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Mean square stability for discrete linear stochastic systems

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ABSTRACT. Several results concerning asymptotical mean square stability of the null solution of specific linear stochastic systems are presented and proven. It is shown that the mean square stability of the implicit Euler method, taken from the monography of Kloeden and Platen (1992) and applied to linear stochastic differential equations, is necessary for the mean square stability of the corresponding implicit Milstein method (using the same implicitness parameter). Furthermore, a sufficient condition for the mean square stability of the implicit Euler method can be varified for autonomous systems. Additionally, the principle of 'monotonous inclusion' of the sequel of mean square stability domains holds for linear systems. The paper generalizes the results due to Schurz (1993) where one-dimensional linear complex systems with respect to asymptotical p–th mean stability have been investigated. Finally, a simple example confirms these assertions. The results can also be used to deduce recommendations for the practical implementation of numerical methods solving nonlinear systems by orienting on their linearization.

1. Two iterated functional systems and mean square stability

Although the requirement of mean square stability represents a relatively strong stability assertion we are going to examine discrete stochastic systems with respect to mean square stability. The examination goes back to linear systems, keeping in mind that they are obtained as the linearization of corresponding nonlinear stochastic systems. It is provided that the original nonlinear systems have components (drift and diffusion) vanishing at the origin. This is the case, e.g. in population dynamics, if Lotka–Volterra systems are disturbed by multiplicative noise. Note that we will not discuss and enlighten the relation between the original nonlinear system and its corresponding linearized one. Khas'minskij [5] has done it for stochastic differential equation in some extent, but it is still an open question for the discrete systems to be mentioned here.
Consider the following two iterated systems for nonrandom real–valued $d \times d$ - matrices A_n and B_n^j, $(d = 1, 2, \ldots)$ and standard Gaussian distributed random variables $\xi_n^j (i.i.d., n = 0, 1, 2, \ldots, j = 1, \ldots, m)$

$$Y_{n+1}^{(M)} = Y_n^{(M)} + \{\alpha A_{n+1} Y_{n+1}^{(M)} + (1-\alpha) A_n Y_n^{(M)}\}\Delta + \sum_{j=1}^{m} B_n^j Y_n^{(M)} \xi_n^j \sqrt{\Delta} + + \sum_{j,k=1}^{m} B_n^j B_n^{k^T} Y_n^{(M)} \int_0^{\Delta} \int_0^{s} d\xi_n^j(r) d\xi_n^k(s) \Delta \tag{1.1}$$

and

$$Y_{n+1}^{(E)} = Y_n^{(E)} + \{\alpha A_{n+1} Y_{n+1}^{(E)} + (1-\alpha) A_n Y_n^{(E)}\}\Delta + \sum_{j=1}^{m} B_n^j Y_n^{(E)} \xi_n^j \sqrt{\Delta}, \tag{1.2}$$

starting at $Y_0 \in \mathbb{R}^d$ for a given fixed $\Delta > 0$. System (1.1) is often called implicit Milstein method and (1.2) implicit Euler method with implicitness parameter $\alpha \in [0, 1]$. For the resolution of the system of algebraic equations (1.1) as well as (1.2) we have to require the existence of the inverse of $I - \alpha \Delta A_{n+1}$ at any step n in the case of $\alpha > 0$. For the same parameter value α we also call the method (1.2) the Euler method corresponding to the Milstein method (1.1). System (1.1) as well as (1.2) can be interpreted as numerical solution of the stochastic differential equation

$$dX_t = A(t) X_t dt + \sum_{j=1}^{m} B^j(t) X_t dW_t^j \tag{1.3}$$

at time $t_n = n \cdot \Delta$. Here $(W_t^j)_{j=1,2,\ldots,m}$ are independent identically distributed Wiener processes with $W_t^j = \xi_t^j \sqrt{t}$ and $\xi_{t_n}^j = \sum_{k=0}^{n-1} (\xi_{k+1}^j - \xi_k^j)$; $(W_0^j = \xi_0^j = 0)$. The solution of (1.3) always exists and is unique under appropriate boundedness conditions on the matrices $A(t)$ and $B^j(t)$ over the time interval $[0, T]$. For references, see Arnold [1], Gikhman and Skorokhod [4], Milstein [8], Kloeden and Platen [6], Talay [11] or Kloeden, Platen and Schurz [7]. The methods (1.1) and (1.2) enable us to construct the simplest numerical solution of (1.3). Corresponding convergence results for them justify their application, and the right to be considered as numerical solution of (1.3) at all. In L^1 (the space of absolutely mean integrable functions) or in L^2 (the space of mean square integrable functions) the method (1.1) possesses the same (by definition) convergence order Δ and the method (1.2) order $\Delta^{1/2}$, but in the distributional or weak sense both methods are converging with the same order Δ. Our main result is independent of these convergence notions. Convergence and stability together yield reasonable and well–behaving numerical solutions, as in ordinary numerical analysis. In stochastic analysis there are several stability notions and concepts (cp. Khas'minskij [5]). A pioneer work for stochastic stability theory of such system equations (1.3) has been done by Khas'minskij [5]. Our results base on the concept of mean square stability. To motivate the reader we remind that there are two basic methods to examine the stability of general nonlinear systems (as well as linear ones), the method of stochastic Lyapunov functions and the method of examination of the corresponding linearized system, provided that the drift and diffusion functions vanish at the origin (or, in general, at a stationary point). We are following the latter method.

