, we have derived a contradiction. An analogous argumentation to get a contradiction can be used if $w(x_1,t_1) = w_- - \delta$.

Hence, we have derived a contradiction for all cases we have to consider by (3.18). Therefore, we have proved (3.17). Recalling (1.6) and (3.9), we get also uniform bounds for u and q, and (3.16) is proved. □

Lemma 3.5. *There are positive constant $C_6, \ldots C_{10}$ such that*

$$\max_{1\leq i\leq 4} \sup_{0\leq t} \left(\|\mathcal{H}_i[u_x,w](\cdot,t)\|_\infty\right) \leq C_6, \tag{3.28}$$

$$0 \leq \sup_{0\leq t} \int_0^1 \left(-\mathcal{F}_2[u_x,w](x,t)\right)\,\mathrm{d}x\ \leq\ C_7, \tag{3.29}$$

$$\max_{1\leq i\leq 4} |(\mathcal{H}_i[u_x,w])_t| + |(\mathcal{F}_1[u_x,w])_t| \leq C_8\left(|u_{xt}| + \sqrt{w_t\left(\mathcal{G}[w]\right)_t}\right) \tag{3.30}$$

$$\leq C_9\left(|u_{xt}| + |w_t|\right), \quad \textit{a.e. in} \quad \Omega_\infty, \tag{3.31}$$

$$|\sigma| + |w_t|\ \leq\ C_{10}(1+\theta), \quad \textit{a.e. in} \quad \Omega_\infty. \tag{3.32}$$

Proof. Because of (3.16), we have uniform bounds for u_x and w. Thanks to **(H6)** and (3.5), we see that (3.28)–(3.30) are satisfied. Recalling (3.16) and **(H4)**ii), we deduce that

$$0 \le w_t(t) \left(\mathcal{G}[w]\right)_t(t) \le C_{11} w_t(t)^2, \quad \text{for a.e. } t \in (0,\infty).$$

From (3.30), we get therefore (3.31). Combining (1.2), (1.5), (1.4), and (3.28), we find that (3.32) holds. □

Lemma 3.6. *We have a.e. on* Ω

$$\begin{aligned} &(\mathcal{F}_1[u_x,w])_t - \sigma(x,t)u_{xt} \\ &= -\left|(\mathcal{G}[w])_t w_t\right| - \left|\mathcal{D}_1[u_x,w]\right| - \theta\left(\mathcal{H}_2[u_x,w]u_{xt} + (\mathcal{G}[w])_t \mathcal{H}_4[u_x,w]\right). \end{aligned} \tag{3.33}$$

Proof. We apply **(H4)**i) and (1.2) to conclude that a.e. on Ω_∞ holds

$$(\mathcal{F}_1[u_x,w])_t - \sigma(x,t)u_{xt} = (\mathcal{G}[w])_t \mathcal{H}_3[u_x,w] - |\mathcal{D}_1[u_x,w]| - \theta \mathcal{H}_2[u_x,w]u_{xt}.$$

Now, applying (1.5), (1.4), and **(H4)**ii) leads to (3.33). □

Lemma 3.7. *We have a positive constant* C_{12} *such that*

$$\begin{aligned} &\int_0^\infty \left(\left\| \frac{\theta_x}{\theta}(\cdot,t)\right\|_2^2 + \left\| \frac{u_{xt}}{\sqrt{\theta}}(\cdot,t)\right\|_2^2 + \left\| \frac{(\mathcal{G}[w])_t w_t}{\theta}(\cdot,t)\right\|_1 + \|\mathcal{D}_2[u_x,w](\cdot,t)\|_1 \right) \mathrm{d}t \\ &+ \sup_{0\le t} \|\ln\theta(\cdot,t))\|_1 \le C_{12}. \end{aligned} \tag{3.34}$$

Proof. Testing (1.3) by $-1/\theta$ and using (1.6), (3.33), **(H2)**, and **(H4)**i), we observe that

$$\begin{aligned} &-\frac{\partial}{\partial t}\int_\Omega \ln\theta(x,t)\,\mathrm{d}x + \int_\Omega \left(\left(\frac{\theta_x(x,t)}{\theta(x,t)}\right)^2 + \frac{u_{xt}^2(x,t)}{\theta(x,t)} \right) \mathrm{d}x \\ \le &-\frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x - \int_\Omega \frac{\left|(\mathcal{G}[w])_t(x,t)w_t(x,t)\right| + |\mathcal{D}_1[u_x,w](x,t)|}{\theta(x,t)}\,\mathrm{d}x \\ &+ \int_\Omega \left(-|\mathcal{D}_2[u_x,w](x,t)| + |g_2(x,t)|\right)\,\mathrm{d}x\,. \end{aligned}$$

Now, we integrate this equation over time and observe that (3.34) follows by applying (3.29), **(H2)**, **(H5)**, (3.5), and the inequality

$$|\ln s| \le s - \ln s + C_{13}, \quad \forall\, s > 0,$$

that can be shown by elementary analysis. □

Lemma 3.8. *We have a positive constant* C_{14} *such that*

$$\begin{aligned} \int_0^\infty \Big(&\|u_{xt}(\cdot,t)\|_1^2 + \|u_t(\cdot,t)\|_\infty^2 + \|p(\cdot,t)\|_\infty^2 + \|(\mathcal{G}[w])_t(\cdot,t)\|_1^2 \\ &+ \|(\mathcal{F}_1[u_x,w])_t\|_1^2 + \left\|\left(\sqrt{\theta}\right)_x(\cdot,1)\right\|_1^2 \Big)\,\mathrm{d}t \le C_{14}. \end{aligned} \tag{3.35}$$

Proof. Since $\theta > 0$ a.e. on Ω_∞, we can apply Schwarz's inequality and (3.5) to show that for all $t > 0$

$$\|u_{xt}(\cdot,t)\|_1 = \int_\Omega \frac{|u_{xt}(x,t)|}{\sqrt{\theta(x,t)}} \sqrt{\theta(x,t)}\,\mathrm{d}x \le C_{15} \left\| \frac{u_{xt}}{\sqrt{\theta}}(\cdot,t) \right\|_2 . \tag{3.36}$$

Recalling now (3.34) leads to the estimate for u_{xt} in (3.35). Using that, by (1.6) and (2.23), $u_t(y,t) = \int_0^y u_{xt}(x,t)\,\mathrm{d}x$ for all $y \in \overline{\Omega}$, we get the estimate for u_t. Combining this estimate with (3.9) leads to the estimate for p.

