Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

An approximation method for Navier-Stokes equations based on probabilistic approach

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submitted: 3rd April 2002

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No. 733
Berlin 2002

2000 *Mathematics Subject Classification.* 76D05, 60H30, 65M99.

Key words and phrases. Numerical analysis of Navier-Stokes equations; probabilistic representations for equations of mathematical physics; weak approximation of solutions of stochastic differential equations.

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ABSTRACT. A new layer method solving the space-periodic problem for the Navier-Stokes equations is constructed by using probabilistic representations of their solutions. The method exploits the ideas of weak sense numerical integration of stochastic differential equations. Despite its probabilistic nature this method is nevertheless deterministic. A convergence theorem is proved.

1. INTRODUCTION

We propose an approximation method based on probabilistic approach for the three-dimensional system of Navier-Stokes equations (NSEs) with spatial periodic boundary conditions (for a review and numerical analysis of NSEs see e.g. [3], [12], [4], and references therein). Probabilistic techniques are successfully used for investigation of different partial differential equations. Comparatively few probabilistic works are devoted to NSEs (see among them [2], [6], [1]). To the authors' knowledge there is no numerical method based on a probabilistic representation of solutions to NSEs. In the next section we construct a new layer method solving the space-periodic problem for NSEs by using probabilistic representations of their solutions. The method exploits the ideas of weak sense numerical integration of stochastic differential equations (see [8]). Despite the probabilistic nature this method is nevertheless deterministic. Such methods have been derived for nonlinear equations of parabolic type in [9], [10], [11]. The probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. In Section 3 a convergence theorem for the layer method for NSEs is proved. Here we restrict ourselves to the space-periodic problem. Applications of the developed approach to other problems for Navier-Stokes equations will appear elsewhere. Numerical implementations of the obtained method will be considered in a separate work.

2. CONSTRUCTION OF THE APPROXIMATION METHOD

Let us consider the system of Navier-Stokes equations for velocity v and pressure p in a viscous incompressible flow

$$\frac{\partial v}{\partial s} + (v, \nabla)v = \frac{\sigma^2}{2}\Delta v - \nabla p, \ -T \le s \le 0, \ x, v \in \mathbf{R}^3, \tag{2.1}$$

$$divv = 0, \tag{2.2}$$

with initial condition (2.3) and spatial periodic condition (2.4):

$$v(-T, x) = \varphi(x), \tag{2.3}$$

$$v(s, x + Le_i) = v(s, x), \ i = 1, 2, 3, \ -T \le s \le 0, \tag{2.4}$$

where $\{e_i\}$ is the canonical basis in $\mathbf{R}^3$ and $L > 0$ is the period in i-th direction. Denote by $Q = (0, L)^3$ the cube of the period. Of course, one may consider different periods $L_1, \ L_2, \ L_3$ in the different directions.

System (2.1) is autonomous and, consequently, its solution does not depend on shift of time. The choice of the interval $[-T, 0]$ is convenient for a probabilistic representation of the solution to the problem (2.1)-(2.4). Introducing the new time $t = -s$ and the new function $u(t,x) = v(-t,x)$, $0 \le t \le T$, we get

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\Delta u - (u,\nabla)u - \nabla p = 0,\ 0 \le t \le T,\ x \in \mathbf{R}^3, \tag{2.5}$$

$$div u = 0, \tag{2.6}$$

$$u(T,x) = \varphi(x), \tag{2.7}$$

$$u(t, x + Le_i) = u(t,x),\ i = 1,2,3,\ 0 \le t \le T. \tag{2.8}$$

To study the solutions to this system, we need some functional spaces and the Helmholz decomposition (see, e.g. [7], [12], [4]). We denote by $\mathbf{L}^2(Q)$ the Hilbert space of functions on Q with the scalar product and the norm

$$(u,v) = \int_Q \sum_{i=1}^{3} u^i(x)v^i(x)dx,\ ||u|| = (u,u)^{1/2}.$$

We keep the notation $|\cdot|$ for the absolute value of numbers and for the length of three-dimensional vectors, for example,