Definition 1.1. Let $X_t(x_0)$ denote the solution of equation (1.3) started in x_0 at time $t_0 = 0$. Then the null solution $X \equiv 0$ is called **(asymptotically) mean square stable** iff

$$\exists \delta > 0 \; \forall x_0 \in \mathbb{R}^d \quad ||x_0|| < \delta \quad : \quad \lim_{t \to 0} \mathbb{E}||X_t(x_0)||^2 = 0 \tag{1.4}$$

where $||\cdot||$ denotes the Euclidean vector norm in $\mathbb{R}^d$. Furthermore, it is said to be **exponentially mean square stable** iff

$$\exists c_1, c_2 > 0 \quad : \quad \mathbb{E}||X_t(x_0)||^2 \leq c_1 ||x_0||^2 \exp(-c_2(t - t_0)). \tag{1.5}$$

The following theorem proved by Khas'minskij [5] and stated in Arnold [1] yields necessary and sufficient conditions on the matrices A and B^j in order to guarantee an exponentially mean square stable null solution of (1.3). It also ensures sufficient conditions for the existence of asymptotically mean square stable solutions of linear systems, and shows for which systems it makes sense to look at them concerning mean square stability.

Theorem 1.1. *Assume that the matrix-valued functions $A(t)$ and $B^j(t)$ in equation (1.3) are bounded on $[t_0, \infty)$. Then, for exponential stability of the null solution in the mean square sense it is necessary that for any, and sufficient that for a symmetrical, positive definite, continuous and bounded $d \times d$ - matrix $C(t)$ with $x^T C(t) x \geq k_1 |x|^2$ $(k_1 > 0)$ the matrix–valued differential equation*

$$\frac{dD(t)}{dt} + A^T(t)D(t) + D(t)A(t) + \sum_{j=1}^{m} B^{j^T}(t)D(t)B^j(t) = -C(t) \tag{1.6}$$

possesses a solution matrix $D(t)$ with the same properties as the matrix $C(t)$.

Remark. For autonomous systems, equation (1.6) obtains a simpler structure

$$\mathcal{H}D := A^T D + DA + \sum_{j=1}^{m} B^{j^T} D B^j = -Q, \tag{1.7}$$

i.e. the bounded linear operator $\mathcal{H}$ possesses always a positive definite inverse for any positive definite matrix $Q \in \mathfrak{S}_{d\times d}$ (the space of symmetrical $d \times d$ - matrices). These facts in mind, we introduce the notion of the mean square stable null solution of discrete time systems, such as (1.1) or (1.2), in an analogous way.

Definition 1.2. A numerical solution $(Y_n)_{n\in\mathbb{N}}$ started in y_0 at time $t_0 = 0$ has an **(asymptotically) mean square stable** null solution iff

$$\exists \delta > 0 \ \forall y_0 \in \mathbb{R}^d \quad ||y_0|| < \delta : \lim_{n\to\infty} \mathbb{E}\,||Y_n||^2 = 0 \tag{1.8}$$

where we understand the limit in (1.8) taken only at discrete time t_n.

$()^T$ denotes the transposed of the inscribed vector or matrix. In the following our examination draws back to look at the time evolution of the symmetrical matrices

$$P(t) = \mathbb{E}X_t(x_0)X_t^T(x_0) = (\mathbb{E}X_t^iX_t^j) \tag{1.9}$$

for the continuous system (1.3), and for the discrete systems (1.1) or (1.2)

$$P_n = \mathbb{E}Y_nY_n^T = (\mathbb{E}Y_n^iY_n^j). \tag{1.10}$$

From there we can gain assertions about the mean square stability of the null solution. Obviously, mean square stability is equivalent to the stability of the corresponding matrix system (1.9) or (1.10) because of

$$||P(t)||_{d\times d} \leq K \cdot \mathbb{E}||X_t(x_0)||^2 \quad \text{and} \quad \mathbb{E}||X_t(x_0)||^2 = \text{ trace } (P(t)) = \sum_{i=1}^{d} P_{ii}(t) \tag{1.11}$$

($||\cdot||$ is the Euclidean vector norm on $\mathbb{R}^d$ and $||\cdot||_{d\times d}$ any compatible matrix norm on $\mathbb{R}^{d\times d}, K > 0$).

2. A necessary condition for mean square stability of system (1.1)

After the introductory words the following theorem can be formulated.

