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On Polynomial Collocation for Second Kind Integral Equations with Fixed Singularities of Mellin Type

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Abstract

We consider a polynomial collocation for the numerical solution of a second kind integral equation with an integral kernel of Mellin convolution type. Using a stability result by Junghanns and one of the authors, we prove that the error of the approximate solution is less than a logarithmic factor times the best approximation and, using the asymptotics of the solution, we derive the rates of convergence. Finally, we describe an algorithm to compute the stiffness matrix based on simple Gauß quadratures and an alternative algorithm based on a recursion in the spirit of Monegato and Palamara Orsi. All together an almost best approximation to the solution of the integral equation can be computed with $\mathcal{O}(n^2[\log n]^2)$ resp. $\mathcal{O}(n^2)$ operations, where n is the dimension of the polynomial trial space.

1 Introduction

A lot of mathematical problems in mathematics, physics, and engineering can be reduced to the solution of a second kind integral equation over the interval with a kernel function in form of a Mellin convolution or, more generally, to a Cauchy singular integral equation with additional terms of Mellin convolution type (cf. e.g. [8, 3]). For the numerical solution of such equations, many different methods have been proposed including spline methods (cf. [33]), quadrature methods based on composite rules (cf. [16, 23]), $h-p$ methods (cf. [10]), as well as discretization schemes based on polynomial approximation (cf. [7, 21, 33]). The polynomial methods are the best possible if weighted polynomials form the eigenfunctions of the integral operators. Usually, these p-methods converge exponentially fast. In applications where exponential convergence is not possible or not realistic, we believe that, nevertheless, the polynomial methods are best even if asymptotic error estimates predict the same rate of convergence for h or for h-p methods. This fact has been observed in many similar situations (cf. [31, 30, 36]). Moreover, inside the class of polynomial methods, we expect the Nyström method to be faster than collocation and collocation to be faster than Galerkin. The errors, however, should be smaller for Galerkin than for collocation resp. smaller for collocation than for the Nyström method, the last fact being confirmed in [25]. So the polynomial collocation seems to be a good compromise between low complexity and high approximation order.

Motivated by applications to complex iterative procedures for non-linear problems, Junghanns, Roch, Silbermann, Weber, and one of the authors have started to analyze the collocation for Cauchy singular equations in the case that no invariance property for the polynomial trial spaces is satisfied [22, 20, 18]. To combine the collocation method with transformation techniques and to treat new kind of equations, Junghanns and one of the authors have derived a quite general stability and convergence result for Cauchy singular equations including perturbations of Mellin convolution type. On the other hand, Russo and one of the authors have obtained new results on the Marcinkiewicz inequality and on the error of polynomial interpolation [26]. The goal of this paper is to combine the stability and the approximation result to derive an error estimate for the collocation at least for the special case of second kind equations including Mellin convolutions. Note that a collocation for a special equation of this type has been proposed in [25], but no proof has been given so far.

The decisive point for the applicability of such a collocation scheme is the computation of the collocation matrix. A naive approach to compute the integrals of the global trial functions in the entries of the collocation matrix may lead to an operation count of $\mathcal{O}(n^3)$, where n denotes the number of degrees of freedom. A more sophisticated approach based on composite quadrature rules over geometrically graded meshes and on Gauß quadratures for oscillatory weight functions leads to a complexity of $\mathcal{O}(n^2[\log n]^2)$. Note that the proof of the fact that the quadrature approximated collocation is a small perturbation of the exact collocation and has the same order of convergence is a rather standard. Indeed, one only has to apply exponential error estimates of Gauß rules for analytic functions. However, we abstain from repeating these arguments here. Following an approach by Monegato and Palamara Orsi, we consider special kind of rational Mellin convolution kernels and propose a recursive algorithm for the assembling of the matrix. Unfortunately, this recursion of [29] is unstable for integrals of smooth Mellin kernels, i.e. the recursion applied to almost singular Mellin kernels is to be combined with a quadrature algorithm for integrals of smooth Mellin kernels. Of course, the quadrature algorithm is more time consuming than the recursion. So we present a new modification of the recursive algorithm which is stable. All together we arrive at an algorithm with no more than $\mathcal{O}(n^2)$ arithmetic operations. Moreover, we hope that the presented ideas can be useful also for the design of fast $\mathcal{O}(n)$ algorithms, which are analyzed so far for h-methods, only.

For simplicity sake, we restrict ourselves throughout this paper to Mellin convolutions with singularities fixed at the left end-point of the interval. The case of additional Mellin kernels with singularities fixed at the right end-point of the interval can be analyzed in the same manner. Moreover, the results remain true if the Mellin operators are multiplied not only from the left but also from the right by a smooth function, or if a finite sum of such operators appears. The smoothness assumptions on the right-hand side, on the kernels and on their Mellin transforms can, obviously, be relaxed. For instance, the $\mathbf{C}^\infty$ regularity can be replaced by a differentiability condition of finite degree depending on the rate of convergence. The error estimate and algorithm for the Cauchy singular operator equation will be considered in a forthcoming paper.

The plan of this paper is as follows. First, we shall introduce the integral equation and describe well-known invertibility results on the corresponding integral operator in Section 2. In Section 3 we shall recall the well-known asymptotic expansion for the solution and the transformation technique to improve the asymptotics. The polynomial collocation and a corresponding stability result from a previous paper will be introduced in Section 4. This stability result will be combined with the asymptotics and with results from approximation theory to derive rates of convergence in Section 5. In Section 6 we shall describe an algorithm based on a simple quadrature and an alternative algorithm based on the exploitation of recurrence relations. In particular, we shall show that a polynomial approximation of degree n can be computed with no more than $\mathcal{O}(n^2)$ arithmetic operations. Numerical experiments will be presented in Section 7. All technical proofs are shifted to Section 8.

2 The Integral Equation

We consider the second kind integral equation $Au = f$ with fixed singularities of Mellin convolution type given by

$$u(x) + b(x) \int_{-1}^{1} \mathbf{k}\left(\frac{1+x}{1+y}\right) \frac{u(y)}{1+y}\,\mathrm{d}y + \int_{-1}^{1} \mathbf{k}_0(x,y)\,u(y)\,\mathrm{d}y \;=\; f(x), \;\; -1 < x < 1. \quad (2.1)$$

Here b is a function from the class $\mathbf{C}^\infty[-1,1]$. The Mellin kernel $\mathbf{k}$ is supposed to be a continuous functions over the half axis $\mathbb{R}_+ := (0,\infty)$. For two real numbers α and β with $\alpha < \beta$, we suppose that the Mellin symbol

$$\widehat{\mathbf{k}}(z) \;:=\; \int_0^\infty y^{z-1}\mathbf{k}(y)\,\mathrm{d}y$$

is analytic in the strip $\{z \in \mathbb{C} : \;\alpha < \Re\mathrm{e}\, z < \beta\}$. Moreover, we assume the differentiability and decay condition

$$\sup_{z:\,\alpha<\Re\mathrm{e}\,z<\beta} \left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} \widehat{\mathbf{k}}(z)(1+|z|)^{1+k} \right| < \infty, \quad k = 0,1,2,\ldots\;. \qquad (2.2)$$

For simplicity, the kernel function $\mathbf{k}_0 : \;[-1,1] \times [-1,1] \longrightarrow \mathbb{C}$ and the right-hand side $f : \;[-1,1] \longrightarrow \mathbb{C}$ are supposed to be $\mathbf{C}^\infty$ smooth. We consider (2.1) as an operator equation in the weighted Lebesgue space $\mathbf{L}^2_\sigma$ and seek the unknown solution u in $\mathbf{L}^2_\sigma$. Here, the norm and the inner product in the Hilbert space $\mathbf{L}^2_\sigma$ is given by

$$\langle u, v\rangle_\sigma \;:=\; \int_{-1}^{1} u(y)\overline{v(y)}\sigma(y)\,\mathrm{d}y, \quad \|u\|_\sigma \;:=\; \sqrt{\langle u,u\rangle_\sigma}\,, \qquad (2.3)$$

where σ is a Jacobi weight $\sigma(y) := (1-y)^{\sigma_+}(1+y)^{\sigma_-}$ with exponents σ_+ and σ_- such that $-1 < \sigma_\pm < 1$ and $\alpha < \sigma_-/2 + 1/2 < \beta$. According to the three terms on the left-hand side of (2.1) we split the operator A into the sum $I + bK + K_0$.

The class of equations (2.1) resp. the class of systems of such equations includes important boundary integral equations (cf. e.g. [2, 3]) and equations with fixed singularities (cf. e.g. [8]). It is well known that the Mellin convolution operator $I + b(-1)K$ over the interval can be transformed into a Wiener-Hopf operator over the half axis $\mathbb{R}_+$ simply by substituting the variables $x = 2e^{-t} - 1$ and $y = 2e^{-s} - 1$. The Fourier transform of the arising Wiener-Hopf kernel is the Mellin transform $\widehat{\mathbf{k}}$ of the Mellin convolution kernel $\mathbf{k}$. The perturbation operator $bK - b(-1)K + K_0$ is compact. Hence, we get

Theorem 2.1 *The operator A on the right-hand side of (2.1) is Fredholm if and only if its symbol does not vanish, i.e. if*

$$1 + b(-1)\widehat{\mathbf{k}}\left(\frac{1}{2} + \frac{\sigma_-}{2} + \mathbf{i}\xi\right) \;\neq\; 0, \quad \xi \in \mathbb{R}.$$

The index of A is zero if and only if the winding number around zero of the symbol curve

$$\left\{1 + b(-1)\widehat{\mathbf{k}}\left(\frac{1}{2} + \frac{\sigma_-}{2} + \mathbf{i}\xi\right) : \;\xi \in \mathbb{R}\right\}$$

vanishes. Operator A is invertible if and only if it is Fredholm with index zero and if the homogeneous equation $Av = 0$ has no solution different from zero.

Throughout this paper we shall assume that A is invertible. To guarantee the convergence of the collocation method, to derive the rate of convergence, and to enable a simple algorithm for the assembling of the stiffness matrix, we need further assumptions on $\mathbf{k}$ and σ. These conditions will be introduced in the corresponding Sections 4, 5, and 6.

