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Stochastic interacting particle systems and nonlinear kinetic equations

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Abstract

We present the stochastic approach to nonlinear kinetic equations (without gradient terms) in a unifying general framework, which covers many interactions important in applications, like coagulation, fragmentation, inelastic collisions, as well as source and efflux terms. We provide conditions for the existence of corresponding stochastic particle systems in the sense of regularity (non-explosion) of a jump process with unbounded intensity. Using an appropriate space of measure-valued functions, we prove relative compactness of the sequence of processes and characterize the weak limits in terms of solutions to the nonlinear equation. As a particular application, we derive existence theorems for Smoluchowski's coagulation equation with fragmentation, efflux and source terms, and for the Boltzmann equation with dissipative collisions.

Contents

1. Introduction

Studies of the connection between stochastic interacting particle systems and nonlinear kinetic equations have a long history. The earliest references seem to be the papers by Leontovich [22] and Kac [20], where the Boltzmann equation (cf. [6], [7]) from rarefied gas dynamics was considered. In the simplest (spatially homogeneous) case this equation describes the time evolution of the velocity distribution of gas molecules that change their velocities during collisions. The stochastic approach to the Boltzmann equation has been further developed in [25], [32]. The practically relevant (unbounded) hard sphere collision kernel was treated in [31]. We refer to [35] for more comments and references concerning this field. Algorithms based on the corresponding stochastic interacting particle systems are presently the most widely used numerical tools in kinetic theory [5].

Stochastic particle systems related to Smoluchowski's coagulation equation (cf. [34], [8]) were used in [24], [14], [23] in the context of various applications. In the spatially homogeneous case this equation describes the time evolution of the size distribution of particles moving in a physical medium and merging during collisions. The stochastic approach to Smoluchowski's coagulation equation has been reviewed in [1]. We refer to [10], [12] for comments and references concerning applications of the particle systems in numerics. Note that the coagulation process can be considered as a chemical system with infinitely many species (characterized by size) and simple reactions (merging of two partners). The study of the relationship between stochastic and deterministic models for chemical systems with a finite number of species and reactions goes back to [21] (cf. [15] concerning numerical applications).

Developing the stochastic approach to the Boltzmann equation, systems with a general binary interaction between particles and a general (Markovian) single particle evolution (including spatial motion) were considered in [26], [17]. Results concerning the approximation of the solution to the corresponding nonlinear kinetic equation by the particle system were obtained in the case of bounded intensities and a constant (in time) number of particles. The weak law of large numbers for stochastic particle systems related to Smoluchowski's coagulation equation with general kernels has attracted attention only recently (cf. [1, Problem 10(a)]). Meanwhile, rigorous results of this type are contained, e.g., in [18] (discrete coagulation-fragmentation equation with bounded kernels), [19] (discrete coagulation-fragmentation equation), [28] (continuous coagulation equation), [11] (continuous coagulation-fragmentation equation). Most of the practically relevant coagulation kernels are unbounded. Moreover, if the kernel grows sufficiently fast, solutions to the limiting equation show the so-called gelation effect (loss of mass in finite time). It has been observed that the stochastic approach provides new existence results for the deterministic limiting equation (cf. the discussion in [11]), besides the approximation results that were the original motivation.

The **purpose of this paper** is to present the stochastic approach to nonlinear kinetic equations (without gradient terms) in a unifying general framework, which covers the cases mentioned above and allows one to include other effects important in applications, like multiple fragmentation, structured clusters, inelastic collisions, internal degrees of freedom, sources and efflux, etc. (cf., e.g., [36, Sections 3.3, 3.7], [29]). To this end we use an arbitrary locally compact separable metric space as the type space of a single particle and consider rather general multiple interactions with unbounded rates. The state space of the particle system consists of appropriately normalized discrete measures on the type space. The limiting equation is considered in a weak form so that solutions are functions of time taking values in some space of measures on the type space.

The paper is organized as follows. The main results are given in **Section 2**. The first theorem provides conditions for the existence of the particle system in the sense of regularity (non-explosion) of a jump process. The second theorem studies the property of relative compactness of the sequence (with respect to the normalization parameter) of processes. The third theorem characterizes the weak limits of the sequence in terms of solutions to a deterministic nonlinear equation. An existence theorem for the limiting equation is given in form of a special corollary. In **Section 3** the general results are applied to some specific models. First, the coagulation-fragmentation equation with source and efflux is considered. In this case new existence results are obtained. Second, a generalized Boltzmann equation with dissipative collisions is considered. Such equations have attracted considerable interest in recent years in connection with the study of granular materials (cf., e.g., [16], [4]). An existence result is obtained that covers the known results in the classical Boltzmann case. The rest of the paper is concerned with the proofs of the main results. In **Section 4** we consider the so-called minimal jump process, with a compactly bounded kernel on some locally compact separable metric space, and prove a theorem concerning its regularity (non-explosion). In **Section 5** we give proofs of our main theorems, using the results from the previous section and applying techniques from [13]. Some auxiliary results are collected in an appendix, in order to make the paper self-contained.

In conclusion we note that convergence of the particle system to the solution of the limiting deterministic equation (weak law of large numbers) is obtained under the assumption of uniqueness of that solution. So far no general uniqueness result have been obtained (cf. [28] concerning the coagulation case). However, the general framework proposed in this paper provides a basis for the derivation and justification of stochastic algorithms in many fields of application. The results cover both unbounded kernels (thus avoiding any truncation leading to unnecessary numerical errors) and a variable number of particles (possibly unbounded in time). As to concrete applications, we worked out only two specific models in order to keep the length of the paper reasonable. However, any combinations of these interaction models, and many others, can be considered.

2. Main results

Let E and E' be metric and separable spaces. Let $M(E)$, $B(E)$, $C(E)$, $C_b(E)$ and $C_c(E)$ denote the sets of functions on E that are measurable, bounded measurable, continuous, bounded continuous, and continuous with compact support, respectively. For E locally compact, let $C_0(E)$ denote the set of continuous functions on E vanishing at infinity as the closure of $C_c(E)$ with respect to the sup-norm $\|.\|$. Furthermore, the sets of Borel measures, bounded Borel measures and probability measures on the Borel-σ-algebra $\mathcal{B}(E)$ are denoted by $\mathcal{M}(E)$, $\mathcal{M}_b(E)$ and $\mathcal{P}(E)$, respectively. The Dirac measure on $\xi \in E$ is denoted by δ_ξ. Vague and weak convergence of Borel measures are denoted by $\mu_n \stackrel{v}{\to} \mu$ and $\mu_n \stackrel{w}{\to} \mu$, respectively, whereas the sign $\Rightarrow$ is used for convergence in distribution.

Let $C([0,\infty),E)$ be the space of continuous paths and $D([0,\infty),E)$ the Skorohod space of cadlag paths. For $\varphi \in M(E)$ and $\mu \in \mathcal{M}(E)$ we use the notation $\langle \varphi, \mu \rangle = \int \varphi \, d\mu$. Finally, let 1_A denote the indicator function of a set A. A **kernel** from E to E' (on E if $E = E'$) is a function $\lambda : E \times \mathcal{B}(E') \to [0,\infty)$ such that

$$\lambda(\cdot, B) \in M(E)\,, \quad \forall\, B \in \mathcal{B}(E') \qquad \text{and} \qquad \lambda(\xi, \cdot) \in \mathcal{M}_b(E')\,, \quad \forall\, \xi \in E\,.$$

A kernel λ is called **compactly bounded** if

$$\sup_{\xi \in C} \lambda(\xi, E') \;<\; \infty\,, \qquad \text{for any compact} \quad C \subset E\,.$$

We consider particles with types from a locally compact separable metric space $\mathcal{Z}$ and weights $\frac{1}{N}$. Define the **state space of the particle system** as

$$E^N \;=\; \left\{ \frac{1}{N} \sum_{i=1}^{n} \delta_{x_i} \;:\; n \geq 0\,,\; x_i \in \mathcal{Z}\,,\; i = 1, \ldots, n \right\}, \qquad N = 1, 2, \ldots\,. \tag{2.1}$$

Any event in the system consists in the interaction of at most R particles and produces as a result at most K particles. This includes, for example, the generation of new particles from a source, the extinction or transformation of single particles, and the collision of two particles. The admissible subsequent states of $\mu \in E^N$ are denoted by

$$\begin{aligned} J_0(\mu, \xi) &= \mu + \frac{1}{N}\, \xi\,, \\ J_1(\mu, i, \xi) &= \mu + \frac{1}{N}\, [\xi - \delta_{x_i}]\,, \\ J_r(\mu, i_1, \ldots, i_r, \xi) &= \mu + \frac{1}{N}\, [\xi - \delta_{x_{i_1}} - \ldots - \delta_{x_{i_r}}]\,, \qquad r = 2, \ldots, R\,, \end{aligned} \tag{2.2}$$

where $i_1, \ldots, i_r$ are pairwise distinct indices from $\{1, \ldots, n\}$ and $\xi \in E_K$, with

$$E_K \;=\; \left\{ \sum_{i=1}^{n} \delta_{x_i} \;:\; 0 \leq n \leq K\,,\; x_i \in \mathcal{Z}\,,\; i = 1, \ldots, n \right\}, \tag{2.3}$$

for some given natural numbers R and K. Both spaces E^N and E_K are equipped with the weak topology.

The **rates** for the different events are determined by a measure q_0 and kernels $q_1, \ldots, q_R$ such that

$$\begin{aligned} &q_0 \;\in\; \mathcal{M}_b(E_K) \quad \text{and} \\ &q_r : \mathcal{Z}^r \times \mathcal{B}(E_K) \to [0,\infty)\,, \quad r = 1, \ldots, R\,, \quad \text{are compactly bounded.} \end{aligned} \tag{2.4}$$

Thus, transitions (jumps) in the system are governed by the kernel

$$\begin{aligned} \lambda^N(\mu, B) = N \int_{E_K} 1_B(J_0(\mu,\xi))\, q_0(d\xi) + \sum_{i=1}^{n} \int_{E_K} 1_B(J_1(\mu,i,\xi))\, q_1(x_i, d\xi) + \\ \sum_{r=2}^{R} \frac{1}{N^{r-1}} \tilde{\sum_{1 \leq i_1, \ldots, i_r \leq n}} \int_{E_K} 1_B(J_r(\mu, i_1, \ldots, i_r, \xi))\, q_r(x_{i_1}, \ldots, x_{i_r}, d\xi)\,, \qquad B \in \mathcal{B}(E^N)\,, \end{aligned} \tag{2.5}$$

where $\mu \in E^N$ and $\tilde{\sum}$ denotes summation over pairwise distinct indices.

We first provide conditions for the regularity (non-explosion) of the system.

Theorem 2.1 (Regularity) *Consider a locally compact separable metric space $\mathcal{Z}$ and a function*

$$H \in C(\mathcal{Z}) \quad \text{such that} \quad H > 0 \quad \text{and} \quad \frac{1}{H} \in C_0(\mathcal{Z}). \tag{2.6}$$

Suppose $\nu_0^N \in \mathcal{P}(E^N)$ satisfies

$$\int_{E^N} \langle H, \mu \rangle \, \nu_0^N(d\mu) \;\; \leq \;\; c_0 \,, \tag{2.7}$$

for some $c_0 \geq 0$. Suppose $q_0, q_1, \ldots, q_R$ satisfy (2.4) and are such that the kernel (2.5) satisfies

$$\int_{E^N} [\langle H, \mu_1 \rangle - \langle H, \mu \rangle] \, \lambda^N(\mu, d\mu_1) \;\; \leq \;\; c_1 \, [\langle H, \mu \rangle + c_1'] \,, \qquad \forall \, \mu \in E^N \,, \tag{2.8}$$

for some $c_1 \,,\, c_1' \geq 0$.

Then there exists a random process X^N with sample paths in $D([0,\infty), E^N)$ that is indistinguishable from the minimal jump process, corresponding to the kernel λ^N and the initial distribution ν_0^N.

Next we study asymptotic properties of the sequence X^N. To this end, we construct an appropriate common state space. Consider two functions

$$h, H \in C(\mathcal{Z}) : \quad 0 \leq h(x) \leq c \, H(x) \,, \quad \forall \, x \in \mathcal{Z} \,, \quad \text{for some} \quad c > 0 \,, \tag{2.9}$$

the set

$$\mathcal{M}(\mathcal{Z}, H) \;\; = \;\; \Big\{ \mu \in \mathcal{M}(\mathcal{Z}) \,:\, \langle H, \mu \rangle < \infty \Big\} \tag{2.10}$$

and the metric

$$d_h(\mu, \nu) \;\; = \;\; d_0(\mu, \nu) + \min\{1, |\langle h, \mu \rangle - \langle h, \nu \rangle|\} \,, \qquad \mu, \; \nu \in \mathcal{M}(\mathcal{Z}, H) \,, \tag{2.11}$$

where d_0 is a metric generating the vague topology. Introduce the space

$$\mathcal{M}(\mathcal{Z}, H, h) \;\; = \;\; \Big(\mathcal{M}(\mathcal{Z}, H), d_h \Big). \tag{2.12}$$

Note that $\mathcal{M}(\mathcal{Z}, 0) = \mathcal{M}(\mathcal{Z})$, $\mathcal{M}(\mathcal{Z}, 1) = \mathcal{M}_b(\mathcal{Z})$ and

$$\mathcal{M}(\mathcal{Z}, H) \subset \mathcal{M}_b(\mathcal{Z}) \quad \text{if} \quad \inf_{x \in \mathcal{Z}} H(x) > 0 \,. \tag{2.13}$$

According to [2, Theorem 45.7], the metric d_1 generates the weak topology on $\mathcal{M}_b(\mathcal{Z})$.

Theorem 2.2 (Relative Compactness) *Consider a locally compact separable metric space $\mathcal{Z}$ and functions H satisfying (2.6) and*

$$h \in C(\mathcal{Z}) \;\; \text{such that} \;\; h \geq 0 \;\; \text{and} \;\; \tfrac{h}{H} \; \in \; C_0(\mathcal{Z}) \,. \tag{2.14}$$

Suppose $\nu_0^N \in \mathcal{P}(E^N)$ satisfy (2.7) uniformly in N. Suppose $q_0, q_1, \ldots, q_R$ are such that (2.8) is satisfied uniformly in N and

$$q_0(E_K) \leq c_2 \quad \text{and} \quad q_r(x, E_K) \leq c_2 \, H(x_1) \ldots H(x_r) \,, \qquad \forall \, x = (x_1, \ldots, x_r) \in \mathcal{Z}^r \,, \tag{2.15}$$

for $r = 1, \ldots, R$ and some $c_2 \geq 0$.

Then the processes X^N form a relatively compact sequence of $D([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$-valued random variables.

Theorem 2.3 (Characterization of Weak Limits) *Consider a locally compact separable metric space $\mathcal{Z}$ and functions H, h satisfying (2.6) and (2.14). Suppose $\nu_0^N \in \mathcal{P}(E^N)$ satisfy (2.7) uniformly in N and are such that*

$$X^N(0) \;\Rightarrow\; \mu_0\,, \qquad \text{for some} \quad \mu_0 \in \mathcal{M}(\mathcal{Z}, H)\,. \tag{2.16}$$

Suppose $q_0, q_1, \ldots, q_R$ are such that (2.8) is satisfied uniformly in N,

$$q_0(E_K) \le c_2 \quad \text{and} \quad q_r(x, E_K) \le c_2\, h(x_1) \ldots h(x_r)\,, \qquad \forall\, x = (x_1, \ldots, x_r) \in \mathcal{Z}^r\,, \tag{2.17}$$

for $r = 1, \ldots, R$ and some $c_2 \ge 0$, and

$$q_r(., E_K) \in C(\mathcal{Z}^r)\,, \qquad \int_{E_K} \langle \varphi, \xi \rangle\, q_r(., d\xi) \in C(\mathcal{Z}^r)\,, \qquad r = 1, \ldots, R\,, \tag{2.18}$$

for any $\varphi \in C_c(\mathcal{Z})$.

