Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

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submitted: 29th November 1993

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Preprint No. 73
Berlin 1993

Key words and phrases. Stochastic PDE, Kolmogorov equation, travelling wave.

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Abstract

We consider the one-dimensional Kolmogorov equation driven by a particular space-time white noise term and show that there exist stochastic wavelike solutions which travel with a linear limiting speed.

Keywords: Stochastic PDE, Kolmogorov equation, travelling wave.

1 Introduction

We consider the one-dimensional stochastic partial differential equation,

$$u_t = u_{xx} + \theta u - u^2 + |u|^{1/2}\dot{W}. \tag{1}$$

Here W is a space-time white noise and $\theta \geq 0$ is a parameter measuring the mass creation rate. Without the white noise term the equation is the well studied Kolmogorov equation which has a family of non-negative travelling wave solutions $u(t,x) = w(x - At)$ with speeds $A \geq 2\theta^{1/2}$ (see Bramson [2]). The form of the noise term in (1) arises from particle branching in a particle approximation. The same noise term appears in the stochastic PDE which describes the density of one-dimensional super Brownian motion (see Konno and Shiga [8]). Perkins [11] has shown that equation (1) arises as the high density limit of particle systems which undergo branching random walks and have an extra death mechanism due to overcrowding (see also Mueller and Tribe [10]). The existence and uniqueness of solutions to (1) for which $u(0,x)$ is integrable is proved in Evans and Perkins [6]. In section 2 we give the proof of existence and uniqueness for solutions to (1) with infinite initial mass (satisfying certain growth conditions).

Define

$$R_0(u(t)) := \sup(x : u(t,x) > 0).$$

In contrast to the deterministic equation, if u is a solution to (1) for which $R_0(u(0)) < \infty$ then $R_0(u(t)) < \infty$ for all $t \geq 0$. Thus $R_0(u(t))$ is a natural marker for the front of a travelling wave. The preservation of this compact support property has been studied for stochastic P.D.E's in [9], [12] and will follow for our equation from an argument of Iscoe [7].

We look for solutions u to (1) which have the following properties:

$$\begin{aligned} &i. \quad R_0(u(t)) \in (-\infty, \infty) \ \text{ for all } t \geq 0, \\ &ii. \quad u(t, \cdot - R_0(u(t))) \ \text{ is a stationary process in time.} \end{aligned}$$

We call such a solution a travelling wave.

The behavior of solutions to (1) started from initial conditions with compact support is studied in Mueller and Tribe [10]. It is shown that there is a critical value $\theta_c > 0$ below which all solutions die in finite time and above which solutions have non-zero probability of surviving forever. The main result of this paper is in section 3 where we show that for $\theta > \theta_c$ there exist travelling wave solutions to (1). We do not prove the uniqueness of the travelling wave or investigate convergence to the travelling wave. Travelling waves have been constructed for discrete time interacting particle systems. To construct our solutions we follow the proof in Durrett [4] where a travelling wave for oriented percolation in one dimension is studied.

In section 4 we show (using the ergodic theorem) that any travelling wave solution has an asymptotic wave speed

$$R_0(u(t))/t \stackrel{a.s.}{\to} A \in [-\infty, 2\theta^{1/2}] \quad \text{as} \quad t \to \infty.$$

Note that the wave speed is no faster than the slowest corresponding deterministic wave speed. If $\theta > \theta_c$ then the (possibly random) wavespeed is non-negative. By using the coupling of solutions to (1) with an oriented percolation process established in [10] we show that, as $\theta \to \infty$, the normalised wavespeed $A/\theta^{1/2}$ approaches its largest possible value 2. We do not prove that the wavespeed is constant or that it is strictly increasing in θ.

In the remainder of this section we give the notation for the spaces on which our solutions will live and state tightness criteria.

Notation. Let $\phi_\lambda(x) = \exp(-\lambda|x|)$. For continuous functions $f : \mathrm{R} \to \mathrm{R}$ let $||f||_\lambda = \sup(|f(x)\phi_\lambda(x)| : x \in \mathrm{R})$. Let $C_\lambda^+ = (f \geq 0 : |f(x)\phi_\lambda(x)| \to 0$ as $x \to \pm\infty)$. $C_{\mathrm{tem}}^+ = \bigcap_{\lambda > 0} C_\lambda^+$ is the space of continuous functions with slower than exponential growth. C_λ has the topology given by the norm $|| \cdot ||_\lambda$ and C_{tem}^+ that given by the family $(|| \cdot ||_\lambda, \lambda > 0)$.

Let $\Phi = (f : ||f||_\lambda < \infty$ for some $\lambda < 0)$ be the space of functions with exponential decay. We write (f, g) for the integral $\int f(x)g(x)dx$ whenever this is defined (i.e. when $f \in C_{\mathrm{tem}}^+, g \in \Phi$). Let $(C([0,\infty), C_{\mathrm{tem}}^+), \mathcal{U}, \mathcal{U}_t, U(t))$ be continuous path space, the canonical right continuous filtration and the coordinate variables. Finally C_c^∞ is the space of infinitely differentiable functions on R with compact support.

We state the Arzela Ascoli theorem and Kolmogorov tightness criterion for the spaces $C_\lambda^+, C_{\text{tem}}^+$. $K \subseteq C_\lambda^+$ is relatively compact if and only if

i. $(f : f \in K)$ are equicontinuous on compacts.

ii. $\lim_{R\to\infty} \sup_{f\in K} \sup_{|x|\geq R} |f(x)\phi_\lambda(x)| = 0$.

$K \subseteq C_{\text{tem}}^+$ is (relatively) compact if and only if it is (relatively) compact in C_λ^+ for all $\lambda > 0$. For $C < \infty, \gamma, \delta > 0, \mu < \lambda$ define

$$K(C,\delta,\gamma,\mu) = (f : |f(x) - f(x')| \leq C|x - x'|^\gamma e^{\mu|x|} \text{ for all } |x - x'| \leq \delta).$$

Then using the above conditions one can show that

$$K(C,\delta,\gamma,\mu) \cap (f : (f,\phi_1) \leq a) \text{ is compact in } C_\lambda^+.$$

If $X_n(\cdot)$ are C_λ^+ valued processes, with (X_n, ϕ_1) tight and with $C_0 < \infty, p > 0, \gamma > 1, \mu < \lambda$ such that for all $n \geq 1$

$$E(|X_n(x) - X_n(x')|^p) \leq C_0|x - x'|^\gamma e^{\mu p|x|} \text{ for all } |x - y| \leq 1 \tag{2}$$

then (X_n) are tight. Indeed if (2) holds and $\bar\gamma < (\gamma - 1)/p, \mu < \bar\mu < \lambda$ then there exist deterministic $C = C(\bar\gamma, \bar\mu) < \infty, \rho = \rho(\gamma, \bar\gamma, p) > 0$ and random $\delta(\omega)$ such that

$$X_n(\omega) \in K(C,\delta,\bar\gamma,\bar\mu) \text{ and } E(\delta^{-\rho}) \leq \text{Constant}(C_0, \mu, \bar\mu, \gamma, \bar\gamma, p) < \infty.$$

Similarly if $X_n(\cdot,\cdot)$ are $C([0,T], C_\lambda^+)$ valued, $(X_n(0), \phi_1)$ are tight and there are $C_0 < \infty, p > 0, \gamma > 2, \mu < \lambda$ such that for all $n \geq 1$

$$\begin{aligned} &E(|X_n(x,t) - X_n(x',t')|^p) \leq C_0(|x - x'|^\gamma + |t - t'|^\gamma)e^{\mu p|x|} \\ &\text{for all } |x - y| \leq 1, |t - t'| \leq 1, t, t' \in [0,T] \end{aligned} \tag{3}$$

then (X_n) are tight. C_{tem}^+ (respectively $C([0,\infty), C_{\text{tem}}^+)$ valued processes (X_n) are tight if and only if they are tight as C_λ^+ (respectively $C([0,\infty), C_\lambda^+)$) processes for each $\lambda > 0$.