Theorem 2.1. *Assume that* $\mathbb{E}Y_0^{(E)}Y_0^{(E)^T} = \mathbb{E}Y_0^{(M)}Y_0^{(M)^T} = \mathbb{E}X_0X_0^T \in \mathbb{R}^{d\times d}$ *is a positive semi–definite matrix with* Y_0 *being independent of* ξ_n*, and that the matrices*

$$C_n := (I - \alpha\Delta A_{n+1})^{-1} \quad (n = 0, 1, 2, \dots)$$

always exist (e.g., this holds if $||\alpha\Delta A(t)|| < 1$ *uniformly in* $t \in [0, \infty)$*, or if the matrices* $A(t)$ *have only nonpositive eigenvalues, as it is the case in mean square stable, autonomous systems (1.3)).*
Then, for the linear stochastic systems (1.1) and (1.2) the following inequality holds

$$\forall n \in \mathbb{N} \quad : \quad \mathbb{E}\,Y_n^{(E)}Y_n^{(E)^T} \leq \mathbb{E}\,Y_n^{(M)}Y_n^{(M)^T}. \tag{2.1}$$

Thereby it can be immediately concluded that the implicit Milstein method possesses a mean square stable null solution if the corresponding Euler method possesses it too. The inequality character in (2.1) is understood in the sense of positive semi–definite matrices S, i.e.

$$x^T S x \geq 0 \quad \forall x \in \mathbb{R}^d$$

or in other words, if between two positive semi–definite matrices S_1 and S_2 the symbolic relation $S_1 \leq S_2$ is used the following relation is meant

$$\forall x \in \mathbb{R}^d : x^T(S_2 - S_1)x = x^T S_2 x - x^T S_1 x \geq 0 \Longleftrightarrow S_1 \leq S_2 \,.$$

In the following $\mathfrak{S}$ denotes the space of symmetrical real–valued $d \times d$ matrices S, and $\mathfrak{S}^+$ the space of positive semi-definite $d \times d$ matrices (also symmetrical by definition). The proof of the theorem 2.1 is done via induction.

Proof. For $n = 0$ the assertion is obviously valid by assumption. Suppose the relation

$$P_n^{(E)} := \mathbb{E}\, Y_{n-1}^{(E)^T} \leq \mathbb{E}\, Y_{n-1}^{(M)} Y_{n-1}^{(M)^T} =: P_n^{(M)} \tag{2.2}$$

is satisfied for a fixed $n \geq 1$ where $P_n^{(E)}$ and $P_n^{(M)}$ are positive semi–definite. Now we show the validity for $n+1$. Systems (1.1) and (1.2) can be rewritten in the equivalent form

$$Y_{n+1}^{(M)} = C_n(I + (1-\alpha)\Delta A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta} + \sum_{j,k=1}^{m} B_n^j B_n^{k^T} V_n^{j,k} \Delta) Y_n^{(M)} \tag{2.3}$$

and

$$Y_{n+1}^{(E)} = C_n(I + (1-\alpha)\Delta A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta}) Y_n^{(E)} \tag{2.4}$$

where

$$C_n = (I - \alpha \Delta A_{n+1})^{-1} \quad \text{and} \quad V_n^{j,k} = \int_0^{\Delta}\int_0^{s} d\xi_n^j(r) d\xi_n^k(s) \,. \tag{2.5}$$

Consequently, one obtains the matrix equations

$$
\begin{aligned}
P_{n+1}^{(M)} =& \mathbb{E}\, Y_{n+1}^{(M)} Y_{n+1}^{(M)^T} = \mathbb{E}\, C_n \left(I + (1-\alpha)A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta} \right.\\
&\left. + \sum_{j,k=1}^{m} B_n^j B_n^{k^T} V_n^{j,k} \Delta \right) Y_n^{(M)} Y_n^{(M)^T} \left(I + (1-\alpha)\Delta A_n^T \right.\\
&\left. + \sum_{j=1}^{m} B_n^{j^T} \xi_n^j \sqrt{\Delta} + \sum_{j,k=1}^{m} B_n^k B_n^{j^T} V_n^{j,k} \Delta \right) C_n^T \\
=& C_n (I + (1-\alpha)\Delta A_n) P_n^{(M)} (I + (1-\alpha)\Delta A_n)^T C_n^T \\
&+ \sum_{j=1}^{m} C_n B_n^j P_n^{(M)} B_n^{j^T} C_n^T \Delta \\
&+ \sum_{j_1,k_1=1}^{m} C_n B^{j_1} B^{k_1^T} P_n^{(M)} \mathbb{E} \left(I + (1-\alpha)\Delta A_n^T \right.\\
&\left. + \sum_{j=1}^{m} B_n^{j^T} \xi_n^j \sqrt{\Delta} + \sum_{j_2,k_2=1}^{m} B_n^{k_2} B_n^{j_2^T} V_n^{j_2,k_2} \Delta \right) C_n^T V_n^{j_1,k_1} \Delta \\
=& C_n (I + (1-\alpha)\Delta A_n) P_n^{(M)} (I + (1-\alpha)\Delta A_n)^T C_n^T \\
&+ \sum_{j=1}^{m} C_n B_n^j P_n^{(M)} B_n^{j^T} C_n^T \Delta + \sum_{j,k=1}^{m} C_n B^j B^{k^T} P_n^{(M)} B_n^k B_n^{j^T} C_n^T \Delta^2/2
\end{aligned}
\tag{2.6}
$$