$$|u(t,x)| = [(u^1(t,x))^2 + (u^2(t,x))^2 + (u^3(t,x))^2]^{1/2}.$$

We denote by $\mathbf{H}^m_p(Q)$, $m = 0, 1, ...$, the Sobolev space of functions which are in $\mathbf{L}^2(Q)$, together with all their derivatives of order $\le m$, and which are periodic functions with the period Q. The space $\mathbf{H}^m_p(Q)$ is a Hilbert space with the scalar product and the norm

$$(u,v)_m = \int_Q \sum_{i=1}^{3} \sum_{[\alpha^i]\le m} D^{\alpha^i}u^i(x)D^{\alpha^i}v^i(x)dx,\ ||u||_m = [(u,u)_m]^{1/2},$$

where $\alpha^i = (\alpha^i_1, \alpha^i_2, \alpha^i_3)$, $\alpha^i_j \in \{0, ...m\}$, $[\alpha^i] = \alpha^i_1 + \alpha^i_2 + \alpha^i_3$, and

$$D^{\alpha^i} = D_1^{\alpha^i_1} D_2^{\alpha^i_2} D_3^{\alpha^i_3} = \frac{\partial^{[\alpha^i]}}{\partial x_1^{\alpha^i_1}\partial x_2^{\alpha^i_2}\partial x_3^{\alpha^i_3}},\ i = 1,2,3.$$

Clearly $\mathbf{H}^0_p(Q) = \mathbf{L}^2(Q)$.

In connection with the Helmholz decomposition, introduce the Hilbert subspaces of $\mathbf{H}^m_p(Q)$:

$$\mathbf{V}^m_p = \{v : v \in \mathbf{H}^m_p(Q),\ div v = 0\},\ m > 0,$$

$$\mathbf{V}^0_p = \text{the closure of } \mathbf{V}^m_p,\ m > 0,\ \text{in } \mathbf{L}^2(Q).$$

Clearly

$$\mathbf{V}^{m_1}_p = \text{the closure of } \mathbf{V}^{m_2}_p \text{ in } \mathbf{H}^{m_1}_p(Q) \text{ for any } m_2 \ge m_1.$$

Denote by P the orthogonal projection in $\mathbf{H}_p^m(Q)$ onto $\mathbf{V}_p^m$ (we omit m in the notation for P here). The Helmholz decomposition consists in the fact that any $u \in \mathbf{H}_p^m(Q)$ is equal to

$$u = Pu + \nabla g, \ divPu = 0,$$

where $g = g(x)$ is a scalar Q-periodic function such that $\nabla g \in \mathbf{H}_p^m(Q)$. In other words,

$$(\mathbf{V}_p^m)^\perp = \{v : v \in \mathbf{H}_p^m(Q), \ v = \nabla g\}.$$

Let $u(t,x), \ p(t,x)$ be a solution of the problem (2.5)-(2.8). The following representation holds (we use here the probabilistic representation of solutions of the Cauchy problem for equations of parabolic type (see, e.g. [5], [6])):

$$u(t,x) = E[\varphi(X_{t,x}(T)) - \int_t^T \nabla p(s, X_{t,x}(s))ds], \tag{2.9}$$

where $X_{t,x}(s)$ solves the Ito system of stochastic differential equations

$$dX = -u(s,X)ds + \sigma dW(s), \ X(t) = x, \tag{2.10}$$

W is a standard 3-dimensional Wiener process.

Let $0 = t_0 < t_1 < ... < t_N = T$ be a uniform partition of the interval $[0,T]$ and $h = T/N$ (we restrict ourselves to the uniform partition for simplicity only). Clearly, analogously to (2.9) one can write

$$u(t_k,x) = E[u(t_{k+1}, X_{t_k,x}(t_{k+1})) - \int_{t_k}^{t_{k+1}} \nabla p(s, X_{t_k,x}(s))ds], \tag{2.11}$$

where $X_{t_k,x}(s)$ solves (2.10) with the initial data $X(t_k) = x$.