Applying (3.30), **(H4)**ii), (3.34), and Young's inequality, we deduce that

$$\int_0^\infty \left(\left\| \frac{(\mathcal{G}[w])_t}{\sqrt{\theta}}(\cdot,t) \right\|_2^2 + \left\| \frac{(\mathcal{F}_1[u_x,w])_t}{\sqrt{\theta}}(\cdot,t) \right\|_2^2 \right) \mathrm{d}t \le C_{16}.$$

Considering now (3.36) with u_{xt} replaced by $(\mathcal{G}[w])_t$, we get the estimate for $(\mathcal{G}[w])_t$ in (3.35), and the estimate for $(\mathcal{F}_1[u_x,w])_t$ is derived analogous. Thanks to Schwarz's inequality, we have

$$\left\| \left(\sqrt{\theta}\right)_x (\cdot,t) \right\|_1 = \int_\Omega \frac{|\theta_x(x,t)|}{\sqrt{\theta(x,t)}}\,\mathrm{d}x \le \left\| \frac{|\theta_x|}{\theta}(\cdot,t) \right\|_2 \left\| \sqrt{\theta}(\cdot,t) \right\|_2 .$$

In the light of (3.5) and (3.34), we see that also the estimate for $\sqrt{\theta}_x$ in (3.35) is shown. □

Lemma 3.9. *For $\overline{\theta}$ and I_1 as in* (2.31) *and* (3.4) *there are positive constant C_{17}, C_{18}, and C_{19} such that*

$$|I_1(t)| \le C_{17}, \quad C_{18} < \overline{\theta}(t) < C_{19}, \quad \forall\, t \ge 0, \tag{3.37}$$

$$\left\|\theta(\cdot,t) - \overline{\theta}(t)\right\|_\infty \le \|\theta_x(\cdot,t)\|_1 \le \|\theta_x(\cdot,t)\|_2 , \quad \forall\, t \ge 0. \tag{3.38}$$

Proof. Combining (3.4), (3.6), (1.7), (3.5), and Hölder's inequality, we see that

$$|I_1(s)| \le C_{20} + \left| \int_\Omega (u(x,s) - u_0(x)) f_\infty(x)\,\mathrm{d}x \right| + \int_0^s \|f(\cdot,t) - f_\infty(t)\|_2 \|u_t(\cdot,t)\|_2 \,\mathrm{d}t .$$

Recalling (3.16), (3.5), **(H7)**, and **(H1)**, we get the uniform bound for I_1 in (3.37). Since $s \mapsto -\ln s$ is a convex function on $(0,\infty)$, we get by (2.26) and Jensen's inequality that

$$-\ln \int_\Omega \theta(x,t)\,\mathrm{d}x \le -\int_\Omega \ln(\theta(x,t))\,\mathrm{d}x , \quad \forall\, t \ge 0.$$

Invoking now (3.34), (2.31), and (3.5), we get (3.37). The first inequality in (3.38) follows the from the definition in (2.31), and the second by applying Schwarz's inequality and $\int_\Omega 1\,\mathrm{d}x = 1$. □

Lemma 3.10. *We have a positive constant C_{21} such that*

$$\int_0^\infty \left(\|\theta_x(\cdot,t)\|_2^2 + \left\| \frac{\partial}{\partial x}\left((u_t)^2\right)(\cdot,t)\right\|_2^2 + \left(\frac{\partial I_1(t)}{\partial t}\right)^2 \right) \mathrm{d}t$$
$$+ \sup_{0\le t}\left(\|u_t(\cdot,t)\|_4 + \|\theta(\cdot,t)\|_2\right) \le C_{21}. \tag{3.39}$$

Proof. We test (3.1) by $\theta + \frac{1}{2}u_t^2$ and (1.1) by αu_t^3 where $\alpha > 0$ will be fixed later. Summing the resulting equations and using (1.6) and (3.4), we observe that for all $t \ge 0$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|\theta(\cdot,t) + \frac{1}{2}u_t^2(\cdot,t)\right\|_2^2 + \|\theta_x(\cdot,t)\|_2^2 + \frac{\alpha}{4}\frac{\partial}{\partial t}\|u_t(\cdot,t)\|_4^4 + (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2$$
$$\le \overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} + I_2(t) + I_3(t) + I_4(t), \tag{3.40}$$

with

$$I_2(t) := \int_\Omega \big(-\left(\mathcal{F}_1[u_x,w]\right)_t(x,t) + g(x,t,\theta(x,t)) + u_t(x,t)f(x,t)\big)$$
$$\big(\theta(x,t) - \overline{\theta}(t)\big)\,\mathrm{d}x\,, \tag{3.41}$$

$$I_3(t) := -\int_\Omega \left(\frac{1}{2}\left(\mathcal{F}_1[u_x,w]\right)_t u_t^2 + 2\theta_x u_t u_{tx} + u_t\sigma\theta_x + (1+3\alpha)u_t^2 u_{tx}\sigma\right)\mathrm{d}x\,, \tag{3.42}$$

$$I_4(t) := \int_\Omega \left(g + (1+2\alpha)u_t f\right)\frac{1}{2}u_t^2\,\mathrm{d}x\,. \tag{3.43}$$

