Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

On safe crack shapes in elastic bodies

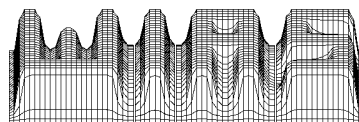Dietmar Hömberg[1] and Alexander M. Khludnev[2]

submitted: 20 Feb 2002

[1] Weierstrass Institute for Applied Analysis and Stochastics
    Mohrenstraße 39, D – 10117 Berlin, Germany
    E-Mail: hoemberg@wias-berlin.de
        [2] Lavrentyev Institute of Hydrodynamics
            of the Russian Academy of Sciences
            Novosibirsk 630090, Russia
            E-Mail: khlud@hydro.nsc.ru

Abstract

According to the Griffith criterion, a crack propagation occurs provided that the derivative of the energy functional with respect to the crack length reaches some critical value. We consider a generalization of this criterion to the case of nonlinear cracks satisfying a non-penetration condition and investigate the dependence of the shape derivative of the energy functional on the crack shape. In the paper, we find the crack shape which gives the maximal deviation of the energy functional derivative from a given critical value and, in particular, prove that this optimality problem admits a solution.

Key words: *Griffith criterion, nonlinear crack, shape derivative*
AMS subject classifications: *74G65, 74P99, 74R99*

1 Introduction

The well-known Griffith criterion of a crack propagation says that a propagation occurs provided that the derivative of the energy functional with respect to the crack length reaches some critical value. This derivative depends, in particular, on the crack shape. In this work we analyse the dependence of the derivative of the energy functional on the crack shape for the nonlinear crack theory. Having in mind the usual application of the Griffith criterion, our goal is to find the crack shape providing the maximal deviation of the derivative of the energy functional from the critical value. We prove the existence of such a crack shape. The result obtained in the paper is new both for the classical linear crack problem and for the nonlinear crack problem.

Note that the classical linear approach to the crack problem is characterized by the equality type boundary condition at the crack faces. This approach does not exclude the mutual penetration of crack faces which has been remarked in many works. In contrast with this linear approach, the nonlinear models considered in the present work do not allow the mutual penetration between crack faces, and consequently, from the standpoint of applications these nonlinear models are more suitable. Boundary problems describing nonlinear cracks with the nonpenetrationon conditions for many constitutive laws are widely presented in [6]. In this case inequality type restrictions are imposed on the solution which implies the nonlinearity of the analysed problems.

Dependence of solutions on parameters for different domain perturbations has been analysed in many works. The case of smooth domains was considered in [16]. Nonsmooth domains are analysed in [17]. Results on differentiability of the energy

1

functional for domains with cuts (cracks) in elastic problems can be found in [11], [15]. General problems related to solution singularities for nonsmooth domains are presented in [3]–[5], [10] and [13]. For concrete solutions and theory applications we refer to [2], [12] and [14].

The differentiability of the energy functional for the nonlinear crack theory is connected with the necessity to exclude the material derivative of the solution with respect to the perturbation parameter. Such an analysis was provided in [7], [8]. Optimal control in boundary value problems for elastic bodies with restrictions imposed on the solutions can be found in [9]. For classical approaches to optimal control problems in the linear elasticity we refer the reader to [1].

In the next section, we consider perturbations of the equilibrium problem of an elastic body with a crack through a family of domains $\{\Omega_\delta\}$ and present the corresponding shape derivative of the energy functional. In Section 3 we formulate an optimal control problem related to the generalization of the Griffith criterion and prove existence of a solution to this problem.

2 Perturbation of the equilibrium problem

Let $\Omega \subset I\!\!R^2$ be a bounded domain with smooth boundary Γ_0, and Γ_0^c be a smooth curve without selfintersections such that $\Gamma_0^c \subset \Omega$ (see Figure 1). Denote $\Omega_0 = \Omega \backslash \bar{\Gamma}_0^c$.
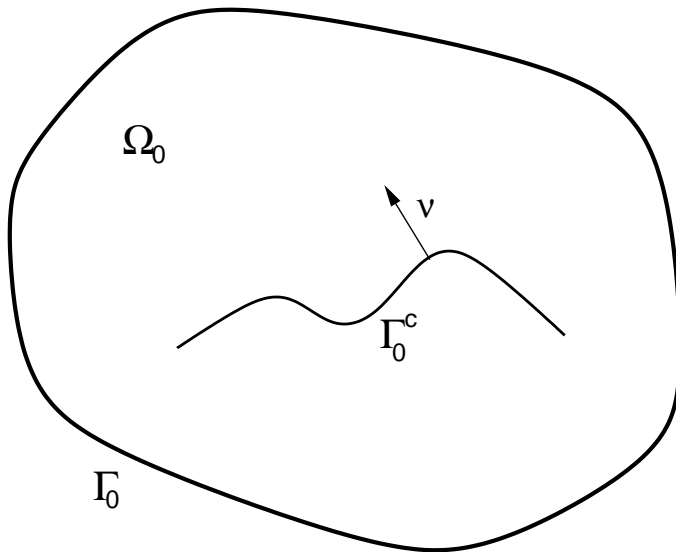

Figure 1: The unperturbed domain Ω_0 with crack Γ_0^c.