For the latter two conclusions the following relations have been used. It is well–known that

$$
\mathbb{E}\, \xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\, V_n^{j,k} = 0 \quad \forall j, k = 1, \ldots, m
$$

as a property of the Itô–integral being a martingale. Furthermore, in [6] (Lemma 5.7.2, p. 191) one finds the equation

$$
\begin{aligned}
\mathbb{E}\, I_{j_1,k_1}(\Delta) I_{j_2,k_2}(\Delta) &= \mathbb{E} \left(\int_0^{\Delta} \int_0^{s} dW^{j_1}(r) dW^{k_1}(s) \cdot \int_0^{\Delta} \int_0^{s} dW^{j_2}(r) dW^{k}(s) \right) \\
= \Delta^2 \mathbb{E} \left(\int_0^{\Delta} \int_0^{s} d\xi^{j_1}(r) d\xi^{k_1}(s) \cdot \int_0^{\Delta} \int_0^{s} d\xi^{j_2}(r) d\xi^{k_2}(s) \right) &= \delta_{j_1,j_2} \cdot \delta_{k_1,k_2} \cdot \Delta^2/2.
\end{aligned}
$$

(cp. also p. 223 in [6]). Analogously one argues with the remainder terms. Using Lemma 5.12.3 from [6] (p. 221) one encounters with

$$
\begin{aligned}
I_j(\Delta) I_{j_1,k_1}(\Delta) &= \int_0^{\Delta} I_{j_1,k_1}(s) dW_s^j + \int_0^{\Delta} I_{j_1}(s) I_j(s) dW_s^{k_1} \\
&+ \int_0^{\Delta} I_{j_1}(s) \cdot 1\!\!\mathrm{I}_{\{j=k_1\neq 0\}} ds
\end{aligned}
$$

where $1\!\!\mathrm{I}_{\{j=k_1\neq 0\}}$ is the indicator function of the inscribed set.

Thereby $\mathbb{E}\,(I_j \cdot I_{j_1,k_1}) = 0$ follows for all $j, j_1, k_1 = 1, \ldots, m$. Thus, relation (2.6) is confirmed after rearranging the matrix products and applying the moment properties mentioned above. Returning to (2.6) we introduce the abbreviation

$$P_{n+1}^{(M)} = \mathcal{L}P_n^{(M)} \tag{2.7}$$

as the operator equation defined via right hand side of (2.6). This operator $\mathcal{L}$ mapping $\mathfrak{S}_{d\times d}$ onto $\mathfrak{S}_{d\times d}$ is linear and bounded. Furthermore $\mathcal{L}$ is nonnegative, i.e. $\mathcal{L}\mathfrak{S}^+ \subseteq \mathfrak{S}^+$. This can be easily varified. According to the assumption $P_n^{(M)}$ is positive semi-definite, and any positive semi-definite matrix can be decomposed by the Cholesky factorization such that

$$P_n^{(M)} = L_n^{(M)} L_n^{(M)^T}$$

where $L_n^{(M)}$ is triangular. Thus, matrices Q_n^l satisfying

$$\begin{aligned}
P_{n+1}^{(M)} &= \mathcal{L}P_n^{(M)} = \sum_{l=0}^{m(m+1)} Q_n^l \cdot Q_n^{l^T} \\
\text{with} \quad Q_n^0 &= C_n(I + (1-\alpha)\Delta A_n)L_n^{(M)}, \quad Q_n^j = C_n B_n^j L_n^{(M)} \cdot \sqrt{\Delta} \\
\text{and} \quad Q_n^l &\in \{C_n B_n^i B_n^{k^T} L_n^{(M)} \cdot \Delta\frac{\sqrt{2}}{2} \;:\; i, k = 1, 2, \ldots, m\}
\end{aligned}$$

$(j = 1, 2, \ldots, m, \quad l = m+1, \ldots m(m+1))$ exist. Now, because a sum of positive semi-definite matrices is again positive semi-definite, we know the validity of the inclusion $\mathcal{L}\mathfrak{S}^+ \subseteq \mathfrak{S}^+$. The difference $P_n^{(M)} - P_n^{(E)}$ must be positive semi-definite according to the induction assumption, hence the relation

$$\mathcal{L}(P_n^{(M)} - P_n^{(E)}) \geq 0 \qquad \left(\Leftrightarrow \mathcal{L}P_n^{(M)} \geq \mathcal{L}P_n^{(E)}\right)$$

follows. Finally, we conclude that

$$\begin{aligned}
P_{n+1}^{(M)} = \mathcal{L}P_n^{(M)} &= \mathcal{L}P_n^{(M)} - \mathcal{L}P_n^{(E)} + \mathcal{L}P_n^{(E)} \\
&= \mathcal{L}P_n^{(E)} + \mathcal{L}(P_n^{(M)} - P_n^{(E)}) \geq \mathcal{L}P_n^{(E)} \geq P_{n+1}^{(E)}\,,
\end{aligned} \tag{2.8}$$

i.e. the validity of the Theorem 2.1 has been established. Note, in (2.8) we used the identity

$$\mathcal{L}P_n^{(E)} = P_{n+1}^{(E)} + \sum_{j,k=1}^{m} C_n B_n^j B_n^{k^T} P_n^{(E)} B_n^k B_n^{j^T} C_n^T \Delta^2/2$$

which follows from (2.6) via the definition of the operator $\mathcal{L}$ in (2.7). □

Conclusion.