In the sequel, the generic constants C_i will be independent of α. Applying (3.41), Hölder's inequality, **(H7)**, (3.38), and Young's inequality, we get

$$I_2(t)$$
$$\le \left(\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1 + \|g(\cdot,t,\theta(\cdot,t))\|_1 + \|u_t(\cdot,t)\|_\infty\|f(\cdot,t)\|_1\right)\left\|\theta(\cdot,t) - \overline{\theta}(t)\right\|_\infty$$
$$\le \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2 + C_{22}\left(\|(\mathcal{F}_1[u_x,w])_t(\cdot,t)\|_1^2 + \|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_\infty^2\right). \tag{3.44}$$

Invoking (3.42), (3.31), (3.32), Hölder's inequality, and Young's inequality, we deduce that

$$I_3(t) \le C_{23}\left((1+\alpha)\left\|u_{tx}(\cdot,t)u_t^2(\cdot,t)\right\|_1 + \left\|u_t^2(\cdot,t)\right\|_1 + \left\|u_t^2(\cdot,t)\theta(\cdot,t)\right\|_1\right)$$
$$+ 2\|\theta_x(\cdot,t)u_t(\cdot,t)u_{tx}(\cdot,t)\|_1 + C_{24}\|u_t(\cdot,t)\theta_x(\cdot,t)\|_1$$
$$+ C_{25}\|\theta_x(\cdot,t)u_t(\cdot,t)\theta(\cdot,t)\|_1 + (1+\alpha)C_{26}\left\|u_t^2(\cdot,t)u_{tx}(\cdot,t)\theta(\cdot,t)\right\|_1$$
$$\le C_{27}\left(\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + \left(1+\alpha^2\right)\|u_t(\cdot,t)\|_2^2 + \left(1+\alpha^2\right)\|u_t(\cdot,t)\|_\infty^2\|\theta(\cdot,t)\|_2^2\right)$$
$$+ \frac{1}{6}\|\theta_x(\cdot,t)\|_2^2\,. \tag{3.45}$$

Using (3.43), **(H2)**, Hölder's inequality, (3.5), **(H7)**, (3.38), (3.37), and Young's inequality, we conclude

$$
\begin{aligned}
&2I_4(t)\\
&\leq \|g_1(\cdot,t)\|_2 \|u_t(\cdot,t)\|_2 \|u_t(\cdot,t)\|_\infty + \left(\overline{\theta}(t) + \left\|\theta(\cdot,t)-\overline{\theta}(t)\right\|_\infty\right) \|g_2(\cdot,t)\|_\infty \|u_t(\cdot,t)\|_2^2\\
&\quad + (1+2\alpha)\|u_t(\cdot,t)\|_2 \|f(\cdot,t)\|_2 \|u_t(\cdot,t)\|_\infty^2\\
&\leq \frac{1}{6}\|\theta_x\|_2^2 + C_{28}\left(\|g_1(\cdot,t)\|_2^2 + \|g_2(\cdot,t)\|_\infty + \|g_2(\cdot,t)\|_\infty^2\right) + C_{29}(1+\alpha^2)\|u_t(\cdot,t)\|_\infty^2 .
\end{aligned}
\tag{3.46}
$$

Because of (3.2) and Young's inequality, we have

$$
\overline{\theta}(t)\frac{\partial I_1(t)}{\partial t} \leq I_0\frac{\partial I_1(t)}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}\left(I_1(t)\right)^2 + \frac{1}{4}\|u_t(\cdot,t)\|_2^4 + \frac{1}{4}\left(\frac{\partial I_1(t)}{\partial t}\right)^2 . \tag{3.47}
$$

From (3.4), we get by using Hölder's inequality, Young's inequality, **(H7)**, and **(H2)** that

$$
\left(\frac{\partial I_1(t)}{\partial t}\right)^2 \leq C_{30}\left(\|g(\cdot,t,\theta(\cdot,t))\|_1^2 + \|u_t(\cdot,t)\|_2^2 + \|\left(\mathcal{F}_1[u_x,w]\right)_t(\cdot,t)\|_1^2\right). \tag{3.48}
$$

Now, we integrate the sum of (3.40) and (3.48) over time, and use (1.7), **(H1)**, (3.44)–(3.48), (3.6), **(H2)**, (3.5), (3.35), (3.37), and $\theta > 0$ a.e. on Ω to show that

$$
\begin{aligned}
&\frac{1}{2}\|\theta(\cdot,s)\|_2^2 + \frac{\alpha}{4}\|u_t(\cdot,s)\|_4^4\\
&\qquad + \int_0^s \left(\frac{1}{2}\|\theta_x(\cdot,t)\|_2^2 + (1+3\alpha)\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + \frac{3}{4}\left(\frac{\partial I_1(t)}{\partial t}\right)^2\right) \mathrm{d}t\\
&\leq \quad C_{31}\left(1+\alpha^2 + \int_0^s \left(\|u_t(\cdot,t)u_{tx}(\cdot,t)\|_2^2 + \left(1+\alpha^2\right)\|u_t(\cdot,t)\|_\infty^2 \|\theta(\cdot,t)\|_2^2\right)\mathrm{d}t\right)
\end{aligned}
$$

holds for all $s > 0$. Next, we define $\alpha := C_{31}$, apply Gronwall's Lemma, and recall (3.35) to show that (3.39) is satisfied. □