It is assumed that an elastic body occupies the domain Ω_0, and Γ_0^c corresponds to the crack in the body. The equilibrium problem for the body can be formulated as follows. We have to find the displacement vector $u = (u_1, u_2)$ such that

$$-\sigma_{ij,j} = f_i\,,\ i = 1, 2\,, \quad \text{in} \quad \Omega_0\,, \tag{2.1}$$

$$\sigma_{ij} = b_{ijkl}\varepsilon_{kl}(u)\,,\ i, j = 1, 2\,, \quad \text{in} \quad \Omega_0\,, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_0\,, \tag{2.3}$$

$$[u]\nu \geq 0\,,\ \sigma_\nu \leq 0\,,\ [\sigma_\nu] = 0\,,\ \sigma_\tau = 0\,,\ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_0^c\,. \tag{2.4}$$

Here $\varepsilon_{kl} = \varepsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k})$ are the strain tensor components, $u_{k,l} = \frac{\partial u_k}{\partial y_l}$, $y = (y_1, y_2) \in \Omega_0$; $\sigma_{ij} = \sigma_{ij}(u)$ are the stress tensor components,

$$\{\sigma_{ij}\nu_j\}_{i=1}^2 = \sigma_\nu \cdot \nu + \sigma_\tau\,,\ \sigma_\nu = \sigma_{ij}\nu_j\nu_i\,.$$

The brackets $[v] = v^+ - v^-$ mean the jump of the function v through Γ_0^c, and $v^\pm$ fit the positive and negative crack faces $\Gamma_0^{c\pm}$ with respect to the unit normal vector ν on Γ_0^c.

As usually, we assume the coefficients b_{ijkl} to satisfy the conditions

$$b_{ijkl} = b_{klij} = b_{jikl}\,,\ b_{ijkl}\xi_{kl}\xi_{ij} \geq c|\xi|^2\,,\ c > 0\,,\ \xi_{ij} = \xi_{ji}\,.$$

To simplify the formula below we consider the case $b_{ijkl} = \text{const}$. Finally, we assume $f = (f_1, f_2) \in [C_{\text{loc}}^1(\mathbb{R}^2)]^2$.

Boundary conditions (2.4) correspond to the mutual nonpenetration between the crack faces without friction (see [6]).

Problem (2.1)–(2.4) is uniquely solvable, and it admits the variational formulation. Namely, let $H^{1,0}(\Omega_0)$ be the Sobolev space of functions having the first square integrable derivatives and equal to zero at the external boundary Γ_0. Consider the closed convex set

$$K_0 = \left\{ v = (v_1, v_2)\mid v_i \in H^{1,0}(\Omega_0)\,,\ i = 1, 2\,; [v]\nu \geq 0 \quad \text{on} \quad \Gamma_0^c \right\}\,.$$

Then the problem (2.1)–(2.4) is equivalent to minimizing the functional

$$\frac{1}{2}\int_{\Omega_0} b_{ijkl}\varepsilon_{kl}(v)\varepsilon_{ij}(v) - \int_{\Omega_0} fv$$

over the set K_0, and it can be written in the variational inequality form

$$u \in K_0 : \int_{\Omega_0} b_{ijkl}\varepsilon_{kl}(u)(\varepsilon_{ij}(\bar{u}) - \varepsilon_{ij}(u)) \geq \int_{\Omega_0} f(\bar{u} - u) \quad \forall \bar{u} \in K_0\,. \tag{2.5}$$

We can define the energy functional

$$\Pi(\Omega_0) = \frac{1}{2}\int_{\Omega_0} b_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(u) - \int_{\Omega_0} fu$$

3

for the problem (2.5).