Then the processes X^N form a relatively compact sequence of $D([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$-valued random variables and every weak limit X satisfies, almost surely,

$$\langle \varphi, X(t) \rangle \;=\; \langle \varphi, \mu_0 \rangle + \int_0^t \mathcal{G}(\varphi, X(s))\, ds\,, \qquad \forall\, t \ge 0\,, \quad \varphi \in C_c(\mathcal{Z})\,, \tag{2.19}$$

where, for $\mu \in \mathcal{M}(\mathcal{Z}, H)$,

$$\mathcal{G}(\varphi, \mu) = \int_{E_K} \langle \varphi, \xi \rangle\, q_0(d\xi) + \tag{2.20}$$
$$\sum_{r=1}^{R} \int_{\mathcal{Z}} \ldots \int_{\mathcal{Z}} \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_1) - \ldots - \varphi(x_r)]\, q_r(x_1, \ldots, x_r, d\xi)\, \mu(dx_1) \ldots \mu(dx_r)\,.$$

Corollary 2.4 (Continuity) *Under the assumptions of Theorem 2.2, every weak limit X satisfies*

$$P\Big(X \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))\Big) \;=\; 1\,. \tag{2.21}$$

Corollary 2.5 (Moments) *Under the assumptions of Theorem 2.2, every weak limit X satisfies*

$$\mathbb{E}\,\langle H, X(t) \rangle \;\le\; (c_0 + c_1')\exp(c_1\, t)\,, \qquad \forall\, t \ge 0\,. \tag{2.22}$$

Corollary 2.6 (Existence) *Consider a locally compact separable metric space $\mathcal{Z}$ and functions H, h satisfying (2.6) and (2.14). Let $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$. Suppose $q_0, q_1, \ldots, q_R$ are such that (2.8) is satisfied uniformly in N, and assumptions (2.17), (2.18) are fulfilled.*

Then there exists some $\mu \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$ solving the **macroscopic equation**

$$\langle \varphi, \mu(t) \rangle \;=\; \langle \varphi, \mu_0 \rangle + \int_0^t \mathcal{G}(\varphi, \mu(s))\, ds\,, \qquad \forall\, t \ge 0\,, \quad \varphi \in C_c(\mathcal{Z})\,. \tag{2.23}$$

Corollary 2.7 (Convergence) *Let the assumptions of Theorem 2.3 be fulfilled. If there is a unique* $\mu \in C([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$ *satisfying equation (2.23), then the stochastic processes* X^N *converge in distribution to* μ *.*

Corollary 2.8 *For nonnegative* $g \in C(\mathcal{Z})$ *and* $\gamma > 0$ *consider the set*

$$\mathcal{M}_\gamma(\mathcal{Z},g) \quad = \quad \{\mu \in \mathcal{M}(\mathcal{Z}) \,:\, \langle g,\mu\rangle \le \gamma\}\,. \tag{2.24}$$

Suppose $q_0,q_1,\ldots,q_R$ *are such that, for all* $N=1,2,\ldots$ *,*

$$\lambda^N(\mu,E^N\cap\mathcal{M}_\gamma(\mathcal{Z},g)) \quad = \quad \lambda^N(\mu,E^N)\,, \qquad \forall\,\mu \in E^N\cap\mathcal{M}_\gamma(\mathcal{Z},g)\,. \tag{2.25}$$

Then the Theorems 2.1, 2.2 and 2.3 hold, when the spaces E^N *and* $\mathcal{M}(\mathcal{Z},H,h)$ *are replaced by* $E^N\cap\mathcal{M}_\gamma(\mathcal{Z},g)$ *and* $(\mathcal{M}(\mathcal{Z},H)\cap\mathcal{M}_\gamma(\mathcal{Z},g),d_h)$ *, respectively.*

Corollary 2.9 *Consider a locally compact separable metric space* $\mathcal{Z}$ *and functions* H,h *satisfying (2.6) and (2.14). Let* $\mu_0 \in \mathcal{M}(\mathcal{Z},H)$ *be such that* $\langle g,\mu_0\rangle < \infty$ *, for some nonnegative* $g \in C(\mathcal{Z})$ *, and consider* $\gamma = \langle g,\mu_0\rangle + 1$ *. Suppose* $q_0,q_1,\ldots,q_R$ *are such that assumptions (2.17), (2.18) and (2.25) are fulfilled, and (2.8) is satisfied uniformly in* N *, with* E^N *replaced by* $E^N\cap\mathcal{M}_\gamma(\mathcal{Z},g)$ *.*

Then there exists some $\mu \in C([0,\infty),(\mathcal{M}(\mathcal{Z},H)\cap\mathcal{M}_\gamma(\mathcal{Z},g),d_h))$ *solving the equation (2.23).*

We finish this section by providing some basic properties of the objects under consideration, which will be used at several parts of the paper.

Remark 2.10 *Let* $\mu_k = N^{-1}\sum_{i=1}^{n_k}\delta_{x_i^k}$*,* $k\ge 1$*, and* $\mu = N^{-1}\sum_{i=1}^{n}\delta_{x_i}$*. Then* $d_1(\mu_k,\mu)\to 0$ *if and only if there are an* $l\ge 1$ *and permutations* π_k *on* $\{1,\ldots,n\}$*,* $k\ge l$*, such that*

$$n_k \;=\; n\,, \quad k\ge l, \qquad \text{and} \qquad \lim_{k\to\infty,\;k\ge l} x^k_{\pi_k(i)} \;=\; x_i\,, \quad i=1,\ldots,n\,.$$

Note that the kernel (2.5) satisfies

$$\begin{aligned}
&\int_{E^N}\Psi(\nu)\,\lambda^N(\mu,d\nu) = \\
&\qquad N\int_{E_K}\Psi(J_0(\mu,\xi))\,q_0(d\xi) + \sum_{i=1}^n\int_{E_K}\Psi(J_1(\mu,i,\xi))\,q_1(x_i,d\xi) \\
&\qquad + \sum_{r=2}^R\frac{1}{N^{r-1}}\tilde{\sum_{1\le i_1,\ldots,i_r\le n}}\int_{E_K}\Psi(J_r(\mu,i_1,\ldots,i_r,\xi))\,q_r(x_{i_1},\ldots,x_{i_r},d\xi)\,,
\end{aligned} \tag{2.26}$$

for $\mu \in E^N$ and appropriate test functions, e.g., $\Psi \in C(E^N)$. In particular, one obtains

$$\lambda^N(\mu,E^N) \;\le\; N\left[q_0(E_K) + \sum_{r=1}^R\int_{\mathcal{Z}}\cdots\int_{\mathcal{Z}}q_r(x_1,\ldots,x_r,E_K)\,\mu(dx_1)\ldots\mu(dx_r)\right]. \tag{2.27}$$

Remark 2.11 *If* $\varphi \in C(\mathcal{Z})$ *, then the function*

$$\Phi \; : \; E^N \to \mathbb{R} \,, \qquad \Phi(\mu) \;=\; \langle \varphi, \mu \rangle \,, \tag{2.28}$$

is continuous, according to **Remark 2.10**.

Using (2.26) and (2.2), one obtains

$$\int_{E^N} [\langle \varphi, \nu \rangle - \langle \varphi, \mu \rangle]^k \, \lambda^N(\mu, d\nu) = \tag{2.29}$$
$$N^{1-k} \Bigg[\int_{E_K} \langle \varphi, \xi \rangle^k \, q_0(d\xi) + \frac{1}{N} \sum_{i=1}^{n} \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_i)]^k \, q_1(x_i, d\xi) +$$
$$\sum_{r=2}^{R} \frac{1}{N^r} \tilde{\sum_{1 \le i_1, \ldots, i_r \le n}} \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_{i_1}) - \ldots - \varphi(x_{i_r})]^k \, q_r(x_{i_1}, \ldots, x_{i_r}, d\xi) \Bigg] \,,$$

for $\mu \in E^N$, $k = 1, 2$, and $\varphi \in C(\mathcal{Z})$. Introduce the notation

$$\begin{aligned} Q_0(\varphi) &= \int_{E_K} \langle \varphi, \xi \rangle \, q_0(d\xi) \,, \\ Q_r(\varphi, x) &= \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_1) - \ldots - \varphi(x_r)] \, q_r(x, d\xi) \,, \qquad r = 1, \ldots, R \,, \end{aligned} \tag{2.30}$$

for $x = (x_1, \ldots, x_r) \in \mathcal{Z}^r$ and $\varphi \in C(\mathcal{Z})$. Using (2.29) with $\varphi = H$ and $k = 1$, condition (2.8) takes the form

$$Q_0(H) + \int_{\mathcal{Z}} Q_1(H, x) \, \mu(dx) + \sum_{r=2}^{R} \frac{1}{N^r} \tilde{\sum_{1 \le i_1, \ldots, i_r \le n}} Q_r(H, x_{i_1}, \ldots, x_{i_r}) \le c_1 \, [\langle H, \mu \rangle + c_1'] \,. \tag{2.31}$$

It follows from the definition (2.3) that

$$|\langle \varphi, \xi \rangle| \;\le\; \|\varphi\| \, K \,, \qquad \forall \, \xi \in E_K \,, \tag{2.32}$$

and

$$\begin{aligned} \int_{E_K} |\langle \varphi, \xi \rangle|^k \, q_0(d\xi) &\le K^k \, \|\varphi\|^k \, q_0(E_K) \,, \\ \int_{E_K} |\langle \varphi, \xi \rangle - \varphi(x_1) - \ldots - \varphi(x_r)|^k q_r(x, d\xi) &\le (K + r)^k \, \|\varphi\|^k \, q_r(x, E_K) \,, \end{aligned} \tag{2.33}$$

for $k = 1, 2$, $x = (x_1, \ldots, x_r) \in \mathcal{Z}^r$, $r = 1, \ldots, R$ and $\varphi \in C_b(\mathcal{Z})$. Using (2.33) with $k = 1$, one obtains (cf. (2.30))

$$\begin{aligned} |Q_0(\varphi)| &\le K \, \|\varphi\| \, q_0(E_K) \,, \\ |Q_r(\varphi, x)| &\le (K + r) \, \|\varphi\| \, q_r(x, E_K) \,, \qquad \forall \, x \in \mathcal{Z}^r \,, \quad r = 1, \ldots, R \,. \end{aligned} \tag{2.34}$$

3. Applications

In this section we apply the general results, in particular Corollaries 2.6 and 2.9, to several special cases. We consider $R = 2$. One has to check conditions (2.8), (2.17), (2.18) and (2.25) (in case of Corollary 2.9), for appropriate functions H, h satisfying (2.6) and (2.14).

According to (2.31), **condition (2.8)** is satisfied for all $\mu \in E^N$, if

$$Q_0(H) \; = \; \int_{E_K} \langle H, \xi \rangle \, q_0(d\xi) \; \leq \; c_1 \,, \tag{3.1}$$

$$Q_1(H,x) \; = \; \int_{E_K} [\langle H, \xi \rangle - H(x)] \, q_1(x, d\xi) \; \leq \; c_1 \, H(x) \,, \qquad \forall x \in \mathcal{Z} \,, \tag{3.2}$$

and

$$Q_2(H,x,y) \; = \; \int_{E_K} [\langle H, \xi \rangle - H(x) - H(y)] \, q_2(x, y, d\xi) \; \leq \; 0 \,, \qquad \forall x, y \in \mathcal{Z} \,. \tag{3.3}$$

Using (2.26) with $\Psi = 1_{E^N \cap \mathcal{M}_\gamma(\mathcal{Z},g)}$ and (2.2), one observes that **condition (2.25)** is satisfied provided that

$$q_0 = 0 \,, \qquad \langle g, \xi \rangle \leq g(x) \,, \quad q_1(x, d\xi) \text{ a.e.} \\ \text{and} \quad \langle g, \xi \rangle \leq g(x) + g(y) \,, \quad q_2(x,y,d\xi) \text{ a.e.} \,, \qquad \forall x, y \in \mathcal{Z} \,. \tag{3.4}$$

In this case, **condition (2.8)** is satisfied for all $\mu \in E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$, if (3.2) holds and

$$Q_2(H,x,y) \; \leq \; c_1 \, [H(x) \, g(y) + g(x) \, H(y) + g(x) \, g(y)] \,, \qquad \forall x, y \in \mathcal{Z} \,. \tag{3.5}$$

Indeed, (3.5) implies

$$\frac{1}{N^2} \sum_{1 \leq i \neq j \leq n} Q_2(H, x_i, x_j) \leq c_1(\gamma) \, [\langle H, \mu \rangle + 1] \,, \qquad \forall \, \mu \in \mathcal{M}_\gamma(\mathcal{Z}, g) \,.$$

3.1. Source and efflux

Any **source term** $q_0 \in \mathcal{M}_b(E_K)$ (cf. **condition (2.17)**) satisfying (3.1) is covered by the results. In particular, we consider

$$q_0(B) = \int_{\mathcal{Z}} 1_B(\delta_x) \, S(dx) \,, \qquad B \in \mathcal{B}(E_K) \,,$$

where $S \in \mathcal{M}_b(\mathcal{Z})$. **Condition (3.1)** takes the form

$$\int_{\mathcal{Z}} H(x) \, S(dx) \; < \; \infty \,. \tag{3.6}$$

Note that (cf. (2.30))

$$Q_0(\varphi) \; = \; \langle \varphi, S \rangle \,. \tag{3.7}$$

Next we consider the **efflux term**

$$q_1(x,B) = 1_B(0)\,E(x)\,, \qquad x\in\mathcal{Z}\,, \quad B\in\mathcal{B}(E_K)\,,$$

where 0 denotes the zero measure and $E\in C(\mathcal{Z})$ is such that

$$0 \;\le\; E(x) \;\le\; c_2\,h(x)\,, \qquad \forall\, x\in\mathcal{Z}\,. \tag{3.8}$$

Conditions (2.17), (2.18) are satisfied, and **condition (3.2)** is fulfilled with $c_1=0$, since the left-hand side is non-positive. Note that (cf. (2.30))

$$Q_1(\varphi,x) \;=\; -\varphi(x)\,E(x)\,. \tag{3.9}$$

3.2. Coagulation and fragmentation

Let $\mathcal{Z}=\mathbb{N}$ or $\mathcal{Z}=(0,\infty)$. Consider the **coagulation term**

$$q_2(x,y,B) = 1_B(\delta_{x+y})\,K(x,y)\,, \qquad x,y\in\mathcal{Z}\,, \quad B\in\mathcal{B}(E_K)\,,$$

where $K\in C(\mathcal{Z}\times\mathcal{Z})$ is non-negative, and note that (cf. (2.30))

$$Q_2(\varphi,x,y) \;=\; [\varphi(x+y)-\varphi(x)-\varphi(y)]\,K(x,y)\,. \tag{3.10}$$

Consider the **fragmentation term**

$$q_1(x,B) = \int_{\mathcal{Z}} 1_B(\delta_{x-y}+\delta_y)\,F(x,dy)\,, \qquad x\in\mathcal{Z}\,, \quad B\in\mathcal{B}(E_K)\,,$$

where F is a kernel on $\mathcal{Z}$ satisfying

$$F(x,[x,\infty)) \;=\; 0\,, \qquad \forall x\in\mathcal{Z}\,, \tag{3.11}$$

and note that (cf. (2.30))

$$Q_1(\varphi,x) \;=\; \int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)-\varphi(x)]\,F(x,dy)\,. \tag{3.12}$$

With the terms (3.7), (3.9), (3.10) and (3.12), equation (2.23) takes the form

$$\begin{aligned}\int_{\mathcal{Z}}\varphi(x)\,\mu(t,dx) = \int_{\mathcal{Z}}\varphi(x)\,\mu_0(dx) + \int_0^t \Big[\int_{\mathcal{Z}}\varphi(x)\,S(dx) - \int_{\mathcal{Z}}\varphi(x)\,E(x)\,\mu(s,dx) + \\ \int_{\mathcal{Z}}\int_{\mathcal{Z}}[\varphi(x+y)-\varphi(x)-\varphi(y)]\,K(x,y)\,\mu(s,dx)\,\mu(s,dy) + \\ \int_{\mathcal{Z}}\int_{\mathcal{Z}}[\varphi(x-y)+\varphi(y)-\varphi(x)]\,F(x,dy)\,\mu(s,dx)\Big]\,ds\,, \qquad \forall\, t\ge 0\,, \quad \varphi\in C_c(\mathcal{Z})\,.\end{aligned} \tag{3.13}$$

Theorem 3.1 *Let $\mathcal{Z}=\mathbb{N}$ or $\mathcal{Z}=(0,\infty)$. Consider functions H satisfying (2.6) and*

$$\frac{H(x)}{x} \;\ge\; \frac{H(y)}{y}\,, \qquad \forall\, 0<x\le y<\infty\,, \tag{3.14}$$

and h satisfying (2.14). Let $\mu_0, S \in \mathcal{M}(\mathcal{Z},H)$ and $E \in C(\mathcal{Z})$ such that (3.8) holds. Consider a function $K \in C(\mathcal{Z}\times\mathcal{Z})$ such that

$$0 \;\le\; K(x,y) \;\le\; c_2\, h(x)\, h(y)\,, \qquad \forall\, x,y \in \mathcal{Z}\,, \tag{3.15}$$

and a kernel F satisfying (3.11),

$$F(x_n,.) \;\stackrel{w}{\to}\; F(x,.)\,, \qquad \text{if} \quad x_n \to x \in \mathcal{Z}\,, \tag{3.16}$$

$$\int_{\mathcal{Z}} [H(x-y)+H(y)-H(x)]\, F(x,dy) \;\le\; c_1\, H(x)\,, \qquad \forall x \in \mathcal{Z}\,, \tag{3.17}$$

and

$$0 \;\le\; F(x,\mathcal{Z}) \;\le\; c_1\, h(x)\,, \qquad \forall\, x \in \mathcal{Z}\,. \tag{3.18}$$

Then there exists some $\mu \in C([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$ satisfying equation (3.13).