2 Existence and uniqueness of solutions

To prove existence and uniqueness for (1) we consider a more general equation. Choose $\alpha, \beta, \gamma \in C([0,\infty), C_{\text{tem}}^+)$. In the following equation we may interpret α as the immigration rate, $\theta - \beta$ as the mass creation-annihilation rate and γ as the overcrowding rate.

$$u_t = u_{xx} + \alpha + \theta u - \beta u - \gamma u^2 + |u|^{1/2}\dot W. \tag{4}$$

A solution to (4) consists of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, an adapted white noise W and an adapted continuous C^+_{tem} valued process $u(t)$ such that for all $\phi \in C_c^\infty$

$$\begin{aligned}(u(t), \phi) &= (u(0), \phi) + \int_0^t (u(s), \phi_{xx} + (\theta - \beta(s) - \gamma(s)u(s))\phi)ds \qquad (5)\\ &\quad + \int_0^t (\alpha(s), \phi)ds + \int_0^t \int |u(s,x)|^{1/2}\phi(x)dW_{x,s}.\end{aligned}$$

If in addition $P(u(0,x) = f(x)) = 1$ then we say the solution u starts at f. Let u be any solution to (4) started at f. Approximating ϕ_λ with functions in C_c^∞ and taking expectations carefully in (5) we may derive the bound

$$E(\sup_{t \leq T}(u(t), \phi_\lambda)^p) < \infty \quad \text{for all } T, p, \lambda > 0,\ f \in C^+_{\text{tem}}. \qquad (6)$$

Using this bound we may extend (5) to test functions $\psi(t,x)$ so that if ψ has one continuous derivative in t and two continuous derivatives in x and

$$\sup_{t \in [0,T]} |\psi(t,\cdot)| \wedge |\psi_t(t,\cdot)| \wedge |\psi_{xx}(t,\cdot)| \in \Phi \qquad (7)$$

then for $t \in [0,T]$

$$\begin{aligned}&(u(t), \psi(t))\\ &= (f, \psi(0)) + \int_0^t (u(s), \psi_t + \psi_{xx}(s) + (\theta - \beta(s) - \gamma(s)u(s))\psi(s))ds\\ &\quad + \int_0^t (\alpha(s), \psi(s))ds + \int_0^t \int |u(s,x)|^{1/2}\psi(s,x)dW_{x,s}. \qquad (8)\end{aligned}$$

Let $p_t(x) = (4\pi t)^{-1/2}\exp(-x^2/4t)$ and $p_t^\theta(x) = e^{\theta t}p_t(x)$. Write P_t, P_t^θ for the semigroups generated by convolution with these functions. Applying (8) with $\psi(s,x) = p^\theta_{t-s+\delta}(x)$ for $s \leq t$ and then letting $\delta \to 0+$ we obtain the Green's function representation

$$\begin{aligned}u(t,x) &= (f, p_t^\theta(x - \cdot)) - \int_0^t (u(s), p^\theta_{t-s}(x-\cdot)(\beta(s) + \gamma(s)u(s)))ds\\ &\quad + \int_0^t (\alpha(s), p^\theta_{t-s}(x - \cdot))ds + \int_0^t \int |u(s,y)|^{1/2} p^\theta_{t-s}(x-y)dW_{y,s}.\end{aligned}$$

The following lemma estimates how fast the support of solutions to (4) can spread. It is a similar estimate to Dawson, Iscoe and Perkins [3] theorem 3.3.

Lemma 2.1 *Let u be a solution to (4) started at f. Suppose for some $R > 0$ that f and $(\alpha(s,\cdot) : s \le t)$ are supported outside $(-R-2, R+2)$. Then for $t \ge 1$*

$$\begin{aligned} & P\Big(\int_0^t \int_{-R}^R u(s,x)dx\,ds > 0\Big) \\ \le\ & 48(1+\theta)e^{\theta t} \int \exp(-(|x|-(R+1))^2/4t)(f(x) + \int_0^t \alpha(s,x)ds)dx. \end{aligned}$$

If f and $(\alpha(s,\cdot) : s \le t)$ are supported outside $(R-2,\infty)$ then the same bound holds for $P(\int_0^t \int_R^\infty u(s,x)dx\,ds > 0)$.

Proof. We indicate the changes necessary in the proof of [3] theorem 3.3 to prove the first part of our lemma. The second part is proved in the same way. Let $0 \le \phi_0 \in C_c^\infty$ satisfy $(x : \phi_0(x) > 0) = (-R, R)$. For $\lambda \ge 0$ let $\xi(\lambda, t, x)$ be the unique non-negative solution to

$$\xi_t = \xi_{xx} + \theta\xi - (1/2)\xi^2 + \lambda\phi_0, \quad \xi(0,x) = 0.$$

Then ξ satisfies (7). Apply (8) with $\psi(s,x) = \xi(t-s,x)$ for $s \in [0,t]$. Using Ito's formula to expand

$$\zeta_s := \exp(-(u(s),\xi(\lambda,t-s)) - \lambda\int_0^s (u(r),\phi_0)dr + \int_0^s (\alpha(r),\xi(\lambda,t-r))dr)$$

we obtain for $s \in [0,t]$ that ζ_s equals

$$\zeta_0 + \int_0^s \int \zeta_r \xi(\lambda,t-r,x)|u(r,x)|^{1/2}dW_{x,r} + \int_0^s \zeta_r(\beta(r) + \gamma(r)u(r),\xi(\lambda,t-r))dr$$

so that

$$E(\exp(-\lambda\int_0^t (u(s),\phi_0)ds) \ge \exp(-(f,\xi(\lambda,t)) - \int_0^t (\alpha(s),\xi(\lambda,t-s))ds). \quad (9)$$

We now claim that

$$\xi(\lambda,t,x) \le h(x) := 2\theta + 12\left(|x|-R\right)^{-2} \quad \text{for all } |x| > R, \lambda \ge 0. \qquad (10)$$

We argue as in a proof of a maximum principle. Suppose, to the contrary, that $\xi(\lambda,t,x) > h(x)$ for some $(t,x) \in [0,T]\times(R,\infty)$. Note that $\xi(t,x) \le \int_0^t P_s^\theta\phi_0(x)ds \to 0$ as $x \to \infty$. Since $\xi(\lambda,0,x) \le h(x)$ for $x \in (R,\infty)$ and

$$\lim_{x\to\infty} \xi(\lambda,s,x) - h(x) = -2\theta, \quad \lim_{x\to R+} \xi(\lambda,s,x) - h(x) = -\infty$$

then there exists $(t_0,x_0) \in (0,T]\times(R,\infty)$ at which $\xi(\lambda,\cdot,\cdot) - h(\cdot)$ achieves its maximum in $[0,T]\times[R,\infty)$. Then

$$\xi_t(\lambda,t_0,x_0) \ge 0, \quad (\xi_{xx} - h_{xx})(t_0,x_0) \le 0$$

and $\xi(\lambda, t_0, x_0) > h(x_0) > 2\theta$. h is chosen so that $h_{xx} + \theta h - (1/2)h^2 \leq 0$ on (R, ∞). Then at (t_0, x_0)

$$\begin{aligned} 0 \leq \xi_t &= \xi_{xx} + \theta\xi - (1/2)\xi^2 \\ &\leq (\xi_{xx} - h_{xx}) + (\theta\xi - (1/2)\xi^2) - (\theta h - (1/2)h^2) \\ &\leq (\theta\xi - (1/2)\xi^2) - (\theta h - (1/2)h^2) < 0 \end{aligned}$$

since $f(z) = \theta z - (1/2)z^2$ is strictly decreasing when $z > \theta$. This contradiction proves the claim (10).

The arguments in [3] lemma 3.5 and theorem 2.3 now improve (10) to the bound

$$\xi(\lambda, t, x) \leq 48(1+\theta)e^{\theta t} \exp(-(|x| - (R+1))^2/4t) \text{ for } |x| \geq R+2, t \geq 1, \lambda \geq 0.$$

Then substituting this bound into (9) and letting $\lambda \to \infty$ proves the lemma. •

Theorem 2.2 *a) For all $f \in C^+_{tem}$ there is a solution to (4) started at f.*

b) All solutions to (4) started at f have the same law which we denote $Q^{f,\alpha,\beta,\gamma}$. The map $(f, \alpha, \beta, \gamma) \to Q^{f,\alpha,\beta,\gamma}$ is continuous. The laws $Q^{f,\alpha,\beta,\gamma}$ for $f \in C^+_{tem}$ form a strong Markov family.

c) For $R, T > 0$ let $\mathcal{U}_{R,T} = \sigma(U(t,x) : t \leq T, |x| \leq R)$. Then the two laws $Q^{f,\alpha,\beta,\gamma}, Q^{f,\alpha,0,0}$ are mutually absolutely continuous on $\mathcal{U}_{R,T}$.

As indicated in the proof, the only new result in this theorem is uniqueness in the presence of overcrowding when the initial condition f is not integrable.

Proof: Existence can be proved using the techniques of Shiga [12]. We sketch the steps. Set $F_n(u) = |u|^{1/2} \wedge n|u|$, $G_n(u) = u^2 \wedge n$, $\alpha_n = \alpha \wedge n, \beta_n = \beta \wedge n$, $\gamma_n = \gamma \wedge n$. Then there exists a pathwise unique solution $u^n(t)$ in C^+_{tem} solving

$$u^n_t = u^n_{xx} + \alpha_n + \theta u^n - \beta_n u^n - \gamma G_n(u^n) + F_n(u^n)\dot{W}.$$

This is proved in Shiga [12] theorems 2.1, 2.2. Adapting the proof of Shiga [12] theorem 2.5, we may check the Kolmogorov tightness criterion (3) to show that $(u^n : n \geq 1)$ are tight and that any limit point is a solution to (4).

For uniqueness we consider first the case $\beta = \gamma = 0$. We sketch the standard argument (see [1] for this case when f is integrable). Fixing $0 \leq \phi \in \Phi$, let $V_t\phi(x)$ be the unique non-negative bounded solution to

$$v_t = v_{xx} + \theta v - (1/2)v^2, \;\; v(0,x) = \phi(x).$$

Fix $T > 0$ and let $\psi(r,x) = V_{t-r}\phi(x)$ for $r \in [0,t]$. Then ψ satisfies (7). Let u be any solution to (4) with $\beta = \gamma = 0, u(0,x) = f$. Apply (8) with this choice of ψ. Ito's formula shows that

$$\exp(-(u(t), \phi) - \int_0^t (\alpha(r), V_{t-r}\phi)dr)$$

is a martingale. Taking conditional expectations we obtain

$$E(\exp(-(u(t),\phi))|\mathcal{F}_s) = \exp(-(u(s),V_{t-s}\phi) - \int_s^t (\alpha(r),V_{t-s}\phi)dr). \qquad (11)$$

Induction using (11) shows that $E(\exp(-(u(t_1),\phi_1)-\ldots-(u(t_n),\phi_n))$ and hence also the law of u are determined.