Lemma 3.11. *There are positive constants C_{32}, C_{33} such that*

$$
\int_0^\infty \left(\|u_{xt}(\cdot,t)\|_2^2 + \|\left(\mathcal{G}[w]\right)_t(\cdot,\tau)w_t(\cdot,t)\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,\tau)\|_1\right)\mathrm{d}t \leq C_{32}, \tag{3.49}
$$

$$
\begin{aligned}
&\int_0^\infty \Big(\|p_{xx}(\cdot,t)\|_2^2 + \|(p+q)_t(\cdot,t)\|_2^2 + \|u_t(\cdot,t)\|_\infty^2 + \|\left(\mathcal{F}_1[u_x,w]\right)_t(\cdot,t)\|_2^2\\
&\qquad + \sum_{i=1}^4 \|\left(\mathcal{H}_i[u_x,w]\right)_t(\cdot,t)\|_2^2 + \|\left(\mathcal{G}[w]\right)_t(\cdot,t)\|_2^2\Big)\mathrm{d}t \leq C_{33}.
\end{aligned}
\tag{3.50}
$$

Proof. Integrating (1.3) over Ω, and applying (1.6), (2.31), (3.33), and **(H4)**i), we derive

$$\|u_{xt}(\cdot,t)\|_2^2 \le \frac{\partial\overline{\theta}(t)}{\partial t} + \|g(\cdot,t,\theta(\cdot,t))\|_1 - \left\|(\mathcal{G}[w])_t(\cdot,t)w_t(\cdot,t)\right\|_1 - \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1$$
$$- \int_\Omega \left(\theta(x,t)-\overline{\theta}(t)\right)\left(\mathcal{H}_2[u_x,w](x,t)u_{xt}(x,t) + (G[w])_t(x,t)\mathcal{H}_4[u_w,w](x,t)\right)\mathrm{d}x$$
$$- \overline{\theta}(t)\frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x\,.$$

We multiply this inequality by $1/\overline{\theta}(t)$ and use (3.28), Hölder's inequality, and Young's inequality to prove

$$\frac{1}{\overline{\theta}(t)}\left(\|u_{xt}(\cdot,t)\|_2^2 + \left\|(G[w])_t(\cdot,t)w_t(\cdot,t)\right\|_1 + \|\mathcal{D}_1[u_x,w](\cdot,t)\|_1\right)$$
$$\le \frac{\partial\ln\overline{\theta}(t)}{\partial t} + \frac{1}{\overline{\theta}(t)}\|g(\cdot,t,\theta(\cdot,t))\|_1 - \frac{\partial}{\partial t}\int_\Omega \mathcal{F}_2[u_x,w](x,t)\,\mathrm{d}x$$
$$+ \frac{C_{34}}{\overline{\theta}(t)}\left(\left\|\theta(\cdot,t)-\overline{\theta}(t)\right\|_\infty^2 + \|u_{xt}(\cdot,t)\|_1^2 + \left\|(G[w])_t(\cdot,t)\right\|_1^2\right).$$

Integrating this inequality over time, and using (3.6), (3.37), (3.38), (3.35), and (3.39), we observe that (3.49) is proved. The estimates in (3.50) follow by applying (3.9), (1.6), (3.30), **(H4)**, and (3.16). ☐

Lemma 3.12. *There is positive constant C_{35} such that*

$$\int_0^\infty \left(\|\tilde{\sigma}(\cdot,\tau)\|_2^2 + \|p_t(\cdot,\tau)\|_2^2\right)\mathrm{d}\tau \le C_{35}. \tag{3.51}$$

Proof. Let $J(x,t):\Omega_\infty \to \mathbb{R}$ be defined by

$$J(x,t) := \mathcal{H}_2[u_x,w](x,t)\left(\overline{\theta}(t)-\theta(x,t)+\frac{1}{2}\|u_t(\cdot,t)\|_2^2 + \int_1^x \left(f_\infty(\xi)-f(\xi,t)\right)\mathrm{d}\xi\right)$$
$$+ \tilde{\sigma}(x,t), \quad \text{a.e. on} \quad \Omega. \tag{3.52}$$

Utilizing (3.11) two times, we get

$$(\tilde{\sigma}(x,t))^2 = p_t(x,t)\tilde{\sigma}(x,t) - p_{xx}(x,t)\tilde{\sigma}(x,t)$$
$$= p_t(x,t)J(x,t) + (\tilde{\sigma}(x,t)+p_{xx}(x,t))(\tilde{\sigma}(x,t)-J(x,t)) - p_{xx}(x,t)\tilde{\sigma}(x,t).$$

Integrating this equation over Ω, and using Young's inequality, (3.52), (3.28), and (3.38), we observe that

$$\frac{1}{2}\|\tilde{\sigma}(\cdot,t)\|_2^2 \le \frac{\partial}{\partial t}\int_\Omega p(x,t)J(x,t)\,\mathrm{d}x - \int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x$$
$$+ C_{36}\left(\|p_{xx}(\cdot,t)\|_2^2 + \|\theta_x(\cdot,t)\|_2 + \|u_t(\cdot,t)\|_2^4 + \|f(\cdot,t)-f_\infty(\cdot,t)\|_1^2\right). \tag{3.53}$$

Applying (3.52), (3.10), (1.2), **(H7)**, and (3.2), we observe that

$$J(x,t) = \mathcal{H}_1[u_x,w](x,t) + \mathcal{H}_2[u_x,w](x,t)\,(I_1(t)+I_0) + \int_1^x f_\infty(\xi)\,\mathrm{d}\xi\,. \tag{3.54}$$