Remark 3.2 *In the case $\mathcal{Z} = (0,\infty)$ and $F \equiv 0$, any continuous coagulation kernel satisfying (3.15) is covered, provided that H satisfies (3.14). Note there is no restriction on K at zero.*

Corollary 3.3 *Let $\mathcal{Z} = (0,\infty)$,*

$$H(x) \;=\; x^{-\alpha} + x\,, \qquad \alpha \in (0,1)\,,$$

and

$$h(x) \;=\; x^{-\alpha+\varepsilon} + x^{1-\varepsilon}\,, \qquad \varepsilon \in (0,\alpha]\,.$$

Let $\mu_0, S \in \mathcal{M}(\mathcal{Z},H)$ and $E \in C(\mathcal{Z})$ such that (3.8) holds. Consider a function $K \in C(\mathcal{Z}\times\mathcal{Z})$ satisfying (3.15). Assume F has the form

$$F(x,dy) \;=\; 1_{(0,x)}(y)\, f(x,y)\, dy\,, \tag{3.19}$$

where f is continuous with respect to the first argument and satisfies

$$0 \;\le\; f(x,y) \;\le\; \frac{R(x)\, y^{-\beta}}{x^{1-\beta}}\,, \qquad \beta \in [0,1-\alpha)\,, \qquad R(x) = c\,(1+x^{1-\varepsilon})\,.$$

Then there exists some $\mu \in C([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$ satisfying equation (3.13).

To our knowledge, the most general existence result in the continuous case, including source and efflux terms, is contained in [9, Theorem 2.2]. There it is assumed that both K and f have compact support, and that both the source term and the initial distribution have a finite moment of some order $r \ge 1$. Thus, Corollary 3.3 provides a new existence result for unbounded K and f.

Corollary 3.4 *Let $\mathcal{Z} = \mathbb{N}$ and $H(x) = x\,, \quad h(x) = o(x)$. Consider $\mu_0, S \in \mathcal{M}(\mathcal{Z},H)$ and E such that (3.8) holds. Suppose K satisfies (3.15) and F satisfies (3.11), (3.18). Then there exists some $\mu \in C([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$ satisfying equation (3.13).*

To our knowledge, the most general existence result in the discrete case, including source and efflux terms, is contained in [30]. The corresponding assumptions there are $K(x,y) = o(x)\,o(y)$, $E(x) = O(x)$, $\sum_{x=1}^{\infty} x\,S(x) < \infty$ and boundedness of $F(x,\mathcal{Z})$ in x. Thus, Corollary 3.4 provides a new existence result for an unbounded total fragmentation rate.

Theorem 3.5 *Let $\mathcal{Z} = \mathbb{N}$ or $\mathcal{Z} = (0,\infty)$. Consider functions H, h satisfying (2.6), (2.14). Let $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$ such that*

$$\int_{\mathcal{Z}} x\,\mu_0(dx) \quad < \quad \infty\,, \tag{3.20}$$

$S = 0$ and $E \in C(\mathcal{Z})$ such that (3.8) holds. Consider a function $K \in C(\mathcal{Z}\times\mathcal{Z})$ satisfying (3.15) and

$$[H(x+y) - H(x) - H(y)]\,K(x,y) \quad \le \quad c_1\,[H(x)\,y + x\,H(y) + x\,y]\,, \tag{3.21}$$

and a kernel F satisfying (3.11), (3.16), (3.17) and (3.18). Then there exists some $\mu \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$ satisfying equation (3.13).

Corollary 3.6 *Let $\mathcal{Z} = (0,\infty)$,*

$$H(x) \quad = \quad x^{-\alpha} + x^2\,, \qquad \alpha \in (0,1)\,,$$

and

$$h(x) \quad = \quad x^{-\alpha+\varepsilon} + x^{2-\varepsilon}\,, \qquad \varepsilon \in (0,\alpha]\,.$$

Let $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$, $S = 0$ and $E \in C(\mathcal{Z})$ such that (3.8) holds. Consider a function $K \in C(\mathcal{Z}\times\mathcal{Z})$ satisfying

$$K(x,y) \quad \le \quad c_1(1 + x + y)\,. \tag{3.22}$$

Suppose F has the form (3.19), where f is continuous with respect to the first argument and satisfies

$$0 \;\le\; f(x,y) \;\le\; \frac{R(x)\,y^{-\beta}}{x^{1-\beta}}\,, \qquad \beta \in [0, 1-\alpha)\,, \qquad R(x) = c\,(1 + x^{2-\varepsilon})\,.$$

Then there exists some $\mu \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$ satisfying equation (3.13).

Corollary 3.7 *Consider $\mathcal{Z} = \mathbb{N}$, and the functions $H(x) = x^r$, $h(x) = x^{r-\varepsilon}$ with some $r = 2, 3, \ldots$ and $\varepsilon \in (0,1]$. Let $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$, $S = 0$ and E such that (3.8) holds. Suppose K satisfies*

$$K(x,y) \quad \le \quad c_1(x+y)\,, \tag{3.23}$$

and F satisfies (3.11), (3.18). Then there exists some $\mu \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$ satisfying equation (3.13).

Remark 3.8 *Consider* $\mathcal{Z} = (0,\infty)$. *Suppose* F *has the form (3.19) and*

$$\mu(t,dx) \quad = \quad c(t,x)\,dx\,, \qquad S(dx) = S(x)\,dx\,.$$

Then, using the identity

$$\int_0^\infty \int_0^\infty \psi(x,y)\,dy\,dx = \int_0^\infty \int_0^x \psi(x-y,y)\,dy\,dx\,,$$

equation (3.13) takes the form

$$\begin{aligned}\int_0^\infty \varphi(x)\; c(t,x)\; dx \;=\; & \int_0^\infty \varphi(x)\; c_0(x)\; dx \;+\; \int_0^\infty dx\,\varphi(x) \int_0^t ds\,\Big[S(x) - \\ & E(x)\,c(s,x) + 2\int_0^\infty f(x+y,y)\,c(s,x+y)\,dy - \int_0^x f(x,y)\;c(s,x)\;dy\;+ \\ & \int_0^x K(x-y,y)\,c(s,x-y)\,c(s,y)\,dy - \int_0^\infty [K(x,y)+K(y,x)]\;c(s,x)\;c(s,y)\;dy\Big]\,.\end{aligned}$$

Removing the test functions, one obtains

$$\begin{aligned}\frac{\partial}{\partial t}\,c(t,x) \;=\; & S(x) - E(x)\,c(t,x) + \\ & 2\int_0^\infty f(x+y,y)\,c(t,x+y)\,dy - c(t,x)\,F(x,\mathcal{Z})\;+ \\ & \int_0^x K(x-y,y)\,c(t,x-y)\,c(t,y)\,dy - \int_0^\infty [K(x,y)+K(y,x)]\;c(t,x)\;c(t,y)\;dy\,.\end{aligned} \tag{3.24}$$

In the discrete case $\mathcal{Z} = \mathbb{N}$, *analogous transformations of equation (3.13) lead to the form (3.24), with integrals replaced by sums. In this case both forms are equivalent, without any additional assumptions.*

Lemma 3.9 *If (3.16) then condition (2.18) is satisfied. If* F *has the form (3.19), for some non-negative function* f, *which is continuous with respect to the first argument, then condition (3.16) is fulfilled.*

Proof. Condition (2.18) reduces to

$$F(.,\mathcal{Z}) \in C(\mathcal{Z})\,, \qquad \int_{\mathcal{Z}} [\varphi(.-y)+\varphi(y)]\,F(.,dy) \in C(\mathcal{Z})\,. \tag{3.25}$$

Note that

$$\begin{aligned}&\left|\int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x,dy) - \int_{\mathcal{Z}} [\varphi(x_n-y)+\varphi(y)]\,F(x_n,dy)\right| \le \\ &\qquad \left|\int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x,dy) - \int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x_n,dy)\right| \\ &\qquad + \left|\int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x_n,dy) - \int_{\mathcal{Z}} [\varphi(x_n-y)+\varphi(y)]\,F(x_n,dy)\right| \\ &\quad \le \;\left|\int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x,dy) - \int_{\mathcal{Z}} [\varphi(x-y)+\varphi(y)]\,F(x_n,dy)\right| \\ &\qquad + F(x_n,\mathcal{Z})\,\sup_{y\in\mathcal{Z}} |\varphi(x-y)-\varphi(x_n-y)|\end{aligned}$$

and

$$\lim_{n\to\infty} \sup_{y\in\mathcal{Z}} |\varphi(x-y)-\varphi(x_n-y)| = 0\,, \qquad \forall\, \varphi \in C_c(\mathcal{Z})\,,$$

if $x_n \to x$. Thus, (3.25) follows from (3.16).

If F has the form (3.19), then condition (3.16) takes the form

$$\int_{\mathcal{Z}} 1_{(0,x)}(y)\,\varphi(y) f(x,y)\,dy \in C(\mathcal{Z})\,, \qquad \forall\, \varphi \in C_b(\mathcal{Z})\,. \tag{3.26}$$

Note that

$$1_{(0,x_n)}(y)\,\varphi(y) f(x_n,y) \;\le\; 1_{(0,\bar{x})}(y)\,\|\varphi\| \sup_n \frac{R(x_n)}{x_n^{1-\beta}}\, y^{-\beta}\,, \qquad \bar{x} = \sup_n x_n\,,$$

and

$$\lim_{n\to\infty} 1_{(0,x_n)}(y)\,\varphi(y) f(x_n,y) \;=\; 1_{(0,x)}(y)\,\varphi(y) f(x,y)\,, \qquad \forall\, y \neq x\,,$$

if $\lim_{n\to\infty} x_n = x$. Thus, (3.26) follows from the dominated convergence theorem, since $f(x,y)$ is continuous in x. ∎

Lemma 3.10 *If (3.14) then* $H(x)+H(y) \ge H(x+y)$. *If*

$$\frac{H(x)}{x} \;\le\; \frac{H(y)}{y}\,, \qquad \forall\, 0 < x \le y < \infty\,,$$

then $H(x)+H(y) \le H(x+y)$.

Proof. Since

$$H(x)+H(y) = \frac{H(x)}{x}\,x + \frac{H(y)}{y}\,y\,, \qquad H(x+y) = \frac{H(x+y)}{x+y}\,x + \frac{H(x+y)}{x+y}\,y\,,$$

the assertions follow. ∎

Lemma 3.11 *Let* $\mathcal{Z} = (0,\infty)$, $H_1(x) = x^{-\alpha}$, $\alpha \in (0,1)$, *and*

$$f_1(x,y) \;=\; \frac{y^{-\beta}}{x^{1-\beta}}\,, \qquad \beta \in [0, 1-\alpha)\,. \tag{3.27}$$

Then

$$\int_0^x [H_1(x-y)+H_1(y)-H_1(x)]\, f_1(x,y)\,dy \;\le\; c(\alpha,\beta)\, H_1(x)\,, \qquad \forall\, x > 0\,. \tag{3.28}$$

Proof. Using

$$\begin{aligned}
\int_0^x (x-y)^{-\alpha}\, y^{-\beta}\,dy &= \int_0^{x/2} (x-y)^{-\alpha}\, y^{-\beta}\,dy + \int_{x/2}^x (x-y)^{-\alpha}\, y^{-\beta}\,dy \\
&\le \int_0^{x/2} (x/2)^{-\alpha}\, y^{-\beta}\,dy + \int_{x/2}^x (x-y)^{-\alpha}\, (x/2)^{-\beta}\,dy \\
&= (x/2)^{-\alpha}\,(x/2)^{1-\beta}\,\frac{1}{1-\beta} + (x/2)^{-\beta}\,(x/2)^{1-\alpha}\,\frac{1}{1-\alpha} \\
&= (x/2)^{1-\alpha-\beta}\left[\frac{1}{1-\beta}+\frac{1}{1-\alpha}\right]
\end{aligned}$$

and (3.27), one obtains

$$\int_0^x [H_1(x-y)+H_1(y)-H_1(x)]\, f_1(x,y)\, dy =$$
$$\frac{1}{x^{1-\beta}} \int_0^x \left[(x-y)^{-\alpha}+y^{-\alpha}-x^{-\alpha}\right] y^{-\beta}\, dy$$
$$\leq \; x^{-\alpha} \left\{ \frac{1}{2^{1-\alpha-\beta}} \left[\frac{1}{1-\beta} + \frac{1}{1-\alpha} \right] + \frac{1}{1-\alpha-\beta} - \frac{1}{1-\beta} \right\}$$

so that (3.28) follows. ■

Lemma 3.12 *Let* $\mathcal{Z} = (0,\infty)$,

$$H(x) \;=\; x^{-\alpha} + x^{\gamma}, \qquad \alpha \in (0,1), \qquad \gamma \geq 1,$$

and

$$h(x) \;=\; x^{-\alpha+\varepsilon} + x^{\gamma-\varepsilon}, \qquad \varepsilon \in (0,\alpha].$$

Assume F *has the form (3.19), where* f *satisfies*

$$0 \;\leq\; f(x,y) \;\leq\; \frac{R(x)\, y^{-\beta}}{x^{1-\beta}}, \qquad \beta \in [0, 1-\alpha), \qquad R(x) = c\,(1+x^{\gamma-\varepsilon}). \quad (3.29)$$

Then conditions (3.17) and (3.18) are fulfilled.

Proof. Using **Lemma 3.10**, (3.29), and **Lemma 3.11**, one obtains

$$\int_{\mathcal{Z}} [H(x-y)+H(y)-H(x)]\, F(x,dy) \leq \int_0^x [H_1(x-y)+H_1(y)-H_1(x)]\, f(x,y)\, dy$$
$$\leq R(x) \int_0^x [H_1(x-y)+H_1(y)-H_1(x)]\, f_1(x,y)\, dy \leq c(\alpha,\beta)\, R(x)\, H_1(x)\,,$$

and condition (3.17) follows. Moreover, (3.29) implies

$$F(x,\mathcal{Z}) \leq \frac{R(x)}{x^{1-\beta}} \int_0^x y^{-\beta}\, dy = \frac{R(x)}{1-\beta} \;\leq\; c_1\, h(x)$$

so that condition (3.18) is satisfied. ■

Proof of Theorem 3.1. The statement is a consequence of Corollary 2.6. Indeed, condition **(2.17)** follows from (3.8), (3.15) and (3.18), and condition **(2.18)** is satisfied due to the continuity assumptions and **Lemma 3.9**. Furthermore, condition **(3.1)** follows from $S \in \mathcal{M}(\mathcal{Z}, H)$ (cf. (3.6)), condition **(3.2)** follows from (3.17), and condition **(3.3)** is fulfilled, according to **Lemma 3.10** and (3.14). ■

Proof of Corollary 3.3. The statement is a consequence of Theorem 3.1. Note that (2.6), (2.14) and condition (3.14) are satisfied. Moreover, condition (3.16) is fulfilled, according to **Lemma 3.9**, and conditions (3.17), (3.18) follow from **Lemma 3.12**, with $\gamma = 1$. ■

Proof of Corollary 3.4. The statement follows immediately from Theorem 3.1. ■

Proof of Theorem 3.5. The statement is a consequence of Corollary 2.9, with $g(x) = x$. Note that condition **(3.4)** is satisfied. Moreover, condition **(2.17)** follows from (3.8), (3.15) and (3.18), and condition **(2.18)** is satisfied due to the continuity assumptions and **Lemma 3.9**. Furthermore, condition **(3.2)** follows from (3.17), and condition **(3.5)** is fulfilled, according to (3.21). ■

Proof of Corollary 3.6. The statement is a consequence of Theorem 3.5. Note that (2.6), (2.14) and (3.20) are satisfied, and (3.15) follows from (3.22). Using **Lemma 3.10** and (3.22), one obtains

$$[H(x+y) - H(x) - H(y)]\, K(x,y) \le \left[(x+y)^2 - x^2 - y^2\right] K(x,y) =$$
$$2\,x\,y\,K(x,y) \le 2\,c_1\,[x\,y + x^2\,y + x\,y^2] \le 2\,c_1\,[x\,y + H(x)\,y + x\,H(y)]$$

so that (3.21) is fulfilled. Moreover, condition (3.16) is fulfilled, according to **Lemma 3.9**, and conditions (3.17), (3.18) are consequences of **Lemma 3.12**, with $\gamma = 2$. ■