We now consider the case when the functions $(\beta(t,\cdot),\gamma(t,\cdot) : t \geq 0)$ have compact support. This case is handled by change of measure techniques (see Evans and Perkins [6] for our equation when f is integrable). We sketch the argument. Let u be a solution to (4) with $u(0,x) = f(x)$ and $(\beta(t,\cdot),\gamma(t,\cdot) : t \geq 0)$ supported in $[-K,K]$. Then

$$\int_0^t \int (\beta(s,x) + \gamma(s,x)u(s,x))^2 u(s,x)dx\,ds < \infty, \ \ \forall t \geq 0, \ P-a.s.$$

So the stochastic integral

$$\int_0^t (\beta(s,x) + \gamma(s,x)u(s,x))|u(s,x)|^{1/2}dW_{x,s}$$

is well defined. Let R_t be the exponential martingale derived from this stochastic integral. The arguments of [6] theorem 3.10 show that R_t is a true martingale and that defining a new measure by $(dQ/dP)|_{\mathcal{F}_t} = R_t$ then under Q, u solves the equation

$$u_t = u_{xx} + \alpha + \theta u + |u|^{1/2}\dot{\tilde{W}}$$

for some new adapted white noise $\tilde{W}$. Thus under Q, u has law $Q^{f,\alpha,0,0}$. Approximating $(\beta(s,x)-\gamma(s,x)u(s,x))$ by a sequence of simple predictable integrands shows that R_t is adapted to $\sigma(u_s : s \leq t)$. Hence if $A \in \mathcal{U}_t$

$$P(u \in A) = Q(\mathrm{I}(u \in A)R_t^{-1})$$

which is determined thus proving uniqueness.

Suppose that $(\beta_n(t,\cdot),\gamma_n(t,\cdot) : n \geq 1, t \geq 0)$ are all supported in $[-K,K]$. Suppose also that $(f_n,\alpha_n,\beta_n,\gamma_n) \to (f,\alpha,\beta,\gamma)$. If u^n solve (4) and have law $Q^{f_n,\alpha_n,\beta_n,\gamma_n}$ then it may be chacked that the Kolmogorov tightness criterion holds and any limit point u has law $Q^{f,\alpha,\beta,\gamma}$. This proves that $Q^{f_n,\alpha_n,\beta_n,\gamma_n} \to Q^{f,\alpha,\beta,\gamma}$. The same argument will also prove continuity in the general case once uniqueness is established.

We now prove uniqueness in the general case. Let u be a solution to (4) started at f and defined on $(\Omega,\mathcal{F},\mathcal{F}_t,P)$.We shall couple u with a solution to

$$w_t = w_{xx} + \alpha + \theta w - \beta_K w - \gamma_K w^2 + |w|^{1/2}\dot{W} \qquad (12)$$

where $\beta_K = \beta I_K, \gamma_K = \gamma I_K$ and

$$I_K(x) = \begin{cases} 1 & \text{if } \ |x| \leq K \\ K+1-|x| & \text{if } \ K \leq |x| \leq K+1 \\ 0 & \text{if } \ |x| \geq K+1. \end{cases}$$

To do this we use the coupling technique of Barlow, Evans and Perkins [1] theorem 5.1. Define random C^+_{tem} valued processes A_K, B_K by

$$\begin{aligned} A_K(\omega)(t,x) &= (u(t,x)\beta(t,x) + u^2(t,x)\gamma(t,x))(1 - I_K(x)) \\ B_K(\omega)(t,x) &= \beta_K(t,x) + 2u(t,x)\gamma_K(t,x). \end{aligned}$$

Define

$$\Omega' = \Omega \times C([0,\infty), C^+_{\text{tem}}),\ \mathcal{F}' = \mathcal{F} \times \mathcal{U},\ \mathcal{F}'_t = \mathcal{F}_t \times \mathcal{U}_t$$

$$u'(\omega, f) = u(\omega),\ v(\omega, f) = U(f),\ w(\omega, f) = u'(\omega, f) + v(\omega, f).$$

There is a unique probability P'_K on $(\Omega', \mathcal{F}')$ such that for $F \in \mathcal{F}, G \in \mathcal{U}$

$$P'_K(F \times G) = \int_\Omega \mathrm{I}(\omega \in F) Q^{0, A_K(\omega), B_K(\omega), \gamma_K}(G)\, P(d\omega). \tag{13}$$

The integrand on the right hand side of (13) is measurable by the continuity of the map $(f, \alpha, \beta, \gamma) \to Q^{f,\alpha,\beta,\gamma}$ when β, γ are supported in $[-K-1, K+1]$. The techniques of [1] then show that w solves (12) (on a possibly enlarged probability space). The idea is that the process v has immigration which exactly matches the mass lost from u due to annihilation and overcrowding outside $[-K, K]$. The process v itself has annihilation and overcrowds itself inside $[-K, K]$. The extra annihilation term $2u(t,x)\gamma_K(t,x)$ has the same effect as overcrowding between the u and v processes inside $[-K, K]$.

Fix $R, T \geq 1$ and $A \in \mathcal{U}_{R,T}$. The next step is to show that for large K the processes u' and w agree on $[0,T] \times [-R, R]$ with large probability. If $K \geq R+2$, using lemma 2.1,

$$\begin{aligned} & |P(u \in A) - Q^{f,\alpha,\beta_K,\gamma_K}(A)| \\ = \ & |P'_K(u' \in A) - P'_K(w \in A)| \\ \leq \ & P'_K(\exists\, (t,x) \in [0,T] \times [-R,R] \text{ such that } u(t,x) \neq w(t,x)) \\ = \ & P'_K\Big(\int_0^T \int_{-R}^R v(s,x)dx\, ds > 0\Big). \\ = \ & \int_\Omega Q^{0, A_K(\omega), B_K(\omega), \gamma_K}\Big(\int_0^T \int_{-R}^R U_s(x) dx\, ds > 0\Big) P(d\omega) \\ \leq \ & 48(1+\theta)e^{\theta T} E\Big(\int_0^T \int_{-R}^R \exp(-(|x| - (R+1))^2/4T) \\ & \qquad (u(s,x)\beta(s,x) + u^2(s,x)\gamma(s,x))(1 - I_K(x)) dx ds\Big) \\ \to \ & 0 \text{ as } K \to \infty. \end{aligned} \tag{14}$$

Hence $P(u \in A)$ is determined to arbitrary precision proving uniqueness. The strong Markov property follows from uniqueness by the standard argument.

To prove part c) we rewrite (14) as

$$|Q^{f,\alpha,\beta,\gamma}(A) - Q^{f,\alpha,\beta_K,\gamma_K}(A)| \to 0 \text{ as } K \to \infty.$$

Using the mutual absolute continuity of $Q^{f,\alpha,0,0}$ and $Q^{f,\alpha,\beta,\gamma}$ when β, γ are compactly supported

$$\begin{aligned} Q^{f,\alpha,0,0}(A) = 0 \quad &\Rightarrow \quad Q^{f,\alpha,\beta_K,\gamma_K}(A) = 0 \text{ for all } K \\ &\Rightarrow \quad Q^{f,\alpha,\beta,\gamma}(A) = 0 \end{aligned} \tag{15}$$

proving $Q^{f,\alpha,\beta,\gamma} \ll Q^{f,\alpha,0,0}$ on $\mathcal{U}_{R,T}$. For continuity in the opposite direction we reverse the above arguments. Letting $\beta^K = \beta(1-I_K), \gamma^K = \gamma(1-I_K)$ check that $Q^{f,\alpha,\beta,\gamma}$ and $Q^{f,\alpha,\beta^K,\gamma^K}$ are mutually absolutely continuous on $\mathcal{U}$. Check also that for $A \in \mathcal{U}_{R,T}$

$$|Q^{f,\alpha,0,0}(A) - Q^{f,\alpha,\beta^K,\gamma^K}(A)| \to \infty \quad \text{as } k \to \infty.$$

Now the argument in (15) may be reversed to finish the proof. •

3 Construction of the travelling wave

We now write Q^f for the law of the unique solution to (1) such that $u(0) = f$. If ν is a probability on C^+_{tem} we define

$$Q^\nu(A) = \int_{C^+_{\text{tem}}} Q^f(A)\nu(df).$$

We consider as markers of the wavefront $R_\lambda : C^+_{\text{tem}} \to [-\infty, \infty]$ for $\lambda \geq 0$ defined as follows:

$$\begin{aligned} R_0(f) &= \sup(x : f(x) > 0) \\ R_\lambda(f) &= \lambda^{-1} \ln((e_\lambda, f)) \quad \text{where } e_\lambda(x) = \exp(\lambda x). \end{aligned}$$

Then for $\lambda \geq 0$ we have $R_\lambda(f(\cdot - c)) = R_\lambda(f) + c$ (with obvious conventions involving $\pm\infty$). We adopt the convention that $U(t, +\infty) = 1, U(t, -\infty) = 0$. Define on $C([0,\infty), C^+_{\text{tem}})$

$$\begin{aligned} R_\lambda(t) &= R_\lambda(U(t)), \\ V_\lambda(t) &= U(t, \cdot + R_\lambda(t)) \end{aligned}$$

so that $V_\lambda(t)$ is the wave U shifted so that its wavefront lies at the origin. Note that whenever $R_0(f) < \infty$, then the compact support property in lemma 2.1 implies that $R_\lambda(U(t)) < \infty$, $\forall t \geq 0$, Q^f-a.s.