Because we know about the meaning of the relation (2.1), we have obtained that even the difference $P_n^{(M)} - P_n^{(E)}$ is positive semi-definite for all $n \in \mathbb{N}$, provided that the assumption of Theorem 2.1 is satisfied. The property of positive semi-definiteness yields nonnegative diagonal elements of the considered matrix. Consequently, Theorem 2.1 also implies that the relation

$$\mathbb{E}\,(Y_{n,i}^{(M)})^2 \geq \mathbb{E}\,(Y_{n,i}^{(E)})^2$$

holds for each component of the subscribed vectors. Furthermore, the assumption that the matrix $\mathbb{E}\ X_0 X_0^T$ is positive semi–definite can be considered as reasonable and naturally fulfilled. It involves such cases as independent initial random variables X_0^i with $\mathbb{E}\ X_0^i = 0$. Moreover, Theorem 1.1 justifies that the requirement of positive semi–definiteness of the initial moment matrix $P(0)$ is not restrictive. For linear systems, it is also confirmed by Theorem 8.5.5 in Arnold [1], as consequence of a property of covariance operators.

3. A SIMPLE COMPLEX EXAMPLE

The one–dimensional complex model equation

$$dX_t = \lambda X_t dt + \gamma X_t dW_t\ , \quad X_0 = x_0 \tag{3.1}$$

has the exact solution

$$X_t = X_0 \cdot \exp((\lambda - \gamma^2/2)t + \gamma W_t)$$

with its second moment

$$\begin{aligned} \mathbb{E}\ X_t X_t^* &= \mathbb{E}\ \exp(2(\lambda - \gamma^2/2)_r t + 2\gamma_r W_t) \cdot ||x_0||^2 \\ &= ||x_0||^2 \cdot \exp(2(\lambda_r - \gamma_r^2/2 + \gamma_i^2/2)t + 2\gamma_r^2 t) \\ &= ||x_0||^2 \cdot \exp((2\lambda_r + ||\gamma||^2)t) \end{aligned}$$

where $x_0 \in \mathbb{C}$ is nonrandom (z_r is the real part, z_i the imaginary part of $z \in \mathbb{C}$). * denotes the conjugate complex value. The trivial solution $X \equiv 0$ of (3.1) is mean square stable for the process $\{X_t : t \geq 0\}$ iff $2\lambda_r + ||\gamma||^2 < 0$. Applied to equation (3.1) the implicit Milstein (1.1) and Euler methods (1.2) are given by

$$Y_{n+1}^{(M)} = \frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta} + \gamma^2(\xi_n^2 - 1)\Delta/2}{1 - \alpha\lambda\Delta} \cdot Y_n^{(M)} \tag{3.2}$$

and

$$Y_{n+1}^{(E)} = \frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta}}{1 - \alpha\lambda\Delta} \cdot Y_n^{(E)}\ , \tag{3.3}$$

respectively. Their second moments satisfy the relations

$$\begin{aligned} \mathbb{E}\ Y_{n+1}^{(M)} Y_{n+1}^{(M)*} \quad &= \quad \mathbb{E}\ ||\frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta}}{1 - \alpha\lambda\Delta}||^2 \cdot \mathbb{E}\ Y_n^{(M)} Y_n^{(M)*} \\ &\quad + \mathbb{E}\ ||\frac{\gamma^2(\xi_n^2 - 1)}{1 - \alpha\lambda\Delta}||^2 \cdot \mathbb{E}\ Y_n^{(M)} Y_n^{(M)*} \cdot \Delta^2/4 \\ = \mathbb{E}\ Y_0^{(M)} Y_0^{(M)*} \quad &\cdot \quad \left(\frac{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta + ||\gamma||^4\Delta^2/2}{||1 - \alpha\lambda\Delta||^2}\right)^{n+1} \\ > \mathbb{E}\ Y_0^{(E)} Y_0^{(E)*} \quad &\cdot \quad \left(\frac{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta}{||1 - \alpha\lambda\Delta||^2}\right)^{n+1} \\ = \mathbb{E}\ Y_{n+1}^{(E)} Y_{n+1}^{(E)*} \quad & \qquad \left(\text{provided that } \mathbb{E}\ Y_0^{(M)} Y_0^{(M)*} \leq Y_0^{(E)} Y_0^{(E)*}\right), \end{aligned}$$

or equivalently

$$
\begin{aligned}
\mathbb{E}\, Y_{n+1}^{(M)} Y_{n+1}^{(M)*} &= \mathbb{E}\, Y_{n+1}^{(E)} Y_{n+1}^{(E)*} \cdot \left(\frac{||1+(1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta + ||\gamma||^4\Delta^2/2}{||1+(1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta} \right)^{n+1} \\
&= \mathbb{E}\, Y_{n+1}^{(E)} Y_{n+1}^{(E)*} \cdot \left(1 + \frac{||\gamma||^4\Delta^2/2}{||1+(1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta} \right)^{n+1} .
\end{aligned}
$$

Hence the implicit Milstein method (3.2) possesses a mean square stable null solution if the corresponding Euler method (3.3) possesses it too. Moreover, the mean square stability domain of (3.2) is smaller than the corresponding mean square stability domain of (3.3) for any implicitness $\alpha \in [0,1]$. This special result has been already proven in [10]. Also, it can be concluded that the implicit Euler method (3.3) has a mean square stable null solution if $\alpha \geq \frac{1}{2}$ and $2\lambda_r + ||\gamma||^2 < 0$. The latter condition coincides with the necessary and sufficient condition for the mean square stability of the null solution for the complex equation (3.1), as already mentioned.