Proof of Corollary 3.7. The statement is a consequence of Theorem 3.5. Note that (2.6), (2.14) and (3.20) are satisfied, and (3.15) follows from (3.23). Since

$$x^k\,y^l = x\,(x/y)^{k-1}\,y^{l+k-1} \le x\,y^{l+k-1}\,, \qquad \text{if} \quad x \le y\,,$$

one obtains

$$x^k\,y^l \;\le\; x\,y^{k+l-1} + y\,x^{k+l-1}\,, \qquad \forall\, x,y \ge 0\,, \quad k,l \ge 1\,. \tag{3.30}$$

Using (3.23) and (3.30), one obtains

$$[(x+y)^r - x^r - y^r]\, K(x,y) = \sum_{l=1}^{r-1} C_l^r\, x^l\, y^{r-l}\, K(x,y)$$
$$\le\; c_1 \left[\sum_{l=1}^{r-1} C_l^r\, x^{l+1}\, y^{r-l} + \sum_{l=1}^{r-1} C_l^r\, x^l\, y^{r-l+1}\right] \le 2\,c_1 \left(\sum_{l=1}^{r-1} C_l^r\right) [x\,y^r + y\,x^r]\,,$$

and condition (3.21) follows. Finally, condition (3.17) is a consequence of **Lemma 3.10**.
■

3.3. Dissipative collisions

Here we consider the case $\mathcal{Z} = \mathbb{R}^d$, $d \ge 1$. Denote

$$v'(v,w,e,\theta) \;=\; \frac{v+w}{2} + \varepsilon(v,w,\theta)\,\frac{\|v-w\|\,e}{2} \tag{3.31}$$

and

$$w'(v,w,e,\theta) \;=\; \frac{v+w}{2} - \varepsilon(v,w,\theta)\,\frac{\|v-w\|\,e}{2}\,, \tag{3.32}$$

where $v, w \in \mathbb{R}^d$, $e \in \mathbb{S}^{d-1}$ (unit sphere), $\theta \in \Theta$, for some measurable space Θ, and ε is some measurable function. Note that

$$\|v'\|^2 + \|w'\|^2 \;=\; \|v\|^2 + \|w\|^2 - \frac{1-\varepsilon(v,w,\theta)^2}{2}\,\|v-w\|^2\,, \tag{3.33}$$

i.e. energy is dissipated if $\varepsilon^2 < 1$, conserved if $\varepsilon^2 = 1$, and created if $\varepsilon^2 > 1$. Transformation (3.31), (3.32) generalizes the one-dimensional model proposed in [33], with

$$\varepsilon(v,w,\theta) \;=\; \frac{1}{1+\theta\,\|v-w\|^a}\,, \qquad \theta \in [0,\infty)\,, \quad a>0\,.$$

In the special case $\varepsilon \equiv 1$, we use the notation

$$v^*(v,w,e) = \frac{v+w}{2} + \frac{\|v-w\|\,e}{2}\,, \qquad w^*(v,w,e) = \frac{v+w}{2} - \frac{\|v-w\|\,e}{2}\,. \tag{3.34}$$

Theorem 3.13 *Consider the functions*

$$H(v) \;=\; \|v\|^2 + 1\,, \qquad v \in \mathbb{R}^d, \tag{3.35}$$

and h satisfying (2.14). Let $\beta_1\,,\,\beta_2\,:\,\mathbb{R}^d \times \mathbb{R}^d \times \mathcal{B}(\mathbb{S}^{d-1}) \to [0,\infty)$ be compactly bounded kernels satisfying

$$\beta_1(v,w,\mathbb{S}^{d-1}) \;\le\; c\,h(v)\,H(w)\,, \qquad \forall\, v,w \in \mathbb{R}^d, \tag{3.36}$$

$$\beta_2(v,w,\mathbb{S}^{d-1}) \;\le\; c\,h(v)\,h(w)\,, \qquad \forall\, v,w \in \mathbb{R}^d, \tag{3.37}$$

$$\beta_1(v_n,w,.) \;\stackrel{w}{\to}\; \beta(v,w,.)\,, \qquad \forall\, w \in \mathbb{R}^d, \tag{3.38}$$

and

$$\beta_2(v_n,w_n,.) \;\stackrel{w}{\to}\; \beta(v,w,.)\,, \tag{3.39}$$

when $v_n \to v$ and $w_n \to w$ in $\mathbb{R}^d$. Consider $\mu_0 \in \mathcal{M}(\mathbb{R}^d, H)$, $\pi \in \mathcal{P}(\Theta)$, and

$$M \in \mathcal{P}(\mathbb{R}^d) \quad \text{such that} \quad \int_{\mathbb{R}^d} \|w\|^4\, M(dw) \;<\; \infty\,. \tag{3.40}$$

Suppose

$$\int_\Theta \varepsilon(v,w,\theta)^2\,\pi(d\theta) \;\le\; 1\,, \qquad \forall\, v,w \in \mathbb{R}^d, \tag{3.41}$$

and

$$\varepsilon(.,.,\theta) \in C(\mathbb{R}^d \times \mathbb{R}^d)\,, \qquad \forall\,\theta \in \Theta\,. \tag{3.42}$$

Then there exists some $\mu \in C([0,\infty), \mathcal{M}(\mathbb{R}^d, H, h))$ satisfying the equation

$$\begin{aligned}
&\langle \varphi, \mu(t)\rangle = \langle \varphi, \mu_0\rangle + \\
&\quad \int_0^t \Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big[\varphi(v^*(v,w,e)) - \varphi(v)\Big]\beta_1(v,w,de)\,M(dw)\,\mu(s,dv) \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_\Theta\int_{\mathbb{S}^{d-1}} \Big[\varphi(v'(v,w,e,\theta)) + \varphi(w'(v,w,e,\theta)) - \varphi(v) - \varphi(w)\Big] \times \\
&\quad \beta_2(v,w,de)\,\pi(d\theta)\,\mu(s,dv)\,\mu(s,dw)\Big]\,ds\,, \qquad t \ge 0\,,
\end{aligned} \tag{3.43}$$

for any $\varphi \in C_c(\mathbb{R}^d)$.

Remark 3.14 *The probability measure $\pi \in \mathcal{P}(\Theta)$ introduces some randomness into the collision events. The probability measure $M \in \mathcal{P}(\mathbb{R}^d)$ represents the influence of some background gas. Condition (3.40) is fulfilled, for example, when M is a Maxwellian.*

Remark 3.15 *Consider the special case*

$$\varepsilon \equiv 1\,, \qquad \beta_1(v,w,de) = \|v-w\|\,de\,, \qquad \beta_2(v,w,de) = \frac{1}{2}\,\|v-w\|\,de\,,$$

and suppose $\mu(t,dv) = f(t,v)\,dv$, $M(dv) = M(v)\,dv$. Then equation (3.43) takes the form

$$\int_{\mathbb{R}^d} \varphi(v)\,f(t,v)\,dv = \int_{\mathbb{R}^d} \varphi(v)\,f_0(v)\,dv +$$
$$\int_{\mathbb{R}^d} dv\,\varphi(v) \int_0^t ds\,\Big[\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \|v-w\|\Big[M(w^*)\,f(s,v^*) - M(w)\,f(s,v)\Big]\,de\,dw +$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \|v-w\|\Big[f(s,w^*)\,f(s,v^*) - f(s,w)\,f(s,v)\Big]\,de\,dw\Big]\,.$$

Removing the test functions, one obtains

$$\begin{aligned}\frac{\partial}{\partial t}\,f(t,v) \;&=\; \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \|v-w\|\Big[M(w^*)\,f(t,v^*) - M(w)\,f(t,v)\Big]\,de\,dw \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \|v-w\|\Big[f(t,w^*)\,f(t,v^*) - f(t,w)\,f(t,v)\Big]\,de\,dw\,.\end{aligned}$$

Proof of Theorem 3.13. Introducing the **background collision term**

$$q_1(v,B) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} 1_B(\delta_{v^*(v,w,e)})\,\beta_1(v,w,de)\,M(dw)\,, \qquad v \in \mathbb{R}^d, \quad B \in \mathcal{B}(E_K)\,,$$

one obtains (cf. (2.30))

$$Q_1(\varphi,v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big[\varphi(v^*(v,w,e)) - \varphi(v)\Big]\beta_1(v,w,de)\,M(dw)\,. \tag{3.44}$$

With (3.35), **condition (3.2)** takes the form

$$\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \big[\|v^*(v,w,e)\|^2 - \|v\|^2\big]\,\beta_1(v,w,de)\,M(dw) \;\le\; c_1\,\big[\|v\|^2+1\big]\,. \tag{3.45}$$

Note that (3.33) (with $\varepsilon = 1$) implies

$$\|v^*(v,w,e)\|^2 - \|v\|^2 \;\le\; \|w\|^2\,,$$

so that assumptions (3.36) and (3.40) are sufficient for condition (3.45).

Introducing the **binary collision term**

$$q_2(v,w,B) = \int_{\Theta}\int_{\mathbb{S}^{d-1}} 1_B(\delta_{v'(v,w,e,\theta)} + \delta_{w'(v,w,e,\theta)})\,\beta_2(v,w,de)\,\pi(d\theta)\,,$$

where $v,w\in\mathbb{R}^d$ and $B\in\mathcal{B}(E_K)$, one obtains (cf. (2.30))

$$Q_2(\varphi,v,w)= \tag{3.46}$$
$$\int_\Theta\int_{\mathbb{S}^{d-1}}\Big[\varphi(v'(v,w,e,\theta))+\varphi(w'(v,w,e,\theta))-\varphi(v)-\varphi(w)\Big]\beta_2(v,w,de)\,\pi(d\theta)\,.$$

In view of (3.35) and (3.33), **condition (3.3)** takes the form

$$\frac{\|v-w\|^2}{2}\left[\int_\Theta \varepsilon(v,w,\theta)^2\,\pi(d\theta)-1\right]\beta_2(v,w,\mathbb{S}^{d-1})\ \le\ 0\,,$$

and follows from (3.41).

Condition (2.17) takes the form (3.37) and

$$\int_{\mathbb{R}^d}\beta_1(v,w,\mathbb{S}^{d-1})\,M(dw)\ \le\ c\,h(v)\,,\qquad \forall\, v\in\mathbb{R}^d\,,$$

and follows from (3.36) and (3.40).

Condition (2.18) reduces to $\beta_2(.,.,\mathbb{S}^{d-1})\in C(\mathbb{R}^d,\mathbb{R}^d)$,

$$\int_\Theta\int_{\mathbb{S}^{d-1}}\Big[\varphi(v'(.,.,e,\theta))+\varphi(w'(.,.,e,\theta))\Big]\beta_2(.,.,de)\,\pi(d\theta)\in C(\mathbb{R}^d,\mathbb{R}^d)\,, \tag{3.47}$$

$$\int_{\mathbb{R}^d}\beta_1(.,w,\mathbb{S}^{d-1})\,M(dw)\quad\text{and}\quad \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\varphi(v^*(.,w,e))\,\beta_1(.,w,de)\,M(dw)\in C(\mathbb{R}^d) \tag{3.48}$$

Note that (cf. (3.31))

$$\|v'(v,w,e,\theta)-v'(v_n,w_n,e,\theta)\|\le\frac{\|v-v_n\|+\|w-w_n\|}{2}+$$
$$|\varepsilon(v,w,\theta)-\varepsilon(v_n,w_n,\theta)|\,\frac{\|v-w\|}{2}+\frac{\varepsilon(v_n,w_n,\theta)}{2}\Big|\|v-w\|-\|v_n-w_n\|\Big|$$

implies

$$\sup_{e\in\mathbb{S}^{d-1}}\|v'(v,w,e,\theta)-v'(v_n,w_n,e,\theta)\|\ \to\ 0\qquad\text{if}\quad (v_n,w_n)\to(v,w)\,,$$

according to assumption (3.42). Since $\varphi\in C_c(\mathbb{R}^d)$, it follows that

$$\sup_{e\in\mathbb{S}^{d-1}}\Big|\varphi(v'(v,w,e,\theta))-\varphi(v'(v_n,w_n,e,\theta))\Big|\ \to\ 0\qquad\text{if}\quad (v_n,w_n)\to(v,w)\,. \tag{3.49}$$

One obtains

$$\Big|\int_{\mathbb{S}^{d-1}}\varphi(v'(v,w,e,\theta))\,\beta_2(v,w,de)-\int_{\mathbb{S}^{d-1}}\varphi(v'(v_n,w_n,e,\theta))\,\beta_2(v_n,w_n,de)\Big|$$
$$\le\ \Big|\int_{\mathbb{S}^{d-1}}\varphi(v'(v,w,e,\theta))\,\beta_2(v,w,de)-\int_{\mathbb{S}^{d-1}}\varphi(v'(v,w,e,\theta))\,\beta_2(v_n,w_n,de)\Big|$$
$$+\sup_{e\in\mathbb{S}^{d-1}}\Big|\varphi(v'(v,w,e,\theta))-\varphi(v'(v_n,w_n,e,\theta))\Big|\,\beta_2(v_n,w_n,\mathbb{S}^{d-1}) \tag{3.50}$$

and the continuity of

$$\int_{\Theta}\int_{\mathbb{S}^{d-1}} \varphi(v'(.,.,e,\theta))\,\beta_2(.,.,de)\,\pi(d\theta)$$

follows from (3.50), (3.49), assumption (3.39), and the dominated convergence theorem. The other terms in (3.47), (3.48) are treated in an analogous way, using also assumption (3.38). Finally, the assertion is a consequence of Corollary 2.6, since, with the terms (3.44), (3.46), equation (2.23) takes the form (3.43). ■

4. The minimal jump process

Let λ be a compactly bounded kernel on a locally compact separable metric space E. The **minimal jump process** X^{Δ}, corresponding to λ and some initial distribution $\nu_0 \in \mathcal{P}(E)$, is constructed on the one-point compactification E^{Δ} in the following way (cf. [13, p.263], [27, p.69]). Let $Y_0, Y_1, \ldots$ be a Markov chain in E with initial distribution ν_0 and transition function $p : E \times \mathcal{B}(E) \to [0,1]$ defined by

$$p(\xi, B) \;=\; \begin{cases} \frac{\lambda(\xi,B)}{\lambda(\xi,E)} & : \quad \lambda(\xi,E) > 0\,, \\ 1_B(\xi) & : \quad \lambda(\xi,E) = 0\,. \end{cases}$$

Let $T_0, T_1, \ldots$ be independent and exponentially distributed random variables with mean 1 that are also independent of (Y_k), all defined on some probability space $(\Omega, \mathcal{F}, P)$. Introduce the jump and explosion times

$$\tau_0 \;=\; 0\,, \qquad \tau_l \;=\; \sum_{k=0}^{l-1} \frac{T_k}{\lambda(Y_k,E)}, \quad l = 1,2,\ldots\,, \qquad \tau_\infty \;=\; \sum_{k=0}^{\infty} \frac{T_k}{\lambda(Y_k,E)}\,, \tag{4.1}$$

where $T_k/0 := \infty$, and define

$$X^{\Delta}(t) \;=\; \begin{cases} Y_l & : \quad \tau_l \le t < \tau_{l+1} \\ \Delta & : \quad t \ge \tau_\infty \end{cases}\,, \qquad t \ge 0\,. \tag{4.2}$$

Note that X^{Δ} is an E^{Δ}-valued stochastic process, since

$$\mathcal{B}(E^{\Delta}) \;=\; \mathcal{B}(E) \cup \Big\{ B \cup \{\Delta\} \,:\; B \in \mathcal{B}(E) \Big\}$$

and

$$\big\{X^{\Delta}(t) \in B\big\} \;=\; \bigcup_{l=0}^{\infty} \{\tau_l \le t < \tau_{l+1}\} \cap \{Y_l \in B\} \;\in\; \mathcal{F}$$

and

$$\Big\{X^{\Delta}(t) \in B \cup \{\Delta\}\Big\} \;=\; \big\{X^{\Delta}(t) \in B\big\} \cup \{t \ge \tau_\infty\} \;\in\; \mathcal{F}\,,$$

for all $B \in \mathcal{B}(E)$.

Theorem 4.1 *Let λ be a compactly bounded kernel on a locally compact separable metric space E, and $\nu_0 \in \mathcal{P}(E)$. Suppose there exists a nonnegative continuous function η such that*

$$\frac{1}{\eta+1} \in C_0(E)\,, \tag{4.3}$$

$$\int_E \eta(\xi)\, \nu_0(d\xi) \;\; \leq \;\; c_0 \tag{4.4}$$

and

$$\int_E [\eta(\xi_1) - \eta(\xi)]\, \lambda(\xi, d\xi_1) \;\; \leq \;\; c_1\, [\eta(\xi) + c_1']\,, \qquad \forall\, \xi \in E\,, \tag{4.5}$$

for some $c_0, c_1, c_1' \geq 0$.