Hence, using (3.28), (3.37), **(H7)**, Hölder's inequality, Young's inequality, (3.35), (3.50), and (3.39), we get uniform bounds for J and, for all $s \geq 0$,

$$-\int_0^s \int_\Omega p(x,t)\frac{\partial J(x,t)}{\partial t}\,\mathrm{d}x\ \mathrm{d}t \ \leq \int_0^s \left(\|p(\cdot,t)\|_\infty^2 + \left\|\frac{\partial J(\cdot,t)}{\partial t}\right\|_2^2 \right) \mathrm{d}t\ \leq C_{37}.$$

Integrating now (3.53) with respect to time and using (3.15), (3.50), (3.39), (3.5), (3.35), and **(H7)**, we have shown the estimate for $\tilde{\sigma}$ in (3.51). Combining this estimate with (3.11) and (3.50), we get the estimate for p_t. □

Lemma 3.13. *Let $\zeta \in L^2_{\mathrm{loc}}(0,\infty;H^2(\Omega)) \cap H^1_{\mathrm{loc}}(0,\infty;L^2(\Omega))$ be the solution to the parabolic initial-boundary value problem*

$$\zeta_t - \zeta_{xx} = \tilde{\sigma}_t, \quad a.e.\ in \quad \Omega_\infty, \tag{3.55}$$

$$\zeta_x(0,t) = \zeta(1,t) = 0, \quad \forall\, t \geq 0, \qquad\qquad \zeta(\cdot,0) \equiv 0. \tag{3.56}$$

Then we have a positive constant C_{38} such that, for all $t \geq 0$,

$$\|\zeta(\cdot,t)\|_\infty^2 \leq C_{38}\left(1 + \max_{0\leq\tau\leq t}\|\theta(\cdot,\tau)\|_\infty^{3/2} + \left(\int_0^t \|\theta_t(\cdot,\tau)\|_2^2\right)^{3/4}\right). \tag{3.57}$$

Proof. Multiplying (3.55) by ζ, integrating over $\Omega\times(0,T)$, performing partial integrations, and using (3.56), we get for all $t > 0$

$$\begin{aligned}\frac{1}{2}\|\zeta(\cdot,t)\|_2^2 + \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\ &= \int_0^t\int_\Omega \tilde{\sigma}_t(x,\tau)\zeta(x,\tau)\,\mathrm{d}x\ \mathrm{d}\tau \\ = \int_\Omega \tilde{\sigma}(x,t)\zeta(x,t)\,\mathrm{d}x\ &- \int_0^t\int_\Omega \tilde{\sigma}(x,\tau)\zeta_t(x,\tau)\,\mathrm{d}x\ \mathrm{d}\tau\,. \end{aligned} \tag{3.58}$$

Because of (3.10), (3.32), (3.39), and **(H7)**, we have a uniform upper bound for $\|\tilde{\sigma}(\cdot,t)\|_2$. Hence, we get from (3.58) by applying Hölder's inequality, Young's inequality, and (3.51) that

$$\frac{1}{4}\|\zeta(\cdot,t)\|_2^2 + \int_0^t \|\zeta_x(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\ \leq C_{39}\left(\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\right)^{1/2}. \tag{3.59}$$

Formally, we test (3.55) with ζ_t, use (3.56), integrate over time, and apply Young's inequality to deduce that

$$\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\ + \|\zeta_x(\cdot,t)\|_2^2\ \leq \frac{1}{2}\int_0^t \|\zeta_t(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\ + \frac{1}{2}\int_0^t \|\tilde{\sigma}_t(\cdot,\tau)\|_2^2\ \mathrm{d}\tau\,. \tag{3.60}$$

For a rigorous derivation of this inequality, one has to consider (3.55) with $\tilde{\sigma}_t$ replaced by some smooth approximation, perform this computation for the corresponding solutions and consider afterwards the limit.

Inserting (3.59) into the left-hand side of (3.60) and using (3.10), (1.2), (3.35), Hölder's inequality, Young's inequality, (3.28), (3.50), and **(H7)**, we observe that

$$\frac{1}{4^2 C_{39}^2 2}\|\zeta(\cdot,t)\|_2^4 + \|\zeta_x(\cdot,t)\|_2^2 \leq \frac{1}{2}\int_0^t \|(\mathcal{H}_1[u_x,w] + \theta\mathcal{H}_2[u_x,w] + F)_t(\cdot,\tau)\|_2^2 \, \mathrm{d}\tau$$
$$\leq C_{40} + C_{41}\max_{0\leq\tau\leq t}\|\theta(\cdot,\tau)\|_\infty^2 + C_{42}\int_0^t \|\theta_t(\cdot,\tau)\|_2^2 \, \mathrm{d}\tau\,. \tag{3.61}$$

Thanks to the Gagliardo-Nirenberg inequality (see below) and Young's inequality, we conclude that

$$\begin{aligned}\|\zeta(\cdot,t)\|_\infty^2 &\leq \left(C_{43}\|\zeta_x(\cdot,t)\|_2^{1/2}\|\zeta(\cdot,t)\|_2^{1/2} + C_{44}\|\zeta(\cdot,t)\|_2\right)^2 \\ &\leq C_{45}\left(1 + \|\zeta_x(\cdot,t)\|_2^{3/2} + \|\zeta(\cdot,t)\|_2^3\right).\end{aligned}$$

Now, we apply (3.61) and Young's inequality to prove that (3.57) holds. □

The following version of the Gagliardo-Nirenberg inequality is a special case, more general formulations can be fond, e.g., in [BS96, Zhe95].