Fix k for a while and assume the function $u(t_{k+1}, x)$ as a function of x to be known, i.e., we assume the solution on the layer $t = t_{k+1}$ to be known. Our nearest aim is to find an approximation of $u(t_k, x)$, i.e., of the solution on the previous k-th layer $t = t_k$. Applying a slightly modified explicit Euler scheme with the simplest noise simulation to the system (2.10), we obtain

$$\bar{X}_{k+1} = x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi, \tag{2.12}$$

where $\xi = (\xi^1, \xi^2, \xi^3)^\top, \ \xi^1, \xi^2, \xi^3$ are i.i.d. random variables with the law $P(\xi^i = -1) = P(\xi^i = 1) = 1/2, \ i = 1,2,3.$

From the theory of weak numerical methods for stochastic differential equations [8], it follows that if $u(t_{k+1}, x)$ and $p(s,x), \ t_k \le s \le t_{k+1}$, are sufficiently smooth, then

$$Eu(t_{k+1}, X_{t_k,x}(t_{k+1})) = Eu(t_{k+1}, \bar{X}_{k+1}) + O(h^2), \tag{2.13}$$

and

$$E\int_{t_k}^{t_{k+1}} \nabla p(s, X_{t_k,x}(s))ds = \nabla p(t_{k+1}, x)h + O(h^2). \tag{2.14}$$

The remainder $O(h^2)$ in both (2.13) and (2.14) is a function of $k,\ x,\ h$ and it is of the second order of smallness uniformly with respect to these variables, i.e.,

$$|O(h^2)| \leq Kh^2,$$

where $K > 0$ does not depend on $k,\ x$, and h. Below we use the letters K and C without any index for various constants which do not depend on $k,\ x,\ h$.

Hence it follows from (2.11)-(2.12) that

$$\begin{aligned} u(t_k, x) &= Eu(t_{k+1}, \bar{X}_{k+1}) - \nabla p(t_{k+1}, x)h + O(h^2) \\ &= Eu(t_{k+1}, x^1 - u^1(t_{k+1}, x)h + \sigma\sqrt{h}\xi^1, ..., x^3 - u^3(t_{k+1}, x)h + \sigma\sqrt{h}\xi^3) \\ &\quad -\nabla p(t_{k+1}, x)h + O(h^2) \\ &= v(t_k, x) - \nabla p(t_{k+1}, x)h + O(h^2), \end{aligned} \tag{2.15}$$

where

$$v(t_k, x) = Eu(t_{k+1}, \bar{X}_{k+1}) = \frac{1}{8}\sum_{q=1}^{8} u(t_{k+1}, x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q), \tag{2.16}$$

$\xi_1 = (1,1,1)^\top, ...,\ \xi_8 = (-1,-1,-1)^\top$.

Using the Helmholz decomposition, we derive

$$v(t_k, x) = Pv(t_k, x) + \nabla g(t_k, x). \tag{2.17}$$

Taking into account that $divu(t_k, x) = 0,\ divPv(t_k, x) = 0$, comparing (2.15) with (2.17), and using the Helmholz decomposition again, we get

$$\nabla g(t_k, x) = \nabla p(t_{k+1}, x)h + \tilde{O}(h^2), \tag{2.18}$$

$$u(t_k, x) = Pv(t_k, x) + \tilde{O}(h^2). \tag{2.19}$$

We should underline that the remainder $\tilde{O}(h^2)$ unlike $O(h^2)$ is of the second order of smallness in the sense of the space $\mathbf{L}^2(Q)$, i.e.

$$||\tilde{O}(h^2)|| \leq Kh^2,$$

where $K > 0$ does not depend on k and h.

Heuristically, a method, based on a one-step approximation of the second order, converges and has the first order of smallness with respect to h. So we can propose the following method

$$\bar{u}(t_N, x) = \varphi(x),\ \bar{u}(t_k, x) = P\bar{v}(t_k, x),\ k = N-1, ..., 0, \tag{2.20}$$

where

$$\bar{v}(t_k, x) = \frac{1}{8}\sum_{q=1}^{8} \bar{u}(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q). \tag{2.21}$$