Consider next the family of perturbations of the domain Ω_0,

$$x = \varphi_\delta(y)\,, \ y \in \bar{\Omega}\,.$$

We assume that φ_δ establishes a one-to-one correspondence between $\bar{\Omega}$ and $\varphi_\delta(\bar{\Omega})$, $\varphi_0(y) = y$, and the Jacobian $\left|\frac{\partial\varphi_\delta(y)}{\partial y}\right|$ is positive. Also, the smoothness $\varphi, \varphi^{-1} \in C^2(-\delta_0, \delta_0; C^1_{\text{loc}}(I\!\!R^2))$ is assumed, where $\delta_0 > 0$ is a given number. For any fixed $\delta \in (-\delta_0, \delta_0)$ we can consider the perturbation of the problem (2.1)–(2.4). In fact, let $\Gamma_\delta = \varphi_\delta(\Gamma_0), \Gamma_\delta^c = \varphi_\delta(\Gamma_0^c), \Omega_\delta = \varphi_\delta(\Omega_0)$. Then the perturbed problem can be formulated in the following form. We have to find the displacement vector $u^\delta = (u_1^\delta, u_2^\delta)$ such that

$$-\sigma_{ij,j}^\delta = f_i\,, \ i = 1, 2\,, \quad \text{in} \quad \Omega_\delta\,, \tag{2.6}$$

$$\sigma_{ij}^\delta = b_{ijkl}\varepsilon_{kl}(u^\delta)\,, \ i, j = 1, 2\,, \quad \text{in} \quad \Omega_\delta\,, \tag{2.7}$$

$$u^\delta = 0 \quad \text{on} \quad \Gamma_\delta\,, \tag{2.8}$$

$$[u^\delta]\nu^\delta \geq 0\,, \ \sigma_{\nu^\delta}^\delta \leq 0\,, \ \left[\sigma_{\nu^\delta}^\delta\right] = 0\,, \ \sigma_{\tau^\delta}^\delta = 0\,, \ \sigma_{\nu^\delta}^\delta \cdot [u^\delta]\nu^\delta = 0 \quad \text{on} \quad \Gamma_\delta^c\,. \tag{2.9}$$

Here ν^δ is the unit normal vector to Γ_δ^c, $\varepsilon_{kl}(u^\delta) = \frac{1}{2}(u_{k,l}^\delta + u_{l,k}^\delta)$, τ^δ is a tangential vector to Γ_δ^c.

As before, the problem (2.6)–(2.9) admits the variational formulation. If

$$K_\delta = \left\{ v = (v_1, v_2) \middle| \ v_i \in H^{1,0}(\Omega_\delta)\,, \ i = 1, 2\,; \ [v]\nu^\delta \geq 0 \quad \text{on} \quad \Gamma_\delta^c \right\}$$

then the relations (2.6)–(2.9) are equivalent to the variational inequality

$$u^\delta \in K_\delta : \int\limits_{\Omega_\delta} b_{ijkl}\varepsilon_{kl}(u^\delta)(\varepsilon_{ij}(\bar{u}) - \varepsilon_{ij}(u^\delta)) \geq \int\limits_{\Omega_\delta} f(\bar{u} - u^\delta) \quad \forall \bar{u} \in K_\delta\,. \tag{2.10}$$

The Sobolev space $H^{1,0}(\Omega_\delta)$ is introduced similar to $H^{1,0}(\Omega_0)$, in particular, functions from $H^{1,0}(\Omega_0)$ are equal to zero on Γ_δ.

Observe that the problem (2.6)–(2.9) (or the problem (2.10)) reduces to (2.1)–(2.4) as $\delta = 0$.

As it was proved in [7], the energy functional

$$\Pi(\Omega_\delta) = \frac{1}{2} \int\limits_{\Omega_\delta} b_{ijkl}\varepsilon_{kl}(u^\delta)\varepsilon_{ij}(u^\delta) - \int\limits_{\Omega_\delta} fu^\delta$$

has the derivative $\mathcal{R}$ with respect to δ as $\delta = 0$ provided that φ_δ establishes a one-to-one correspondence between K_0 and K_δ for small δ. Moreover, the following formula holds,

$$\mathcal{R} = \left.\frac{d\Pi(\Omega_\delta)}{d\delta}\right|_{\delta=0} =$$

4

$$\int\limits_{\Omega_0} \left\{ \frac{1}{2}\sigma_{ij}\varepsilon_{ij}(u) \ \mathrm{div}\,\Lambda - \sigma_{kl}u_{k,p}\Lambda^p_{,l} \right\} - \int\limits_{\Omega_0} u_k \ \mathrm{div}\,(f_k\Lambda), \qquad (2.11)$$

where the vector-field $\Lambda(y)$ is defined by the relation

$$\Lambda(y) = \left. \frac{d\varphi_\delta(y)}{d\delta} \right|_{\delta=0}$$

and $\sigma_{ij} = \sigma_{ij}(u)$.