3 The Asymptotics of the Solution and a Transformation Technique

Clearly, the solution u is a $\mathbf{C}^\infty$ function over $(-1,1]$. The technique to establish the asymptotics of the solution u to (2.1) at -1 is well known for a long time. An abstract setting for a quite general equation including (2.1) is treated in [9]. From this paper we infer

Theorem 3.1 *Suppose the assumptions of the last section are satisfied and that the Mellin transform* $\widehat{\mathbf{k}}(z)$ *is meromorphic in* $\{z \in \mathbb{C} : \Re\mathrm{e}\, z < \beta\}$ *with no more than a finite number of poles* ζ_k^p, $k = 1,2,\ldots,k_m^p$ *and zeros* ζ_k^z, $k = 1,2,\ldots,k_m^z$ *in each complex strip* $\{z \in \mathbb{C} : -m < \Re\mathrm{e}\, z < \beta\}$, $m = 1,2,\ldots$. *Moreover, we suppose, for any* $\varepsilon > 0$,

$$\sup_{\substack{z:\ -m+\varepsilon<\Re\mathrm{e}\, z<\beta-\varepsilon \\ |z-\zeta_{k'}^p|>\varepsilon,\ k'=1,2,\ldots,k_m^p}} \left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} \widehat{\mathbf{k}}(z)(1+|z|)^{1+k} \right| < \infty, \quad k = 0,1,2,\ldots\ .$$

Then the solution u of (2.1) admits the asymptotic expansion

$$u(x) \quad \sim \quad \sum_{i=0}^{\infty} \sum_{k=0}^{m_i} \lambda_{i,k} (1+x)^{\kappa_i} [\log(1+x)]^k, \quad x \longrightarrow -1. \tag{3.1}$$

Here, the $\lambda_{i,k}$ *are complex coefficients, the* m_i *are non-negative integers, and the exponents* κ_i *are complex numbers with* $\Re\mathrm{e}\,\kappa_0 \le \Re\mathrm{e}\,\kappa_1 \le \Re\mathrm{e}\,\kappa_2 \le \ldots$ *and* $\Re\mathrm{e}\,\kappa_i \longrightarrow \infty$ *for* $i \longrightarrow \infty$.

The numbers $-\kappa_i$ are the poles of the Mellin transform $\widehat{f}$ of the right-hand side f in (2.1), the zeros of the Mellin transform $1 + b(-1)\widehat{\mathbf{k}}$, and the shifts of these points by negative integers. The m_i correspond to the multiplicity of these poles and zeros. Note that, for smooth right-hand sides f, the zeros of $\widehat{f}$ are the non-positive integers. Since we consider solutions in $\mathbf{L}^2_\sigma$ only, we may suppose $\Re\mathrm{e}\,\kappa_0 > -1/2 - \sigma_-/2$.

Unfortunately, the singular terms in (3.1) are difficult to approximate by polynomials and by other continuous functions if κ_i is not an integer and if $\Re\mathrm{e}\,\kappa_i$ is small. Therefore, we recommend to apply the following transformation technique to produce an equivalent equation with larger exponents in the asymptotic expansion (3.1): We simply introduce a bijective transformation of the interval $\Phi : [-1,1] \longrightarrow [-1,1]$, substitute the variables x and y in (2.1) by $x = \Phi(t)$ and $y = \Phi(s)$, and multiply the equation by the expression $\sqrt{\Phi'(t)(1+\Phi(t))^{\sigma_-}/(1+t)^{\sigma_-}}$. Setting

$$\tilde{u}(t) := u\Big(\Phi(t)\Big)\sqrt{\Phi'(t)\frac{(1+\Phi(t))^{\sigma_-}}{(1+t)^{\sigma_-}}}, \quad \tilde{f}(t) := f\Big(\Phi(t)\Big)\sqrt{\Phi'(t)\frac{(1+\Phi(t))^{\sigma_-}}{(1+t)^{\sigma_-}}} \tag{3.2}$$

and assuming $\Phi(t) \sim t$ for a neighbourhood of the point 1, we get $\|u\|_\sigma \sim \|\tilde{u}\|_\sigma$, and (2.1) is equivalent to

$$\begin{aligned}
\tilde{u}(t) + \int_{-1}^{1} \widetilde{\mathbf{k}}(t,s)\,\tilde{u}(s)\,\mathrm{d}s \;&=\; \tilde{f}(t), \quad -1 < t < 1, \qquad (3.3)\\
\widetilde{\mathbf{k}}(t,s) := \Big\{ b\Big(\Phi(t)\Big)\mathbf{k}\left(\frac{1+\Phi(t)}{1+\Phi(s)}\right)\frac{1}{1+\Phi(s)} &+ \mathbf{k}_0\Big(\Phi(t),\Phi(s)\Big)\Big\} \times \\
\sqrt{\Phi'(t)\frac{(1+\Phi(t))^{\sigma_-}}{(1+t)^{\sigma_-}}}&\sqrt{\Phi'(s)\frac{(1+\Phi(s))^{-\sigma_-}}{(1+s)^{-\sigma_-}}}.
\end{aligned}$$

If the interval transformation satisfies $1+\Phi(t) \sim (1+t)^q$ with q a real number greater or equal to one, then (3.3) is of the same structure as (2.1). Though, for practical implementations, we recommend a transformation in the spirit of [23], we now consider the simplest transformation $1+\Phi(t) = 2^{1-q}(1+t)^q$ with an integer parameter $q \geq 1$, i.e. we set $\Phi(t) = 2^{1-q}(1+t)^q - 1$. In this case the kernel $\widetilde{\mathbf{k}}$ takes the form

$$\begin{aligned}
\widetilde{\mathbf{k}}(t,s) \;&:=\; \Big\{ b\Big(\Phi(t)\Big)\mathbf{k}\left(\frac{(1+t)^q}{(1+s)^q}\right)\frac{1}{2^{1-q}(1+s)^q} + \mathbf{k}_0\Big(\Phi(t),\Phi(s)\Big)\Big\} \times \\
&\quad q2^{1-q}(1+t)^{(q-1)/2}\frac{(1+t)^{q\sigma_-/2}}{(1+t)^{\sigma_-/2}}(1+s)^{(q-1)/2}\frac{(1+s)^{-q\sigma_-/2}}{(1+s)^{-\sigma_-/2}} \\
&=\; b\Big(\Phi(t)\Big)\mathbf{m}\left(\frac{1+t}{1+s}\right)\frac{1}{1+s} + \\
&\quad \mathbf{k}_0\Big(\Phi(t),\Phi(s)\Big)q2^{1-q}(1+t)^{(q-1)(1+\sigma_-)/2}(1+s)^{(q-1)(1-\sigma_-)/2}, \\
\mathbf{m}(t) \;&:=\; q\mathbf{k}\left(t^q\right)t^{(q-1)(1+\sigma_-)/2}, \quad \widehat{\mathbf{m}}(z) \;=\; \widehat{\mathbf{k}}\left(\frac{z+(q-1)(1+\sigma_-)/2}{q}\right).
\end{aligned}$$

Though the kernel of the non Mellin part is not perfectly smooth, we get the asymptotic expansion from (3.2) and (3.1).

$$\begin{aligned}
\tilde{u}(t) \;&\sim\; \sum_{i=0}^{\infty}\sum_{k=0}^{m_i}\widetilde{\lambda}_{i,k}(1+t)^{\widetilde{\kappa}_i}[\log(1+t)]^k, \quad t \longrightarrow -1, \qquad (3.4)\\
\widetilde{\kappa}_i \;&:=\; q\left[\kappa_i + \frac{1+\sigma_-}{2}\right] - \frac{1+\sigma_-}{2}.
\end{aligned}$$

Hence, due to $\Re\mathrm{e}\,\kappa_i > -1/2 - \sigma_-/2$, we can make $\Re\mathrm{e}\,\widetilde{\kappa}_i$ as large as we like if we choose q sufficiently large. Since (3.3) is of the same type as (2.1), the convergence results and the results on the assembling of the stiffness matrix from the subsequent sections apply to the transformed equation (3.3) and provide improved orders of convergence. Moreover, the presented transformation technique is also useful to enforce the condition (4.3) for the convergence of the collocation.

Sometimes it is helpful to choose the parameter q of the transformation such that $\widetilde{\kappa}_0$ is not large but a non-negative integer. If the multiplicity m_0 is zero and if the real parts of $\widetilde{\kappa}_i,\ i > 0$ are sufficiently large, then the resulting function $\tilde{u}$ with the asymptotic expansion (3.4) can be approximated by polynomials with high rates of convergence. On the other hand, if κ_0 is the unique exponent κ_i in (3.1) with minimal real part and if its multiplicity index m_i is zero, then often (2.1) is multiplied by $(1+x)^{-\kappa_0}$ and a new unknown function $\tilde{u}(x) := u(x)(1+x)^{-\kappa_0}$ is introduced. This way the badly behaving first term $\lambda_{i,0}(1+x)^{\kappa_0}$ is transformed into a constant term which is easy to approximate by polynomial functions. We note, however, that, in contrary to the transformation technique leading to (3.3), such a multiplication step may change the invertibility properties of the equation and the new solution may be a completely different one.

4 Collocation Based on Polynomial Trial Functions

We set $\vartheta_\pm := 1/4 - \sigma_\pm/2$ and consider the new Jacobi weight $\vartheta(x) := (1-x)^{\vartheta_+}(1+x)^{\vartheta_-}$. For $n \in \mathbb{N}$, we introduce the trial space $\vartheta\mathbb{P}_n$ of complex valued polynomials of degree less than n multiplied by ϑ. The collocation points are the zeros of the nth orthogonal polynomial corresponding to the the Chebyshev weight $\varphi(x) := \sqrt{1-x^2}$, i.e. they are defined by $x^\varphi_{kn} := \cos(\pi k/(n+1))$, $k = 1, 2, \ldots, n$. Now the collocation method consists in seeking an approximate solution $u_n \in \vartheta\mathbb{P}_n$ of u the solution to $Au = f$ by solving

$$(Au_n)\left(x^\varphi_{kn}\right) = f\left(x^\varphi_{kn}\right), \; k = 1, 2, \ldots, n. \tag{4.1}$$

This system can be written equivalently as $A_n u_n := M_n f$, where $A_n := M_n A|_{\vartheta\mathbb{P}_n}$ and where M_n denotes the interpolation projection given by $M_n f \in \vartheta\mathbb{P}_n$ and $M_n f(x^\varphi_{kn}) = f(x^\varphi_{kn})$, $k = 1, \ldots, n$. Introducing the Chebyshev polynomial of the second kind $U_n(x) := \sqrt{2/\pi}\sin((n+1)\arccos(x))/\sin(\arccos(x))$ and choosing the Lagrange basis

$$\tilde{\ell}^\varphi_{kn}(x) := \frac{\vartheta(x)}{\vartheta(x^\varphi_{kn})}\,\frac{U_n(x)}{(x - x^\varphi_{kn})U_n'(x^\varphi_{kn})}, \tag{4.2}$$

the projection M_n is given by $M_n f := \sum_{k=1}^n f(x^\varphi_{kn})\tilde{\ell}^\varphi_{kn}$.