Clearly

$$div\bar{u}(t_k, x) = 0. \tag{2.22}$$

Knowing $\bar{v}(t_k,x)$, it is not difficult to find $\bar{u}(t_k,x)$. Indeed, due to the Helmholz decomposition

$$\bar{v}(t_k,x) = P\bar{v}(t_k,x) + \nabla\bar{g}(t_k,x) = \bar{u}(t_k,x) + \nabla\bar{g}(t_k,x). \tag{2.23}$$

The functions $\bar{v}(t_k,x)$, $\bar{u}(t_k,x)$, and $\bar{g}(t_k,x)$ can be expanded in Fourier series (for the sake of simplicity the dependences on k are omitted in the right hand sides of (2.24)):

$$\bar{v}(t_k,x) = \sum_{\mathbf{n}\in Z^3} \bar{v}_{\mathbf{n}} e^{i\frac{2\pi}{L}(\mathbf{n},x)}, \ \bar{u}(t_k,x) = \sum_{\mathbf{n}\in Z^3} \bar{u}_{\mathbf{n}} e^{i\frac{2\pi}{L}(\mathbf{n},x)}, \tag{2.24}$$

$$\bar{g}(t_k,x) = \sum_{\mathbf{n}\in Z^3} \bar{g}_{\mathbf{n}} e^{i\frac{2\pi}{L}(\mathbf{n},x)}.$$

We have from (2.23)

$$\bar{v}^j_{\mathbf{n}} = \bar{u}^j_{\mathbf{n}} + i\frac{2\pi}{L}\mathbf{n}^j\bar{g}_{\mathbf{n}}, \ j = 1,2,3. \tag{2.25}$$

Now (2.22) and (2.25) yield

$$(\bar{u}_{\mathbf{n}},\mathbf{n}) = 0, \ (\bar{v}_{\mathbf{n}},\mathbf{n}) = i\frac{2\pi}{L}(\mathbf{n},\mathbf{n})\bar{g}_{\mathbf{n}}$$

and consequently

$$\bar{u}_{\mathbf{n}} = \bar{v}_{\mathbf{n}} - \frac{(\bar{v}_{\mathbf{n}},\mathbf{n})}{(\mathbf{n},\mathbf{n})}\mathbf{n}, \ \mathbf{n} \neq \mathbf{0}. \tag{2.26}$$

Clearly

$$\bar{u}_{\mathbf{0}} = \bar{v}_{\mathbf{0}}.$$

Thus, the method (2.20)-(2.21) can be realized. In the next section we prove that this method converges and has the order of accuracy $\tilde{O}(h)$. More exactly, we prove that $\bar{u}(t_k,x) = u(t_k,x) + \tilde{O}(h)$. In this paper we do not consider approximation of pressure p in detail. Most likely, the proposed method gives a good approximation for derivatives of $u(t_k,x)$ with respect to x^i as well, i.e., $\partial\bar{u}/\partial x^i$ are sufficiently close to $\partial u/\partial x^i$. In such a case, the pressure can approximately be found from the well known equation

$$\Delta\bar{p}(t_k,x) = -\sum_{i,j=1}^{3} \frac{\partial\bar{u}^j(t_k,x)}{\partial x^i}\frac{\partial\bar{u}^i(t_k,x)}{\partial x^j}. \tag{2.27}$$

Let us propose an additional method which is based on the one-step approximations of u and p given by formulas (2.15) and (2.27):

$$\bar{u}(t_N,x) = \varphi(x) \tag{2.28}$$

$\bar{p}(t_{k+1},x)$ is found from (2.27), $\bar{u}(t_k,x) = \bar{v}(t_k,x) - \nabla\bar{p}(t_{k+1},x)h, \ k = N-1,...,0,$

where $\bar{v}(t_k, x)$ is expressed by (2.21). We should note that $div\bar{u}(t_k, x) \neq 0$. Due to (2.17)-(2.19), the one-step error of this method is estimated in the following way

$$u(t_k,x) - (v(t_k,x) - \nabla p(t_{k+1},x)h)$$
$$= u(t_k,x) - (Pv(t_k,x) + \nabla g(t_k,x) - \nabla p(t_{k+1},x)h)$$
$$= u(t_k,x) - Pv(t_k,x) + \tilde{O}(h^2) = \tilde{O}(h^2),$$

i.e., the one-step error is of $\tilde{O}(h^2)$. Therefore we can expect the method (2.28) to be convergent with accuracy $\tilde{O}(h)$ just as the method (2.20)-(2.21). Here we do not give a complete justification of the method (2.28).