We now summarise the method for constructing a travelling wave. Take an initial condition $f_0(x) = 1 \wedge (-x \vee 0)$. Define

$$\nu_T \text{ to be the law of } T^{-1} \int_0^T V_1(s)ds \text{ under } Q^{f_0}.$$

We shall show (lemma 3.7) that the sequence ν_T, $T = 1, 2 \ldots$ is tight. We shall also show and that any limit point is nontrivial (which seems easier using the wavefront marker R_1 rather than R_0). We then check that for any limit point ν, Q^ν is the law of a travelling wave. The two ingredients that go into the proof of tightness are the Kolmogorov tightness criterion for the unshifted waves (see lemma 3.4) and control on the movement of the wavefront to ensure the shifting doesn't destroy the tightness (see lemma 3.6).

The following coupling lemma may be proved by the method used in [10] lemmas 2.1.3 and 2.1.4. Note that the process v defined below in (16) is the density of a one-dimensional super Brownian motion with mass creation rate θ.

Lemma 3.1 *a) There is a coupling of u a solution to (1) started at $f \in C^+_{tem}$ with v a solution to*

$$v_t = v_{xx} + \theta v + |v|^{1/2}\dot{W}, \quad v(0) = f \tag{16}$$

so that $u \le v$.

b) Fix $f, \bar{f} \in C^+_{tem}$ with $f \le \bar{f}$. Then we may construct coupled solutions $u, \bar{u}$ to (1) with initial conditions $u(0) = f, \bar{u}(0) = \bar{f}$ and satisfying $u(t,x) \le \bar{u}(t,x)$ for all $t \ge 0, x \in \mathbf{R}$.

Lemma 3.2 *For $\theta \ge 0, n \in \mathbf{N}, T > 0$ there exists $C(\theta, n, T)$ such that for all $t \le T, f \in C^+_{tem}, x \in \mathbf{R}$*

$$Q^f(U^n(t,x)) \le C(\theta, n, T)((f, p_t(x - \cdot)) + (f, p_t(x - \cdot))^n).$$

Proof. By lemma 3.1 a) it is enough to prove the bound for a solution v to (16). Moments for the superprocess density v are studied in Konno and Shiga [8] where a slightly smaller class of initial conditions is considered but the arguments are valid for C^+_{tem}. Setting $X^h_t(x) = (U(t), p^\theta_h(x - \cdot))$ for $h \in (0,1)$ we have from [8] lemmas 2.2, 2.1

$$Q^f((X^h_t(x))^n) = \sum_{k=0}^{n-1} \binom{n-1}{k} (f, \nu^{(n-k)}(t, \cdot, h)) Q^f((X^h_t(x))^k) \tag{17}$$

where $\nu^{(j)}(t) = \nu^{(j)}(t, \cdot, h) \in \Phi$ satisfy $\nu^{(1)}(t) = p^\theta_{t+h}(x - \cdot)$ and for $n \ge 2$

$$\nu^{(n)}(t) = \sum_{k=1}^{n-1} \binom{n-1}{k} \int_0^t P^\theta_{t-s}(\nu^{(k)}(s)\nu^{(n-k)}(s))ds. \tag{18}$$

From (18) we have $P^\theta_s \nu^{(n)}(t) \le \nu^{(n)}(t+s)$ so that

$$\nu^{(n)}(t) \le \sum_{k=1}^{n-1} \binom{n-1}{k} \int_0^t \|\nu^{(n-k)}(s)\|_\infty ds\, \nu^{(k)}(t). \tag{19}$$

By induction from (19) there exists $C(n,T,\theta)$ such that for all $t \leq T, n \geq 1$

$$\begin{aligned} |\nu^{(n)}(t,x,h)| &\leq C(n,T,\theta)t^{-1/2} \\ (f,\nu^{(n)}(t,\cdot,h)) &\leq C(n,T,\theta)(f,p_{t+h}(x-\cdot)). \end{aligned}$$

Using this bound in (17), letting $h \to 0$ and using induction on n then proves the lemma.•

Lemma 3.3 *Suppose $\phi > 0$ has two continuous derivatives with $\phi, \phi_{xx} \in \Phi$ and that $\alpha := \sup(\phi_{xx}(y)/\phi(y) : y \in \mathrm{R}) < \infty$. Set $\beta = \sup(\phi(y) : y \in \mathrm{R})$,$\gamma = \int \phi(x)dx$. Then for all $p \geq 2, t > 0, f \in C^+_{tem}$*

$$Q^f((U(t),\phi)^p) \leq (2\gamma)^p t^{-p} \vee 1 \vee (2\gamma(\alpha+\theta+p\beta))^p.$$

Note that the bound is independent of the initial condition f. Such a bound is possible since the overcrowding term $-u^2$ will drag down arbitrarily large initial conditions in a finite time.

Proof. Let u be a solution to (1) started at f. From (8) and Ito's formula we have

$$\begin{aligned} & d(u(t),\phi)^p \\ = \;& p(u(t),\phi)^{p-1}(u(t),\phi_{xx}+\theta\phi-u(t)\phi)dt \qquad (20) \\ & +(p/2)(p-1)(u(t),\phi)^{p-2}(u(t),\phi^2)dt + p(u(t),\phi)^{p-1}|u(t,x)|^{1/2}\phi(x)dW_{x,t}. \end{aligned}$$

From (6) the stochastic integral in (20) is a true martingale. Set $g(t) = E((u(t),\phi)^p)$ which is continuous in t from (6). Note that $(u(t),\phi u(t)) \geq (1/\gamma)(u(t),\phi)^2$. So we may bound

$$\begin{aligned} Q^f((u(t),\phi)^{p-1}(u(t),\phi_{xx})) &\leq \alpha g(t) \\ Q^f((u(t),\phi)^{p-2}(u(t),\phi^2)) &\leq \beta g(t)^{1-(1/p)} \\ Q^f((u(t),\phi)^{p-1}(u(t),u(t)\phi)) &\geq (1/\gamma)g(t)^{1+(1/p)}. \end{aligned}$$

Taking expectations in (20) shows that $g(t)$ is continuously differentiable and that

$$g'(t) \leq (1/2)p(p-1)\beta g(t)^{1-(1/p)} + p(\alpha+\theta)g(t) - (p/\gamma)g(t)^{1+(1/p)}.$$

Setting $T(n) = \inf(t \geq 0 : g(t) \leq n)$ we have for $t \leq T(1)$

$$g'(t) \leq -(p/\gamma)g(t)^{1+(1/p)} + p(\alpha+\theta+p\beta)g(t).$$

For $t \leq T(1 \vee (2\gamma(\alpha+\theta+p\beta)))$ we have $g'(t) \leq -(p/2\gamma)g(t)^{1+(1/p)}$. A comparison with $h(t) = (2\gamma)^p t^{-p}$ (which solves $h'(t) = -(p/2\gamma)h(t)^{1+(1/p)}$ on $(0,\infty)$) finishes the proof.•

Lemma 3.4 *For $\theta \geq 0, p \geq 2, t > 0$ there exists $C(\theta, p, t)$ such that for all $|x - x'| \leq 1, f \in C^+_{tem}$*

$$Q^f(|U(t,x) - U(t,x')|^p) \leq C(\theta, p, t)|x - x'|^{(p/2)-1}.$$

Proof. Take a solution u to (1) started at f. From the Green's function representation for $u(t,x)$ we have for $x' > x, |x - x| \leq 1$

$$\begin{aligned}
& E(|u(s,x) - (s,x')|^p) \\
\leq\ & 3^{p-1}|P^\theta_s(f)(x) - P^\theta_s f(x')|^p \\
& +3^{p-1}E(\left|\int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))u^2(r,y)dy\,dr\right|^p) \\
& +3^{p-1}E(\left|\int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))|u(r,y)|^{1/2}dW_{y,r}\right|^p) \\
=\ & I + II + III.
\end{aligned}$$

We shall use the bounds, for $|x - x'| \leq 1$

$$\begin{aligned}
\int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))^2 dy\,dr &\leq C(\theta,s)|x - x'| \\
\int_0^s \int |p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y)| dy\,dr &\leq C(\theta,s)|x - x'|^{1/2}.
\end{aligned}$$

Then using Burkholder's inequality we bound the term III by

$$\begin{aligned}
& C(p)E(\left|\int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))^2 u(r,y)dy\,dr\right|^{p/2}) \\
\leq\ & C(p)\left(\int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))^2 dy\,dr\right)^{(p/2)-1} \\
& \int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))^2 E(u^{p/2}(r,y))dy\,dr \\
\leq\ & C(\theta,p,s)|x - x'|^{(p/2)-1} \\
& \int_0^s \int (p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y))^2 E(u^{p/2}(r,y))dy\,dr.
\end{aligned}$$