Now the collocation is called convergent if, for any $f \in \mathbf{L}^2_\sigma$ and for any $f_n \in \vartheta\mathbb{P}_n$ with $f_n \longrightarrow f$, the solution u_n of $A_n u_n = f_n$ exists uniquely at least for sufficiently large n and if this u_n tends to the exact solution u of $Au = f$ in the norm of $\mathbf{L}^2_\sigma$. Endowing the trial space $\vartheta\mathbb{P}_n$ with the norm induced from the space $\mathbf{L}^2_\sigma$, we call the collocation method stable if the approximate operators A_n are invertible at least for n sufficiently large and if the norms $\|A_n\|$ and $\|[A_n]^{-1}\|$ are bounded uniformly with respect to n. Note that this notion is equivalent to the boundedness of the condition numbers of the matrices of A_n with respect to the scaled basis $\{\tilde{\ell}^\varphi_{kn}/\omega_{kn}\}$ with $\omega_{kn} := \sqrt{\pi/(n+1)}\sqrt{\varphi(x^\varphi_{kn})\sigma(x^\varphi_{kn})}$, $\varphi(x) := \sqrt{1-x^2}$ (cf. the subsequent Equation (8.3)).

Next we recall the main result from [19] for the special case of our equation (2.1). To this end we need the following notation. We suppose the condition (recall the conditions $\alpha < 1/2 + \sigma_-/2 < \beta$ and $-1 < \sigma_- < 1$ from Section 2)

$$\alpha < \frac{\sigma_-}{2} < 1 + \frac{\sigma_-}{2} < \beta. \tag{4.3}$$

Furthermore, we introduce the operator $A_- := (a^-_{j,k})^\infty_{j,k=0} \in \mathcal{L}(\ell^2)$, which is a limit of the discretized operators A_n, as follows. We denote the matrix of A_n with respect to the Lagrange basis $\{\tilde{\ell}^\varphi_{kn}\}$ by $(a^n_{j,k})^n_{j,k=1}$ and define the entry $a^-_{j,k}$ of A_- by the limit relation

$$a^-_{j,k} \;:=\; \lim_{n\longrightarrow\infty}(j+1)^{1/2-\sigma_-}a^n_{(j+1),(k+1)}(k+1)^{-1/2+\sigma_-}.$$

In [19] the existence of the limit A_- has been shown and an integral representation has been derived. Moreover, it has been proved that the limit belongs to the Gohberg-Krupnik algebra of operators generated by Toeplitz matrices with piecewise continuous generator functions. For operators of this algebra, Fredholm property and index are known. The main result of [19] applied to the case of second kind equations with Mellin convolution kernels claims

Theorem 4.1 *Suppose the conditions of Section 2 and (4.3) are satisfied. Then the collocation method is convergent and stable if and only if*

i) The operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ is invertible (cf. Theorem 2.1).
ii) The null space of the operator $A_- \in \mathcal{L}(\ell^2)$ is trivial.

Remark 4.1 *In principle condition i) is easy to check. The Fredholm property with index zero follows according to Theorem 2.1 if the Mellin symbol $\widehat{\mathbf{k}}$ is known. The additional condition on the triviality of the null space of A follows usually from the physical background or from an equivalence to a boundary value problem for a partial differential equation. To check condition ii), however, seems to be hopeless in the general case. Therefore, it is good to know that condition ii) is not "essential" in the sense that the case of condition i) fulfilled but condition ii) violated is very rare and exceptional: Indeed, fix a Mellin kernel $\mathbf{k}_M$, and a smooth kernel $\mathbf{k}_0$, as well as σ. Choose $\mathbf{k} = z\mathbf{k}_M$ with $z \in \mathbb{C}$. Consider the set Σ of all complex numbers z such that condition i) holds. The set of points in Σ such that condition ii) is violated is at most countable, and the accumulation points of this set are in the complement $\mathbb{C} \setminus \Sigma$. If the exceptional case should occur, then the numerical method should be modified slightly. One way to do this is the so called i_* modification introduced in [16] and used also e.g. in [23, 33].*

Note that the additional assumption $\sigma_\pm \neq 1/2$ in (2.7) of [19] is dropped in the last theorem. For our special case (2.1), this assumption is redundant. Indeed, let us have a look, for instance, at the condition $\sigma_- \neq 1/2$. Since there is no Cauchy singular operator in the equation, we may apply the main theorem of [19] equivalently to the modified equation (with a slightly relaxed continuity assumption on the kernel $\mathbf{k}_0$)

$$\begin{aligned}\tilde{u}(x) + b(x) \int_{-1}^{1} \mathbf{k}\left(\frac{1+x}{1+y}\right) \left[\frac{1+x}{1+y}\right]^{\varepsilon} \frac{\tilde{u}(y)}{1+y}\, \mathrm{d}y \, + \\ \int_{-1}^{1} \mathbf{k}_0(x,y) \left[\frac{1+x}{1+y}\right]^{\varepsilon} \tilde{u}(y)\, \mathrm{d}y = f(x)(1+x)^{\varepsilon}, \quad -1 < x < 1,\end{aligned}$$

where $\varepsilon > 0$ is chosen sufficiently small and where the modified integral operator is considered in $\mathbf{L}^2_{\widetilde{\sigma}}$ with $\widetilde{\sigma}(x) := (1-x)^{\sigma_+}(1+x)^{\sigma_- - 2\varepsilon}$. This way we arrive at the same numerical solution and we measure the error in the same norm.

5 The Rate of Convergence

Next we derive the error estimate. We suppose that the conditions for the convergence of the collocation method (cf. Theorem 4.1) are satisfied and that the formula (3.1) for the asymptotics is valid. Moreover, since the solution u is smooth in the neighbourhood of

1, we have to approximate u by polynomials without weight in the vicinity of 1. In other words, we choose $\sigma_+ := 1/2$. We prepare the estimation of the error of collocation $u - u_n$ by the following three lemmata dealing with the interpolation error of u, with a smoothing property of K, and with the continuity of the interpolation M_n mapping the space $\mathbf{C}_\nu$ into $\mathbf{L}^2_\sigma$. Here, for the Jacobi weight $\nu(x) := (1-x)^{\nu_+}(1+x)^{\nu_-}$, $\nu_\pm := (\sigma_\pm + 1)/2$, the space $\mathbf{C}_\nu$ is the Banach space of all functions v continuous on $(-1,1)$ such that the norm $\|v\|_{\nu,\infty} := \sup_{x:\,-1<x<1} |v(x)\nu(x)|$ is finite. Moreover, from now on we use the letter C to denote a generic constant the value of which varies from instance to instance.

Lemma 5.1 *Suppose $\sigma_+ := 1/2$ and (3.1) for $u \in \mathbf{C}^\infty(-1,1]$. Furthermore, denote by $j \geq 0$ the lowest integer index such that the corresponding exponent κ_j in (3.1) satisfies either $[\kappa_j + \sigma_-/2 - 1/4] \notin \mathbb{Z} \cap [0,\infty)$ or $m_j \neq 0$. Then the interpolation error can be estimated as*

$$\|u - M_n u\|_\sigma \leq C n^{-1-2\Re\mathrm{e}\,\kappa_j - \sigma_-} \begin{cases} [\log n]^{m_j} & \text{if } [\kappa_j + \sigma_-/2 - 1/4] \notin \mathbb{Z} \cap [0,\infty) \\ [\log n]^{m_j - 1} & \text{if } [\kappa_j + \sigma_-/2 - 1/4] \in \mathbb{Z} \cap [0,\infty). \end{cases} \tag{5.1}$$

Lemma 5.2 *Suppose that $\mathbf{k}_0$ is a $\mathbf{C}^\infty$ kernel function and that $\mathbf{k}$ satisfies (2.2) with α and β such that $\alpha < 1/2 + \sigma_-/2 < \beta$. Both the Mellin convolution operator K with the Mellin kernel $\mathbf{k}$ and the integral operator K_0 with the kernel function $\mathbf{k}_0$ map $\mathbf{L}^2_\sigma$ continuously into $\mathbf{C}_\nu$.*

Lemma 5.3 *The interpolation projections M_n are continuous operators from $\mathbf{C}_\nu$ to $\mathbf{L}^2_\sigma$, and, for a constant $C > 0$ independent of n and of $v \in \mathbf{C}_\nu$, we get*

$$\|M_n v\|_\sigma \;\; \leq \;\; C\sqrt{\log n}\, \|v\|_{\nu,\infty}. \tag{5.2}$$

The proofs for the last three lemmata will be given in Section 8.

From the continuous equation $Au = f$ and the approximate equation $A_n u_n = M_n f$ including the stable sequence of approximate operators A_n, we arrive at

$$\begin{aligned} u - u_n &= u - M_n u + M_n u - A_n^{-1} M_n f \\ &= u - M_n u + A_n^{-1}\left[M_n A M_n u - M_n A u\right], \\ \|u - u_n\|_\sigma &\leq \|u - M_n u\|_\sigma + C\|M_n A [M_n u - u]\|_\sigma, \\ & \quad C \;:=\; \sup_n \|A_n^{-1}\| \;<\; \infty. \end{aligned}$$

Using (5.2), $A = I + bK + K_0$, and $M_n I[M_n u - u] = 0$, we continue

$$\|u - u_n\|_\sigma \;\; \leq \;\; \|u - M_n u\|_\sigma + C\sqrt{\log n}\, \|[bK + K_0][M_n u - u]\|_{\nu,\infty}.$$

However, the operator of multiplication by the bounded continuous function b is bounded in $\mathbf{C}_\nu$ and the smoothing properties of K and K_0 in Lemma 5.2 yield

$$\|u - u_n\|_\sigma \;\; \leq \;\; \|u - M_n u\|_\sigma + C\sqrt{\log n}\, \|M_n u - u\|_\sigma \; \leq \; C\sqrt{\log n}\, \|u - M_n u\|_\sigma.$$

This together with (5.1) implies

Theorem 5.1 *Suppose the assumptions in Section 2 and the assumptions and convergence conditions of Theorem 4.1 are fulfilled. Finally, suppose $u \in \mathbf{C}^\infty(-1,1]$ and (3.1). If $j \geq 0$ denotes the lowest integer index such that the corresponding exponent κ_j in (3.1) satisfies either $[\kappa_j + \sigma_-/2 - 1/4] \notin \mathbb{Z} \cap [0,\infty)$ or $m_j \neq 0$, then the error $u - u_n$ of the polynomial collocation method (4.1) satisfies*

$$\|u - u_n\|_\sigma \quad \leq \quad C n^{-1-2\Re\mathfrak{e}\,\kappa_j - \sigma_-} \begin{cases} [\log n]^{m_j+1/2} & \text{if } [\kappa_j + \sigma_-/2 - 1/4] \notin \mathbb{Z} \cap [0,\infty) \\ [\log n]^{m_j-1/2} & \text{if } [\kappa_j + \sigma_-/2 - 1/4] \in \mathbb{Z} \cap [0,\infty). \end{cases}$$

6 The Assembling of the Collocation Equations

6.1 A Remark on the Complexity of the Algorithm

Recall that the collocation equations (4.1) can be written as $A_n u_n := M_n f$ where $A_n := M_n A|_{\vartheta\mathbb{P}_n}$. If we use the Lagrange basis (4.2) and seek u_n in the form $u_n := \sum_1^n u_n(x_{kn}^\varphi)\tilde{\ell}_{kn}^\varphi$, then $A_n u_n := M_n f$ is equivalent to the linear system of equations

$$\left(a_{j,k}^n\right)_{j,k=1}^n \left(u_n(x_{kn}^\varphi)\right)_{k=1}^n \quad = \quad \left(f(x_{jn}^\varphi)\right)_{j=1}^n, \; a_{j,k}^n \; := \; \left(A\tilde{\ell}_{kn}^\varphi\right)(x_{jn}^\varphi). \tag{6.1}$$

Due to the stability result of Theorem 4.1 and the subsequent Marcinkiewicz equality (8.3), the condition numbers of the diagonally preconditioned stiffness matrices $(a_{j,k}^n)_{j,k=1}^n$ are bounded uniformly with respect to n (compare [24]). Thus we can solve (6.1) iteratively by no more than $\mathcal{O}(n^2)$ arithmetic operations.