3. The main theorem

Let us evaluate the one-step error of the method (2.20)-(2.21). This error on the k-th layer (on the $(N-k)$-th step) is equal to $Pv(t_k,x) - u(t_k,x)$ provided $\bar{u}(t_{k+1},x) = u(t_{k+1},x)$:

(3.1) $$Pv(t_k,x) - u(t_k,x) = v(t_k,x) - \nabla g(t_k,x) - u(t_k,x)$$
$$= \frac{1}{8}\sum_{q=1}^{8} u(t_{k+1}, x - u(t_{k+1},x)h + \sigma\sqrt{h}\xi_q) - \nabla g(t_k,x) - u(t_k,x).$$

Due to (2.19), the difference $Pv(t_k,x) - u(t_k,x)$ is of $\tilde{O}(h^2)$. We recall that the derivation of (2.19) is based on probabilistic arguments using the theory of numerical methods in weak sense for stochastic differential equations. Below we give a deterministic proof of this fact.

Lemma 1. *Let the solution to* (2.5)-(2.8) *have the continuous derivatives*

(3.2)
$$\frac{\partial^m u}{\partial t^j \partial (x)^l}(t,x),\ j=0,\ l=1,2,3,4;\ j=1,\ l=0,1,2;\ j=2,\ l=0;\ 0\le t\le T,\ x\in \mathbf{R}^3,$$

where the derivative with respect to $(x)^l$ means a mixed derivative with respect to $x^1,\ x^2,\ x^3$ of order l. Then the one-step error of the method (2.20)-(2.21) *is of second order in $\mathbf{H}_p^0(Q)$, i.e.*

(3.3) $$||Pv(t_k,\cdot) - u(t_k,\cdot)|| \le Ch^2,$$

where the constant C does not depend on h and *k.*

Proof. Expanding the function $u(t_{k+1}, x - u(t_{k+1},x)h + \sigma\sqrt{h}\xi_q)$ at t_k, x in powers of h and $-u^j(t_{k+1},x)h + \sigma\sqrt{h}\xi_q^j,\ j=1,2,3$, we find that the terms with $\sqrt{h}$ and $h\sqrt{h}$ in the sum $\sum_{q=1}^{8} u(t_{k+1}, x - u(t_{k+1},x)h + \sigma\sqrt{h}\xi_q)$ are annihilated. We obtain

(3.4) $$Pv(t_k,x) = u(t_k,x) + \frac{\partial u}{\partial t}(t_k,x)h - (u,\nabla)u(t_k,x)h + \frac{1}{2}\sigma^2\Delta u(t_k,x)h$$
$$-\nabla g(t_k,x) + r(x,h;k),$$

where $|r(x,h;k)| \le Ch^2$ with C independent of $x,\ h,\ k$.

Since $u(t,x)$ solves (2.5), we get from here

$$Pv(t_k,x) - u(t_k,x) = \nabla p(t_k,x)h - \nabla g(t_k,x) + r(x,h;k). \tag{3.5}$$

Using the orthogonality of $Pv(t_k,x) - u(t_k,x)$ and $\nabla p(t_k,x)h - \nabla g(t_k,x)$ in $\mathbf{H}_p^0(Q)$ (we recall that $div(Pv(t_k,x) - u(t_k,x)) = 0$), we attain (3.3). □