Note that if the perturbation φ_δ describes the crack length change, the formula (2.11) provides the derivative of the energy functional with respect to the crack length. Such a derivative is used in the classical Griffith criterion to answer the question on the crack propagation.

3 Choice of a safe crack shape

To simplify the arguments we assume that the curve Γ^c_0 coincides with the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (0,1)$. The function ψ will be a control function. For any fixed ψ we can find the derivative (2.11) and obtain therefore that $\mathcal{R} = \mathcal{R}(\psi)$.

Consider the space

$$H^2_0(0,1) = \left\{ v \in H^2(0,1) | \ v = v_{y_1} = 0 \quad \text{at} \ y_1 = 0,1 \right\},$$

where $v_{y_1} = \frac{\partial v}{\partial y_1}$. Let $\Psi \subset H^2_0(0,1)$ be a bounded and weakly closed set, $\psi \in \Psi$, and $\kappa \in \mathbb{R}$ be a fixed number. In applications κ is used to be a critical value to describe a crack propagation. We assume that for any $\psi \in \Psi$ the graph of the function $y_2 = \psi(y_1)$, $y_1 \in (0,1)$, belongs to Ω.

Consider the optimal control problem

$$\max_{\psi \in \Psi}\{\mathcal{R}(\psi) - \kappa\}. \qquad (3.12)$$

This means that we want to find the crack shape which guarantees the maximal deviation of the derivative $\mathcal{R}(\psi)$ from the critical value κ. In particular, the solution of the problem (3.12) gives the most safe crack shape provided that φ_δ describes the crack length change, and the classical Griffith criterion is used for the crack propagation.

The aim of the arguments below is to prove the existence of a solution to the optimal control problem (3.12). We first establish an auxiliary result concerning the strong convergence of solutions which guarantees the continuity of the derivative with respect to the crack shape.

Assume that we consider the family of cracks described by the graphs Γ^c_λ of functions $y_2 = \lambda\psi(y_1)$, $y_1 \in (0,1)$, where λ is a small parameter converging to zero.
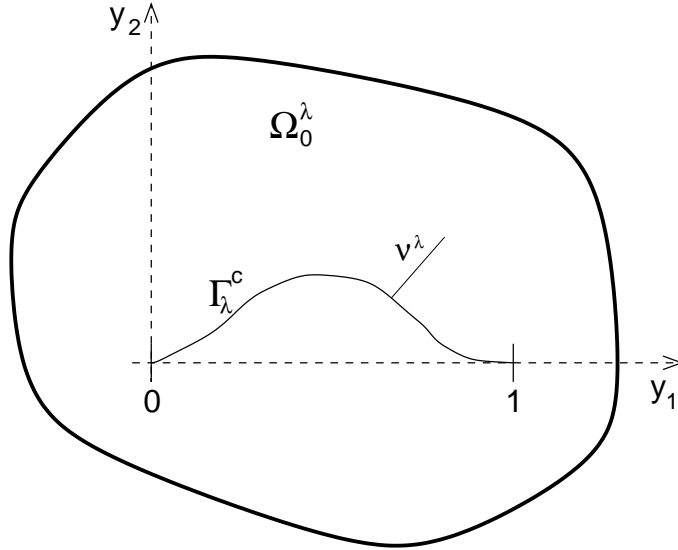
Figure 2: The family of cracks Γ_λ^c in the respective domains Ω_0^λ.

We want to prove that solutions of the problems like (2.1)–(2.4) corresponding to the parameter λ converge strongly as $\lambda \to 0$.

Let Ω_0^λ be a domain corresponding to Γ_λ^c (see Figure 2), i.e., $\Omega_0^\lambda = \Omega \setminus \bar{\Gamma}_\lambda^c$. In this case for $\lambda = 0$ we have $\Omega_0^0 = \Omega_0$, $\Gamma_0^c = (0,1) \times \{0\}$. So, in fact, we consider the perturbation of the crack shape through the parameter λ. Let

$$\nu^\lambda = \frac{(-\lambda\psi', 1)}{\sqrt{1 + (\lambda\psi')^2}}$$

be a unit normal vector to Γ_λ^c,

$$\mathcal{K}_\lambda = \left\{ v = (v_1, v_2) \mid v_i \in H^{1,0}(\Omega_0^\lambda), \ i = 1,2 \, ; \ [v]\nu^\lambda \geq 0 \quad \text{on} \quad \Gamma_\lambda^c \right\}.$$