More precisely, such an iterative solution of the linear system up to the size of the discretization error of Theorem 5.1 requires $\mathcal{O}(\log n)$ iteration steps and $\mathcal{O}(n^2 \log n)$ operations. To get rid of the $\log n$ factor in the asymptotic complexity analysis, one has to consider a sequence of discretization levels with $n = n_k := 2^k$ from the lowest level $n_0 = 1$ to a highest level n_K. On each level $n = n_k$ the corresponding system (6.1) is to be solved iteratively up to an accuracy in the size of the discretization error for the level $n = n_k$ described in Theorem 5.1. The initial solution for the iteration on level n_k is taken to be the interpolation of the final solution of the previous level n_{k-1} if $k > 0$ and to be zero if $k = 0$. This way the number of necessary iteration steps stays bounded independently of the level n_k, and the number of all arithmetic operations is $\mathcal{O}(n^2)$. If the system (6.1) is selfadjoint positive definite or can be transformed to such a system by multiplying with diagonal matrices, then one should choose the conjugate gradient method for the iterations. If the system (6.1) is not of such a structure, then, for example, the GMRes method seems to be a good choice.

In practical computations, however, the iterative solver is usually implemented such that an estimated solution error is less than a prescribed tolerance like the machine precision. Though this requires more operations than the preceding optimal multilevel approach, we prefer the usual way for its simplicity.

In order to get an overall algorithm of complexity $\mathcal{O}(n^2)$ we should be able to assemble the stiffness matrix $(a_{j,k}^n)_{j,k=1}^n$ with no more than $\mathcal{O}(n^2)$ arithmetic operations. For special Mellin kernel functions $\mathbf{k}$, we present two different approaches. First (cf. Section 6.2) we introduce a quadrature algorithm which requires $\mathcal{O}(n^2[\log n]^2)$ arithmetic operations. Second (cf. Section 6.3.1), based on recurrence relations, we propose an algorithm of $\mathcal{O}(n^2)$

arithmetic operations (cf. the analogous methods for special weakly singular operators in [32, 4, 28, 11] and for Mellin operators in [29]). Since the classical recurrence relation becomes seriously unstable, we introduce a non-standard modification in Section 6.3.2. For a precise notion of instability and ill-conditioning, we refer to [5].

6.2 The Assembling by Quadrature

6.2.1. We need the stiffness matrix with respect to the polynomial interpolation basis $\{\tilde{\ell}^{\varphi}_{kn} : \ k = 1, \ldots, n\}$. However, we shall compute the stiffness matrix with respect to the basis of orthonormal polynomials in the trial space, i.e. we consider the orthonormal basis $\{\vartheta U_m : \ m = 0, \ldots, n-1\}$ of the space $\vartheta \mathbb{P}_n \subset \mathbf{L}^2_\sigma$ and determine the first of the two matrices

$$B \;=\; \left(b^n_{j,m}\right)_{\substack{j=1,\ldots,n,\\ m=0,\ldots,n-1}}, \qquad b^n_{j,m} \;:=\; \Big((bK+K_0)\vartheta U_m\Big)\left(x^{\varphi}_{jn}\right),$$

$$C \;=\; \left(c^n_{j,k}\right)_{\substack{j=1,\ldots,n,\\ k=1,\ldots,n}}, \qquad c^n_{j,k} \;:=\; \Big((bK+K_0)\tilde{\ell}^{\varphi}_{kn}\Big)\left(x^{\varphi}_{jn}\right)$$

for K and K_0 defined with the kernels $\mathbf{k}$ and $\mathbf{k}_0$, respectively. To get the matrix C with respect to the interpolation basis $\{\tilde{\ell}^{\varphi}_{kn} : \ k = 1, \ldots, n\}$, one has to apply the simple basis transform T given by

$$T: \ \Big(u_n\left(x^{\varphi}_{kn}\right)\Big)_{k=1}^{n} \mapsto \Big(\eta_m\Big)_{m=0}^{n-1}, \quad u_n \;=\; \sum_{m=0}^{n-1} \eta_m \Big[\vartheta U_m\Big] \;=\; \sum_{k=1}^{n} u_n\left(x^{\varphi}_{kn}\right) \tilde{\ell}^{\varphi}_{kn},$$

$$\eta_m \;=\; \sum_{k=1}^{n} \left[\frac{\sqrt{2\pi}}{n+1} \sin\left(\frac{\pi k(m+1)}{n+1}\right) \varrho\left(x^{\varphi}_{kn}\right)\right] u_n\Big(x^{\varphi}_{kn}\Big), \qquad \varrho \;:=\; \varphi/\vartheta.$$

Obviously, since the evaluation of the matrix product $C = BT$ requires $\mathcal{O}(n^3)$ operations, this product is never computed. Instead, whenever the iterative solution of (6.1) requires the multiplication of a vector by C, we multiply first by T and then by B.

6.2.2. First we consider the case of the kernel $\mathbf{k}_0$. We have to evaluate the entries of the matrix

$$D \;=\; \left(d^n_{j,m}\right)_{\substack{j=1,\ldots,n,\\ m=0,\ldots,n-1}}, \qquad d^n_{j,m} \;:=\; \Big(K_0 \vartheta U_m\Big)\left(x^{\varphi}_{jn}\right) \;=\; \int_{-1}^{1} \mathbf{k}_0\left(x^{\varphi}_{jn}, y\right) U_m(y)\vartheta(y)\, \mathrm{d}y.$$

Clearly, for this integral we use the Gauß quadrature based on the orthogonal polynomials with the weight function ϑ. Suppose this rule is given by

$$\int_{-1}^{1} f(y)\vartheta(y)\, \mathrm{d}y \;\sim\; \sum_{\iota=1}^{N} f(x^{\vartheta}_{\iota N})\omega^{\vartheta}_{\iota N}.$$

Then the entry $d^n_{j,m}$ of our stiffness matrix is approximated by the quadrature sum

$$d^n_{j,m} \;\sim\; \sum_{\iota=1}^{N} \mathbf{k}_0\left(x^{\varphi}_{jn}, x^{\vartheta}_{\iota N}\right) U_m(x^{\vartheta}_{\iota N})\omega^{\vartheta}_{\iota N} \;=\; \sum_{\iota=1}^{N} d^{ny,n}_{j,\iota} d^{tr,n}_{\iota,m}, \tag{6.2}$$

$$
\begin{aligned}
D^{ny} &= \left(d^{ny,n}_{j,\iota}\right)_{\substack{j=1,\dots,n,\\ \iota=1,\dots,N}}, & d^{ny,n}_{j,\iota} &:= \mathbf{k}_0\left(x^{\varphi}_{jn}, x^{\vartheta}_{\iota N}\right)\omega^{\vartheta}_{\iota N},\\
D^{tr} &= \left(d^{tr,n}_{\iota,m}\right)_{\substack{\iota=1,\dots,N\\ m=0,\dots,n-1}}, & d^{tr,n}_{\iota,m} &:= U_m(x^{\vartheta}_{\iota N}).
\end{aligned}
$$

In other words, since we use the same quadrature rule for all entries, the matrix D is approximated by the product of the two matrices D^{ny} and D^{tr}. The matrix D^{ny} is a typical Nyström discretization of K_0 and D^{tr} is a transform which maps the vector of coefficients with respect to the basis of orthogonal polynomials to the vector of function values evaluated at the quadrature knots. To ensure a small quadrature error in (6.2), we choose N such that the Gauß quadrature rule is exact at least for all polynomials U_m with $m < n$, i.e. such that $n/2 < N < Cn$. Using $U_m(x) := \sqrt{2/\pi}\sin((m{+}1)\arccos(x))/\sin(\arccos(x))$ and assuming that each value of the kernel $\mathbf{k}_0$ can be computed with a fixed number of operations, the matrices D^{ny} and D^{tr} can be computed with no more than $\mathcal{O}(n^2)$ arithmetic operations. Similarly to the matrix product $C = BT$ in Subsection 6.2.1, for a fast algorithm, the product $D^{ny}D^{tr}$ must not be computed. Instead, whenever the iterative solution of (6.1) requires the multiplication of a vector by D, we multiply by D^{tr} and by D^{ny} separately.

For the quadrature error in (6.2), we remark that at most $n-1$ degrees of the polynomial exactness degree $2N-1$ of the Gauß quadrature are used to approximate the polynomial U_m in the quadrature. The remaining $N' := 2N-1-(n-1)$ degrees are spent for the approximation of the kernel $\mathbf{k}_0$. If this polynomial approximation error is less than the discretization error (cf. Theorem 5.1) times an additional $\mathcal{O}(n^{-1})$, then the error due to the quadrature approximation in (6.2) is less than the discretization error. In other words the convergence estimate in Theorem 5.1 remains valid for the approximate solution u_n computed with the quadrature discretization (6.2) if the Gauß order N is chosen such that

$$
\sup_{-1\le x\le 1}\ \sup_{p\in\mathbb{P}_{N'}} \left\|\mathbf{k}_0(x,\cdot)-p\right\|_{L^\infty} \ \le\ C\, n^{-2-2\Re\mathfrak{e}\,\kappa_j-\sigma_-}\left[\,\log n\right]^{m_j+1/2}.
$$

No doubt, this is satisfied for $\mathbf{C}^\infty$ kernels $\mathbf{k}_0$ if $N > n(1/2+\varepsilon)$ for any prescribed small real number $\varepsilon > 0$. If ε is sufficiently large, then the presented algorithm works even for a kernel function $\mathbf{k}_0$ with finite degree of smoothness.

Finally, we remark that, for $N = n$ and $\sigma = \varphi^{-1}$, we get $\vartheta = \varphi$ and the Gauß knots of the quadrature rule and the set of collocation points coincide. In this case D^{tr} is the inverse of transform T from Subsection 6.2.1 and the stiffness matrix with respect to the Lagrange basis $\{\tilde{\ell}^{\varphi}_{kn}\}$ of the collocation discretization for K_0 is simply the matrix D^{ny} of the Nyström method.