Then there exists a $D([0,\infty), E)$-valued random variable X such that

$$P(X(t) = X^{\Delta}(t)\,,\; \forall\, t \geq 0) \;\; = \;\; 1 \tag{4.6}$$

and

$$\mathbb{E}\, \eta(X(t)) \;\; \leq \;\; (c_0 + c_1') \exp(c_1\, t)\,, \qquad \forall\, t \geq 0\,. \tag{4.7}$$

Corollary 4.2 *Let X be given by Theorem 4.1. Then*

$$\sigma_{C_m} \;\; = \;\; \inf\, \{t \geq 0 : X(t) \notin C_m\}\,, \tag{4.8}$$

with

$$C_m \;\; = \;\; \{\xi \in E \; : \; \eta(\xi) \leq m\}\,, \tag{4.9}$$

and

$$M(\Psi, t) \;\; = \;\; \Psi(X(t)) - \Psi(X(0)) - \int_0^t \mathcal{A}\Psi(X(s))\, ds\,, \tag{4.10}$$

with

$$\mathcal{A}\Psi(\xi) \;\; = \;\; \int_E [\Psi(\xi_1) - \Psi(\xi)]\, \lambda(\xi, d\xi_1)\,, \qquad \xi \in E\,, \tag{4.11}$$

satisfy

$$P(\sigma_{C_m} \leq t) \;\; \leq \;\; m^{-1}\, (c_0 + c_1') \exp(c_1 t) \tag{4.12}$$

and

$$\mathbb{E} \sup_{s \leq t} |M(\Psi, s \wedge \sigma_{C_m})| \;\; \leq \;\; 2 \left(t \sup_{\xi \in C_m} \int_E [\Psi(\xi_1) - \Psi(\xi)]^2\, \lambda(\xi, d\xi_1) \right)^{1/2}, \tag{4.13}$$

for all $m \geq 1$, $t \geq 0$, and $\Psi \in C_b(E)$.

Remark 4.3 *By construction (4.1), (4.2), X^Δ is right-constant, but left limits may not exist at τ_∞. In particular, one obtains*

$$X^\Delta(t) \in E\,,\ \forall\, t \ge 0 \quad \Leftrightarrow \quad \tau_\infty = \infty \quad \Leftrightarrow \quad X^\Delta \in D([0,\infty), E)\,.$$

Remark 4.4 *For every $A \in \mathcal{B}(E)$, the random variable*

$$\sigma_A^\Delta \;=\; \inf\big\{t \ge 0 : X^\Delta(t) \notin A\big\} \tag{4.14}$$

is an $\{\mathcal{F}_t^{X^\Delta}\}$-stopping time. Indeed, since the paths are right-constant, one obtains

$$X^\Delta(t) \in A\,, \quad \forall t < \sigma_A^\Delta\,, \qquad X^\Delta(\sigma_A^\Delta) \notin A\,, \tag{4.15}$$

and

$$\{\sigma_A^\Delta \le t\} \;=\; \bigcup_{s \in \mathbb{Q} \cap [0,t]} \big\{X^\Delta(s) \notin A\big\} \cup \big\{X^\Delta(t) \notin A\big\} \;\in\; \mathcal{F}_t^{X^\Delta}\,, \qquad t \ge 0\,,$$

where $\mathbb{Q}$ denotes the set of rational numbers.

Lemma 4.5 *If*

$$\sup_{\xi \in A} \lambda(\xi, E) < \infty \tag{4.16}$$

then there exists a process X_A with sample paths in $D([0,\infty), E)$ such that

$$P(X_A(t) = X^\Delta(t \wedge \sigma_A^\Delta)\,,\ \forall\, t \ge 0) \;=\; 1\,. \tag{4.17}$$

Moreover, for $\Psi \in C_b(E)$ and $t \ge 0$, it satisfies

$$\mathbb{E}\,\Psi(X_A(t)) \;=\; \mathbb{E}\,\Psi(X_A(0)) + \mathbb{E}\int_0^t \mathcal{A}_A \Psi(X_A(s))\, ds \tag{4.18}$$

and

$$\mathbb{E}\,\sup_{s \le t} |M_A(\Psi, s)| \;\le\; 2\left(t\, \sup_{\xi \in A} \int_E [\Psi(\xi_1) - \Psi(\xi)]^2\, \lambda(\xi, d\xi_1)\right)^{1/2}, \tag{4.19}$$

where

$$\mathcal{A}_A \Psi(\xi) \;=\; \int_E [\Psi(\xi_1) - \Psi(\xi)]\, \lambda_A(\xi, d\xi_1)\,, \qquad \xi \in E\,, \tag{4.20}$$

with

$$\lambda_A(\xi, B) \;=\; 1_A(\xi)\ \lambda(\xi, B)\,, \qquad \xi \in E\,, \quad B \in \mathcal{B}(E)\,, \tag{4.21}$$

and

$$M_A(\Psi, t) \;=\; \Psi(X_A(t)) - \Psi(X_A(0)) - \int_0^t \mathcal{A}_A \Psi(X_A(s))\, ds\,. \tag{4.22}$$

Proof. We first check that

$$P(\Omega') = 1\,, \qquad \text{where} \qquad \Omega' = \{X^{\Delta}(t \wedge \sigma_A^{\Delta}) \in E\,,\ \forall\, t \geq 0\}\,. \tag{4.23}$$

If there is a $k \geq 0$ such that $T_k(\omega) > 0$ and $Y_k(\omega) \notin A$, then $\sigma_A^{\Delta}(\omega) \leq \tau_k(\omega)$, i.e. there are at most k jumps for this trajectory, and $\omega \in \Omega'$. Thus, for $\omega \notin \Omega'$ one obtains $Y_k(\omega) \in A$ whenever $T_k(\omega) > 0$, and therefore

$$\infty > \tau_\infty(\omega) = \sum_{k=0}^{\infty} \frac{T_k(\omega)}{\lambda(Y_k(\omega), E)} \;\geq\; \frac{1}{\sup_{\xi\in A} \lambda(\xi, E)} \sum_{k=0}^{\infty} T_k(\omega)\,,$$

which implies (cf. **Remark 4.3**)

$$P(\Omega \setminus \Omega') \;\leq\; P(\tau_\infty < \infty) \leq P\left\{\sum_{k=0}^{\infty} T_k < \infty\right\} = 0\,.$$

Therefore, (4.23) is fulfilled, and the process X_A is obtained by redefining $X^{\Delta}(t \wedge \sigma_A^{\Delta})$ on the set $\Omega \setminus \Omega'$ by elements of $D([0,\infty), E)$.

It follows from the explicit construction procedure that the process X_A is equivalent to the minimal jump process corresponding to ν_0 and the kernel (4.21), and therefore is a Markov process with the bounded generator (4.20) (cf., e.g., [13, p.163]). Thus, for $\Psi \in C_b(E)$, the processes (4.22) and

$$M_A(\Psi, t)^2 - \int_0^t \left[\mathcal{A}_A \Psi^2 - 2\,\Psi\, \mathcal{A}_A \Psi\right](X_A(s))\,ds$$

are $\{\mathcal{F}_t^{X_A}\}$-martingales (cf., e.g., [13, p.93 and Proposition 4.1.7]) In particular, one obtains (4.18) and

$$\mathbb{E}\, M_A(\Psi, t)^2 \;=\; \mathbb{E} \int_0^t \left[\mathcal{A}_A \Psi^2 - 2\,\Psi\, \mathcal{A}_A \Psi\right](X_A(s))\,ds\,. \tag{4.24}$$

Using the identity

$$[\mathcal{A}_A \Psi^2 - 2\Psi \mathcal{A}_A \Psi](\xi) \;=\; \int_E [\Psi(\xi_1) - \Psi(\xi)]^2 \lambda_A(\xi, d\xi_1)\,, \qquad \xi \in E\,, \tag{4.25}$$

and Doob's inequality (cf. [13, Corollary 2.2.17])

$$\mathbb{E} \sup_{s\leq t} |M_A(\Psi, s)|^2 \;\leq\; 4\,\mathbb{E}\, M_A(\Psi, t)^2\,, \tag{4.26}$$

one obtains (4.19) from (4.26), (4.24), (4.25) and (4.21). ■

Lemma 4.6 *Let the assumptions of Theorem 4.1 be fulfilled. Then (cf. (4.14), (4.9))*

$$P(\sigma_{C_m}^{\Delta} \leq t) \;\leq\; m^{-1}\,(c_0 + c_1')\exp(c_1 t)\,, \qquad \forall\, t \geq 0\,. \tag{4.27}$$

Proof. Note that

$$C_m \subset O_m := \{\xi \in E \,:\, \eta(\xi) < m+1\} \subset C_{m+1}\,.$$

According to **Lemma A.2**, we choose $e_k \in C_c(E)$, $k \ge 1$, such that

$$e_k(\xi) = 1,\ \xi \in C_k\,, \quad e_k(\xi) = 0,\ \xi \in C_{k+1}^c \qquad \text{and} \qquad 0 \le e_k(\xi) \le 1,\ \xi \in E\,.$$

Note that $e_k(\xi) = 1$, $k \ge m$, $\xi \in C_m$. Thus, the functions $\Psi_k = e_k\,[\eta + c_1'] \in C_c(E)$ satisfy (cf. (4.21))

$$\int_E [\Psi_k(\xi_1) - \Psi_k(\xi)]\,\lambda_{C_m}(\xi, d\xi_1) \;\le\; \int_E [\eta(\xi_1) - \eta(\xi)]\,\lambda_{C_m}(\xi, d\xi_1)\,, \qquad \forall\, k \ge m\,. \tag{4.28}$$

It follows from **assumption (4.3)** and **Lemma A.1** that the sets (4.9) are compact. Thus, since λ is compactly bounded, assumption (4.16) is fulfilled so that **Lemma 4.5** is applicable. Using (4.18), (4.28) and **assumptions (4.4), (4.5)**, one obtains

$$\begin{aligned}
\mathbb{E}\,\Psi_k(X_{C_m}(t)) &= \mathbb{E}\,\Psi_k(X(0)) + \mathbb{E}\int_0^t \int_E [\Psi_k(\xi_1) - \Psi_k(X_{C_m}(s))]\,\lambda_{C_m}(X_{C_m}(s), d\xi_1)\,ds \\
&\le \mathbb{E}\,\Psi_k(X(0)) + \mathbb{E}\int_0^t \int_E [\eta(\xi_1) - \eta(X_{C_m}(s))]\,\lambda_{C_m}(X_{C_m}(s), d\xi_1)\,ds \\
&\le \mathbb{E}\,\Psi_k(X(0)) + c_1\,\mathbb{E}\int_0^t 1_{C_m}(X_{C_m}(s))\;[\eta(X_{C_m}(s)) + c_1']\,ds \\
&\le c_0 + c_1' + c_1 \int_0^t \mathbb{E}\,[\eta(X_{C_m}(s)) + c_1']\,ds\,, \qquad \forall\, t \ge 0\,, \quad k \ge m\,,
\end{aligned}$$

where X_{C_m} is a $D([0,\infty), E)$-valued random variable such that

$$P(X_{C_m}(t) = X^\Delta(t \wedge \sigma^\Delta_{C_m})\,,\ \forall\, t \ge 0) \;=\; 1\,. \tag{4.29}$$

The monotone convergence theorem (with $k \to \infty$) implies

$$\mathbb{E}\eta(X_{C_m}(t)) + c_1' \;\le\; c_0 + c_1' + c_1 \int_0^t \mathbb{E}\,[\eta(X_{C_m}(s)) + c_1']\,ds\,. \tag{4.30}$$

Using (4.29), (4.15) and **assumption (4.5)**, one obtains

$$\begin{aligned}
&\mathbb{E}\,\eta(X_{C_m}(t)) = \mathbb{E}\eta(X_{C_m}(t))\,1_{\{\sigma^\Delta_{C_m} > t\}} + \mathbb{E}\,\eta(X_{C_m}(t))\,1_{\{\sigma^\Delta_{C_m} \le t\}} \\
&\le\; m + \sup_{\xi \in C_m} \int_E \eta(\xi_1)\,\lambda(\xi, d\xi_1) \le m + \sup_{\xi \in C_m} \Big(c_1\,[\eta(\xi) + c_1'] + \eta(\xi)\,\lambda(\xi, E)\Big) \\
&\le\; m + \Big(c_1\,[m + c_1'] + m \sup_{\xi \in C_m} \lambda(\xi, E)\Big).
\end{aligned}$$

An application of Gronwall's inequality to (4.30) yields

$$\mathbb{E}\eta(X_{C_m}(t)) \;\le\; (c_0 + c_1')\exp(c_1\,t)\,, \qquad \forall\, t \ge 0\,. \tag{4.31}$$

Using (4.15) and (4.29), one obtains

$$P(\sigma^\Delta_{C_m} \le t) = P(X_{C_m}(t) \notin C_m) = P(\eta(X_{C_m}(t)) > m) \le m^{-1}\,\mathbb{E}\,\eta(X_{C_m}(t))\,,$$

for all $m \ge 1$ and $t \ge 0$, so that (4.27) follows from (4.31). ∎

Proof of Theorem 4.1. Since $\sigma^{\Delta}_{C_m} \leq \sigma^{\Delta}_{C_{m+1}}$, for all $m \geq 1$, **Lemma 4.6** implies

$$P\left(\lim_{m\to\infty} \sigma^{\Delta}_{C_m} \leq t\right) = 0\,, \qquad \forall\, t \geq 0\,.$$

so that

$$P\left(\lim_{m\to\infty} \sigma^{\Delta}_{C_m} = \infty\right) = 1\,. \tag{4.32}$$

Since $0 \leq \sigma^{\Delta}_{C_m} \leq \tau_\infty = \sigma^{\Delta}_{E}$, (4.32) implies $P(\tau_\infty = \infty) = 1$. Thus, according to **Remark 4.3**, a process X with sample paths in $D([0,\infty),E)$, satisfying (4.6), is obtained by appropriately redefining X^{Δ} on the set $\{\tau_\infty < \infty\}$, e.g., by constant paths. This process is a $D([0,\infty),E)$-valued random variable according to [13, p.128], since E is separable.

Using (4.32) and continuity of η, one obtains

$$P\left(\lim_{m\to\infty} \eta(X(t\wedge\sigma_{C_m})) = \eta(X(t))\right) = 1\,.$$

Thus, Fatou's lemma and (4.31) imply (4.7). ■

Proof of Corollary 4.2. Note that (cf. (4.8), (4.14))

$$P(\sigma_{C_m} = \sigma^{\Delta}_{C_m}) \;=\; 1\,, \tag{4.33}$$

according to (4.6). Thus, (4.12) is a consequence of **Lemma 4.6**.