4. Mean square stability of (1.2) solving autonomous systems (1.3)

Finally, an interesting result concerning mean square stability for the implicit Euler methods (1.2) is to be formulated. For this purpose we only consider autonomous systems, i.e. systems (1.2) and (1.3) with time–independent matrices A and B^j. Furthermore, the validity of relation (1.7) is required. Thereby we examine systems (1.2) where the corresponding differential equation (1.3) has a mean square stable null solution. Assume that all eigenvalues of the matrix A are negative. This requirement is necessary for mean square stability of the null solution of system (1.3). It additionally implies the existence of the inverse of $I - \alpha\Delta A$ for all $\alpha \geq 0$ and $\Delta > 0$. For such autonomous systems, the notation

$$
Y_{n+1} = (I - \alpha\Delta A)^{-1}(I + (1-\alpha)\Delta A + \sum_{j=1}^{m} B^j \sqrt{\Delta}\xi_n^j) Y_n \tag{4.1}
$$

is used for the Euler method with implicitness $\alpha \in [0,1]$. Then the system of second moments $\mathbb{E}\, Y_n^i Y_n^j$ of this method satisfies the inequality

$$
\begin{aligned}
0 \leq P_{n+1} =& \mathbb{E}\, Y_{n+1}Y_{n+1}^T = (\mathbb{E}\, Y_{n+1}^i Y_{n+1}^j) \\
=& (I-\alpha\Delta A)^{-1}\Bigg(P_n + (1-\alpha)^2\Delta^2 A P_n A^T + (1-\alpha)\Delta(AP_n + P_n A^T) \\
& + \Delta\sum_{j=1}^{m} B^j P_n B^{j^T}\Bigg)(I-\alpha\Delta A^T)^{-1} \\
=& (I-\alpha\Delta A)^{-1}(P_n + \alpha^2\Delta^2 A P_n A^T - \alpha\Delta(AP_n + P_n A^T))(I-\alpha\Delta A^T)^{-1} \\
& + \Delta(I-\alpha\Delta A)^{-1}(AP_n + P_n A^T + \sum_{j=1}^{m} B^j P_n B^{j^T})(I-\alpha\Delta A^T)^{-1} \\
& + (1-2\alpha)\Delta^2(I-\alpha\Delta A)^{-1} A P_n A^T (I-\alpha\Delta A^T)^{-1} \\
=& (I-\alpha\Delta A)^{-1}(I-\alpha\Delta A)P_n(I-\alpha\Delta A)(I-\alpha\Delta A^T)^{-1} \\
& - \Delta(I-\alpha\Delta A)^{-1} Q (I-\alpha\Delta A^T)^{-1} \\
& + (1-2\alpha)\Delta^2(I-\alpha\Delta A)^{-1} A P_n A^T (I-\alpha\Delta A^T)^{-1} \\
<& P_n + (1-2\alpha)\Delta^2(I-\alpha\Delta A)^{-1} A P_n A^T (I-\alpha\Delta A)^{-1^T}
\end{aligned}
\tag{4.2}
$$

where the positive definiteness of matrix Q follows from the condition of mean square stability for (1.3) stated in relation (1.7). The inequality in (4.2) is understood once again in terms of positive definiteness of the $d \times d$–matrices Q, i.e. $0 < x^T Q_1 x < x^T Q_2 x \iff Q_1 < Q_2$ holds for all vectors $x \in \mathbb{R}^d$. Suppose that system (4.1) starts with a positive definite matrix of second moments $(\mathbb{E}\, Y_0^i Y_0^j)$ what is naturally fulfilled for mean square stable systems (1.3). Now, if one chooses $\alpha \geq \frac{1}{2}$ the relation

$$0 \leq \mathcal{L}P_n = P_{n+1} < P_n < P_0 \tag{4.3}$$

follows for all $n = 1, 2, \ldots$. Consequently, $\lim_{k\to\infty} \mathcal{L}P_k < \mathcal{L}P_0 < P_0$ is valid. The limit moment matrix $P := \lim_{k\to\infty} P_k$ must be positive semi–definite because the space $\mathfrak{S}^+$ of positive semi–definite $d \times d$–matrices is closed. Therefore we obtain

$$x^T P x < x^T P_n x < x^T P_0 x$$

for all vectors $x \in \mathbb{R}^d$. Any real positive semi–definite matrix P can be factorized by Cholesky such that $P = LL^T$ where L is a lower triangular matrix with real nonnegative diagonal elements. Furthermore, for any matrix A the symmetrical product AA^T possesses a complete set of orthonormal vectors $e_i \in \mathbb{R}^d$ satisfying the equation

$$AA^T e_i = \lambda_i e_i$$

with $\lambda_i \geq 0$. For further details in linear algebra, for example see the books of Gantmacher [3] or Usmani [12]. Consider now the operator $\mathcal{L}$ on the unit ball $\mathfrak{K}$ in $\mathfrak{S}^+$ defined by