6.2.3. Next we turn to the integration of the Mellin kernels which are typically analytic with the exception of fixed singularities at ± 1. The additional multiplication operator b in $A = I + bK + K_0$ results in a simple multiplication of $b^n_{j,m}$ by $b(x^{\varphi}_{jn})$. Hence, for simplicity, we may suppose $b \equiv 1$. Moreover, we restrict our consideration to the following special case. We suppose that the function $\mathbf{k}$ involved in (2.1) takes the form

$$
\mathbf{k}(t) \ := \ t^{\chi}\,\frac{p_N(t)}{p_D(t)}, \tag{6.3}
$$

where χ is a real number and p_N and p_D denote polynomials such that the difference of the degrees $\deg p_D - \deg p_N$ is greater than zero, such that $p_N(0) \neq 0$, and $p_D(t) \neq 0$ for $0 \leq t < \infty$. Note that (2.2) holds for such a $\mathbf{k}$ with $\alpha = -\chi$ and $\beta = -\chi + \deg p_D - \deg p_N$. For such kernels, we have to compute

$$\begin{aligned} E &= \left(e^n_{j,m}\right)_{\substack{j=1,\ldots,n,\\ m=0,\ldots,n-1}}, \\ e^n_{j,m} &:= \left(K\vartheta U_m\right)\left(x^{\varphi}_{jn}\right) = \int_{-1}^{1} \mathbf{k}\left(\frac{1+x^{\varphi}_{jn}}{1+y}\right) \frac{1}{1+y}\, U_m(y)\vartheta(y)\,\mathrm{d}y. \end{aligned}$$

In other words, we only have to set up an efficient quadrature rule for the computation of the integral

$$\int_{-1}^{1} f(y)U_m(y)\vartheta(y)\,\mathrm{d}y, \quad f(y) := \mathbf{k}\left(\frac{1+x^{\varphi}_{jn}}{1+y}\right)\frac{1}{1+y}.$$

Here the integrand function f is analytic on $[-1,1]$ and meromorphic on $\mathbb{C}$ with poles ζ well separated from the set $[-1,1]$, i.e. such that $|\zeta+1| > \varepsilon/n$, that $|\zeta-1| > \varepsilon/n$, and that $|\Re\mathrm{e}\,\zeta| > \varepsilon \min\{|\zeta+1|, |\zeta-1|\}$ holds for a fixed $\varepsilon > 0$. Using the formula $U_m(y) = \sqrt{2/\pi}\sin((m+1)\arccos(y))/\sin(\arccos(y))$ and substituting $y = \cos(t)$, we have to approximate

$$I := \sqrt{\frac{2}{\pi}} \int_0^{\pi} f\Big(\cos(t)\Big)\vartheta\Big(\cos(t)\Big)\sin\Big((m+1)t\Big)\,\mathrm{d}t.$$

We define an appropriate quadrature using the techniques of geometrical mesh refinement (cf. [35]) and of product quadrature for oscillatory integrals (cf. [4]). Thus we first introduce a quadrature partition of $[0,\pi]$ and then we define a quadrature rule for each subinterval.

The quadrature partition is introduced in two steps. First we define a graded mesh with subinterval lengths equal to $\pi/(m+1)$ times a power of two. This special choice of lengths will help us to restrict the number of precomputed quadrature weights and knots. Then we further subdivide the two intervals adjacent to the end-points 0 and π.

The first partition is given by

$$0 =: \widetilde{t}^m_0 < \widetilde{t}^m_1 < \ldots < \widetilde{t}^m_{k'+k''+1} < \widetilde{t}^m_{k'+k''+2} := \pi, \quad \widetilde{t}^m_k := \frac{\pi l^m_k}{m+1}$$

$$l^m_k := \begin{cases} 0 & \text{if } k=0 \\ m'_{k'} & \text{if } k=1 \\ m'_{k'} + \sum_{j=0}^{k-2} 2^{l'_{k'-1-j}} & \text{if } 1 < k \leq k'+1 \\ m'_{k'} + \sum_{j=0}^{k'-1} 2^{l'_j} + \sum_{j=0}^{k-k'-2} 2^{l''_j} & \text{if } k'+2 \leq k \leq k'+k''+1 \\ m+1 & \text{if } k = k'+k''+2. \end{cases}$$

Here the integer numbers $l'_k, l''_k, m'_{k'+1}, m''_{k''+1}$ with $m+1 = 2^{l'_0} + 2^{l'_1} + \ldots + 2^{l'_{k'-1}} + m'_{k'} + 2^{l''_0} + 2^{l''_1} + \ldots + 2^{l''_{k''-1}} + m''_{k''}$ and $l'_0 \geq l'_1 \geq \ldots \geq l'_{k'-1}$, $l''_0 \geq l''_1 \geq \ldots \geq l''_{k''-1}$, are defined as follows. We fix a grading parameter q with $0 < q < 1$. By recursion we define the numbers l'_i, m'_i, $i=1,\dots,k'-1$ and l''_i, m''_i, $i=1,\dots,k''-1$. We choose m'_0 to be the largest integer less or equal to $(m+1)/2$ and set $m''_0=(m+1)-m'_0$. Starting with m'_0, for given m'_k, we define l'_k such that $2^{l'_k}\le q\cdot m'_k<2^{l'_k+1}$ and $m'_{k+1}:=m'_k-2^{l'_k}$. We proceed for $k=0,\dots,k'-1$ until we get $q\cdot m'_{k'}<1$. Analogously, we define the numbers l''_i and m''_k beginning with m''_0. The resulting partition $\widetilde{t}^m_k$ is graded towards 0 and π. Indeed, we observe

$$\begin{aligned} l^m_l &= m'_{k'+1-l},\ l=1,\dots,k'+1,\\ (1-q)m'_i\le m'_{i+1}=m'_i-2^{l'_i}&\le m'_i-0.5\cdot 2^{l'_i+1}\le(1-q/2)m'_i,\quad i=0,\dots,k'-1\end{aligned}$$

and, from these and from the analogous estimates with m''_i and l''_i, we conclude the grading conditions

$$\begin{aligned} (1-q)\widetilde{t}^m_i &\le \widetilde{t}^m_{i-1} &&\le (1-q/2)\widetilde{t}^m_i, && i=2,\dots,k'+1,\\ (1-q)[\pi-\widetilde{t}^m_i] &\le [\pi-\widetilde{t}^m_{i+1}] &&\le (1-q/2)[\pi-\widetilde{t}^m_i], && i=k'+1,\dots,k'+k''.\end{aligned}$$

The resulting subintervals $[\widetilde{t}^m_{i-1},\widetilde{t}^m_i]$ of the first partition have lengths of the desired form power of two times $\pi/(m+1)$ by construction.

To complete the quadrature partition in a second step, we subdivide the two intervals $[\widetilde{t}^m_0,\widetilde{t}^m_1]$ and $[\widetilde{t}^m_{k'+k''+1},\widetilde{t}^m_{k'+k''+2}]$ adjacent to the end-points 0 and 1. We add the points of

$$\Big\{(1-q)^l\widetilde{t}^m_1\,:\,l=1,2,\dots,l'\Big\}\cup\Big\{\pi-(1-q)^l\Big[\pi-\widetilde{t}^m_{k'+k''+1}\Big]\,:\,l=1,2,\dots,l''\Big\}$$

to the previous partition $\{\widetilde{t}^m_l\,:\,l=0,\dots,k'+k''+2\}$. Here the integer numbers l' and l'' are chosen such that

$$\begin{aligned}(1-q)^{l'}\,\widetilde{t}^m_1 &\le \frac{\pi}{n} < (1-q)^{l'-1}\,\widetilde{t}^m_1,\\ (1-q)^{l''}\Big[\pi-\widetilde{t}^m_{k'+k''+1}\Big] &\le \frac{\pi}{n} < (1-q)^{l''-1}\Big[\pi-\widetilde{t}^m_{k'+k''+1}\Big].\end{aligned}$$

We denote the final partition by $\{t^{m,n}_k\,:\ k=0,\dots,N_{m,n}\}$. According to the quadrature partition we split the integral I into the sum of integrals over the subdomains.

$$I=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{N_{m,n}}I_j,\quad I_j:=\int_{t^{m,n}_{j-1}}^{t^{m,n}_j}f\Big(\cos(t)\Big)\vartheta\Big(\cos(t)\Big)\sin\Big((m+1)t\Big)\,\mathrm{d}t.$$

For each of these integrals, we have to introduce a quadrature. We distinguish three cases. First we consider the integrals over the intervals $[\widetilde{t}^m_{k-1},\widetilde{t}^m_k]$, $k=2,\dots,k'+k''+1$, then those over the remaining intervals not adjacent to the end-points, and finally those over the two adjacent intervals $[t^{m,n}_0,t^{m,n}_1]$ and $[t^{m,n}_{N_{m,n}-1},t^{m,n}_{N_{m,n}}]$.

First suppose I_j is the integral over $[\widetilde{t}^m_{k-1},\widetilde{t}^m_k]$, $2\le k\le k'+k''+1$ and, without loss of generality, that $2\le k<k'+1$. Substituting $t=\widetilde{t}^m_{k-1}+s2^{\tilde{l}}/(m+1)$ with $\tilde{l}=l'_{k'-k+1}$, we arrive at

$$I_j=(-1)^{l^m_{k-1}}\frac{2^{\tilde{l}}}{m+1}\int_0^\pi\Big[f\vartheta\Big]\left(\cos\left(\widetilde{t}^m_{k-1}+s\frac{2^{\tilde{l}}}{m+1}\right)\right)\sin\Big(2^{\tilde{l}}s\Big)\,\mathrm{d}s.$$

To the last integral we apply a product quadrature rule of order n_p. More precisely, we choose the function $s\mapsto\sin(2^{\tilde{l}}s)$ as the product weight and the Gauß-Legendre nodes as quadrature knots. If $\{x^1_{l,n_p} : l = 1,\dots,n_p\}$ is the set of zeros of the n_p-th Legendre polynomial, then we get

$$\sigma^k_{l,n_p} := \pi\left(\frac{1}{2}+\frac{1}{2}x^1_{l,n_p}\right), \quad w^k_{l,n_p} := \int_0^\pi \prod_{\substack{l'=1\\ l'\neq l}}^{n_p} \frac{s-\sigma^k_{l',n_p}}{\sigma^k_{l,n_p}-\sigma^k_{l',n_p}} \sin(2^{\tilde{l}}s)\,\mathrm{d}s,$$

$$I_j \sim (-1)^{l^m_{k-1}} \frac{2^{\tilde{l}}}{m+1} \sum_{l=1}^{n_p} \Big[f\vartheta\Big]\left(\cos\left(\widetilde{t}^m_{k-1}+\sigma^k_{l,n_p}\frac{2^{\tilde{l}}}{m+1}\right)\right) w^k_{l,n_p}.$$

We note that the integrand $s \mapsto [f\vartheta](\cos(\widetilde{t}^m_{k-1}+s2^{\tilde{l}}/(m+1)))$ is analytic, and, due to the geometric quadrature partition, the quotient of length of subdomains over distance to singularity point can be estimate by

$$C|\widetilde{t}^m_k - \widetilde{t}^m_{k-1}|/\widetilde{t}^m_{k-1} \le C\widetilde{t}^m_k/\widetilde{t}^m_{k-1} \le C/(1-q).$$

Consequently, we get an error estimate less than any negative power n^{-r}, $r>0$ provided we choose $n_p > c_p \log n$ for a suitable constant $c_p = c_p(r)$. In other words a small order n_p is sufficient. For small n_p, the weights w^k_{l,n_p} can be computed analytically using integration by parts. Alternatively, the weights can be precomputed by quadrature with a larger number of quadrature knots.