Lemma 2. *Let the solution to* (2.5)-(2.8) *have the continuous derivatives* (3.2) *for*

$$j = 0,\ l = 1,...,6;\ j = 1,\ l = 0,1,...,4;\ j = 2,\ l = 0,1,2.$$

Then the one-step error of the method (2.20)-(2.21) *is of second order in the uniform norm, i.e.*

$$|Pv(t_k,x) - u(t_k,x)| \le Ch^2, \tag{3.6}$$

where the constant C does not depend on x, h, and *k.*

Proof. The remainder term $r(x,h;k)$ in (3.4) can be represented as a sum of iterated integrals with limits which are between t_k and $t_k + h$ in t and between x^i and $x^i - u^i(t_{k+1},x)h + \sigma\sqrt{h}\xi_q^i$ in x^i, $i = 1,2,3$. From here and the condition of the lemma we get that $|\partial r(x,h;k)/\partial x^i| \le Ch^2$ and $|\partial^2 r(x,h;k)/\partial x^i \partial x^j| \le Ch^2$ with C independent of $x,\ h,\ k$. Due to (3.5) we have

$$\frac{\partial Pv}{\partial x^i}(t_k,x) - \frac{\partial u}{\partial x^i}(t_k,x) = \nabla \frac{\partial p}{\partial x^i}(t_k,x) - \nabla \frac{\partial g}{\partial x^i}(t_k,x) + \frac{\partial r}{\partial x^i}(x,h;k).$$

As in the proof of the previous lemma, since $div(\partial Pv/\partial x^i - \partial u/\partial x^i) = 0$, we get

$$||\frac{\partial Pv}{\partial x^i}(t_k,\cdot) - \frac{\partial u}{\partial x^i}(t_k,\cdot)|| \le Ch^2,\ i = 1,2,3.$$

Analogously

$$||\frac{\partial^2 Pv}{\partial x^i \partial x^j}(t_k,\cdot) - \frac{\partial^2 u}{\partial x^i \partial x^j}(t_k,\cdot)|| \le Ch^2,\ i,j = 1,2,3.$$

From the last two inequalities we obtain for the norm in $\mathbf{H}_p^2(Q)$:

$$||Pv(t_k,\cdot) - u(t_k,\cdot)||_2 \le Ch^2.$$

Now (3.6) is attained due to the corresponding Sobolev inequality. □

Theorem 1. *Let the assumptions of Lemma 1 be fulfilled. Let*

$$|\bar{u}(t_k,x)| \le K,\ |\partial \bar{u}(t_k,x)/\partial x^i| \le K,$$

where $K > 0$ is independent of $x,\ h,\ k$. Then the method (2.20)-(2.21) *is of first order, i.e.*

$$||\bar{u}(t_k,\cdot) - u(t_k,\cdot)|| \le Ch, \tag{3.7}$$

where the constant C does not depend on h and *k.*

Proof. Denote the error of the method (2.20)-(2.21) on the k-th layer as $\varepsilon(t_k, x) := \bar{u}(t_k, x) - u(t_k, x)$. Thus, we have

$$\bar{u}(t_k, x) = u(t_k, x) + \varepsilon(t_k, x), \ \bar{u}(t_{k+1}, x) = u(t_{k+1}, x) + \varepsilon(t_{k+1}, x).$$

Due to (2.20) we get

$$(3.8) \quad u(t_k, x) + \varepsilon(t_k, x) = \bar{u}(t_k, x) = P\bar{v}(t_k, x) = \frac{1}{8}\sum_{q=1}^{8} P[\bar{u}(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)]$$

$$= \frac{1}{8}\sum_{q=1}^{8} P[u(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)] + \frac{1}{8}\sum_{q=1}^{8} P[\varepsilon(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)].$$

Further

$$(3.9) \quad u(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q) = u(t_{k+1}, x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q) + r_{kq}(x),$$

where

$$(3.10) \quad |r_{kq}(x)| = |u(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q) - u(t_{k+1}, x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)|$$
$$\leq K|\varepsilon(t_{k+1}, x)|h.$$

From (3.8) and (3.9) we get

$$(3.11) \quad u(t_k, x) + \varepsilon(t_k, x) = \frac{1}{8}\sum_{q=1}^{8} P[u(t_{k+1}, x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)] + \frac{1}{8}\sum_{q=1}^{8} Pr_{kq}(x)$$
$$+ \frac{1}{8}\sum_{q=1}^{8} P[\varepsilon(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)].$$