$$\mathfrak{K} := \{S \in \mathfrak{S}^+ : ||S|| \leq 1\} \tag{4.4}$$

where $||S|| = \max\{\lambda_j : \exists e_j \in \mathbb{R}^d : Se_j = \lambda_j e_j\}$. Without loss of generality we can choose an orthonormal set of eigenvectors of the positive semi-definite matrix $\mathcal{L}P_0$, denoted by $\{e_1, \ldots, e_\ell\}$ with $e_j \in \mathbb{R}^d$ and $\ell \leq d$. Thus, using (4.3) we obtain the inequality

$$\lambda_i^n = e_i^T \mathcal{L}^n P_0 e_i < e_i^T \mathcal{L} P_0 e_i = \lambda_i < e_i^T P_0 e_i (= e_i^T p_i e_i) \leq ||P_0|| \leq 1 \qquad (4.5)$$

where $n > 1$ (parenthesis in the case of $\mathcal{L}P_0$ commutes with P_0). Generally, we have

$$\begin{aligned}\lambda_i^n < \lambda_i < \sum_{j=1}^d b_j^i x_j^T P_0 \sum_{k=1}^d b_k^i x_k &= \sum_{j,k=1}^d b_j^i b_k^i x_j^T P_0 x_k = \sum_{j,k=1}^d b_j^i b_k^i \beta_k x_j^T x_k \\ &< \sum_{j=1}^d (b_j^i)^2 \beta_j \leq ||P_0|| \sum_{j=1}^d (b_j^i)^2 = ||P_0|| \leq 1 \end{aligned} \qquad (4.6)$$

where the vectors x_j represent the orthonormal eigenvectors corresponding to the eigenvalues $\beta_j \geq 0$ of the positive semi-definite matrix P_0. Note that the set of vectors x_j has been completed to a full basis of $\mathbb{R}^d$, if necessary. Thereby it follows that all $\lambda_i < 1$ on the unit ball (4.4), but this fact implies that all eigenvalues of $\lim_{k\to\infty} \mathcal{L}^k P_0$ are zero, hence $\lim_{k\to\infty} \mathcal{L}^k P_0 = 0$. The mean square stability of the null solution of the implicit Euler method is obvious, because of the linearity of the operator $\mathcal{L}$, the inclusion $\mathcal{L}\mathfrak{S}^+ \subseteq \mathfrak{S}^+$ and the fact that any eigenvalue of $\mathcal{L}P_0$ must be smaller than one on the ball $\mathfrak{K}$ while (4.6). Thus, the following assertion has been proved.

Theorem 4.1. *Assume that the relation (1.7) is satisfied for the autonomous system (1.3), i.e. the null solution is mean square stable for the stochastic differential equation (1.3) with constant matrices A and B^j.*
Then the implicit Euler method (4.1) with $\alpha \geq \frac{1}{2}$ possesses a mean square stable null solution provided that it starts with a positive definite initial moment matrix $P_0 = \mathbb{E}\, Y_0 Y_0^T$.

Therefore we know numerical methods ($\alpha \geq \frac{1}{2}$) which provide mean square stable solutions under appropriate conditions. Furthermore, it has been proven due to this theorem that for linear systems there is no need to correct stochastically the Euler method by Balanced methods introduced in Milstein et al. [9] in order to achieve mean square stability. A similar result to Theorem 4.1 could be simultanously formulated by Artemiev [2]. In [10] one also finds this result for one-dimensional linear complex equations which are numerically solved by Balanced methods.

5. Monotonous inclusion principle for linear autonomous systems

In this section we present a further interesting result for linear autonomous systems. For the numerical methods defined by the iterated systems (1.1) and (1.2), the **principle of 'monotonous inclusion'** of the sequel of mean square stability domains $(\Gamma_\alpha)_{\alpha\geq 0}$ is discovered, i.e. if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then the mean square stability domain Γ_{α_2} includes the domain Γ_{α_1}. These domains can be expressed via the operator $\mathcal{L}$ having eigenvalues smaller than one. The following result is established.

Theorem 5.1. *Consider autonomous system (1.1) or (1.2) with its mean square operator $\mathcal{L}_\alpha$. Then, for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ the implication*

$$\mathcal{L}_{\alpha_1} P \leq P \Longrightarrow \mathcal{L}_{\alpha_2} P \leq P \tag{5.1}$$

holds, provided that $P \in \mathfrak{S}^+$, and the matrix A has only nonpositive eigenvalues.

Proof. We prove the assertion only for the implicit Euler methods. The proof for the implicit Milstein methods follows similarily. From section 2 and 4 we recall that for the implicit Euler method (1.2)

$$\begin{aligned}\mathcal{L}_\alpha P_n^{(\alpha)} = P_{n+1}^{(\alpha)} = \\ &(I-\alpha\Delta A)^{-1}(I+(1-\alpha)\Delta A)P_n^{(\alpha)}(I+(1-\alpha)\Delta A^T)(I-\alpha\Delta A^T)^{-1} \\ &+\Delta\sum_{j=1}^{m}(I-\alpha\Delta A)^{-1}B^jP_n^{(\alpha)}B^{j^T}(I-\alpha\Delta A^T)^{-1}\end{aligned} \tag{5.2}$$