Lemma 3.14 (Gagliardo-Nirenberg inequality). *For all $p \geq 1$ there are positive constants C_{46}, C_{47} such that*

$$\|v\|_\infty \leq C_{46}\|v_x\|_2^{2/(p+2)}\|v\|_p^{p/(p+2)} + C_{47}\|v\|_p\,, \quad \forall\, v \in H^1(\Omega). \tag{3.62}$$

Lemma 3.15. *There is a positive constant C_{48} such that*

$$\|u_{xt}(\cdot,t)\|_\infty^2 \leq C_{48}\left(1 + \max_{0\leq\tau\leq t}\|\theta(\cdot,\tau)\|_\infty^2 + \left(\int_0^t \|\theta_t(\cdot,\tau)\|_2^2 \, \mathrm{d}\tau\right)^{3/4}\right). \tag{3.63}$$

Proof. Let $z_1, z_2 : \overline{\Omega_\infty} \to \mathbb{R}$ be the solutions to the parabolic initial-boundary value problems

$$z_{i,t} - z_{i,xx} = 0, \quad \text{a.e. in} \quad \Omega_\infty, \quad \forall\, i \in \{1,2\}, \tag{3.64}$$
$$z_i(1,t) = z_{i,x}(0,t) = 0, \quad \text{for a.e.}\ t > 0, \quad \forall\, i \in \{1,2\}, \tag{3.65}$$
$$z_1(x,0) = u_{1,x}(x,0), \quad z_2(x,0) = \tilde{\sigma}(x,0) \quad \text{a.e. in} \quad \Omega. \tag{3.66}$$

Let $z_3 : \overline{\Omega_\infty} \to \mathbb{R}$ be defined by

$$z_3(x,t) = \int_1^x \int_0^y z_1(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y + \int_0^t (z_2(x,\tau) + \zeta(x,\tau))\,\mathrm{d}\tau\,, \quad \forall\,(x,t) \in \Omega_\infty. \tag{3.67}$$

Recalling (3.64), (3.65), (3.66), (3.55), (3.56), and **(H1)**, we observe that

$$z_{3,t} = z_1 + z_2 + \zeta, \quad z_{3,xx} = z_1 + z_2 + \zeta - \tilde{\sigma}, \quad \text{a.e. in} \quad \Omega_\infty, \tag{3.68}$$
$$z_3(1,t) = 0 = z_{3,x}(0,t), \quad \text{for a.e. } t \geq 0, \quad z_3(x,0) = \int_1^x u_1(\xi)\,\mathrm{d}\xi\,, \quad \forall\, x \in \Omega.$$

Hence, we see that z_3 is a solution to the linear parabolic initial-boundary value problem considered in (3.11), (3.12). Since p is the unique solution to this problem, we have $p = z_3$ a.e. on Ω_∞. Therefore, recalling $u_{xt} = p_{xx}$ and (3.68), we have

$$u_{xt} = z_{3,xx} = z_1 + z_2 + \zeta - \tilde{\sigma}, \quad \text{a.e. in} \quad \Omega_\infty. \tag{3.69}$$

Using (3.66), **(H1)**, (3.10), (1.2), (1.6), **(H6)**, and **(H7)**, we get uniform bounds for $z_1(\cdot,0)$ and $z_2(\cdot,0)$. Applying the maximum principle for linear parabolic equations, we get uniform bounds for z_1 and z_2. Because of (3.10), **(H7)**, and (3.32), we have

$$\tilde{\sigma} \leq C_{49} + C_{50}\theta, \quad \text{a.e. in} \quad \Omega_\infty.$$

Thus, applying (3.69), (3.57), and Young's inequality, yields that (3.63) holds. □

Lemma 3.16. *There is a positive constant C_{51} such that*

$$\sup_{0\leq t} \|\theta_x(\cdot,\tau)\|_2 + \int_0^t \|\theta_t(\cdot,\tau)\|_2^2 \,\mathrm{d}\tau \,\leq C_{51}. \tag{3.70}$$

Proof. Testing (1.3) by θ_t, using (1.6), **(H2)**, Young's inequality, Hölder's inequality, and (3.32), we see that

$$\begin{aligned}
&\frac{1}{2}\|\theta_t(\cdot,t)\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2 \\
&\leq \frac{1}{2}\left\|u_{xt}^2(\cdot,t) + \sigma u_{xt}(\cdot,t) - (\mathcal{F}_1[u_x,t])_t(\cdot,t) + g(\cdot,t,\theta(\cdot,t))\right\|_2^2 \\
&\leq C_{52}\|u_{xt}(\cdot,t)\|_2^2\left(\|u_{xt}(\cdot,t)\|_\infty^2 + 1 + \|\theta(\cdot,t)\|_\infty^2\right) + C_{53}\|(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2 \\
&\quad + C_{54}\|g_1(\cdot,t)\|_2^2 + C_{55}\|g_2(\cdot,t)\|_2^2\|\theta(\cdot,t)\|_\infty^2 .
\end{aligned} \tag{3.71}$$

Integrating this equation over time, using (1.7), **(H1)**, **(H2)**, Hölder's inequality, (3.49), (3.50), and (3.63), we see that

$$\begin{aligned}
&\int_0^s \|\theta_t(\cdot,t)\|_2^2 \,\mathrm{d}t \,+ \|\theta_x(\cdot,s)\|_2^2 \leq C_{56} + C_{57}\max_{0\leq t\leq s}\left(\|u_{xt}(\cdot,t)\|_\infty^2 + \|\theta(\cdot,t)\|_\infty^2\right) \\
&\leq C_{58} + C_{59}\left(\int_0^s \|\theta_t(\cdot,t)\|_2^2\,\mathrm{d}t\right)^{\frac{3}{4}} + C_{60}\max_{0\leq t\leq s}\|\theta(\cdot,t)\|_\infty^2 .
\end{aligned} \tag{3.72}$$