Consider a solution u^λ of the problem

$$u^\lambda \in \mathcal{K}_\lambda : \int\limits_{\Omega_0^\lambda} b_{ijkl} u_{k,l}^\lambda (\bar{u}_{i,j}^\lambda - u_{i,j}^\lambda) \geq \int\limits_{\Omega_0^\lambda} f(\bar{u}^\lambda - u^\lambda) \quad \forall \bar{u}^\lambda \in \mathcal{K}_\lambda. \qquad (3.13)$$

Analogously, for $\lambda = 0$ we can consider the solution u of the unperturbed problem

$$u \in \mathcal{K}_0 : \int\limits_{\Omega_0^0} b_{ijkl} u_{k,l} (\bar{u}_{i,j} - u_{i,j}) \geq \int\limits_{\Omega_0^0} f(\bar{u} - u) \quad \forall \bar{u} \in \mathcal{K}_0 \qquad (3.14)$$

with a convex and closed set

$$\mathcal{K}_0 = \left\{ v = (v_1, v_2) \mid v_i \in H^{1,0}(\Omega_0^0), \ i = 1,2 \, ; \ [v]\nu^0 \geq 0 \quad \text{on} \quad \Gamma_0^c \right\}.$$

6

Note that $\mathcal{K}_0$ coincides with K_0 if Γ^c is a straight line.

It is possible to establish a one-to-one correspondence between the domains Ω_0^λ and Ω_0^0. To this end, we introduce a transformation of the independent variables

$$x_1 = y_1\,, \ \ x_2 = y_2 - \lambda\xi(y)\psi(y_1)\,, \ \ x \in \Omega_0^0\,, \ y \in \Omega_0^\lambda\,, \tag{3.15}$$

with $\xi \in C_0^\infty(\Omega)$, $\xi = 1$ in a neighbourhood of Γ_0^c. Let us recall that $\psi \in \Psi \subset H_0^2(0,1)$. Hence, we can extend the function ψ beyond $(0,1)$ by zero to have a correct definition of the map (3.15).

Let $u^\lambda(y) = u_\lambda(x), y \in \Omega_0^\lambda, x \in \Omega_0^0$. Denoting $\mathbf{H}$ the vector-valued counterpart of any real-valued Banach space H (i.e. $\mathbf{H} = H \times H$), we prove the following assertion:

Lemma 3.1 *Let u be a solution of the problem (3.14). Then as $\lambda \to 0$, $u_\lambda \to u$ strongly in $\mathbf{H}^{1,0}(\Omega_0^0)$.*

Proof. The difficulty in proving the strong convergence is that the transformation (3.15) does not provide the one-to-one correspondence between $\mathcal{K}_\lambda$ and $\mathcal{K}_0$. Denote

$$\mathcal{K}_{0\lambda} = \left\{v = (v_1, v_2)\vert\ v_i \in H^{1,0}(\Omega_0^0)\,, \ i = 1,2\,, \ [v]\nu^\lambda \geq 0 \quad \text{on} \quad \Gamma_0^c\right\}\,.$$

Then that the transformation (3.15) maps $\mathcal{K}_\lambda$ onto the set $\mathcal{K}_{0\lambda}$.

It was proved in [6] that as $\lambda \to 0$

$$u_\lambda \to u \quad \text{weakly in} \quad \mathbf{H}^{1,0}(\Omega_0^0)\,. \tag{3.16}$$

Moreover, it was shown that for any fixed $w \in \mathcal{K}_0$ there exists a sequence $w_\lambda \in \mathcal{K}_{0\lambda}$ such that

$$w_\lambda \to w \quad \text{strongly in} \quad \mathbf{H}^{1,0}(\Omega_0^0)\,. \tag{3.17}$$

Now introduce the bilinear form on the space $[\mathbf{H}^{1,0}(\Omega_0^0)]^2$

$$B_\lambda(v,w) = \int\limits_{\Omega_0^0} b_{ijkl}v_{k,l}w_{i,j}g_\lambda^{-1}\,,$$

where $g_\lambda(y) = \left\vert\frac{\partial x(y)}{\partial y}\right\vert$ is the Jacobian of the transformation (3.15). It is clear that $g_\lambda(y) = 1 - \lambda\psi\xi_{y_2} > 0$ for small λ. We change the domain integration Ω_0^λ by Ω_0^0 in (3.13) in accordance with (3.15). This provides the relation

$$u_\lambda \in \mathcal{K}_{0\lambda} : B_\lambda(u_\lambda, \bar{u}_\lambda - u_\lambda) + \int\limits_{\Omega_0^0} F(u_{\lambda x}^2, \bar{u}_{\lambda x}u_{\lambda x}, \lambda, (\psi\xi)_y)g_\lambda^{-1}$$

$$\geq \int\limits_{\Omega_0^0} \tilde{f}(\bar{u}_\lambda - u_\lambda)g_\lambda^{-1} \quad \forall\bar{u}_\lambda \in \mathcal{K}_{0\lambda}\,. \tag{3.18}$$