Second, we suppose that the integration interval $[t^{m,n}_{j-1}, t^{m,n}_j]$ of I_j is not adjacent to the endpoints and that it is contained in $[\widetilde{t}^m_0, \widetilde{t}^m_1]$ or $[\widetilde{t}^m_{k'+k''+1}, \widetilde{t}^m_{k'+k''+2}]$. For the sake of definiteness, we assume $[t^{m,n}_{j-1}, t^{m,n}_j] \subseteq [\widetilde{t}^m_0, \widetilde{t}^m_1]$, i.e. $1 < j \le l'$. Now the length of the interval is relatively small such that the factor $t \mapsto \sin((m+1)t)$, $t^{m,n}_{j-1} < t < t^{m,n}_j$ is not oscillatory anymore. Again the integrand is analytic and we simply apply a Gauß-Legendre rule of order $n_g > c_g \log n$. We get

$$\begin{aligned} I_j \;&\sim\; \sum_{l=1}^{n_g} \Big[f\vartheta\Big]\left(\cos\left(\frac{t^{m,n}_j+t^{m,n}_{j-1}}{2} + x^1_{l,n_g}\frac{t^{m,n}_j-t^{m,n}_{j-1}}{2}\right)\right) \times \\ &\quad \sin\left((m+1)\left(\frac{t^{m,n}_j+t^{m,n}_{j-1}}{2} + x^1_{l,n_g}\frac{t^{m,n}_j-t^{m,n}_{j-1}}{2}\right)\right)\frac{t^{m,n}_j-t^{m,n}_{j-1}}{2}\, w^g_{l,n_g}, \\ w^g_{l,n_g} \;&:=\; \int_{-1}^1 \prod_{\substack{l'=1\\ l'\neq l}}^{n_g} \frac{s-x^1_{l',n_g}}{x^1_{l,n_g}-x^1_{l',n_g}}\,\mathrm{d}s. \end{aligned}$$

The efficient computation of the Gauß-Legendre nodes x^1_{l,n_g}, $l = 1,\dots,n_g$ and of the corresponding quadrature weights w^g_{l,n_g}, $l=1,\dots,n_g$ is well established.

Third, we suppose the integration interval is $[t^{m,n}_0, t^{m,n}_1]$ or $[t^{m,n}_{N_{m,n}-1}, t^{m,n}_{N_{m,n}}]$. For the sake of definiteness, we consider $[t^{m,n}_0, t^{m,n}_1]$. This time the factors $t \mapsto f(\cos(t))$ and $t \mapsto \sin((m+1)t)$, $t^{m,n}_0 < t < t^{m,n}_1$ are analytic and non-oscillatory, but $t \mapsto \vartheta(\cos(t))$ is weakly singular. The singular factor in the last function is $t \mapsto t^{2\vartheta_+}$. Hence, we apply the Gauß-Jacobi quadrature rule for the weight function $[-1,1] \ni s \mapsto \widetilde{\vartheta}(s) := (1+s)^{2\vartheta_+}$. Setting $\vartheta^\#(s) := \widetilde{\vartheta}([\arccos(s) - t^{m,n}_1/2]\,2/t^{m,n}_1)^{-1}$, the function $[f\vartheta\vartheta^\#]$ turns out to be smooth. With an order $n'_p > c'_p \log n$ and the Gauß-Jacobi nodes $x^{\widetilde{\vartheta}}_{l,n'_p}$, $l = 1,\dots,n'_p$

corresponding to the weight $\widetilde{\vartheta}$, we obtain

$$
\begin{aligned}
I_1 &= \frac{t_1^{m,n}}{2} \int_{-1}^{1} \Big[f\vartheta \Big] \left(\cos\left(\frac{t_1^{m,n}}{2} + s\frac{t_1^{m,n}}{2} \right) \right) \sin\left((m+1)\left(\frac{t_1^{m,n}}{2} + s\frac{t_1^{m,n}}{2} \right) \right) \mathrm{d}s \\
&= \frac{t_1^{m,n}}{2} \int_{-1}^{1} \Big[f\vartheta\vartheta^{\#} \Big] \left(\cos\left(\frac{t_1^{m,n}}{2} + s\frac{t_1^{m,n}}{2} \right) \right) \times \\
&\qquad\qquad \sin\left((m+1)\left(\frac{t_1^{m,n}}{2} + s\frac{t_1^{m,n}}{2} \right) \right) \widetilde{\vartheta}(s)\,\mathrm{d}s \\
&\sim \sum_{l=1}^{n_p'} \Big[f\vartheta\vartheta^{\#} \Big] \left(\cos\left(\frac{t_1^{m,n}}{2} + x_{l,n_p'}^{\widetilde{\vartheta}} \frac{t_1^{m,n}}{2} \right) \right) \times \\
&\qquad\qquad \sin\left((m+1)\left(\frac{t_1^{m,n}}{2} + x_{l,n_p'}^{\widetilde{\vartheta}} \frac{t_1^{m,n}}{2} \right) \right) \frac{t_1^{m,n}}{2}\, w_{l,n_p'}^{p}\,, \\
w_{l,n_p'}^{p} &:= \int_{-1}^{1} \prod_{\substack{l'=1\\ l'\neq l}}^{n_p'} \frac{s - x_{l',n_p'}^{\widetilde{\vartheta}}}{x_{l,n_p'}^{\widetilde{\vartheta}} - x_{l',n_p'}^{\widetilde{\vartheta}}}\, \widetilde{\vartheta}(s)\,\mathrm{d}s.
\end{aligned}
$$

Again the computation of the Gauß-Jacobi nodes $x_{l,n_p'}^{\widetilde{\vartheta}}$, $l = 1,\ldots,n_p'$ and of the corresponding quadrature weights $w_{l,n_p'}^{k}$, $l = 1,\ldots,n_p'$ is well established.

All together we can compute each entry of the stiffness matrix with no more than $\mathcal{O}([\log n]^2)$ quadrature knots and arithmetic operations. For the whole matrix we need no more than $\mathcal{O}(n^2[\log n]^2)$ operations. The error is less than any prescribed power Cn^{-r}, $r > 0$ if the constants c_p, c_g, and c_p' for the quadrature rules are chosen suitably large. Hence, the discretized polynomial collocation converges with the same rate as described in Theorem 5.1.

6.3 The Assembling by Recurrence Relations

6.3.1. In Subsection 6.2.2 we have seen how to compute the part of the matrix corresponding to the smooth kernel $\mathbf{k}_0$ with no more than $\mathcal{O}(n^2)$ arithmetic operations. Hence, we need a fast recursive algorithm only for the part of the matrix corresponding to the Mellin kernel $\mathbf{k}$. To employ the recursion technique, we need the special type of kernels introduced in (6.3). Clearly, we can split the kernel of (6.3) into the sum

$$
\mathbf{k}(t) := \sum_{i=1}^{I} \sum_{l=1}^{L_i} B_{i,l} \frac{t^{\chi}}{(t-\zeta_i)^l} \tag{6.4}
$$

with constant coefficients $B_{i,l}$, with ζ_i the zeros of p_D, and with L_i the multiplicity of ζ_i. Therefore, we can restrict our consideration to kernels of the form

$$
\mathbf{k}_l(t) := \frac{t^{\chi}}{(t-\zeta)^l}, \quad \widehat{\mathbf{k}}_l(z) := -\pi\, \zeta^{z+\chi-l} \binom{z+\chi-1}{l-1} \frac{e^{-\mathbf{i}\pi(z+\chi)}}{\sin\Big(\pi(z+\chi)\Big)}, \tag{6.5}
$$

$$
\zeta^z := e^{z[\log|\zeta| + \mathbf{i}\arg\zeta]}, \quad 0 < \arg\zeta < 2\pi
$$

with integer parameters $l = 1, \ldots, L$. We derive the subsequent formulae for this elementary type of kernel. Using (6.4), it is easy to compose the recursion for general kernels $\mathbf{k}$ of the type (6.3).

For $\mathbf{k} = \mathbf{k}_l$, we have to compute $b_{j,m}^{n,l} = (1 + x_{jn}^{\varphi})^{\chi} M_l^{m,l}(x_{jn}^{\varphi})$ with the so called modified moments $M_l^{m,l}(x_{jn}^{\varphi})$, i.e.

$$M_l^{m,l'}(x_{jn}^{\varphi}) \quad := \quad \int_{-1}^{1} \frac{\vartheta(y)}{(1+y)^{\chi-l+1}} \frac{U_m(y)}{\Big[[1 + x_{jn}^{\varphi}] - \zeta[1+y]\Big]^{l'}} \,\mathrm{d}y. \tag{6.6}$$

To compute these by recursion, we replace m by $m+1$, insert the recurrence relation $U_{m+1}(y) = 2yU_m(y) - U_{m-1}(y)$ valid for $m = 1, 2, \ldots$, replace the factor y before U_m by the expression $\zeta^{-1}[(1 + x_{jn}^{\varphi}) - \zeta] - \zeta^{-1}[(1 + x_{jn}^{\varphi}) - \zeta(1+y)]$, and obtain

$$\begin{aligned} M_l^{m+1,l'}(x_{jn}^{\varphi}) \quad &= \quad 2\Big[\zeta^{-1}(1 + x_{jn}^{\varphi}) - 1\Big] M_l^{m,l'}(x_{jn}^{\varphi}) - M_l^{m-1,l'}(x_{jn}^{\varphi}) - 2\zeta^{-1} M_l^{m,l'-1}(x_{jn}^{\varphi}), \\ & \qquad\qquad m = 1, 2, \ldots \,. \end{aligned} \tag{6.7}$$