Moreover, (4.6) and (4.33) imply

$$P(X^{\Delta}(t\wedge\sigma^{\Delta}_{C_m}) = X(t\wedge\sigma_{C_m})\,,\ \forall\, t\geq 0) \;=\; 1$$

and (cf. (4.17))

$$P(X_{C_m}(t) = X(t\wedge\sigma_{C_m})\,,\ \forall\, t\geq 0) \;=\; 1\,. \tag{4.34}$$

Note that (cf. (4.10))

$$M(\Psi, t\wedge\sigma_{C_m}) \;=\; \Psi(X(t\wedge\sigma_{C_m})) - \Psi(X(0)) - \int_0^{t\wedge\sigma_{C_m}} \mathcal{A}\Psi(X(s))\,ds \tag{4.35}$$

and (cf. (4.22), (4.20), (4.21), (4.11))

$$M_{C_m}(\Psi,t) \;=\; \Psi(X_{C_m}(t)) - \Psi(X_{C_m}(0)) - \int_0^t 1_{C_m}(X_{C_m}(s))\,\mathcal{A}\Psi(X_{C_m}(s))\,ds\,. \tag{4.36}$$

Since $X_{C_m}(s) \notin C_m$ a.e., for $s \geq \sigma_{C_m}$, one obtains, using (4.34), (4.35) and (4.36),

$$P(M(\Psi, t\wedge\sigma_{C_m}) = M_{C_m}(\Psi,t)\,,\ \forall\, t\geq 0) \;=\; 1\,,$$

so that (4.13) follows from (4.19), with $A = C_m$. ■

5. Proofs of the main results

5.1. Regularity

Lemma 5.1 *Let $\mathcal{Z}$ be a locally compact separable metric space. Then the space (E^N, d_1) (cf. (2.1), (2.11)) is separable and locally compact. If, in addition, assumption (2.4) holds, then the kernel λ^N (cf. (2.5)) on (E^N, d_1) is compactly bounded.*

Proof. Let $\mathcal{Z}'$ be a countable dense set in $\mathcal{Z}$. Then the set

$$\left\{\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i} \,:\, n\geq 0\,,\ x_i\in\mathcal{Z}'\,,\ i=1,\ldots,n\right\}$$

is countable and dense in (E^N, d_1), according to **Remark 2.10**. Choose compact and open sets Γ_m, Ω_m, $m\geq 1$, according to **Lemma A.3**. Then the sets

$$G_m \quad = \quad \left\{\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i} \,:\, 0\leq n\leq m\,,\ x_i\in\Gamma_m\,,\ i=1,\ldots,n\right\} \tag{5.1}$$

and

$$O_m \quad = \quad \left\{\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i} \,:\, 0\leq n\leq m\,,\ x_i\in\Omega_m\,,\ i=1,\ldots,n\right\}$$

are, respectively, compact and open (this is easily established using **Remark 2.10**). They satisfy

$$G_m\subset O_m\subset G_{m+1} \quad \text{and} \quad E^N=\bigcup_m G_m\,. \tag{5.2}$$

In particular, every $\mu\in E^N$ has a compact neighbourhood, which proves local compactness of the space E^N. Any compact set $C\subset E^N$ is covered by a finite number of sets O_m, according to (5.2). Thus, it is contained in some G_m, and (2.27) implies

$$\sup_{\mu\in C}\lambda^N(\mu,E^N) \quad\leq\quad N\left[q_0(E_K)+\sum_{r=1}^{R}\sup_{x\in\Gamma_m^r}q_r(x,E_K)\left(\frac{m}{N}\right)^r\right]. \tag{5.3}$$

The right-hand side of (5.3) is finite by **assumption (2.4)**, so that the kernel is compactly bounded. ∎

Remark 5.2 *For any $\mu=\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i}\in E^N$ one obtains*

$$\frac{n}{N}=\mu(E^N)\leq\frac{\langle H,\mu\rangle}{\inf H} \qquad \textit{and} \qquad H(x_i)\leq N\,\langle H,\mu\rangle\,,\quad \forall\, i=1,\ldots,n\,. \tag{5.4}$$

Lemma 5.3 *If assumption (2.6) holds, then the function Φ defined in (2.28), with $\varphi=H$, satisfies*

$$\frac{1}{\Phi+1} \quad\in\quad C_0(E^N)\,. \tag{5.5}$$

Proof. It follows from **assumption (2.6)** that the set $\Gamma = \{x \in \mathcal{Z} \,:\, H(x) \le m\,N\}$ is compact, according to **Lemma A.1**, and that $\inf_{x\in\mathcal{Z}} H(x) > 0$. As a consequence of **Remark 5.2**, the set

$$\left\{\mu \in E^N \,:\, \Phi(\mu) \le m\right\}, \qquad m > 0\,, \tag{5.6}$$

is contained in the set

$$\left\{\frac{1}{N}\sum_{i=1}^{n}\delta_{x_i} \,:\, 0 \le n \le \frac{m\,N}{\inf H}\,,\; x_i \in \Gamma\right\},$$

which is compact (cf. (5.1)). Since the function Φ is continuous, according to **assumption (2.6)** and **Remark 2.11**, the set (5.6) is closed, and therefore compact. Finally, property (5.5) follows from **Lemma A.1** and the first part of **Lemma 5.1**. ∎

Proof of Theorem 2.1. According to assumption **(2.4)** and **Lemma 5.1**, the kernel λ^N is compactly bounded on the separable and locally compact space (E^N, d_1). Condition (4.3), with $\eta = \Phi$, follows from assumption **(2.6)** and **Lemma 5.3**. Assumptions **(2.7), (2.8)** take the form (4.4), (4.5). Thus, Theorem 2.1 follows from **Theorem 4.1**. ∎

We finish this section by providing further consequences of the assumptions of Theorem 2.1, which will be used later.

Remark 5.4 *The process X^N provided by Theorem 4.1 is a $D([0,\infty), E^N)$-valued random variable. Note that $E^N \subset \mathcal{M}(\mathcal{Z}, H)$ and the embedding $(E^N, d_1) \to \mathcal{M}(\mathcal{Z}, H, h)$ is continuous (cf. Remark 2.10). Thus, in view of Lemma A.4, the process X^N can be considered as a $D([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$-valued random variable.*

Using (4.7), one obtains

$$\mathbb{E}\,\langle H, X^N(t)\rangle \;\le\; (c_0 + c_1')\exp(c_1\,t)\,, \qquad \forall\, t \ge 0\,. \tag{5.7}$$

Moreover, **Corollary 4.2** implies

$$P\left(\sigma_m^N > T\right) \;\ge\; 1 - m^{-1}(c_0 + c_1')\exp(c_1\,T)\,, \qquad \forall\, T \ge 0\,, \quad m \ge 1\,, \tag{5.8}$$

and

$$\mathbb{E}\sup_{t\le T}\left|M^N(\Psi, t\wedge\sigma_m^N)\right| \;\le\; 2\left(T\sup_{\mu\in C_m^N}\int_{E^N}\left[\Psi(\nu)-\Psi(\mu)\right]^2\lambda^N(\mu, d\nu)\right)^{1/2}, \tag{5.9}$$

for all $m \ge 1$, $T \ge 0$, and $\Psi \in C_b(E^N)$, where

$$\sigma_m^N \;=\; \inf\left\{t \ge 0 : X^N(t) \notin C_m^N\right\}, \tag{5.10}$$

$$C_m^N \;:=\; \left\{\mu \in E^N \,:\, \langle H, \mu\rangle \le m\right\}, \qquad m > 0\,, \tag{5.11}$$

$$M^N(\Psi, t) \;=\; \Psi(X^N(t)) - \Psi(X^N(0)) - \int_0^t \mathcal{A}^N\Psi(X^N(s))\,ds\,, \qquad t \ge 0\,, \tag{5.12}$$

and

$$\mathcal{A}^N\Psi(\mu) \;=\; \int_{E^N}\left[\Psi(\nu)-\Psi(\mu)\right]\lambda^N(\mu, d\nu)\,, \qquad \mu \in E^N\,. \tag{5.13}$$

Lemma 5.5 *Suppose the assumptions of Theorem 2.1 are satisfied. Let* $\varphi \in C_b(\mathcal{Z})$ *and* Φ *be defined in (2.28). Then*

$$\mathbb{E} \sup_{t \le T} \left| M^N(\Phi, t \wedge \sigma_m^N) \right| \;\le\; 2 \left(T \sup_{\mu \in C_m^N} \int_{E^N} [\langle \varphi, \nu \rangle - \langle \varphi, \mu \rangle]^2 \, \lambda^N(\mu, d\nu) \right)^{1/2}, \tag{5.14}$$

for any $T \ge 0$ *and* $m \ge 1$.

Proof. The function (2.28) is continuous but, in general, unbounded. However, the stopped process reaches only a set, on which the function is bounded. Indeed, introduce the sets $\hat{C}_m^N$ of all $\nu \in E^N$ such that $\nu = J_0(\mu, \xi)$ or $\nu = J_r(\mu, i_1, \ldots, i_r, \xi)$, for some $\mu \in C_m^N$, pairwise distinct indices $i_1, \ldots, i_r$ from $\{1, \ldots, n\}$, $r = 1, \ldots, R$ and $\xi \in E_K$. Using

$$|\Phi(\mu)| \le \|\varphi\| \, \mu(E^N) \;\le\; \|\varphi\| \frac{\langle H, \mu \rangle}{\inf H}, \qquad \forall\, \mu \in E^N,$$

and (2.32), one obtains (cf. (2.2))

$$|\Phi(\nu)| \;\le\; \|\varphi\| \left(\frac{m}{\inf H} + \frac{K}{N} \right), \qquad \forall\, \nu \in C_m^N \cup \hat{C}_m^N. \tag{5.15}$$

Note that

$$P\left(X^N(t \wedge \sigma_m^N) \in C_m^N \cup \hat{C}_m^N, \quad \forall\, t \ge 0 \right) \;=\; 1.$$

Consequently, (5.15) implies

$$P\left(\Phi(X^N(t \wedge \sigma_m^N)) = \Phi_m(X^N(t \wedge \sigma_m^N)), \quad \forall\, t \ge 0 \right) \;=\; 1, \tag{5.16}$$

where

$$\Phi_m(\mu) \;:=\; \Phi(\mu) \wedge \|\varphi\| \left(\frac{m}{\inf H} + \frac{K}{N} \right), \qquad \mu \in E^N,$$

is a bounded function. Moreover, (2.26) and (5.15) imply, for $\mu \in C_m^N$,

$$\int_{E^N} [\Phi(\nu) - \Phi(\mu)]^k \, \lambda^N(\mu, d\nu) = \int_{E^N} [\Phi_m(\nu) - \Phi_m(\mu)]^k \, \lambda^N(\mu, d\nu), \qquad k = 1, 2. \tag{5.17}$$

It follows from (5.16) and (5.17), with $k = 1$, that

$$P\left(M^N(\Phi, t \wedge \sigma_m^N) = M^N(\Phi_m, t \wedge \sigma_m^N), \quad \forall\, t \ge 0 \right) \;=\; 1. \tag{5.18}$$

Finally, (5.14) follows from (5.9), (5.18) and (5.17), with $k = 2$. ∎

5.2. Relative compactness

Lemma 5.6 *Suppose assumption (2.15) is satisfied. Then (cf. (5.11))*

$$\sup_{\mu\in C_m^N}\left|\int_{E^N}[\langle\varphi,\nu\rangle-\langle\varphi,\mu\rangle]^k\,\lambda^N(\mu,d\nu)\right| \;\le\; c_2\,N^{1-k}\,(K+R)^k\,\|\varphi\|^k\sum_{r=0}^R m^r\,,$$

for any $\varphi\in C_b(\mathcal{Z})$, $k=1,2$, *and* $m\ge 1$.

Proof. Using (2.29), (2.33) and **assumption (2.15)**, one obtains

$$\left|\int_{E^N}[\langle\varphi,\nu\rangle-\langle\varphi,\mu\rangle]^k\,\lambda^N(\mu,d\nu)\right|\le$$
$$N^{1-k}\,(K+R)^k\,\|\varphi\|^k\left[q_0(E_K)+\sum_{r=1}^R\int_{\mathcal{Z}}\ldots\int_{\mathcal{Z}}q_r(x_1,\ldots,x_r,E_K)\,\mu(dx_1)\ldots\mu(dx_r)\right]$$
$$\le\; c_2\,N^{1-k}\,(K+R)^k\,\|\varphi\|^k\,\left[1+\langle H,\mu\rangle+\ldots+\langle H,\mu\rangle^R\right],$$

and the assertion follow from the definition (5.11). ■

Corollary 5.7 *Suppose the assumptions (2.6), (2.7), (2.8) and (2.15) are satisfied. Let* $\varphi\in C_b(\mathcal{Z})$ *and* Φ *be defined in (2.28). Then (cf. (5.10))*

$$\mathbb{E}\sup_{t\le T}\left|M^N(\Phi,t\wedge\sigma_m^N)\right| \;\le\; 2\,\|\varphi\|\,(K+R)\left(\frac{c_2\,T}{N}\sum_{r=0}^R m^r\right)^{1/2}, \tag{5.19}$$

for any $T\ge 0$ *and* $m\ge 1$.

Proof. Property (5.19) follows from **Lemma 5.5** and **Lemma 5.6**, with $k=2$. ■

Lemma 5.8 *Suppose the assumptions (2.6), (2.15) are satisfied, and the assumptions (2.7), (2.8) hold uniformly in* N. *Consider*

$$\varphi\in C(\mathcal{Z})\quad\textit{such that}\quad\frac{\varphi}{H}\in C_0(\mathcal{Z}). \tag{5.20}$$

Then, for any $T>0$ *and* $\varepsilon>0$,

$$\exists\;\Delta t,N_0>0\;:\quad\sup_{N\ge N_0}P\left(\sup_{|s-t|\le\Delta t,\,t\le T}\left|\langle\varphi,X^N(s)\rangle-\langle\varphi,X^N(t)\rangle\right|\ge\varepsilon\right)\;\le\;\varepsilon\,. \tag{5.21}$$

Proof. Consider $T>0$ and $\varepsilon>0$ fixed. According to **Lemma A.1**, the set

$$\Gamma\;=\;\left\{x\in\mathcal{Z}:\frac{|\varphi(x)|}{H(x)}\ge\frac{\varepsilon}{8\,m}\right\} \tag{5.22}$$

is compact, for any $m\ge 1$. Choosing ψ according to **Lemma A.2**, we find $\tilde\varphi=\varphi\,\psi\in C_c(\mathcal{Z})$ such that

$$\tilde\varphi(x)\;=\;\varphi(x)\,,\quad x\in\Gamma\qquad\text{and}\qquad|\tilde\varphi(x)|\;\le\;|\varphi(x)|\,,\quad x\in\mathcal{Z}\,. \tag{5.23}$$

Consider (cf. (5.10), (5.11))

$$0 \le s \le t < \sigma_m^N \tag{5.24}$$

so that $X^N(s) \in C_m^N$ and $\langle H, X^N(s)\rangle \le m$. Using (5.23) and (5.22), we obtain

$$\begin{aligned}
&\left|\langle \varphi, X^N(s)\rangle - \langle \varphi, X^N(t)\rangle\right| \\
&\quad \le \left|\langle \varphi - \tilde\varphi, X^N(s)\rangle\right| + \left|\langle \varphi - \tilde\varphi, X^N(t)\rangle\right| + \left|\langle \tilde\varphi, X^N(s)\rangle - \langle \tilde\varphi, X^N(t)\rangle\right| \\
&\quad \le \int_{\Gamma^c} \frac{2\,|\varphi(z)|}{H(z)}\, H(z)\, X^N(s,dz) + \int_{\Gamma^c} \frac{2\,|\varphi(z)|}{H(z)}\, H(z)\, X^N(t,dz) + \left|\langle \tilde\varphi, X^N(s)\rangle - \langle \tilde\varphi, X^N(t)\rangle\right| \\
&\quad \le \frac{\varepsilon}{2} + \left|\langle \tilde\varphi, X^N(s)\rangle - \langle \tilde\varphi, X^N(t)\rangle\right| .
\end{aligned} \tag{5.25}$$

Furthermore, setting $\tilde\Phi(\mu) = \langle \tilde\varphi, \mu\rangle$, we obtain (cf. (5.24), (5.12), (5.13))

$$\begin{aligned}
\left|\langle \tilde\varphi, X^N(s)\rangle - \langle \tilde\varphi, X^N(t)\rangle\right| &\le \left|M^N(\tilde\Phi, s) - M^N(\tilde\Phi, t)\right| + \int_s^t \left|\mathcal{A}^N \tilde\Phi(X^N(r))\right| dr \\
&\le \left|M^N(\tilde\Phi, s) - M^N(\tilde\Phi, t)\right| + c\,(t-s) ,
\end{aligned} \tag{5.26}$$

where $c = c(m, \varphi, K, R)$, according to **Lemma 5.6**, with $k = 1$. Choose m according to (5.8) such that

$$\inf_N P\left(\sigma_m^N > T+1\right) \;\ge\; 1 - \frac{\varepsilon}{2} , \tag{5.27}$$

and let $0 < \Delta t < \frac{\varepsilon}{4c} \wedge 1$. Using (5.25), (5.26) (which hold under condition (5.24)), (5.27), and Tschebyscheff's inequality, we obtain

$$\begin{aligned}
&P\left(\sup_{|s-t|\le \Delta t,\, t\le T} \left|\langle \varphi, X^N(s)\rangle - \langle \varphi, X^N(t)\rangle\right| \ge \varepsilon\right) \\
&\quad\le P\left(\sup_{|s-t|\le \Delta t,\, t\le T} \left|\langle \varphi, X^N(s)\rangle - \langle \varphi, X^N(t)\rangle\right| \ge \varepsilon\,,\ \sigma_m^N > T+1\right) + \frac{\varepsilon}{2} \\
&\quad\le P\left(\frac{3\,\varepsilon}{4} + \sup_{|s-t|\le \Delta t,\, t\le T} \left|M^N(\tilde\Phi, s) - M^N(\tilde\Phi, t)\right| \ge \varepsilon\,,\ \sigma_m^N > T+1\right) + \frac{\varepsilon}{2} \\
&\quad\le P\left(\sup_{t\le T+1} \left|M^N(\tilde\Phi, t)\right| \ge \frac{\varepsilon}{8},\ \sigma_m^N > T+1\right) + \frac{\varepsilon}{2} \le \frac{8}{\varepsilon}\,\mathbb{E}\sup_{t\le T+1}\left|M^N(\tilde\Phi, t\wedge\sigma_m^N)\right| + \frac{\varepsilon}{2} .
\end{aligned}$$

By **Corollary 5.7**, the mean value of the right-hand side becomes smaller than $\varepsilon^2/16$ for sufficiently large N and thus (5.21) is satisfied. ■

Lemma 5.9 *Let $\mathcal{Z}$ be a locally compact separable metric space. Consider*

$$H \in C(\mathcal{Z}) \quad \textit{such that} \quad H > 0 , \tag{5.28}$$

and h satisfying (2.14). Then the sets

$$C_\varepsilon \;=\; \{\mu \in \mathcal{M}(\mathcal{Z}) \,:\, \langle H, \mu\rangle \le \varepsilon\} , \qquad \varepsilon > 0 , \tag{5.29}$$

are compact subsets of the space $\mathcal{M}(\mathcal{Z}, H, h)$.