Similarly we can bound term II by

$$C(\theta,p,s)|x - x'|^{(p-1)/2} \int_0^s \int |p^\theta_{s-r}(x-y) - p^\theta_{s-r}(x'-y)| E(u^{2p}(r,y))dy\,dr.$$

Finally term I equals

$$3^{p-1}|\int \int_x^{x'} ((y-z)/2s)(4\pi s)^{-1/2} e^{\theta s} e^{-(y-z)^2/4s} f(y)dz\,dy|^p$$

$$\leq \quad C(\theta,p,s)\Big(\int\int_x^{x'} s^{-1/2}e^{-(y-z)^2/3s}f(y)dz\,dy\Big)^p$$
$$\leq \quad C(\theta,s,p)|x-x'|^p\int s^{-1/2}e^{-(y-x)^2/3s}f^p(y)dy.$$

Fix t,θ,p,μ as in the lemma and set $s=t/3$. Applying the Markov property at time $2s$ and the above estimates gives

$$\begin{aligned}
& E(|u(t,x)-u(t,x')|^p) \\
\leq \quad & C(\theta,p,t)|x-x'|^{(p/2)-1}\Bigg(\int s^{-1/2}e^{-(y-x)^2/3s}E(u^p(2s,y))dy \\
& + \int_{2s}^t\int |p^\theta_{t-r}(x-y)-p^\theta_{t-r}(x'-y)|E(u^{2p}(r,y))dy\,dr \\
& + \int_{2s}^t\int (p^\theta_{t-r}(x-y)-p^\theta_{t-r}(x'-y))^2E(u^{p/2}(r,y))dy\,dr\Bigg). \qquad (21)
\end{aligned}$$

From lemma 3.2 we have that for $r\in[0,s], l\in \mathrm{N}$

$$Q^f(U^l(2s+r,x))\leq C(\theta,l,s)(1+Q^f((\langle U(s),p_{2s}(x-\cdot)\rangle)^l)). \qquad (22)$$

Apply lemma 3.3 with $\phi(y)=\exp(-(1+(y-x)^2)^{1/2})$ for which $\alpha\leq 3,\beta=1,\gamma\leq 2$. Note that $p_{2s}(x-y)\leq C(s)\phi(y)$. This shows that the right hand side of (22) is bounded by $C(\theta,l,s)$. Substituting this bound into (21) finishes the proof. •

The following two lemmas give some weak control on the movement of the wavefronts.

Lemma 3.5 *Suppose $\theta>\theta_c$. Recall that $f_0(x)=1\wedge(-x\vee 0)$. Then there exists $C(\theta)<\infty$ such that*

$$Q^{f_0}(R_1(t))\geq -C(\theta)(1+t).$$

Proof. Let $\psi_0:\mathrm{R}\to[0,1]$ be symmetric and satisfy $(x:\psi_0(x)>0)=(-1,1)$. Let $\psi_r(x)=\psi_0(x+r)$. For $r\geq 2$ let $f_r=\sum_{j=1}^\infty \psi_{jr}$. Then $f_0\geq f_r$ and lemma 3.1 shows that $Q^{f_0}(R_1(t))\geq Q^{f_r}(R_1(t))$.

Consider the equation

$$\begin{aligned}
u^{(j)}_t &= u^{(j)}_{xx}+\theta u^{(j)}-(u^{(j)})^2+|u^{(j)}|^{1/2}\dot W \\
& \qquad \text{on } [0,\infty)\times(-(2j+1)r/2,-(2j-1)r/2) \\
u^{(j)}(0,x) &= \psi_{jr}(x) \\
u^{(j)}(t,x) &= 0 \ \text{ for} t\geq 0, x\notin(-(2j+1)r/2,-(2j-1)r/2). \qquad (23)
\end{aligned}$$

Define for $f\in C([0,\infty),C^+_{\mathrm{tem}})$

$T_j(f)=\inf(t\geq 0: f(t,x)>0 \text{ for some } x\notin(-(2j+1)r/2+1,-(2j-1)r/2-1)$.

Let $u^{(j)}$ solve (23). Then $u^{(j)}$ solves (1) "upto time T_j" and uniqueness implies that for $A \in \mathcal{U}_t$

$$P(u^{(j)} \in A, T_j(u^{(j)}) \geq t) = Q^{\psi_{jr}}(U \in A, T_j(U) \geq t). \tag{24}$$

We may couple a solution u to (1) started at f_0 with a sequence of independent processes $u^{(j)}$ solving (23) so that $u \geq \sum_{j=1}^{\infty} u^{(j)}$. See [10] lemma 2.1.5 for this argument. Fix t and set $r = ct$. We claim that for suitable choice of $c = c(\theta)$ there exists $a(\theta) < \infty, \delta(\theta) > 0$ so that

$$P(R_1(u^{(j)}(t)) \geq -a - cjt) \geq \delta \ \text{ for } j = 1, 2 \ldots \tag{25}$$

Assuming this claim then

$$\begin{aligned} Q^{f_0}(R_1(t) \leq -a - cmt) &\leq P(R_1(u^{(j)}(t)) \leq -a - cjt \ \text{ for } j = 1, \ldots, m) \\ &\leq (1-\delta)^m \end{aligned}$$

which implies the lower bound in the lemma.

It remains to prove the claim (25). Let v be a solution to (16) with initial condition f satisfying $(f, 1) < \infty$. The total mass process $V_t = (v(t), 1)$ satisfies the S.D.E. $dV_t = \theta V_t dt + |V_t|^{1/2} dB_t$. From this one may show that

$$P((v(t), 1) = 0) = \exp(-\theta(1 - e^{-\theta t})^{-1}(f, 1)). \tag{26}$$

For $\theta > \theta_c$ there exists $\delta(\theta) > 0$ so that

$$Q^{\psi_0}((U(t), 1) \geq \delta) \geq \delta \quad \text{for all } t \geq 0.$$

To see this suppose the contrary: that for each $\delta > 0$ there exists $t(\delta)$ with $Q^{\psi_0}((U(t), 1) \geq \delta) \leq \delta$. Then using lemma 3.1

$$\begin{aligned} & Q^{\psi_0}(U \text{ dies in finite time}) \\ \geq\ & Q^{\psi_0}((U(t(\delta) + 1), 1) = 0) \\ \geq\ & Q^{\psi_0}(Q^{U(t(\delta))}((U(1), 1) = 0) \mathrm{I}((U(t(\delta)), 1) \leq \delta)) \\ \geq\ & (1 - \delta) \exp(-\theta(1 - e^{-\theta})^{-1}\delta) \\ \rightarrow\ & 1 \text{ as } \delta \rightarrow 0 \end{aligned}$$

which contradicts the definition of θ_c.

By symmetry $Q^{\psi_0}((U(t), \mathrm{I}(0, \infty)) \geq \delta/2) \geq \delta/2$. By lemma 2.1 for $r \geq 4$

$$\begin{aligned} Q^{\psi_0}(T_0(U) \leq t) &\leq 96(1+\theta)e^{\theta t} \int \exp(-((r/2) - 1 - x)^2/4t)\psi_0(x)dx \\ &\leq 192(1+\theta)\exp(\theta t - (r-4)^2/16t). \end{aligned}$$

So we may choose $r = c(\theta)t$ so that $Q^{\psi_0}(T_0(U) \leq t) \leq \delta/4$. Then choosing $a = -\ln(\delta/2)$

$$\begin{aligned}
& P(R_1(u^{(j)}(t)) \geq -a - cjt) \\
\geq \; & P(R_1(u^{(j)}(t)) \geq -a - cjt, T_j(u^{(j)}) \geq t) \\
= \; & P(R_1(u^{(0)}(t)) \geq \ln(\delta/2), T_0(u^{(0)}) \geq t) \\
\geq \; & P((u^{(0)}(t), \mathrm{I}(0,\infty) \geq \delta/2, T_0(u^{(0)}) \geq t) \\
= \; & Q^{\psi_0}((U(t), \mathrm{I}(0,\infty) \geq \delta/2, T_0(U) \geq t) \quad \text{by (24)} \\
\geq \; & \delta/4
\end{aligned}$$

proving the claim. •

Lemma 3.6 *For all $t \geq 0, \theta > \theta_c$ there exists $C(\theta,t) < \infty$ such that*

$$Q^{\nu_T}(|R_1(s)| \geq a) \leq C(\theta,t)/a \;\; \textit{for all} \;\; a \geq 0, s \leq t, T \geq 1.$$

Proof. Let v be a solution to (16) started at f. Then using the coupling from lemma 3.1 a)

$$\begin{aligned}
Q^f(\exp(R_1(s)) &= Q^f((U(s), e_1)) \\
&\leq E((v(s), e_1)) \\
&= (f, P_s^\theta e_1) \\
&= e^{(1+\theta)s}(f, e_1).
\end{aligned}$$