7

Here $\tilde{f}(x) = f(y(x))$, and we have used the following formulae for the first derivatives

$$u_{y_1}^\lambda = u_{\lambda x_1} - \lambda u_{\lambda x_2}(\xi\psi)_{y_1}, \ u_{y_2}^\lambda = u_{\lambda x_2}(1 - \lambda\xi_{y_2}\psi)$$

with the above notations, $u^\lambda(y) = u_\lambda(x)$, $y \in \Omega_0^\lambda$, $x \in \Omega_0^0$. The function F linearly depends on $u_{\lambda x}^2$, $\bar{u}_{\lambda x}u_{\lambda x}$, and it has a quadratic dependence on λ. In particular, as $\lambda \to 0$

$$\int\limits_{\Omega_0^0} F(u_{\lambda x}^2, \ \bar{u}_{\lambda x}u_{\lambda x}, \ \lambda, (\psi\xi)_y)g_\lambda^{-1} \to 0 \qquad (3.19)$$

provided that u_λ, $\bar{u}_\lambda$ are bounded in $\mathbf{H}^{1,0}(\Omega_0^0)$ uniformly in λ.
From (3.13) it follows the uniform in λ estimate

$$\|u^\lambda\|_{\mathbf{H}^{1,0}(\Omega_0^\lambda)} \leq c,$$

consequently, uniformly in λ

$$\|u_\lambda\|_{\mathbf{H}^{1,0}(\Omega_0^0)} \leq c. \qquad (3.20)$$

By (3.17), for the solution $u \in \mathcal{K}_0$ of the problem (3.14) we can find a sequence $\bar{u}_\lambda \in \mathcal{K}_{0\lambda}$ such that

$$\bar{u}_\lambda = u + v_\lambda, \ v_\lambda \to 0 \quad \text{strongly in} \quad \mathbf{H}^{1,0}(\Omega_0^0). \qquad (3.21)$$

Taking these relations into account, the inequality (3.18) implies

$$\begin{aligned}
B_\lambda(u_\lambda - u, u_\lambda - u) \leq\ & B_\lambda(u, u - u_\lambda) + B_\lambda(u, v_\lambda) + B_\lambda(u_\lambda - u, v_\lambda) \\
& + \int\limits_{\Omega_0^0} \tilde{f}(u_\lambda - u)g_\lambda^{-1} - \int\limits_{\Omega_0^0} \tilde{f}v_\lambda g_\lambda^{-1} \\
& + \int\limits_{\Omega_0^0} F(u_{\lambda x}^2, u_{\lambda x}(u_x + v_{\lambda x}), \lambda, (\psi\xi)_y)g_\lambda^{-1}.
\end{aligned} \qquad (3.22)$$

Hence, by (3.16), (3.19), (3.20), (3.21), from (3.22) the needed convergence follows,

$$\|u_\lambda - u\|_{\mathbf{H}^{1,0}(\Omega_0^0)} \to 0, \ \lambda \to 0.$$

Lemma 3.1 is proved. $\blacksquare$

For any fixed λ, in accordance with (2.11), we can find the derivative of the energy functional with respect to the perturbation parameter δ provided that φ_δ establishes a one-to-one correspondence between $\mathcal{K}_\lambda$ and $\mathcal{K}_{\lambda\delta}$ for small δ. Here

$$\mathcal{K}_{\lambda\delta} = \left\{v = (v_1, v_2)|\ v_i \in H^{1,0}(\varphi_\delta(\Omega_0^\lambda)),\ i = 1, 2,\ [v]\nu_\delta^\lambda \geq 0 \quad \text{on} \quad \varphi_\delta(\Gamma_\lambda^c)\right\}$$

8

and ν_δ^λ is a unit normal vector to $\varphi_\delta(\Gamma_\lambda^c)$. Thus, the following formula for the derivative of the energy functional with respect to δ can be obtained, with $\sigma_{ij}^\lambda = \sigma_{ij}(u^\lambda)$,

$$\mathcal{R}(\lambda\psi) = \int\limits_{\Omega_0^\lambda} \left\{ \frac{1}{2}\sigma_{ij}^\lambda \varepsilon_{ij}(u^\lambda) \ \mathrm{div}\,\Lambda - \sigma_{kl}^\lambda u_{k,p}^\lambda \Lambda_{,l}^p \right\} - \int\limits_{\Omega_0^\lambda} u_k^\lambda \ \mathrm{div}\,(f_k\Lambda) \,. \qquad (3.23)$$