Proof. By [3, Corollary 31.3], the set $\tilde{C} = \{\nu \in \mathcal{M}(\mathcal{Z}) : \nu(\mathcal{Z}) \leq \varepsilon\}$ is vaguely compact. Consider the mapping (cf. (2.11))

$$T : (\tilde{C}, d_0) \to (C_\varepsilon, d_h), \qquad T(\nu)(B) = \int_B \frac{1}{H}\, d\nu, \quad B \in \mathcal{B}(\mathcal{Z}),$$

which is invertible and continuous. Indeed, according to **assumption (2.14)** and [3, Theorem 30.6], one obtains

$$\langle h, T(\nu_k)\rangle = \langle \frac{h}{H}, \nu_k\rangle \to \langle \frac{h}{H}, \nu\rangle = \langle h, T(\nu)\rangle,$$

if $\nu_k, \nu \in \tilde{C}$ are such that $d_0(\nu_k, \nu) \to 0$. Hence $C_\varepsilon = T(\tilde{C})$ is compact w.r.t. the topology generated by d_h. ∎

Remark 5.10 *Since $\mathcal{Z}$ is locally compact and separable, there is a countable subset $\{\psi_k\}_{k=1}^\infty$ of $C_c(\mathcal{Z})$ which is dense in $C_c(\mathcal{Z})$ w.r.t. uniform convergence (cf. [3, Lemma 31.4]). According to Lemma A.3 and Lemma A.2, choose compact and open sets satisfying (A.1), and localizing functions $e_m \in C_c(\mathcal{Z})$, $m \geq 1$, such that*

$$e_m(z) = 1,\ z \in \Gamma_m, \quad e_m(z) = 0,\ z \notin \Omega_m \quad \text{and} \quad 0 \leq e_m(z) \leq 1,\ z \in \mathcal{Z}. \tag{5.30}$$

Reorder the elements of the countable set

$$\{\psi_k : k \geq 1\} \cup \{\psi_k \cdot e_m : k, m \geq 1\} \cup \{e_m : m \geq 1\} \tag{5.31}$$

and denote them by $\{\varphi_k\}_{k=1}^\infty$. According to [3, Proof of Theorem 31.5], we introduce the metric

$$d_0(\mu, \nu) = \sum_{k=1}^\infty \frac{1}{2^k} \min\{1, |\langle \varphi_k, \mu\rangle - \langle \varphi_k, \nu\rangle|\}, \qquad \mu, \nu \in \mathcal{M}(\mathcal{Z}), \tag{5.32}$$

generating the vague topology.

Proof of Theorem 2.2. Note that **assumption (2.15)** implies (2.4) so that **Theorem 2.1** is applicable. To prove relative compactness of the sequence (X^N) we apply [13, Theorem 3.7.6] with $E = \mathcal{M}(\mathcal{Z}, H, h)$ (cf. **Remark 5.4**).

The first condition to be checked is the compact containment condition

$$\forall\, T, \varepsilon > 0 \quad \exists \text{ compact } C \subset E \quad : \quad \inf_N P\left(X^N(t) \in C,\ 0 \leq t \leq T\right) \geq 1 - \varepsilon. \tag{5.33}$$

Choose m according to (5.8) such that $\inf_N P\left(\sigma_m^N > T\right) \geq 1 - \varepsilon$. Note that the set C_m (cf. (5.29)) is compact, according to **Lemma 5.9**. One obtains (cf. (5.11), (5.10))

$$\inf_N P\left(X^N(t) \in C_m,\ 0 \leq t \leq T\right) = \inf_N P\left(\sigma_m^N > T\right) \geq 1 - \varepsilon,$$

i.e. (5.33) is satisfied.

The second condition to be checked is

$$\forall\, T, \varepsilon > 0 \quad \exists\, \delta > 0 \quad : \quad \sup_N P\left(w(X^N, \delta, T) \geq \varepsilon\right) \leq \varepsilon \tag{5.34}$$

where the modulus of continuity

$$w(\mu,\delta,T) \;=\; \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} d_h(\mu(s),\mu(t)) \tag{5.35}$$

is defined for $\delta, T > 0$ and $\mu \in D([0,\infty),E)$. Here $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \le t_n$ with $\min_{1\le i\le n}(t_i - t_{i-1}) > \delta$ and $n \ge 1$. Recall the definition (2.11) of the metric d_h and **Remark 5.10**. Let $T,\varepsilon > 0$ and choose $L \ge 0$ such that $\sum_{k=L+1}^{\infty} \frac{1}{2^k} \le \frac{\varepsilon}{2}$. With the notation $\varphi_0 = h$ (cf. (5.32)), we obtain

$$\begin{aligned}
&P\left(\sup_{|s-t|\le\Delta t,\, t\le T} d_h(X^N(s),X^N(t)) \ge \varepsilon\right)\\
&\le\; P\left(\sup_{|s-t|\le\Delta t,\, t\le T} \sum_{k=0}^{L} \left|\langle\varphi_k, X^N(s)\rangle - \langle\varphi_k, X^N(t)\rangle\right| \ge \frac{\varepsilon}{2}\right)\\
&\le\; \sum_{k=0}^{L} P\left(\sup_{|s-t|\le\Delta t,\, t\le T} \left|\langle\varphi_k, X^N(s)\rangle - \langle\varphi_k, X^N(t)\rangle\right| \ge \frac{\varepsilon}{2(L+1)}\right).
\end{aligned} \tag{5.36}$$

Taking into account that (5.20) is fulfilled for $\varphi = h$, according to **assumption (2.14)**, we apply **Lemma 5.8** to the right-hand side of (5.36). Thus, there are $\Delta t, N_0 > 0$ such that

$$\sup_{N\ge N_0} P\left(\sup_{|s-t|\le\Delta t,\, t\le T} d_h(X^N(s),X^N(t)) \ge \varepsilon\right) \;\le\; \varepsilon\,. \tag{5.37}$$

Since (cf. (5.35))

$$w(\mu,\delta,T) \;\le\; \sup_{|s-t|\le\Delta t, s\le T} d_h(\mu(s),\mu(t))\,, \qquad \delta < \Delta t\,,$$

we obtain from (5.37) that

$$\sup_{N\ge N_0} P\left(w(X^N,\delta,T)\ge\varepsilon\right) \le \varepsilon\,, \qquad \delta < \Delta t\,. \tag{5.38}$$

For any N there exists $\delta_N > 0$ such that $P\left(w(X^N,\delta_N,T)\ge\varepsilon\right) \le \varepsilon$, according to [13, Lemma 3.6.2(a)]. Thus, for $0 < \delta < \min\{\Delta t, \delta_1, \cdots, \delta_{N_0-1}\}$, (5.38) implies

$$\sup_N P\left(w(X^N,\delta,T)\ge\varepsilon\right) \;\le\; \varepsilon\,,$$

i.e. condition (5.34) is satisfied. ■

5.3. Characterization of weak limits

Lemma 5.11 *Consider μ, $\mu_n \in \mathcal{M}(\mathcal{Z},H)$ such that*

$$\lim_{n\to\infty} d_h(\mu_n,\mu) = 0\,. \tag{5.39}$$

Let $\mu_n^{(k)}$, $\mu^{(k)}$, $h^{(k)}$, $k = 1, 2, \dots$, *denote the k-fold products of* μ_n , μ , h , *respectively. Then*

$$\lim_{n\to\infty} \langle \psi, \mu_n^{(k)} \rangle \;=\; \langle \psi, \mu^{(k)} \rangle \,, \tag{5.40}$$

for any $\psi \in C(\mathcal{Z}^k)$ *such that*

$$|\psi(x)| \le c\, h^{(k)}(x)\,, \quad \forall\, x \in \mathcal{Z}^k\,, \qquad \textit{for some} \quad c > 0\,. \tag{5.41}$$

Proof. Define the measures ν_n , $\nu \in \mathcal{M}_b(\mathcal{Z})$ by

$$\nu_n(B) \;=\; \int_B h^{(k)}(x)\, \mu_n^{(k)}(dx)\,, \qquad \nu(B) \;=\; \int_B h^{(k)}(x)\, \mu^{(k)}(dx)\,, \qquad B \in \mathcal{B}(\mathcal{Z}^k)\,.$$

Since $\varphi\, h^{(k)} \in C_c(\mathcal{Z}^k)$ for every $\varphi \in C_c(\mathcal{Z}^k)$ and since $\mu_n^{(k)}$ converges vaguely to $\mu^{(k)}$, one obtains

$$\langle \varphi, \nu_n \rangle \;=\; \langle \varphi\, h^{(k)}, \mu_n^{(k)} \rangle \;\to\; \langle \varphi\, h^{(k)}, \mu^{(k)} \rangle \;=\; \langle \varphi, \nu \rangle\,,$$

i.e. ν_n converges vaguely to ν . Using (5.39) one obtains

$$\nu_n(\mathcal{Z}^k) \;=\; \langle h^{(k)}, \mu_n^{(k)} \rangle \;=\; [\langle h, \mu_n \rangle]^k \;\to\; \langle h^{(k)}, \mu^{(k)} \rangle \;=\; \nu(\mathcal{Z}^k)\,,$$

so that ν_n converges weakly to ν . Note that $\mathcal{Z}^k$ is a locally compact separable metric space, and therefore complete according to [2, Th. 44.1]. Using **Lemma A.6** we find, for any $\varepsilon > 0$, a compact $\Gamma \subset \mathcal{Z}^k$ satisfying

$$\nu_n(\Gamma^c) \;\le\; \frac{\varepsilon}{3\,c}\,, \quad n \ge 1 \qquad \text{and} \qquad \nu(\Gamma^c) \;\le\; \frac{\varepsilon}{3\,c}\,.$$

Choose $f \in C_c(\mathcal{Z}^k)$ according to **Lemma A.2** and consider $\psi \in C(\mathcal{Z}^k)$ satisfying (5.41). Then, for sufficiently large n , one obtains

$$\begin{aligned}
\left|\langle \psi, \mu_n^{(k)} \rangle - \langle \psi, \mu^{(k)} \rangle\right| &\le \left|\langle \psi(1-f), \mu_n^{(k)} \rangle - \langle \psi(1-f), \mu^{(k)} \rangle\right| + \left|\langle \psi f, \mu_n^{(k)} \rangle - \langle \psi f, \mu^{(k)} \rangle\right| \\
&\le c \int_{\Gamma^c} h^{(k)}(x)\, \mu_n^{(k)}(dx) + c \int_{\Gamma^c} h^{(k)}(x)\, \mu^{(k)}(dx) + \left|\langle \psi f, \mu_n^{(k)} \rangle - \langle \psi f, \mu^{(k)} \rangle\right| \le \varepsilon\,,
\end{aligned}$$

which proves (5.40). ∎

Lemma 5.12 *Suppose assumptions (2.17) and (2.18) are satisfied. Then the mapping*

$$M_\varphi : D([0,\infty), \mathcal{M}(\mathcal{Z}, H, h)) \to D([0,\infty), \mathbb{R})$$

defined by (cf. (2.20))

$$M_\varphi(\mu)(t) \;=\; \langle \varphi, \mu(t) \rangle - \langle \varphi, \mu(0) \rangle - \int_0^t \mathcal{G}(\varphi, \mu(s))\, ds\,, \qquad t \ge 0\,, \tag{5.42}$$

is continuous, for any $\varphi \in C_c(\mathcal{Z})$.

Proof. According to **Lemma A.5**, the mapping

$$F_1 \,:\, D([0,\infty),\mathcal{M}(\mathcal{Z},H,h)) \to D([0,\infty),\mathbb{R})\,, \qquad F_1(\mu)(t) = \langle \varphi, \mu(0)\rangle$$

is continuous. In view of **Lemma A.4**, and since (cf. [13, p.153, (11.10)]) the mapping

$$F_2 \,:\, D([0,\infty),\mathbb{R}) \to D([0,\infty),\mathbb{R})\,, \qquad F_2(\xi)(t) \;=\; \int_0^t \xi(s)\,ds$$

is continuous, it remains to show that the mappings $f_1(\nu) = \langle\varphi,\nu\rangle$ and $f_2(\nu) = \mathcal{G}(\varphi,\nu)$ from $\mathcal{M}(\mathcal{Z},H,h)$ into $\mathbb{R}$ are continuous. For f_1, this is obvious, since convergence in $\mathcal{M}(\mathcal{Z},H,h)$ implies vague convergence. Using (2.34), continuity of f_2 follows from **assumptions (2.17), (2.18)** and **Lemma 5.11**. ■

Lemma 5.13 *Suppose assumption (2.17) is satisfied. Let $\psi_n,\psi \in C_c(\mathcal{Z})$ be such that $\lim_{n\to\infty} \|\psi - \psi_n\| = 0$ and*

$$\{x\in\mathcal{Z} : |\psi_n(x)| > 0\} \subset \Gamma\,,\; n\geq 1\,, \tag{5.43}$$

for some compact $\Gamma\subset\mathcal{Z}$. Then (cf. (5.42))

$$\lim_{n\to\infty} M_{\psi_n}(\mu)(t) \;=\; M_{\psi}(\mu)(t)\,, \qquad \forall\, t\geq 0\,,$$

for any $\mu\in D([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$.

Proof. Note that

$$\sup_n \|\psi_n\| < \infty \quad\text{and}\quad \langle\psi_n,\xi\rangle \to \langle\psi,\xi\rangle\,,\; \forall\,\xi\in E_K\,. \tag{5.44}$$

The dominated convergence theorem implies, using (2.32) and (5.44),

$$Q_0(\psi_n) \;\to\; Q_0(\psi)\,, \quad Q_r(\psi_n,x) \;\to\; Q_r(\psi,x)\,, \quad \forall\, x\in\mathcal{Z}^r\,, \quad r=1,\ldots,R\,,$$

and, using (2.34) and **assumption (2.17)**,

$$\lim_{n\to\infty} \mathcal{G}(\psi_n,\mu) \;=\; \mathcal{G}(\psi,\mu)\,, \qquad \forall\,\mu\in\mathcal{M}(\mathcal{Z},H)\,. \tag{5.45}$$

Using (5.43), one obtains

$$\lim_{n\to\infty} \langle\psi_n,\mu\rangle \;=\; \langle\psi,\mu\rangle\,, \qquad \forall\,\mu\in\mathcal{M}(\mathcal{Z},H)\,. \tag{5.46}$$

Moreover, it follows from (2.34) and **assumption (2.17)** that (cf. (2.20))

$$\sup_n \sup_{s\leq t} |\mathcal{G}(\psi_n,\mu(s))| \;\leq\; c_2\,(K+R)\,\sup_n\|\psi_n\|\,\sup_{s\leq t}\left[1+\sum_{r=1}^R \langle h,\mu(s)\rangle^r\right] \;<\; \infty\,, \tag{5.47}$$

for any $\mu\in D([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$. Using (5.45), (5.46) and (5.47), a further application of the dominated convergence theorem completes the proof. ■

Lemma 5.14 *Suppose the assumptions (2.14) and (2.17) are satisfied. Let $\varphi \in C_b(\mathcal{Z})$ and Φ be defined in (2.28). Then (cf. (5.13), (2.20))*

$$\lim_{N\to\infty} \sup_{\mu \in C_m^N} \left|\mathcal{A}^N \Phi(\mu) - \mathcal{G}(\varphi,\mu)\right| = 0\,, \qquad \forall\, m \geq 1\,. \tag{5.48}$$