Jensen's inequality then gives that $Q^f(R_1(s)) \leq (1+\theta)s + R_1(f)$. Since $(V_1(t), e_1) \equiv 1$ under Q^{f_0} we have for $s \leq t$

$$\begin{aligned}
Q^{\nu_T}(R_1(s) \geq a) &= T^{-1}\int_0^T Q^{f_0}(Q^{V_1(r)}(R_1(s) \geq a))dr \\
&\leq T^{-1}\int_0^T e^{-a}Q^{f_0}(Q^{V_1(r)}(\exp(R_1(s))))dr \\
&\leq T^{-1}\int_0^T e^{-a+(1+\theta)s}dr \leq e^{-a+(1+\theta)t}.
\end{aligned}$$

This proves half of the lemma.

$$\begin{aligned}
& T^{-1}Q^{f_0}(\int_T^{T+s} R_1(r)dr - \int_0^s R_1(r)dr) \\
= \; & T^{-1}Q^{f_0}(\int_0^T R_1(r+s) - R_1(r)dr) \\
\leq \; & T^{-1}\int_0^T \int_0^\infty Q^{f_0}(R_1(s+r) - R_1(r) \geq y)dy\, dr
\end{aligned}$$

$$
\begin{aligned}
& -aT^{-1}\int_0^T Q^{f_0}(R_1(s+r)-R_1(r)\le -a)dr \\
= & \int_0^\infty Q^{\nu_T}(R_1(s)\ge y)dy - aQ^{\nu_T}(R_1(s)\le -a).
\end{aligned}
$$

Rearranging and using lemma 3.5 gives

$$
\begin{aligned}
& Q^{\nu_T}(R_1(s)\le -a) \\
\le & \; a^{-1}\int_0^\infty Q^{\nu_T}(R_1(s)\ge y)dy + (aT)^{-1}\int_0^s Q^{f_0}(R(r))dr \\
& -(aT)^{-1}\int_T^{T+s} Q^{f_0}(R(r))dr \\
\le & \; a^{-1}\int_0^\infty e^{-y+(1+\theta)t}dy + (aT)^{-1}\int_0^s R_1(f_0)+(1+\theta)rdr \\
& +(aT)^{-1}\int_T^{T+s} C(\theta)(1+s)ds \\
\le & \; C(\theta,t)/a. \bullet
\end{aligned}
$$

Lemma 3.7 *If $\theta > \theta_c$ the sequence $(\nu_T : T = 1,2\ldots)$ is tight.*

Proof. Recall the set $K(C,\delta,\gamma,\mu)$ and the tightness condition (2) in the introduction.

$$
\begin{aligned}
& \nu_T(K(C,\delta,\gamma,\mu)) \\
= & \; T^{-1}\int_0^T Q^{f_0}(U(t,\cdot - R_1(t))\in K(C,\delta,\gamma,\mu))dt \\
\ge & \; T^{-1}\int_1^T Q^{f_0}\left(U(t,\cdot - R_1(t-1))\in K(Ce^{-\mu a},\delta,\gamma,\mu),\right. \\
& \qquad\qquad \left.|R_1(t)-R_1(t-1)|\le a\right)dt \\
\ge & \; T^{-1}\int_1^T Q^{f_0}(Q^{V_1(t-1)}(U(1)\in K(Ce^{-\mu a},\delta,\gamma,\mu)))dt \\
& -T^{-1}\int_1^T Q^{f_0}(|R_1(t)-R_1(t-1)|\ge a)dt \\
:= & \; I - II.
\end{aligned}
$$

Term II is bounded by $Q^{\nu_T}(|R_1(t)|\ge a)\le C(\theta,t)/a$ by lemma 3.6. Lemma 3.4 and the tightness criterion (2) show that given $a,\mu>0$ we can choose C,γ,δ to make term I as close to $(T-1)/T$ as desired. Recall that $\phi_1(x)=\exp(-|x|)\le e_1(x)$ so that

$$\nu_T(f:(f,\phi_1)\le (f,e_1)=1)=1.$$

So given $\mu>0$ we can choose C,δ,γ so that $\nu_T(K(C,\delta,\gamma,\mu)\cap(f:(f,\psi_1)\le 1))$ is as close to one as desired for large T finishing the proof. $\bullet$

Theorem 3.8 *For $\theta > \theta_c$ there is a travelling wave solution to (1).*

Proof. By lemma 3.7 we may take a subsequence $\nu_{T(n)}$ converging to ν. We shall show that Q^ν is the law of a travelling wave solution. We first show four facts:

$$\lim_{a\to\infty}\lim_{T\to\infty}\nu_T(f:(f,\mathrm{I}(a,\infty)=0)=1 \tag{27}$$
$$\nu(f:(f,e_1)=1)=1 \tag{28}$$
$$\nu(f:-\infty<R_0(f)<\infty)=1 \tag{29}$$
$$U(t)\neq 0,\quad Q^\nu\ a.s.\ \forall t\geq 0. \tag{30}$$

Suppose $(g,e_1)=1$. Choose $g_1,g_2\in C^+_{\text{tem}}$ with $g=g_1+g_2$, $(g_1,\mathrm{I}(a/3,\infty))=0$ and $(g_2,\mathrm{I}(-\infty,2a/3)=0)$. Take $v^{(1)},v^{(2)}$ independent solutions to (16) started at g_1,g_2. Then the additive property of superprocesses shows that $v=v^{(1)}+v^{(2)}$ is a solution to (16) started at g. Take $a\geq 6$. Using lemma 3.1

$$\begin{aligned}
&Q^g((U(t),\mathrm{I}(a,\infty))>0)\\
\leq\ & P((v(t),\mathrm{I}(a,\infty))>0)\\
\leq\ & P((v^{(1)}(t),\mathrm{I}(a,\infty))>0)+P((v^{(2)}(t),1)>0)\\
\leq\ & 48(1+\theta)e^{\theta t}\int g_1(x)\exp(-(a-1-x)^2/4t)dx+(g_2,1)\theta(1-e^{-\theta t})^{-1}\\
&\quad\text{(using lemma 2.1 and (26))}\\
\leq\ & 48(1+\theta)e^{(\theta+1)t+1-a}(g,e_1)+e^{-a/3}(g,e_1)\theta(1-e^{-\theta t})^{-1}\\
\leq\ & C(\theta,t)e^{-a/3}.
\end{aligned}\tag{31}$$

So

$$\begin{aligned}
&\nu_T(f:(f,\mathrm{I}(2a,\infty))=0)\\
=\ & T^{-1}\int_0^T Q^{f_0}((V_1(t),\mathrm{I}(2a,\infty))=0)dt\\
\geq\ & T^{-1}\int_1^T Q^{f_0}((U(t),\mathrm{I}(a+R_1(t-1),\infty))=0,|R_1(t)-R_1(t-1)|\leq a)dt\\
\geq\ & T^{-1}\int_1^T Q^{f_0}(Q^{V_1(t-1)}((U(1),\mathrm{I}(a,\infty))=0)dt-Q^{\nu_T}(|R_1(1)|\geq a)\\
\geq\ & (T-1)/T-(C(\theta)/a)
\end{aligned}\tag{32}$$

by (31) and lemma 3.6. This proves (27). Since $\nu_{T(n)}(f:(f,e_1)=1)=1$ we have $\nu(f:(f,e_1)\leq 1)=1$. Let $e_1^a(x)=\exp(a-|x-a|)$. Then

$$\begin{aligned}
\nu(f:(f,e_1)\geq 1)\ &\geq\ \nu(f:(f,e_1^a)\geq 1)\\
&\geq\ \limsup_{n\to\infty}\nu_{T(n)}(f:(f,e_1^a)=1)\\
&=\ \limsup_{n\to\infty}\nu_{T(n)}(f:(f,\mathrm{I}(a,\infty))=0).
\end{aligned}$$

By (27) the right hand side converges to 1 as $a \to \infty$. This proves (28) which in turn implies $\nu(f : R_0(f) > -\infty) = 1$. Choose $0 \leq \psi_a \in \Phi$ with $(\psi_a > 0) = (a, \infty)$. Then

$$\begin{aligned}\nu(f : R_0(f) \leq a) &= \nu(f : (f, \psi_a) = 0) \\ &\geq \limsup_{n\to\infty} \nu_{T(n)}(f : (f, \psi_a) = 0) \\ &= \limsup_{n\to\infty} \nu_{T(n)}(f : (f, \mathrm{I}(a,\infty)) = 0).\end{aligned}$$

Again the right hand side converges to 1 as $a \to \infty$ which proves (29). To show (30) we have

$$\begin{aligned} & Q^{\nu}(U(t) = 0) \\ \leq \;& Q^{\nu}(R_1^a(t) < -a) \\ \leq \;& \liminf_{n\to\infty} Q^{\nu_{T(n)}}(R_1^a(t) < -a) \\ \leq \;& \liminf_{n\to\infty} (Q^{\nu_{T(n)}}(R_1(t) < -a) + Q^{\nu_{T(n)}}((U(t), \mathrm{I}(a,\infty)) > 0)) \\ \leq \;& C(\theta, t)/a \quad \text{(by lemma 3.6 and (32))} \\ \to \;& 0 \text{ as } a \to \infty. \end{aligned}$$