Thanks to the Gagliardo Nirenberg inequality and (3.5), we have

$$\|\theta(\cdot,t)\|_\infty \leq C_{61}\|\theta_x(\cdot,t)\|_2^{2/3}\|\theta(\cdot,t)\|_1^{1/3} + C_{62}\|\theta(\cdot,t)\|_1 \leq C_{63} + C_{64}\|\theta_x(\cdot,t)\|_2^{2/3}$$

Using this inequality to estimate the right-hand side of (3.72), and applying Young's inequality afterwards, we see that (3.70) holds. □

Lemma 3.17. *There are positive constants C_{65}, C_{66} such that*

$$\sup_{0\le t} \left(\|\theta(\cdot,t)\|_\infty + \|u_{xt}(\cdot,t)\|_\infty + \|\sigma(\cdot,t)\|_\infty + \|w_t(\cdot,t)\|_\infty\right) \le C_{65}, \tag{3.73}$$

$$\int_0^\infty \left(\|\sigma_t(\cdot,t)\|_2^2 + \|\psi_t(\cdot,t)\|_2^2 + \|\tilde\sigma_t(\cdot,t)\|_2^2\right)\,\mathrm{d}t \le C_{66}, \tag{3.74}$$

$$\int_0^\infty \left(|\mathcal{D}_1[u_x(x,\cdot), w(x,\cdot)](t)| + |\mathcal{D}_2[u_x(x,\cdot), w(x,\cdot)](t)|\right)\,\mathrm{d}t < \infty, \quad \textit{for a.e. } x\in\Omega. \tag{3.75}$$

Proof. Using (3.38) and (3.70), we get the estimate for θ in (3.74) and applying in addition (3.63) and (3.32) leads to the remaining estimates in (3.73). Invoking (1.2), (1.5), (3.50), (3.73), (3.70), and (3.28), we get the estimates for σ_t and ψ_t. Utilizing also (3.10), **(H7)**, and (3.35), we derive the estimates for $\tilde\sigma_t$. Combining (3.34) and (3.49) and using Fubini's theorem, we see that (3.75) holds. □

4 Proof of the asymptotic results

As in the last section, it will be assumed that **(H1)**–**(H8)** are satisfied, and that a solution (u,θ,w) to (1.1)–(1.2) is given, such that (2.23)–(2.26) holds.

For proving the asymptotic results in Theorem 1 with an argumentation similar to [RZ97, Section 4], the following modification of [SZ93, Lemma 3.1] will be used. In the original formulation, it was assumed that the inequality in (4.1) holds for all t in the considered interval, but the proof in [SZ93] can also be used if this inequality holds only for a.e. t in the considered interval.

Lemma 4.1. *Suppose that y and h are non-negative functions on $(0,\infty)$, with y' locally integrable, such that there are positive positive constants $A_1,\ldots,A_4$ such that*

$$y'(t) \le A_1 y^2(t) + A_2 + h(t), \quad \textit{for a.e. } t\in(0,\infty), \tag{4.1}$$

$$\int_0^\infty y(t)\,\mathrm{d}t \le A_3, \quad \int_0^\infty h(t)\,\mathrm{d}t \le A_4. \tag{4.2}$$

Then

$$\lim_{t\to\infty} y(t) = 0. \tag{4.3}$$

Lemma 4.2. *We have* (2.28) *and*

$$\lim_{t\to\infty}\|p_x(\cdot,t)\|_2 = \lim_{t\to\infty}\|u_t(\cdot,t)\|_2 = 0, \tag{4.4}$$

$$\lim_{t\to\infty}\|\tilde\sigma(\cdot,t)\|_2 = \lim_{t\to\infty}\|q_t\|_2 = 0. \tag{4.5}$$

Proof. Testing (3.11) with $-p_{xx}$, applying (3.12) and Young's inequality, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\|p_x(\cdot,t)\|_2^2 + \|p_{xx}(\cdot,t)\|_2^2 \le \frac{1}{2}\|p_{xx}(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde\sigma(\cdot,t)\|_2^2, \quad \text{for a.e. } t\in(0,\infty).$$

Since $u_t = p_x$ a.e. in Ω_∞, we see by recalling (3.35) and (3.51) that we can apply Lemma 4.1 to show that (4.4) holds. We have, by Young's inequality,

$$\frac{\partial}{\partial t}\|\tilde{\sigma}(\cdot,t)\|_2^2 = 2\int_\Omega \tilde{\sigma}(x,t)\tilde{\sigma}_t(x,t)\,\mathrm{d}x \le \|\tilde{\sigma}(\cdot,t)\|_2^2 + \|\tilde{\sigma}_t(\cdot,t)\|_2^2, \quad \text{for a.e. } t \in (0,\infty).$$

Invoking (3.51), (3.74), and Lemma 4.1, we get the convergence result for $\tilde{\sigma}$ in (4.5). Since (3.13), (3.10), and **(H7)** yield that $q_t = -\tilde{\sigma}$, we also have the result for q_t in (4.5). Combining (4.5), (3.10), **(H7)**, and the definition on F_∞ in (2.31), we get (2.28). □

Lemma 4.3. *We have*

$$\lim_{t\to\infty}\|p_t(\cdot,t)\|_2 = \lim_{t\to\infty}\|p_{xx}(\cdot,t)\|_2 = \lim_{t\to\infty}\|u_{xt}(\cdot,t)\|_2 = \lim_{t\to\infty}\|u_t(\cdot,t)\|_\infty = 0. \tag{4.6}$$