Clearly, this is the recursion for the computation of the second kind Chebyshev polynomials with the additional inhomogeneity $-2\zeta^{-1}M_l^{m,l'-1}(x_{jn}^{\varphi})$. Moreover, the point y from the interval $[-1, 1]$ in the recursion for the Chebyshev polynomials is replaced by $\zeta^{-1}(1 + x_{jn}^{\varphi}) - 1$. To get the initial values $M_l^{1,l'}(x_{jn}^{\varphi})$ and $M_l^{0,l'}(x_{jn}^{\varphi})$ of the recurrence relation, we use $U_1(y) = 2yU_0(y)$ and $U_0(y) = \sqrt{2/\pi}$ and get from (6.6) with $m = 1$ that

$$M_l^{1,l'}(x_{jn}^{\varphi}) \quad = \quad 2\Big[\zeta^{-1}(1 + x_{jn}^{\varphi}) - 1\Big] M_l^{0,l'}(x_{jn}^{\varphi}) - 2\zeta^{-1} M_l^{0,l'-1}(x_{jn}^{\varphi}). \tag{6.8}$$

For $M_l^{0,l'}(x_{jn}^{\varphi})$ with $\vartheta_- - \chi + l - 1 > -1$ and $\vartheta_+ > -1$, we infer from [15, Equation 3.197.8]

$$\begin{aligned} M_l^{0,l'}(x_{jn}^{\varphi}) \quad &:= \quad \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{\vartheta(y)}{(1+y)^{\chi-l+1}} \frac{1}{\Big[[1 + x_{jn}^{\varphi}] - \zeta[1+y]\Big]^{l'}} \,\mathrm{d}y \\ &= \quad \sqrt{\frac{2}{\pi}}\, 2^{\vartheta_+ + \vartheta_- - \chi + l} (1 + x_{jn}^{\varphi})^{-l'} \mathrm{B}\Big(\vartheta_+ + 1, \vartheta_- - \chi + l\Big) \\ & \qquad \times \, {}_2\mathrm{F}_1\Big(l', \vartheta_- - \chi + l; 1 + \vartheta_+ + \vartheta_- - \chi + l; \frac{2\zeta}{1 + x_{jn}^{\varphi}}\Big). \\ \mathrm{B}(\mu, \nu) \quad &:= \quad \int_0^1 x^{\nu-1}(1-x)^{\mu-1} \,\mathrm{d}x, \\ {}_2\mathrm{F}_1(a, b; c; z) \quad &:= \quad \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (a)_k \;:=\; a \cdot (a+1) \cdot \ldots \cdot (a+k-1). \end{aligned} \tag{6.9}$$

Here B is the beta function and ${}_2\mathrm{F}_1$ the hypergeometric series. If no good code for the computation of B and ${}_2\mathrm{F}_1$ is available, then the integrals of $M_l^{0,l'}(x_{jn}^{\varphi})$ can be computed by quadrature. Since the integrand is analytic for $y \neq \pm 1$, a composite quadrature, defined over a partition geometrically graded towards the points ± 1 and with appropriate Gauß rule over each subinterval of the partition, will do the job with $\mathcal{O}(\log n)$ operations (cf.[35] or compare Subsection 6.2.3). For all the n integrals $\mathcal{O}(n \log n)$ arithmetic operations are required.

The recurrence relation (6.7),(6.8) for a fixed l' requires that the values $M_l^{m,l'-1}(x_{j,n}^{\varphi})$, $m = 1, 2, \ldots$ have been computed first. Thus we have to compute $M_l^{m,0}(x_{j,n}^{\varphi})$, $m = 1, 2, \ldots$, and then we use (6.7),(6.8) to get the values $M_l^{m,1}(x_{j,n}^{\varphi})$, $m = 1, 2, \ldots$. Using these we use (6.7),(6.8) to get the values $M_l^{m,2}(x_{j,n}^{\varphi})$, $m = 1, 2, \ldots$, and so on.

Finally, it remains to determine the values of $M_l^{m,0} := M_l^{m,0}(x_{jn}^{\varphi})$ which, in fact, are independent of x_{jn}^{φ}. These modified moments including the Jacobi weight $\vartheta(y)(1+y)^{-\chi+l-1}$ and Chebyshev polynomials of the second kind U_m can be computed analogously to the modified moments including the Jacobi weight $\vartheta(y)(1+y)^{-\chi+l-1}$ and Chebyshev polynomials of the first kind T_n (cf. [32]). We are grateful to D. Occorsio for pointing out this fact and providing us with the formulae. The recursion is

$$\begin{aligned} 0 &= \Big(\vartheta_+ + \vartheta_- - \chi + l + m + 1\Big) M_l^{m+1,0} + 2\Big(\vartheta_+ - \vartheta_- + \chi - l + 1\Big) M_l^{m,0} \\ &\quad + \Big(\vartheta_+ + \vartheta_- - \chi + l - 1 - m\Big) M_l^{m-1,0}, \qquad (6.10) \\ M_l^{1,0} &= 2\frac{\vartheta_- - \chi + l - 1 - \vartheta_+}{\vartheta_+ + \vartheta_- - \chi + l + 1} M_l^{0,0}, \qquad (6.11) \\ M_l^{0,0} &= \sqrt{\frac{2}{\pi}}\, 2^{\vartheta_+ + \vartheta_- - \chi + l}\, \mathrm{B}\Big(\vartheta_+ + 1, \vartheta_- - \chi + l\Big), \qquad (6.12) \end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ stands again for the beta function. If no better code for B is available, then the integral $M_l^{0,0}$ can again be computed by quadrature similarly to the integrals in (6.9). We emphasize that all these steps in the recursion for $l' = 0, \ldots, l$ and for x_{jn}^{φ}, $j = 1, \ldots, n$ require no more than $\mathcal{O}(n^2)$ arithmetic operations.

6.3.2. In Section 8 we shall analyze the stability of the recurrence relations. Unfortunately, the recursive algorithm (6.7) is not stable. Therefore, we shall modify the algorithm. Sometimes, if a recursive computation is not stable, it is recommended to take the recursion in the backward direction starting from an approximate value for the number with the largest index (cf. [5, 12, 32, 27]). Though this is not helpful in our situation, our modification is in the same spirit.

Clearly, setting $z := [\zeta^{-1}(1 + x_{jn}^{\varphi}) - 1]$, the solution of the recurrence relations up to m steps is equivalent to the solution of the linear system of equations $T_m(g)\xi = \eta$ with

$$T_m(g) := \begin{bmatrix} 1 & -2z & 1 & 0 & & & \ldots & & & 0 \\ 0 & 1 & -2z & 1 & 0 & & & & & 0 \\ 0 & 0 & 1 & -2z & 1 & 0 & & & & 0 \\ & & & & \ldots & & & & & \\ & & & & & & 0 & 1 & -2z & 1 \\ & & \ldots & & \ldots & & & 0 & 1 & -2z \\ & & & & & & & & 0 & 1 \end{bmatrix},$$

$$\xi := \Big[M_l^{m+1,l'}(x_{jn}^{\varphi}), M_l^{m,l'}(x_{jn}^{\varphi}), M_l^{m-1,l'}(x_{jn}^{\varphi}), \ldots,$$

$$
\begin{aligned}
& \qquad\qquad M_l^{2,l'}(x_{jn}^\varphi), M_l^{1,l'}(x_{jn}^\varphi), M_l^{0,l'}(x_{jn}^\varphi)\Big]^T, \\
\eta \;&:=\; \Big[-2\zeta^{-1}M_l^{m,l'-1}(x_{jn}^\varphi), -2\zeta^{-1}M_l^{m-1,l'-1}(x_{jn}^\varphi), -2\zeta^{-1}M_l^{m-2,l'-1}(x_{jn}^\varphi),\dots, \\
& \qquad -2\zeta^{-1}M_l^{1,l'-1}(x_{jn}^\varphi), -2\zeta^{-1}M_l^{0,l'-1}(x_{jn}^\varphi), M_l^{0,l'}(x_{jn}^\varphi)\Big]^T.
\end{aligned}
$$

If we suppose that the value $M_l^{m+1,l}(x_{jn}^\varphi)$ is known, then we can delete the first column and the last row and write the recursion in the form $T_{m-1}(\tilde g)\tilde\xi = \tilde\eta$ with

$$
T_{m-1}(\tilde g) \;:=\; \begin{bmatrix}
-2z & 1 & 0 & & \dots & & & 0 \\
1 & -2z & 1 & 0 & & & & 0 \\
0 & 1 & -2z & 1 & 0 & & & 0 \\
 & & & \dots & & & & \\
 & & & & 0 & 1 & -2z & 1 \\
 & \dots & & \dots & & 0 & 1 & -2z
\end{bmatrix},
$$

$$
\begin{aligned}
\tilde\xi \;&:=\; \Big[M_l^{m,l'}(x_{jn}^\varphi), M_l^{m-1,l'}(x_{jn}^\varphi),\dots, M_l^{2,l'}(x_{jn}^\varphi), M_l^{1,l'}(x_{jn}^\varphi), M_l^{0,l'}(x_{jn}^\varphi)\Big]^T, \\
\tilde\eta \;&:=\; \Big[-2\zeta^{-1}M_l^{m,l'-1}(x_{jn}^\varphi) - M_l^{m+1,l'}(x_{jn}^\varphi), -2\zeta^{-1}M_l^{m-1,l'-1}(x_{jn}^\varphi), \\
& \quad -2\zeta^{-1}M_l^{m-2,l'-1}(x_{jn}^\varphi),\dots, -2\zeta^{-1}M_l^{1,l'-1}(x_{jn}^\varphi), -2\zeta^{-1}M_l^{0,l'-1}(x_{jn}^\varphi)\Big]^T.
\end{aligned}
$$

Now the modified recursion consist in the following. We compute the value $M_l^{m+1,l'}(x_{jn}^\varphi)$ by the quadrature approach of Section 6.2.3. The values $M_l^{i,l}(x_{jn}^\varphi)$, $i = 0, 1, \dots, m$ are determined by solving the system $T_{m-1}(\tilde g)\tilde\xi = \tilde\eta$. This stable tridiagonal system (cf. the last proof in Section 8) can be solved by a direct solver (the so-called sweeping method which is called progonka in Russian [34]) in no more than $\mathcal{O}(n)$ operations. Doing so for each x_{jn}^φ, we arrive at an algorithm for the assembling of the matrix with no more than $\mathcal{O}(n^2)$ operations.