We now start the main argument. Let $R_1^a(t) = \ln((U(t), e_1^a))$. Let $F : C_{\text{tem}}^+ \to \mathrm{R}$ be bounded and continuous. Fix $t > 0$. Then

$$\begin{aligned} & |Q^{\nu_{T(n)}}(F(V_1(t))) - Q^{\nu}(F(V_1(t)))| \\ \leq \;& |Q^{\nu_{T(n)}}(F(U(t, \cdot - R_1^a(t)))) - Q^{\nu}(F(U(t, \cdot - R_1^a(t))))| \\ & + \|F(f)\|_{\infty}(Q^{\nu_{T(n)}}(R_1(t) \neq R_1^a(t)) + Q^{\nu}(R_1(t) \neq R_1^a(t))). \end{aligned} \tag{33}$$

Since $\nu_n(f : (f, e_1) = 1) = 1$, (31) shows that

$$Q^{\nu_n}(R_1(t) \neq R_1^a(t)) \leq Q^{\nu_n}((U(t), \mathrm{I}(a,\infty)) > 0) \leq C(\theta, t)/a.$$

The same bound holds for Q^{ν} by (28) so that the second term on the right hand side of (33) is bounded by $C(\theta, t)/a$. By the continuity of $f \to Q^f$ we have $Q^{\nu_{T(n)}} \to Q^{\nu}$. Since $F(U(t, \cdot - R_1^a(t)))$ is a bounded and continuous function on $C([0,\infty), C_{\text{tem}}^+) \cap (U(t) \neq 0)$ the first term on the right hand side of (33) converges to zero as $n \to \infty$. So

$$\begin{aligned} & Q^{\nu}(F(V_1(t))) \\ = \;& \lim_{n\to\infty} Q^{\nu_n}(F(V_1(t))) \\ = \;& \lim_{n\to\infty} T_n^{-1} \int_0^{T_n} Q^{f_0}(F(V_1(s+t)))ds \\ = \;& \lim_{n\to\infty} T_n^{-1} \int_0^{T_n} Q^{f_0}(F(V_1(s)))ds \\ = \;& \nu(F). \end{aligned}$$

We have shown that under Q^ν the one dimensional marginals of $(V_1(t) : t \geq 0)$ have law ν. It is straightforward to check that $(V_1(t) : t \geq 0)$ is also Markov. Hence $(V_1(t) : t \geq 0)$ is stationary and since the map $f \to f(\cdot - R_0(f))$ is measurable on C^+_{tem} the process $(V_0(t) : t \geq 0)$ is also stationary. This together with (29) verifies that Q^ν is the law of a stationary wave. •

4 Wavespeed

This section proves the following result.

Proposition 4.1 *Let $(u(t) : t \geq 0)$ be a travelling wave solution to (1) with parameter θ.*

a) $R_0(u(t))/t \to A \in [-\infty, 2\theta^{1/2}]$ almost surely as $t \to \infty$.

b) If $\theta > \theta_c$ then $A \geq 0$ almost surely.

c) Given $\epsilon > 0$ there there exists $\theta(\epsilon)$ such that if $\theta \geq \theta(\epsilon)$ then

$$P(A \geq (2-\epsilon)\theta^{1/2}) \geq 1 - \epsilon$$

•

Proof. a) Let Q^ν be the law of u. Let $Z(n) = R_0(n) - R_0(n-1)$. The compact support property lemma 2.1 implies that $Q^\nu(Z(n) \vee 0) < \infty$.

$$\begin{aligned}
& Q^\nu((R_0(k+1) - R_0(k), \ldots, R_0(k+m) - R_0(k+m-1)) \in B) \\
= \; & Q^\nu(Q^{U(k)}((R_0(1) - R_0(0), \ldots, R_0(m) - R_0(m-1)) \in B)) \\
= \; & Q^\nu(Q^{V_0(k)}((R_0(1) - R_0(0), \ldots, R_0(m) - R_0(m-1)) \in B)) \\
= \; & Q^\nu((R_0(1) - R_0(0), \ldots, R_0(m) - R_0(m-1)) \in B)
\end{aligned}$$

so that the sequence $(Z(n) : n = 1, 2, \ldots)$ is stationary under Q^ν and the ergodic theorem gives

$$R_0(n)/n = R_0(0)/n + (1/n)\sum_{i=1}^{n} Z(i) \to A \in [-\infty, \infty) \quad \text{a.s. as} \quad n \to \infty.$$

That $A \leq 2\theta^{1/2}$ follows from lemma 2.1. We now interpolate. Define $T_n = \inf(t \geq n : R_0(t) \leq R_0(n) - 2n\epsilon)$. Again using lemma 2.1, applying the strong Markov property at T_n, we have for $\epsilon > 0$ and n large enough

$$\begin{aligned}
& Q^\nu(T_n < n+1, R_0(n+1) \geq R_0(n) - n\epsilon) \\
\leq \; & Q^\nu(Q^{U(T(n)\wedge n+1)}(\sup_{s \leq 1} R_0(s) - R_0(0) \geq n\epsilon)) \\
\leq \; & C(\theta) \int_{-\infty}^{0} \exp(-(n\epsilon - 1 - x)^2/4) dx \\
\leq \; & C(\theta)(n\epsilon - 1)^{-1} \exp(-(n\epsilon - 1)^2/4).
\end{aligned}$$

Since $R_0(n)/n \to A$ we have that $R_0(n+1) \leq R_0(n) - n\epsilon$ for large n. Borel Cantelli then implies that $T_n \geq n+1$ for large n which in turn implies that $\liminf_{n\to\infty} R_0(t)/t \geq A$. The other half of the interpolation is easier and follows again from lemma 2.1.

b) Suppose now that $\theta > \theta_c$ and that the wavespeed is almost surely constant under Q^ν. By translating the initial condition we may assume that $\nu(f : R_0(f) = 0) = 1$. Choose $a, \epsilon > 0$ so that $\nu(f : f \geq \mathrm{I}(-a-\epsilon, -a)) \geq \epsilon$. Define $\bar{\nu}$ by $\bar{\nu}(f \in B) = \nu(f(2a+\epsilon-\cdot) \in B))$. Using the method of [10] lemma 2.1.3 we can construct a coupling of three solutions u^l, u, u^r to (1) with $(x : u(0,x) > 0) = (-a-\epsilon, -a)$, $u^r \stackrel{\mathcal{D}}{=} \nu, u^l \stackrel{\mathcal{D}}{=} \bar{\nu}$ and so that $u(t) \leq u^r(t) \wedge u^l(t)$, $\forall\, t \geq 0$ whenever $u(0) \leq u^r(0) \wedge u^l(0)$. Furthermore we may ask that $u^r(0), u^l(0)$ are independent and that u is independent of $\sigma(u^r(0), u^l(0))$. Define $L_0(u^l(t)) = \inf(x : u^l(t,x) > 0)$. Then, letting $p(\theta) = P(u \text{ lives forever})$, we have

$$\begin{aligned} p(\theta)\epsilon^2 &\leq P(u(0) \leq u^r(0) \wedge u^l(0), u \text{ lives forever}) \\ &\leq P(u(t) \leq u^r(t) \wedge u^l(t), u \text{ lives forever}) \\ &\leq P(R_0(u^r(t)) > L_0(u^l(t))). \end{aligned} \tag{34}$$

If $\theta > \theta_c$ then $p(\theta) > 0$. If $A < 0$ then the right hand side of (34) converges to zero providing a contradiction.

In general if Q^ν is the law of a stationary wave we may decompose it into ergodic parts as follows. Note that $((Z(1), V_0(1)), (Z(2), V_0(2)), \ldots)$ is Markov and stationary and we continue to write Q^ν for its law on $(C^+_{\text{tem}} \times [-\infty, \infty])^{\mathbf{N}}$. Then there exists a probability N_ν on $\mathcal{P}(C^+_{\text{tem}})$ the space of probabilities on C^+_{tem} so that $\mu = \int_{\mathcal{P}(C^+_{\text{tem}})} \mu N_\nu(d\mu)$ and so that $Q^\mu(((Z(1), V_0(1)), (Z(2), V_0(2)), \ldots) \in B) \in \{0,1\}$ for all shift invariant B for N_ν almost all μ. Since $A = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n Z(i)$ is shift invariant, part b) follows from this decomposition and the ergodic case. An ergodic decomposition theorem is proved in [13] section 5.2. That theorem is stated for the case of a measure preserving system. To obtain the decomposition above it is necessary only to check that when the system arises from a Markov process the stationary measure is decomposed into elements of the Markov family.

c) We have to tighten the coupling argument given in [10] in order that we don't drop below the correct wavespeed. We indicate below the changes necessary. However we refer the reader to [10] section 2.2 for the basic steps and definitions. We shall couple a solution to (1) with an N-dependent oriented site percolation with density $1-\rho$. For small enough ρ oriented percolation has a leading occupied site that itself moves with a limiting linear speed. The proof of part c) is based on the fact that as $\theta \to \infty$ the density of the coupled percolation process approaches 1 and the speed of the leading occupied site approaches its maximum value of 1.