Due to (2.16) and (3.5), we get

$$(3.12) \quad \frac{1}{8}\sum_{q=1}^{8} P[u(t_{k+1}, x - u(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)]$$
$$= Pv(t_k, x) = u(t_k, x) + \nabla p(t_k, x)h - \nabla g(t_k, x) + r(x, h; k).$$

From (3.5) and Lemma 1 we have

$$||\nabla p(t_k, \cdot)h - \nabla g(t_k, \cdot) + r(\cdot, h; k)|| = \tilde{O}(h^2).$$

Therefore (3.11) and (3.12) imply

$$(3.13) \quad \varepsilon(t_k, x) = \frac{1}{8}\sum_{q=1}^{8} P[\varepsilon(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)] + \frac{1}{8}\sum_{q=1}^{8} Pr_{kq}(x) + \tilde{O}(h^2).$$

Now introduce

$$\varepsilon_k := ||\varepsilon(t_k, \cdot)||.$$

Let us evaluate the norm $||\cdot||$ of the function $\delta(x) := \varepsilon(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)$. We have

$$||\delta||^2 = \int_Q \sum_{i=1}^{3} [\varepsilon^i(t_{k+1}, x - \bar{u}(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q)]^2 dx$$

$$= \int_Q \sum_{i=1}^{3} [\varepsilon^i(t_{k+1}, y)]^2 \frac{D(x^1, x^2, x^3)}{D(y^1, y^2, y^3)} dy,$$

where

$$y^i = x^i - \bar{u}^i(t_{k+1}, x)h + \sigma\sqrt{h}\xi_q^i, \ i = 1, 2, 3.$$

Due to the condition about the uniform boundedness of $\partial\bar{u}^i/\partial x^j$ and the fact that $div\bar{u} = 0$, we get

$$(3.14) \quad \frac{D(y^1, y^2, y^3)}{D(x^1, x^2, x^3)} = \begin{vmatrix} 1 - h\partial\bar{u}^1/\partial x^1 & -h\partial\bar{u}^1/\partial x^2 & -h\partial\bar{u}^1/\partial x^3 \\ -h\partial\bar{u}^2/\partial x^1 & 1 - h\partial\bar{u}^2/\partial x^2 & -h\partial\bar{u}^2/\partial x^3 \\ -h\partial\bar{u}^3/\partial x^1 & -h\partial\bar{u}^3/\partial x^2 & 1 - h\partial\bar{u}^3/\partial x^3 \end{vmatrix} = 1 + O(h^2).$$

Therefore

$$||\delta|| \leq \varepsilon_{k+1}(1 + Ch^2).$$

Since $||P|| \leq 1$, we obtain from (3.10) and (3.13) (in addition we recall that $\varepsilon(t_N, x) = 0$)

$$(3.15) \qquad \varepsilon_N = 0, \ \varepsilon_k \leq \varepsilon_{k+1} + K\varepsilon_{k+1}h + Ch^2, \ k = N-1, ..., 1, 0.$$

Consequently

$$\varepsilon_k \leq \frac{C}{K}(e^{KT} - 1)h, \ k = N, ..., 0.$$

□

Remark 1. The theorem establishes that the error of method (2.20)-(2.21) on the whole interval is of $\tilde{O}(h)$. This is natural due to Lemma 1 which establishes that the one-step error of the method is of $\tilde{O}(h^2)$. One may expect that under the assumption of Lemma 2 the error of method (2.20)-(2.21) is of $O(h)$, however we have not succeeded in proving this fact rigorously.

Remark 2. The proof of Theorem 1 is not varied if a weaker condition than

$$|\partial\bar{u}(t_k, x)/\partial x^i| \leq K$$

is imposed. Namely it is sufficient to require $|\partial\bar{u}(t_k, x)/\partial x^i| \leq K/\sqrt{h}$. Indeed, in this case we obtain in the right-hand side of (3.14) $1 + O(h)$ instead of $1 + O(h^2)$. Consequently, $||\delta|| \leq \varepsilon_{k+1}(1+Ch)$ and (3.15) follows again. At the same time the conditions of Theorem 1 are natural and they are most likely fulfilled for sufficiently wide class of initial conditions $\varphi(x)$ for the problem (2.5)-(2.8).

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