with $\mathcal{L}_\alpha\mathfrak{S}^+ \subseteq \mathfrak{S}^+$ is valid. Now, suppose that $\mathcal{L}_\alpha P \leq P$ holds for any $P \in \mathfrak{S}^+$ in the sense of positive semi–definite matrices. Then

$$(I-\alpha\Delta A)(\mathcal{L}_\alpha P - P)(I-\alpha\Delta A)^T =$$

$$(I+(1-\alpha)\Delta A)P(I+(1-\alpha)\Delta A)^T + \Delta\sum_{j=1}^{m}B^jPB^{j^T} - (I-\alpha\Delta A)P(I-\alpha\Delta A)^T$$

$$= \Delta(AP+PA^T+\sum_{j=1}^{m}B^jPB^{j^T}) + (1-2\alpha)\Delta^2APA^T \leq 0 \tag{5.3}$$

follows. From here we conclude that

$$\begin{aligned}&(I-\alpha_2\Delta A)(\mathcal{L}_{\alpha_2}P-P)(I-\alpha_2\Delta A)^T \\ =&\Delta(AP+PA^T+\sum_{j=1}^{m}B^jPB^{j^T}) + (1-2\alpha_2)\Delta^2APA^T \\ \leq&\Delta(AP+PA^T+\sum_{j=1}^{m}B^jPB^{j^T}) + (1-2\alpha_1)\Delta^2APA^T \\ =&(I-\alpha_1\Delta A)(\mathcal{L}_{\alpha_1}P-P)(I-\alpha_1\Delta A)^T \leq 0\end{aligned} \tag{5.4}$$

because of positive semi-definiteness of APA^T and the relation $(1-2\alpha_2) \leq (1-2\alpha_1)$. Thereby, we obtain $(I-\alpha_2\Delta A)(\mathcal{L}_{\alpha_2}P-P)(I-\alpha_2\Delta A)^T \leq 0$, what implies the relation $\mathcal{L}_{\alpha_2}P \leq P$. □

Remark. The proof of Theorem 5.1 is mainly based on the fact that transformations CPC^T with any invertible matrix C do not change the positive or negative semi-definiteness of matrices. This fact can be found in linear algebra books like, e.g. Usmani [12]. Consequently, the 'most stable null solution in mean square sense' is provided by the completely drift–implicit Euler method (Milstein method) with $\alpha = 1$ within the class of implicit Euler methods (Milstein methods, resp.) with implicitness $\alpha \in [0, 1]$, at least for linear autonomous systems. The assertion of the Theorem 5.1 can be carried over to nonautonomous systems if one additionally requires monotonously decreasing eigenvalues of the negative semi-definite matrices A_n or $A(t)$, respectively, as well as this is possible for Theorem 4.1.

6. Conclusions and remarks

By Theorem 2.1 one knows that a higher order method does not improve the mean square stability behaviour in comparison with the stability behaviour of a corresponding lower order method. Thus, it is not recommendable to look for a higher order mean square stable numerical solution before the class of lower order methods, such as implicit Euler or more general Balanced methods (see [9]) has not been carefully examined. The proof of Theorem 2.1 can be directly generalized to the case of weak approximations or weak numerical solutions, because it only uses the independence of the random variables ξ_n^i and $V_n^{j,k}$ for $i \neq j, k$, respectively, as well as some moment properties of these random variables. Furthermore, it should be possible to extend this result to other higher order weak and strong numerical solutions arising from Taylor methods proposed in [6].
A further basic result concerning mean square stability of the null solution of the implicit Euler method solving autonomous linear stochastic differential equations could be obtained. For $\alpha \in [\frac{1}{2}, 1]$ the Euler method possesses a mean square stable null solution under the assumption that the corresponding continuous linear system possesses one. That means, it is not necessary to add a stochastic term in Balanced methods proposed by Milstein et al. [9] in order to achieve control in the mean square stable sense. Note, the validity of this fact for simple linear complex systems has been already shown in [10].
Finally, the 'monotonous inclusion principle' of mean square stability domains could be varified. For increasing implicitness $\alpha \in [0, 1]$ the mean square stability domains Γ_α of the implicit Euler method as well as of the implicit Milstein method increase monotonously. Thus, in the situation of mean square stability of the corresponding continuous system the use of completely drift–implicit method ($\alpha = 1$) is recommended for practical implementation. Note, for $\alpha = 0.5$ and autonomous systems we have just the situation that the mean square stability of the linear stochastic differential equation (1.3) is ensured iff the null solution is mean square stable for the corresponding Euler method (1.2).
In passing, Theorems 4.1 and 5.1 are also valid for $\alpha > 1$. This case covers a special class of Balanced methods, the class which does not use stochastic weights (matrices multiplied by the current absolute increment of the Wiener process), and with deterministic weight matrix $(-\alpha)A$ ($\alpha \geq 0$, provided that the matrix A is negative

semi-definite). See in [9] for its structure.
Although progress concerning mean square stability of discrete linear systems could be made it is still necessary to extend the examination to nonlinear systems. However, the reader has already received recommendations through this paper in the nonlinear situation. One linearizes the nonlinear equation, checks the mean square stability of the obtained system, works out appropriate numerical solutions, and finally, one applies a corresponding numerical method (being preferable for the linearized equation) to the nonlinear system.

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