Proof. Using (5.13), (2.29), (2.30), (2.20), one obtains

$$\mathcal{A}^N \Phi(\mu) = \mathcal{G}(\varphi,\mu) - \sum_{r=2}^{R} \frac{1}{N^r} \hat{\sum_{1\leq i_1,\ldots,i_r\leq n}} Q_r(\varphi, x_{i_1}, \ldots, x_{i_r})\,, \tag{5.49}$$

where $\mu \in E^N$ and $\hat{\sum}$ denotes summation over those indices, at least two of which are equal. According to (2.34) and **assumption (2.17)**, (5.49) implies

$$\begin{aligned} \left|\mathcal{A}^N \Phi(\mu) - \mathcal{G}(\varphi,\mu)\right| &\leq \sum_{r=2}^{R} \frac{c_2\,(K+r)\,\|\varphi\|}{N^r} \hat{\sum_{1\leq i_1,\ldots,i_r\leq n}} h(x_{i_1})\ldots h(x_{i_r}) \\ &\leq \sum_{r=2}^{R} \frac{c_2\,(K+r)\,\|\varphi\|\, r\,(r-1)}{2N} \langle h^2,\mu\rangle \,\langle h,\mu\rangle^{r-2} \end{aligned}$$

and

$$\sup_{\mu\in C_m^N} \left|\mathcal{A}^N \Phi(\mu) - \mathcal{G}(\varphi,\mu)\right| \leq \frac{1}{N} \sup_{\mu\in C_m^N} \langle h^2,\mu\rangle\, \frac{c_2\,\|\varphi\|}{2} \sum_{r=2}^{R} (K+r)\, r\,(r-1)\,(c\,m)^{r-2}\,. \tag{5.50}$$

According to **assumption (2.14)** and **Lemma A.1**, the set

$$\Gamma(\varepsilon) = \left\{ x \in \mathcal{Z} : \frac{h(x)}{H(x)} \geq \varepsilon \right\}$$

is compact, for any $\varepsilon > 0$. Using (5.4), one obtains

$$\begin{aligned} \langle h^2,\mu\rangle &= \int_{\Gamma(\varepsilon)} h^2(x)\,\mu(dx) + \int_{\Gamma(\varepsilon)^c} h^2(x)\,\mu(dx) \leq m \sup_{x\in\Gamma(\varepsilon)} \frac{h^2(x)}{H(x)} + \varepsilon^2\,\frac{1}{N}\sum_{i=1}^{n} H^2(x_i) \\ &\leq m \sup_{x\in\Gamma(\varepsilon)} \frac{h^2(x)}{H(x)} + \varepsilon^2\, m^2\, N\,, \end{aligned} \tag{5.51}$$

for any $\mu \in C_m^N$. Finally, (5.50) and (5.51) imply (5.48). ■

Proof of Theorem 2.3. Using (5.42) and (5.12), one obtains

$$M_\varphi(X^N)(t) = M^N(\Phi,t) + \int_0^t \left[\mathcal{A}^N \Phi(X^N(s)) - \mathcal{G}(\varphi, X^N(s))\right] ds\,. \tag{5.52}$$

For $\varepsilon > 0$ and $t \geq 0$, choose $m \geq 1$ according to (5.8) such that

$$\inf_N P\left(\sigma_m^N > t\right) \geq 1 - \frac{\varepsilon}{2}\,. \tag{5.53}$$

By (5.53), (5.52), **Lemma 5.14**, and Tschebyscheff's inequality, one obtains

$$\begin{aligned} P\left(\sup_{s\le t}\left|M_\varphi(X^N)(s)\right| \ge \varepsilon\right) &\le P\left(\sup_{s\le t}\left|M_\varphi(X^N)(s)\right| \ge \varepsilon\,,\ \sigma_m^N > t\right) + \frac{\varepsilon}{2} \\ \le\ \ & P\left(\frac{\varepsilon}{2} + \sup_{s\le t}\left|M^N(\Phi,s)\right| \ge \varepsilon\,,\ \sigma_m^N > t\right) + \frac{\varepsilon}{2} \qquad (5.54) \\ \le\ \ & P\left(\sup_{s\le t}\left|M^N(\Phi, s\wedge\sigma_m^N)\right| \ge \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \ \le\ \frac{2}{\varepsilon}\,\mathbb{E}\sup_{s\le t}\left|M^N(\Phi, s\wedge\sigma_m^N)\right| + \frac{\varepsilon}{2}\,, \end{aligned}$$

for sufficiently large N . By **Corollary 5.7**, the right-hand side of (5.54) becomes smaller than ε for sufficiently large N, i.e.

$$\limsup_N P\left(\sup_{s\le t}\left|M_\varphi(X^N)(s)\right| \ge \varepsilon\right) \le \varepsilon\,, \qquad \forall\,\varepsilon > 0\,,\, t > 0\,.$$

This implies

$$\sup_{s\le t}\left|M_\varphi(X^N)(s)\right| \ \Rightarrow\ 0\,, \qquad \forall\, t > 0\,,$$

and, recalling the definition of the Skorohod metric d ([13, p.117]),

$$d(M_\varphi(X^N), 0) \ \Rightarrow\ 0\,. \qquad (5.55)$$

Suppose $X^{N_l} \Rightarrow X$ for some subsequence N_l . According to **Lemma 5.12**, the mapping M_φ is continuous so that $M_\varphi(X^{N_l}) \Rightarrow M_\varphi(X)$ and $d(M_\varphi(X^{N_l}),0) \Rightarrow d(M_\varphi(X),0)$. Thus, (5.55) implies

$$M_\varphi(X) \ = \ 0\,, \qquad \text{a.e.}\,, \qquad \text{for any fixed } \varphi\,.$$

Using **Remark 5.10**, **Lemma 5.13** and right-continuity of the trajectories, one obtains

$$M_\varphi(X) \ = \ 0\,, \qquad \forall\,\varphi \in C_c(\mathcal{Z})\,, \qquad \text{a.e.}\,. \qquad (5.56)$$

Moreover, it follows from **Lemma A.5** that $X^{N_l}(0) \Rightarrow X(0)$ so that **assumption (2.16)** implies

$$X(0) \ = \ \mu_0\,, \qquad \text{a.e.}\,. \qquad (5.57)$$

According to (5.56), (5.57), X satisfies (2.19) almost everywhere. Note that, by **Theorem 2.2**, the sequence of $D([0,\infty),\mathcal{M}(\mathcal{Z},H,h))$-valued random variables X^N is relatively compact. ■

5.4. Corollaries

Proof of Corollary 2.4. Note that

$$\sup_{t\le T} d_h(X^N(t), X^N(t-)) \ \le \ \sup_{|s-t|\le\Delta t,\, t\le T} d_h(X^N(s), X^N(t))\,, \qquad \forall\,\Delta t > 0\,.$$

Thus, (5.37) implies

$$\sup_{N \geq N_0} P\left(\sup_{t \leq T} d_h(X^N(t), X^N(t-)) \geq \varepsilon\right) \;\leq\; \varepsilon\,,$$

so that

$$\sup_{t \leq T} d_h(X^N(t), X^N(t-)) \;\Rightarrow\; 0\,, \qquad T > 0\,,$$

as $N \to \infty$. An application of [13, Theorem 3.10.2(a)] gives (2.21), for every weak limit X of the sequence (X^N). ■

Proof of Corollary 2.5. Suppose $X^{N_l} \Rightarrow X$ for some subsequence N_l. Then **Corollary 2.4** and [13, Prop. 3.5.2] imply

$$X^{N_l}(t) \Rightarrow X(t)\,, \qquad \forall\, t \geq 0\,. \tag{5.58}$$

For any $\psi \in C_c(\mathcal{Z})$, the mapping $\Psi(\mu) = \langle \psi, \mu \rangle$ from $\mathcal{M}(\mathcal{Z}, H, h)$ into $\mathbb{R}$ is continuous, and

$$\Psi(X^{N_l}(t)) \Rightarrow \Psi(X(t))\,, \qquad \forall\, t \geq 0\,, \tag{5.59}$$

as a consequence of (5.58).

Consider a sequence of localizing functions $e_k \in C_c(\mathcal{Z})$ satisfying (5.30). Then (5.59), with $\psi = H e_k$, implies

$$\langle H e_k, X^{N_l}(t) \rangle \Rightarrow \langle H e_k, X(t) \rangle\,, \qquad \forall\, t \geq 0\,, \quad k = 1, 2, \ldots\,. \tag{5.60}$$

Since, according to (5.7),

$$\mathbb{E}\langle H e_k, X^N(t) \rangle \;\leq\; \mathbb{E}\langle H, X^N(t) \rangle \leq (c_0 + c_1')\exp(c_1\, t)\,, \qquad \forall\, N\,,$$

Fatou's lemma and (5.60) imply

$$\mathbb{E}\langle H e_k, X(t) \rangle \;\leq\; (c_0 + c_1')\exp(c_1\, t)\,, \qquad \forall\, t \geq 0\,, \quad k = 1, 2, \ldots\,.$$

Thus, (2.22) follows from the monotone convergence theorem. ■

Proof of Corollary 2.6. Consider $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$ and note that $\mu_0(\mathcal{Z}) < \infty$, according to **assumption (2.6)** and (2.13). Let $y_1, y_2, \ldots$ be i.i.d. random variables with distribution $\frac{1}{\mu_0(\mathcal{Z})}\, \mu_0(dx)$, and

$$\lim_{N \to \infty} \frac{n_N}{N} = \mu_0(\mathcal{Z})\,. \tag{5.61}$$

Then

$$Y^N = \frac{1}{N} \sum_{i=1}^{n_N} \delta_{y_i} \in E^N \tag{5.62}$$

and, by (5.61) and the law of large numbers,

$$\langle \varphi, Y^N \rangle = \frac{n_N}{N} \frac{1}{n_N} \sum_{i=1}^{n_N} \varphi(y_i) \quad \to \quad \mu_0(\mathcal{Z}) \int_{\mathcal{Z}} \varphi(x) \, \frac{1}{\mu_0(\mathcal{Z})} \, \mu_0(dx) = \langle \varphi, \mu_0 \rangle \quad \text{a.e.}, \tag{5.63}$$

for all nonnegative $\varphi \in M(\mathcal{Z})$ such that $\langle \varphi, \mu_0 \rangle < \infty$. Thus, $\lim_{N\to\infty} d_h(Y^N, \mu_0) = 0$ a.e., and $Y^N \Rightarrow \mu_0$. Moreover,

$$\int_{E^N} \langle H, \mu \rangle \, \nu_0^N(d\mu) = \mathbb{E} \langle H, Y^N \rangle = \frac{n_N}{N} \int_{\mathcal{Z}} H(x) \, \frac{1}{\mu_0(\mathcal{Z})} \, \mu_0(dx) \,, \tag{5.64}$$

where $\nu_0^N \in \mathcal{P}(E^N)$ denotes the distribution of Y^N. Thus, ν_0^N satisfies assumptions (2.7) and (2.16). By **Theorem 2.3** and **Corollary 2.4** there is at least one $\mu \in C([0,\infty), \mathcal{M}(\mathcal{Z}, H, h))$ satisfying equation (2.23). ∎

Proof of Corollary 2.7. By **Theorem 2.3** any weak limit is concentrated on the set of solutions, which now consists only of one element. Thus, all weak limits are the same, and the assertion follows. ∎

Proof of Corollary 2.8. Assumption (2.25) assures that the processes remain in the restricted space, once they have started there. According to **Remark 2.10**, the mapping $\langle g, \mu \rangle$ is continuous so that the subset $E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$ is closed in E^N. According to **Lemma A.7**, the subset $\mathcal{M}(\mathcal{Z}, H) \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$ is closed in $\mathcal{M}(\mathcal{Z}, H, h)$. Thus, all statements about compact sets remain true for the restricted spaces. ∎

Proof of Corollary 2.9. Define (cf. (5.62))

$$\tilde{Y}^N(\omega) \quad = \quad \begin{cases} Y^N(\omega)\,, & \text{if } \langle g, Y^N \rangle \le \langle g, \mu_0 \rangle + 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and let $\tilde{\nu}_0^N$ denote the distribution of $\tilde{Y}^N$. By definition, one obtains $\langle g, \tilde{Y}^N \rangle \le \langle g, \mu_0 \rangle + 1$ and $\mathbb{E}\langle H, \tilde{Y}^N \rangle \le \mathbb{E}\langle H, Y^N \rangle$. This implies $\tilde{\nu}_0^N \in \mathcal{P}(E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g))$, with $\gamma = \langle g, \mu_0 \rangle + 1$, and (2.7), according to (5.64). Moreover, it follows from (5.63) that $\langle g, Y^N \rangle \to \langle g, \mu_0 \rangle$ a.e., so that (5.63) holds for $\tilde{Y}^N$. Consequently, $\tilde{Y}^N \Rightarrow \mu_0$, i.e. (2.16) is fulfilled, and the assertion follows from **Corollaries 2.8** and **2.4**. ∎

Appendix

Lemma A.1 *(cf. [3, Theorem 27.6]) Let E be a locally compact space. Then $\Psi \in C_0(E)$ iff*

$$\Psi \in C(E) \quad \textit{and} \quad \{\xi \in E : |\Psi(\xi)| \ge \varepsilon\} \textit{ is compact for every } \varepsilon > 0\,.$$

Lemma A.2 *(cf. [3, Corollary 27.3]) Let E be a locally compact space and C and O be compact and open subsets such that $C \subset O$. Then there is a $\Psi \in C_c(E)$ such that*

$$\Psi(\xi) = 1,\ \xi \in C\,, \quad \Psi(\xi) = 0,\ \xi \notin O \quad \textit{and} \quad 0 \le \Psi(\xi) \le 1,\ \xi \in E\,.$$

Lemma A.3 *Let $\mathcal{Z}$ be a locally compact separable metric space. Then there are compact and open subsets Γ_m and Ω_m such that*

$$\mathcal{Z} = \bigcup_{m=1}^{\infty} \Gamma_m \qquad \textit{and} \qquad \Gamma_m \subset \Omega_m \subset \Gamma_{m+1}\,, \qquad m \geq 1\,. \tag{A.1}$$

Proof. Since $\mathcal{Z}$ is σ-compact, the statement is given by [3, Lemma 29.8]. ∎

Lemma A.4 *(cf. [13, p.151]) Let E, E_1 be metric spaces. If $f : E \to E_1$ is continuous, then the mapping*

$$F \,:\, D([0,\infty),E) \to D([0,\infty),E_1)\,, \qquad F(\xi)(t) \;=\; f(\xi(t))\,, \quad t \geq 0\,,$$

is continuous.

Lemma A.5 *Let E be a metric space. If $\lim_{n\to\infty} \mu_n = \mu$ in $D([0,\infty),E)$ then $\lim_{n\to\infty} \mu_n(0) = \mu(0)$ in E.*

Proof. The assertion follows from [13, Ch. 3, Prop. 5.2], since 0 is a continuity point for any $\mu \in D([0,\infty),E)$. ∎

Lemma A.6 *Let E be a complete separable metric space, and $\mu\,,\ \mu_n \in \mathcal{M}_b(E)$ such that $\mu_n \xrightarrow{w} \mu$. Then, for each $\varepsilon > 0$, there exists a compact K_ε such that*

$$\mu_n(E \setminus K_\varepsilon) \leq \varepsilon\,, \quad \forall\, n\,, \qquad \mu(E \setminus K_\varepsilon) \leq \varepsilon\,.$$

Proof. Introduce the measures

$$\nu_n(B) = \begin{cases} \frac{1}{\mu_n(E)}\,\mu_n(B)\,, & \text{if} \quad \mu_n(E) > 0\,, \\ \quad 0 \qquad , & \text{otherwise}\,, \end{cases} \qquad \nu(B) = \begin{cases} \frac{1}{\mu(E)}\,\mu(B)\,, & \text{if} \quad \mu(E) > 0\,, \\ \quad 0 \qquad , & \text{otherwise}\,. \end{cases}$$

Note that, if $\mu(E) = 0$ then $\mu_n(E) \leq \varepsilon$, for all n except a finite number. For those one finds the corresponding compact. If $\mu(E) > 0$ then $\nu_n \to \nu$ weakly, and the statement follows from Prohorov's theorem and the boundedness of $\mu_n(E)$. ∎

Lemma A.7 *Let $\mathcal{Z}$ be a locally compact space, and $\mu\,,\ \mu_n \in \mathcal{M}(\mathcal{Z})$ such that $\mu_n \xrightarrow{v} \mu$. Then*

$$\langle H, \mu \rangle \;\leq\; \liminf_{n\to\infty}\, \langle H, \mu_n \rangle\,, \qquad \textit{for any nonnegative } H \in C(\mathcal{Z})\,.$$

Proof. Note that $\nu_n \xrightarrow{v} \nu$, where

$$\nu_n(B) = \int_B H(x)\,\mu_n(dx)\,, \qquad \nu(B) = \int_B H(x)\,\mu(dx)\,, \qquad B \in \mathcal{B}(\mathcal{Z})\,.$$

Thus, the assertion follows from [3, Lemma 30.3]. ∎

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