Lemma 4.2 *For $\epsilon \in (0,1)$ there exists $\rho(\epsilon) > 0$ so that if $(\omega(m,n) : m+n$ even) is a 1-dependent oriented site percolation process with density $1-\rho \geq 1-\rho(\epsilon)$, C is the cluster of occupied sites connected to $(0,0)$ and $s_n = \sup(m : (m,n) \in C)$ then*

$$\liminf_{n\to\infty} P(s_n \geq (1-\epsilon)n) \geq 1-\epsilon.$$

Proof of lemma 4.2. We can extract a proof from Durrett [4]. In section 10 of [4] he shows that $P(|C| < \infty) \to 0$ as $\rho \to 1$ ($|C|$ is the cardinality of C). Let $\tilde{C}$ be the cluster of points connected to $(0,m)$ for some $m = 0,-2,-4,\ldots$ Define also $\tilde{s}_n = \sup(m : (m,n) \in \tilde{C})$. Then on $\{|C| = \infty\}$ we have $s_n = \tilde{s}_n$. In section 11 of [4] he shows that

$$P(\tilde{s}_n \leq (1-\epsilon)n) \leq 3^n(1-\rho)^{\epsilon n/36}(1-3(1-\rho)^{1/36})^{-1}$$

whenever $(1-\rho) < 3^{-36}$. These results imply our lemma. •

Lemma 4.3 *Given $\epsilon \in (0,1/2)$ there exists $M(\epsilon) \geq 1$ such that for all $M \geq M(\epsilon)$, if h is a solution to the following PDE,*

$$\begin{aligned} h_t &= h_{xx} + (1-\epsilon)h \\ h(0,x) &= \delta \mathrm{I}(x \in (-1,1)) \\ h(t,-2M) &= h(t,2M) = 0 \quad \textit{for}\, t \geq 0 \end{aligned}$$

then $h((1+\epsilon)M/2, x) \geq 2\delta \mathrm{I}(-M-1, M+1)$ for all $|x| \leq 2M$.

Proof of lemma 4.3. By linearity we may assume $\delta = 1$. Define $g(t,x) = \exp((1-\epsilon)t)P_t\mathrm{I}(-1,1)(x)$ so that also $g_t = g_{xx} + (1-\epsilon)g$. Note that g is symmetric, unimodal, $g(0) = h(0)$ and $g \geq h$. Using a single Riemann block to approximate the convolution in the definition of g gives the bound, for $|x| \leq M+1$,

$$\begin{aligned} & g((1+\epsilon)M/2, x) \\ \geq\ & g((1+\epsilon)M/2, M+1) \\ \geq\ & (\pi(1+\epsilon)M/2)^{-1/2}\exp((1-\epsilon^2)(M/2) - (M+2)^2(2M(1+\epsilon))^{-1}) \\ \geq\ & 3 \quad \text{for sufficiently large } M. \end{aligned}$$

Also by the maximum principle, for $|x| \leq 3M, t \leq (1+\epsilon)M/2$

$$\begin{aligned} & g(t,x) - h(t,x) \\ \leq\ & \sup_{t \leq (1+\epsilon)M/2} g(t,-3M) \vee g(t,3M) \\ \leq\ & \sup_{t \leq (1+\epsilon)M/2} (\pi t)^{-1/2}\exp(((1-\epsilon)t - ((3M-1)^2/4t)) \\ \leq\ & 1 \quad \text{for sufficiently large } M. \end{aligned}$$

Combining these two bounds completes the proof. •

We now complete the proof of Proposition 4.1 c). Fix $\epsilon \in (0, 1/2)$. Choose $\rho(\epsilon), M(\epsilon)$ as in lemmas 4.2, 4.3. Let u be a solution to (1) with parameter value θ. Define $v(t,x) = \theta^{-1}u(\theta^{-1}t, \theta^{-1/2}x)$. Define $\delta(\epsilon) = \epsilon \exp(-(1-\epsilon^2)M/2)$. Define for $m \in \mathrm{Z}, n \in \mathrm{N}$, $m + n$ even

$$\eta(m,n) = \begin{cases} 1 & \text{if } v((1+\epsilon)Mn/2, \cdot) \geq \delta \mathrm{I}[Mm-1, Mm+1] \\ 0 & \text{otherwise} \end{cases}$$

Further, if $\eta(m-1,n-1) = \eta(m+1,n-1) = 0$ we set

$$\omega(m,n) = \begin{cases} 1 & \text{with probability } 1-\rho \\ 0 & \text{with probability } \rho \end{cases}$$

independently for each m, n. Otherwise we set $\omega(m,n) = \eta(m,n)$. We may now follow the argument in [10] section 2.2 (where our lemma 4.3 is needed in place of lemma 2.2.4) to show that for all sufficiently large θ, say $\theta \geq \theta(\epsilon)$ the process $(\omega(m,n) : m+n \text{ even})$ is a 1-dependent oriented site percolation process of density at least $1 - \rho(\epsilon)$. We now choose our solution u to (1) to have initial condition $u(0) \geq \delta\theta \mathrm{I}(-\theta^{-1/2}, \theta^{1/2})$. Then $v(0) \geq \delta \mathrm{I}(-1, 1)$. Then we have by lemma 4.2, provided $\theta \geq \theta(\epsilon)$

$$\begin{aligned} & P(R_0(u(\theta^{-1}(1+\epsilon)nM/2)) \leq (1-\epsilon)\theta^{-1/2}nM) \\ = \; & P(R_0(v((1+\epsilon)nM/2)) \leq (1-\epsilon)M) \\ \leq \; & P(s_n \leq (1-\epsilon)n) \\ \leq \; & 2\epsilon \quad \text{for large } n. \end{aligned}$$

We now suppose that Q^ν is the law of a travelling wave solution to (1) with parameter value $\theta \geq \theta(\epsilon)$. Define

$$\begin{aligned} T &= \inf(t \geq 0 : V_0(t) > \delta\theta \mathrm{I}[-x - \theta^{-1/2}, -x + \theta^{1/2}], \; \exists x \in \mathrm{N}), \\ X &= \inf(x \in \mathrm{N} : V_0(T) > \delta\theta \mathrm{I}[-x - \theta^{-1/2}, -x + \theta^{1/2}]). \end{aligned}$$

We claim (and prove below) that $Q^\nu(T < \infty) = 1$. Given the claim, we may use the strong Markov property at time T to see that $u(t,x) = U(T + t, x + R_0(U(T)) - X)$ defines a solution whose initial condition satisfies $u(0) \geq \delta\theta \mathrm{I}(-\theta^{-1/2}, \theta^{1/2})$. Then from the above, for all sufficiently large n

$$Q^\nu(R_0(U(T + \theta^{-1}(1+\epsilon)nM/2)) \leq (1-\epsilon)\theta^{-1/2}nM + R_0(U(T)) - X) \leq 2\epsilon.$$

This implies that $Q^\nu(A < 2(1-\epsilon)(1+\epsilon)^{-1}\theta^{1/2}) \leq 2\epsilon$ and completes the proof of part c).

To prove the claim we define $F, \tilde{F} : C^+_{\mathrm{tem}} \to [0,1]$ by

$$\begin{aligned} F(f) &= \mathrm{I}(f > \delta\theta \mathrm{I}[-x - \theta^{-1/2}, -x + \theta^{1/2}], \; \exists x \in \mathrm{N}), \\ \tilde{F}(f) &= Q^f(F(V_0(1))). \end{aligned}$$

By the ergodic theorem $n^{-1}\sum_{i=1}^n F(V_0(i)) \to Z$ almost surely under Q^ν. We shall show that $Z > 0$ which implies the claim.

$$
\begin{aligned}
& n^{-1}\sum_{i=1}^{n} F(V_0(i)) - \tilde{F}(V_0(i)) \\
= \; & n^{-1}\sum_{i=1}^{n-1} (F(V_0(i+1)) - E(F(V_0(i+1))|\mathcal{U}_i)) + F(V_0(1))/n + \tilde{F}(V_0(n))/n \\
= \; & n^{-1}M_n + O(1/n)
\end{aligned}
$$

where $Q^\nu(M_n^2) \leq n$. This shows that $n^{-1}\sum_{i=1}^n \tilde{F}(V_0(i)) \to Z$ in probability and so by the ergodic theorem again $Z = E(\tilde{F}(V_0(1))|\mathcal{I})$ where $\mathcal{I}$ are the invariant sets for the sequence $(V_0(1), V_0(2), \ldots)$. To prove that $Z > 0$ it is now enough to show that $\tilde{F}(f) > 0$ for all $f \neq 0$. By lemma 3.1 b) it is enough to consider $f \in C_c^\infty$. Write $\bar{Q}^f$ for the law of super Brownian motion started at f (i.e. a solution to (16)). By Evans and Perkins [6] theorem 3.10, for $f \in C_c^\infty$ the laws of $U(t,x)dx$ under Q^f and $\bar{Q}^f$ are absolutely continuous on $M(\mathrm{R})$ the space of finite measures. Moreover, by Evans and Perkins [5] corollary 2.3, if $f, g \in C_c^\infty$, $f, g \neq 0$ then the laws of $U(t,x)dx$ under $\bar{Q}^f$ and $\bar{Q}^g$ are also absolutely continuous. We may write $\tilde{F}(f) = Q^f(U(t,x)dx \in B)$ for some measurable $B \subseteq M(\mathrm{R})$. For suitable g, s it is possible by using the existence of jointly continuous densities and the finite speed of propagation of $R_0(t)$ to show that $\bar{Q}^g(U(t,x)dx \in B) > 0$, Then the above absolute continuity results finish the proof. •

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