Proof. Differentiating (3.11) with respect to t, testing it afterwards by p_t, and applying (3.11) and Young's inequality, we see that

$$\frac{\partial}{\partial t}\|p_t(\cdot,t)\|_2^2 + \|p_{xt}(\cdot,t)\|_2^2 \le \frac{1}{2}\|p_t(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde{\sigma}_t(\cdot,t)\|_2^2, \quad \text{for a.e. } t \in (0,\infty).$$

Using (3.51), (3.74), and Lemma 4.1, we get the convergence result for p_t in (4.6). By (3.11), we can combine this with (4.5) to prove the convergence result for p_{xx} in (4.6). Recalling also (3.9), we get the convergence result for u_{xt} and using (1.6), we obtain the result for u_t. □

Lemma 4.4. *We have*

$$\lim_{t\to\infty}\|\theta_x(\cdot,t)\|_2 = \lim_{t\to\infty}\left\|\theta(\cdot,t) - \overline{\theta}(t)\right\|_\infty = 0. \tag{4.7}$$

Moreover, we have some constant $\theta_* > 0$ such that (2.30) *holds.*

Proof. Combining (3.71) with (3.73), we get for a.e. $t \in (0,\infty)$

$$\frac{1}{2}\frac{\partial}{\partial t}\|\theta_x(\cdot,t)\|_2^2 \le C_{67}\left(\|u_{xt}(\cdot,t)\|_2^2 + \|(\mathcal{F}_1[u_x,t])_t(\cdot,t)\|_2^2 + \|g_1(\cdot,t)\|_2^2 + \|g_2(\cdot,t)\|_2^2\right).$$

Because of (3.39), (3.49), (3.50), and **(H2)**, we can now use Lemma 4.1 to get the convergence result for θ_x. Recalling (3.38), we obtain the result for $\theta - \overline{\theta}$. Combining this with (3.37), we get some $t_0 > 0$ such that

$$\theta(x,t) > C_{18}/2, \quad \forall\, x \in \overline{\Omega},\, t \ge t_0.$$

Moreover, (2.24) and (2.26) yield that θ is continuous and positive on $\overline{\Omega}\times[0,t_0]$, and therefore also bounded from below by a positive constant C' on this set. Setting $\theta_* := \min(C_{18}/2, C')$, we see that (2.30) holds. □

Lemma 4.5. *If $\mathcal{G}$ is the identity operator, then we have* (2.33) *and*

$$\lim_{t\to\infty}\|w_t(\cdot,t)\|_2 = \lim_{t\to\infty}\|\psi(\cdot,t)\|_2 = 0. \tag{4.8}$$

Proof. Testing the time derivative of (1.4) by w_t and using Young's inequality, we see that for a.e. $t \in (0,\infty)$

$$\frac{\partial}{\partial t}\|w_t(\cdot,t)\|_2^2 \leq \int_\Omega w_t(x,t)\psi_t(x,t)\,\mathrm{d}x \leq \frac{1}{2}\|w_t(\cdot,t)\|_2^2 + \frac{1}{2}\|\psi_t(\cdot,t)\|_2^2 .$$

By assumption, we have $w_t = (\mathcal{G}[w])_t$ and can therefore apply (3.50), (3.74), Lemma 4.1, and (1.4) to show that (4.8) holds. Using now **(H6)**iii) and (4.6), we get also (2.33) . ☐

Lemma 4.6. *Assume that $\mathcal{H}_1 \equiv \mathcal{H}_3 \equiv \mathcal{F}_1 \equiv 0$, $g \equiv 0$, and $f \equiv 0$. Then, we have*

$$\theta(\cdot,t) \xrightarrow[t\to\infty]{} \|\theta_0\|_1 + \frac{\rho}{2C_V}\|u_1\|_2^2, \quad \textit{in} \quad L^\infty(\Omega). \tag{4.9}$$

and (2.35). *If $\mathcal{G}$ is the identity operator then we have* (2.36).

Proof. Thanks to the assumptions, (3.4), (3.10), (1.2), **(H7)**, and **(H5)**, we see that $I_1 \equiv 0$, that I_0/C_V is equal to the right–hand side of (4.9), and that $\tilde{\sigma} = \theta\mathcal{H}_2[u_x, w]$. Invoking (3.2), (4.6), (4.7), (4.5), and **(H1)**, we get (4.9) and (2.35). If $\mathcal{G}$ is the identity operator then it follows from (4.8), $\psi = \theta\mathcal{H}_4[u_x, w]$, and (4.9) that (2.36) holds. ☐

Lemma 4.7. *If **(H9)** holds then there is a $u_\infty \in W^{1,\infty}(\Omega)$ such that* (2.37)–(2.38) *hold.*

Proof. Owing to (3.75) and **(H9)**, we have a function $\varepsilon_\infty : \Omega \to \mathbb{R}$ such that

$$u_x(x,t) \xrightarrow[t\to\infty]{} \varepsilon_\infty(x), \quad \text{for a.e. } x \in \Omega. \tag{4.10}$$

Invoking (3.16), compactness, and properties of weak-star and weak convergence, we see that

$$u_x(\cdot,t) \xrightarrow[t\to\infty]{} \varepsilon_\infty, \quad \text{weakly-star in} \quad L^\infty(\Omega). \tag{4.11}$$

Defining now $u_\infty(x) := \int_0^x \varepsilon_\infty(\xi)$ and using (1.6), we conclude that $u_\infty \in W^{1,\infty}(\Omega)$ and (2.37)–(2.38) hold. ☐

Lemma 4.8. *If **(H10)** holds then there is a $w_\infty \in L^\infty(\Omega)$ such that* (2.39) *holds.*

Proof. Thanks to (3.75), **(H10)**, (3.16), compactness, and properties of weak convergence, we get a $w_\infty \in L^\infty(\Omega)$ such that (2.39) holds. ☐

This completes the proof of Theorem 1.

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