Note that the inequality (3.14) follows from (3.18) as $\lambda \to 0$. Consequently

$$\mathcal{R}(0) = \int\limits_{\Omega_0^0} \left\{ \frac{1}{2}\sigma_{ij}\varepsilon_{ij}(u) \ \mathrm{div}\,\Lambda - \sigma_{kl}u_{k,p}\Lambda_{,l}^p \right\} - \int\limits_{\Omega_0} u_k \ \mathrm{div}\,(f_k\Lambda) \,. \qquad (3.24)$$

Now change the integration domain from Ω_0^λ to Ω_0^0 in (3.23) in accordance with (3.15). By Lemma 3.1, we derive

$$\mathcal{R}(\lambda\psi) \to \mathcal{R}(0) \,, \ \lambda \to 0 \,. \qquad (3.25)$$

So we have obtained the continuity of the derivative of the energy functional with respect to the crack shape. Let

$$K_0^\psi = \left\{ v = (v_1, v_2) |\ v_i \in H^{1,0}(\Omega_0^\psi) \,, \ i = 1, 2 \,; \ [v]\nu_\psi \geq 0 \quad \text{on} \quad \Gamma_0^\psi \right\} \,,$$

$$K_\delta^\psi = \left\{ v = (v_1, v_2) |\ v_i \in H^{1,0}(\Omega_\delta^\psi) \,, \ i = 1, 2 \,; \ [v]\nu_\psi^\delta \geq 0 \quad \text{on} \quad \Gamma_\delta^\psi \right\} \,,$$

where $\Omega_\delta^\psi = \varphi_\delta(\Omega_0^\psi)$, and ν_ψ, ν_ψ^δ are unit normal vectors to Γ_0^ψ, Γ_δ^ψ, respectively. Here, we have used the obvious notations, Γ_0^ψ, for the graph of the function $y_2 = \psi(y_1)$, $\Gamma_\delta^\psi = \varphi_\delta(\Gamma_0^\psi)$, $\Omega_0^\psi = \Omega\backslash\bar{\Gamma}_0^\psi$, and $\Omega_\delta^\psi = \varphi_\delta(\Omega_0^\psi)$.

Now we are in a position to state the following result:

Theorem 3.1 *Assume that φ_δ establishes a one-to-one correspondence between K_0^ψ and K_δ^ψ for small δ and all $\psi \in \Psi$. Then there exists a solution of the optimal control problem (3.12).*

Proof. Let $\psi^n \in \Psi$ be a miximizing sequence in the problem (3.12). Since Ψ is bounded in $H_0^2(0,1)$ we can assume that as $n \to \infty$

$$\psi^n \to \psi \quad \text{weakly in} \ H_0^2(0,1) \,, \ (\psi^n)' \to \psi' \quad \text{in} \quad C[0,1] \,. \qquad (3.26)$$

For any fixed $n \in N$ we can find the solution u^n of the problem

$$u^n \in K_0^{\psi^n} : \int\limits_{\Omega_0^{\psi^n}} b_{ijkl}u_{k,l}^n(\bar{u}_{i,j} - u_{i,j}^n) \geq \int\limits_{\Omega_0^{\psi^n}} f(\bar{u} - u^n) \quad \forall \bar{u} \in K_0^{\psi^n} \,.$$

Here the domains $\Omega_0^{\psi^n}$ correspond to the graphs of functions $y_2 = \psi^n(y_1)$, respectively.
Consider the change of the variables,

$$x_1 = y_1 \,, \ x_2 = y_2 + \xi(y)(\psi(y_1) - \psi^n(y_1)) \,, \qquad (3.27)$$

9

where $y \in \Omega_0^{\psi^n}$, $x \in \Omega_0^{\psi}$, and the function ξ is being chosen from $C_0^\infty(\Omega)$, $\xi = 1$ in a neighbourhood of the graph of the function $y_2 = \psi(y_1)$. All functions $\psi \in \Psi$ are extended beyond $(0, 1)$ by zero. Hence the definition (3.27) is correct.

Let us find the derivative of the energy functional with respect to δ for a given $n \in N$. This gives

$$\mathcal{R}(\psi^n) = \int_{\Omega_0^{\psi^n}} \left\{ \frac{1}{2}\sigma_{ij}^n \varepsilon_{ij}(u^n) \ \mathrm{div}\,\Lambda - \sigma_{kl}^n u_{k,p}^n \Lambda_{,l}^p \right\} - \int_{\Omega_0^{\psi^n}} u_k^n (\ \mathrm{div}\, f_k \Lambda). \qquad (3.28)$$

Analogously, for the function ψ we can get

$$\mathcal{R}(\psi) = \int_{\Omega_0^{\psi}} \left\{ \frac{1}{2}\sigma_{ij} \varepsilon_{ij}(u) \ \mathrm{div}\,\Lambda - \sigma_{kl} u_{k,p} \Lambda_{,l}^p \right\} - \int_{\Omega_0^{\psi}} u_k \ \mathrm{div}\, (f_k \Lambda),$$

where u is a solution of the problem

$$u \in K_0^{\psi} : \int_{\Omega_0^{\psi}} b_{ijkl} u_{k,l}(\bar{u}_{i,j} - u_{i,j}) \geq \int_{\Omega_0^{\psi}} f(\bar{u} - u) \quad \forall \bar{u} \in K_0^{\psi}.$$

Similar to Lemma 3.1, it can be proved that

$$u_n \to u \quad \text{strongly in} \quad \mathbf{H}^{1,0}(\Omega_0^{\psi}), \qquad (3.29)$$

where $u^n(y) = u_n(x)$, $y \in \Omega_0^{\psi^n}$, $x \in \Omega_0^{\psi}$. We can change the integration domain from $\Omega_0^{\psi^n}$ to Ω_0^{ψ} in (3.28) in accordance with (3.27). Analogously to (3.25) the convergences (3.26), (3.29) allow us to pass to the limit as $n \to \infty$ in the relation obtained. This provides the convergence

$$\mathcal{R}(\psi^n) - \kappa \to \mathcal{R}(\psi) - \kappa.$$

Since $\psi \in \Psi$, $u = u(\psi)$, the limit function ψ solves the problem (3.12). Theorem 3.1 is proved. ∎

Acknowledgements

References

[1] Banichuk, N.V., Optimization of shapes of elastic bodies, Moscow: Nauka, 1980, 255 p.

[2] Cherepanov, G.P., Mechanics of brittle fracture, Moscow: Nauka, 1974, 640 p.

[3] Dauge, M., Elliptic boundary value problems on corner domains, Lecture Notes in Math., Berlin: Springer, 1988, No. 1341, 259 p.

[4] Grisvard, P., Singularities in boundary value problems, Paris: Masson; Berlin: Springer, 1992, 198 p.

[5] Grisvard, P., Elliptic problems in nonsmooth domains, Boston, etc.: Pitman, 1985, 410 p.

[6] Khludnev, A.M., Kovtunenko, V.A., Analysis of cracks in solids, Southampton, Boston: WIT Press, 2000, 408 p.

[7] Khludnev, A.M., Ohtsuka, K., Sokołowski, J., On derivative of energy functional for elastic bodies with cracks and unilaterial conditions, Les prepublications de l'Institut Elie Cartan, 2000, No. 20, 14 p.

[8] Khludnev, A.M., Sokołowski, J., On derivative of the energy functional in the elasticity theory, Appl. Math. Mechs., **64** (2000), No. 3, pp. 464–475.

[9] Khludnev, A.M., Sokołowski, J., Modelling and control in solid mechanics, Basel, etc.: Birkhäuser, 1997, 384 p.

[10] Kondratiev, V.A., Kopaček, I., Oleinik, O.A., On solution behaviour of the second order elliptic equations and elasticity system in a neighbourhood of a boundary point, Proc. Petrovskii seminar, Moscow: Moscow State University, **8**, 1982, pp. 135–152.

[11] Maz'ya, V.G., Nazarov, S.A., Asymptotics of energy integrals for small perturbation of a boundary near corner and conical points, Proc. Moscow Math. Soc., **50**, 1987, pp. 79–129.

[12] Morozov, N.F., Mathematical questions of the crack theory, Moscow: Nauka, 1984, 255 p.

[13] Nazarov, S.A., Plamenevski, B.A., Elliptic problems in piece-wise smooth boundary domains, Moscow: Nauka, 1991, **336** p.

[14] Parton, V.Z., Morozov, E.M., Mechanics of the elastoplastic fracture, Moscow: Nauka, 1985, 502 p.

[15] Ohtsuka, K., Mathematical aspects of fracture mechanics, Lecture Notes Numer. Appl. Anal., **13** (1994), pp. 39–59.

[16] Sokołowski, J., Zolesio, J.P., Introduction to shape optimization. Shape sensitivity analysis, Berlin: Springer, 1992, 250 p.

11

[17] Zakharevich, I.S., On solution variations to integrodifferential equations in mixed problems of the elasticity theory, Appl. Math. Mechs., **49**(1985), No. 6, pp. 961–968.