7 Numerical Test

For our numerical test, we consider the second kind equation of Mellin convolution type (2.1) choosing $b \equiv 1$ and

$$
\mathbf{k}(t) \;:=\; -\frac{1}{\pi}\frac{t}{t^2+1}, \quad \widehat{\mathbf{k}}(z) \;=\; -\frac{1}{2\cos\left(\frac{\pi}{2}z\right)}, \quad \mathbf{k}_0(x,y) \;:=\; \frac{1}{x+y+3}. \tag{7.1}
$$

Note that the main part $I + K$ without the integral operator K_0 corresponds to the double layer kernel over a polygon with a reentrant corner of the size $3\pi/2$. The solution u belongs to the space $\mathbf{L}^\infty$ and (3.1) turns into

$$
u(x) \;\sim\; \lambda_0 + \lambda_1(1+x)^{2/3} + \lambda_2(1+x)^1 + \lambda_2(1+x)^{4/3} + \dots, \quad x \longrightarrow -1. \tag{7.2}
$$

To improve the asymptotics, we apply the transformation technique of Section 3 setting $x = \Phi(t) = 2^{1-q}(1+t)^q - 1$ with $q \in \mathbb{Z} \cap [1, \infty)$. More precisely, we apply the transformation (3.2), (3.3) with $\sigma_- = -1$ since this choice results in an isometric transform of the space $\mathbf{L}^\infty$. This way we arrive at the equation (3.3) with $\widetilde{\mathbf{k}}(x, y) = \mathbf{m}([1+x]/[1+y])/[1+y] + \widetilde{\mathbf{k}}_0(x, y)$ and

$$\mathbf{m}(t) := -q\frac{1}{\pi}\frac{t^q}{t^{2q}+1}, \quad \widehat{\mathbf{m}}(z) = -\frac{1}{2\cos\left(\frac{\pi}{2}\frac{z}{q}\right)},$$
$$\widetilde{\mathbf{k}}_0(x, y) := \frac{q2^{1-q}(1+y)^{q-1}}{[2^{1-q}(1+x)^q - 1] + [2^{1-q}(1+y)^q - 1] + 3}.$$

The Mellin kernel $\mathbf{k}$ splits into the elementary kernels of Subsection 6.3 according to the formulae

$$-q\frac{1}{\pi}\frac{t^q}{t^{2q}+1} = \frac{\mathbf{i}}{2\pi}\sum_{j=0}^{2q-1}\frac{(-1)^j e^{\mathbf{i}\pi[1/2+j]/q}}{t - e^{\mathbf{i}\pi[1/2+j]/q}},$$
$$-\frac{1}{\pi}\frac{t}{t^2+1} = -\frac{1}{\pi}\Re\mathrm{e}\,\frac{1}{t-\mathbf{i}},$$
$$-2\frac{1}{\pi}\frac{t^2}{t^4+1} = -\frac{1}{\pi}\Im\mathrm{m}\,\frac{e^{\mathbf{i}\pi/4}}{t - e^{\mathbf{i}\pi/4}} + \frac{1}{\pi}\Im\mathrm{m}\,\frac{e^{\mathbf{i}3\pi/4}}{t - e^{\mathbf{i}3\pi/4}}.$$

The condition (2.2) for the transformed equation (3.3) is fulfilled with $\alpha = -q$, $\beta = q$ and the asymptotic expansion (3.1) with

$$\tilde{u}(x) \sim \widetilde{\lambda}_0 + \widetilde{\lambda}_1(1+x)^{2q/3} + \widetilde{\lambda}_2(1+x)^q + \widetilde{\lambda}_2(1+x)^{4q/3} + \ldots, \quad x \longrightarrow -1. \quad (7.3)$$

Further, we consider $\mathbf{L}^2_\sigma$ again and we choose $\sigma(x) = \varphi(x) := (1-x)^{1/2}(1+x)^{1/2}$. The solution $\tilde{u} \in \mathbf{L}^2_\sigma$ with (7.3) remains unchanged. Due to this weight σ, we get $\vartheta(x) \equiv 1$, and our trial space is the polynomial space $\mathbb{P}_n$ without weight. This choice of the trial space is the best possible because the first term λ_0 of the asymptotics (7.3) is contained in $\mathbb{P}_n$. The condition (4.3) holds at least for $q \geq 2$. Unfortunately, for $q = 1$, the technical condition (4.3) is violated. Nevertheless, we conjecture that the polynomial collocation is stable also with $q = 1$. For definiteness, we choose the right-hand side f such that the solution u is $u(x) = (1+x)^{2/3}$. In other words

$$\begin{aligned}
f(x) := {} & (1+x)^{2/3} - \frac{(1+x)^{2/3}}{2\pi}\left[\frac{\sqrt{3}}{2}\log\left(\frac{2^{2/3} - \sqrt{3}2^{1/3}(1+x)^{1/3} + (1+x)^{2/3}}{2^{2/3} + \sqrt{3}2^{1/3}(1+x)^{1/3} + (1+x)^{2/3}}\right)\right. \\
& +2\arctan\left(\frac{2^{1/3}}{(1+x)^{1/3}}\right) + \arctan\left(\frac{2\cdot 2^{1/3}}{(1+x)^{1/3}} + \sqrt{3}\right) \\
& \left. + \arctan\left(\frac{2\cdot 2^{1/3}}{(1+x)^{1/3}} - \sqrt{3}\right)\right] \\
& + \frac{(2+x)^{-1/3}}{6}\left[9\cdot 2^{2/3}(2+x)^{1/3} + 6(2+x)\log\left(2^{1/3} + (2+x)^{1/3}\right)\right. \\
& -3(2+x)\log\left(2^{2/3} + (2+x)^{2/3} - 2^{1/3}(2+x)^{1/3}\right) \\
& +6(2+x)\sqrt{3}\arctan\left(\frac{2+x-2\cdot 2^{1/3}(2+x)^{2/3}}{\sqrt{3}(2+x)}\right) + \\
& \left. -\sqrt{3}\pi(2+x)\right].
\end{aligned} \quad (7.4)$$

n	e_n $(q=1)$	o_n $(q=1)$	e_n $(q=2)$	o_n $(q=2)$
1	$4.27{\cdot}10^{-3}$		$2.46{\cdot}10^{-1}$	
2	$5.26{\cdot}10^{-3}$	-0.30	$9.97{\cdot}10^{-3}$	4.63
4	$2.69{\cdot}10^{-3}$	0.97	$8.52{\cdot}10^{-4}$	3.55
8	$8.51{\cdot}10^{-4}$	1.66	$6.71{\cdot}10^{-5}$	3.67
16	$1.99{\cdot}10^{-4}$	2.10	$4.58{\cdot}10^{-6}$	3.87
32	$3.96{\cdot}10^{-5}$	2.33	$2.87{\cdot}10^{-7}$	4.00
64	$7.20{\cdot}10^{-6}$	2.50	$1.69{\cdot}10^{-8}$	4.08
128	$1.24{\cdot}10^{-6}$	2.54	$9.74{\cdot}10^{-10}$	4.12
256	$2.08{\cdot}10^{-7}$	2.58	$5.51{\cdot}10^{-11}$	4.14
512	$3.42{\cdot}10^{-8}$	2.60	$3.09{\cdot}10^{-12}$	4.16
1024	$5.54{\cdot}10^{-9}$	2.63	$1.72{\cdot}10^{-14}$	4.17

Table 1: Convergence of polynomial collocation with $\mathbf{k}_0$ from (7.1).

In accordance with Theorem 5.1, we expect that the error of the collocation method (4.1) is of the size $\mathcal{O}(n^{-3/2-4q/3})$ if q is not a multiple of three.

We have applied the polynomial collocation under the choices $q=1,2$. Using the quadrature algorithm of Section 6.2.3, we have computed the stiffness matrices. The orders of the quadratures n_p, n_g, and n_p' have been chosen to be the minimal numbers such that increasing these values does not improve the discretization errors. E.g., for $n=512$ and $q=1$, we have set $n_p=7$, $n_g=6$, and $n_p'=7$. Clearly, the smaller discretization error for $n=512$ and $q=2$ requires higher values of quadrature orders. In particular, we have chosen $n_p=13$, $n_g=16$, and $n_p'=15$. The linear systems of equations have been solved up to an accuracy of 10^{-16} by a few number of GMRes iterations (for the preconditioning of the iterative solver compare [24]). We have determined the approximate values (cf. the subsequent Equation (8.3))

$$e_n \;:=\; \|M_n\tilde{u}-\tilde{u}_n\|_{\mathbf{L}^2_\sigma} \;=\; \sqrt{\sum_{j=1}^{n}\left|\tilde{u}(x^{\varphi}_{jn})-\tilde{u}_n(x^{\varphi}_{jn})\right|^2 \varphi(x^{\varphi}_{jn})\,\sin\left(\frac{j\pi}{n+1}\right)\frac{\pi}{n+1}}$$

for the weighted norm error $\|\tilde{u}-\tilde{u}_n\|_{\mathbf{L}^2_\sigma}$ of the collocation approximation $\tilde{u}_n$ to the solution of (3.3). Approximate values for the orders of convergence (i.e. minus the exponent of n in the estimate of Theorem 5.1) have been computed by

$$o_n \;:=\; \frac{\log e_{n/2}-\log e_n}{\log 2}\,.$$

The values presented in Table 1 are quite close to the predicted values $2.8333\ldots$ for $q=1$ and $4.1666\ldots$ for $q=2$.

Next we have tested the recurrence relation of Section 6.3.1. First, we have computed the generalized moments

$$M_1^{m,l'}(x) \;:=\; \int_{-1}^{1}\frac{U_m(y)}{\left[(1+x)-\mathbf{i}(1+y)\right]^{l'}}\,\mathrm{d}y,\quad l'=0,1,\ m=0,1,2,\ldots.$$

n	e_n $(q=3)$	o_n $(q=3)$
4	$6.64 \cdot 10^{-4}$	
8	$4.41 \cdot 10^{-5}$	3.91
16	$1.26 \cdot 10^{-6}$	5.13
32	$1.71 \cdot 10^{-8}$	6.20
64	$1.33 \cdot 10^{-10}$	7.01
128	$7.96 \cdot 10^{-13}$	7.38
256	$4.72 \cdot 10^{-15}$	7.40
512	$3.94 \cdot 10^{-16}$	3.58

Table 2: Convergence of polynomial collocation with $\mathbf{k}_0 \equiv 0$.

For $l' = 0$, the recursion yields $M_1^{2m+1,0} = 0$ and $M_1^{2m,0} = \sqrt{2/\pi} \cdot 2/(2m+1)$. Only for the case $l' = 1$, we need to compute by recursion. Choosing $x = -0.9$ (case of almost singular integral in entry of stiffness matrix), we get, for the recursive approximation of $M_1^{30,1}$, an error of $4.22 \cdot 10^{-13}$. In other words, to compute almost singular integrals, the unstable recursion is acceptable. However, if $x = 1$ (case of smooth integrand in entry of stiffness matrix), then we get, for the recursive approximation of $M_1^{30,1}$, an error of $1.71 \cdot 10^3$, and the unstable recursion must not been used. On the other hand, using the stable recursion from Section 6.3.2, the corresponding errors for $M_1^{m,1}$, $m = 0, 1, \dots, 29$ are less than $4.97 \cdot 10^{-15}$ and $4.51 \cdot 10^{-16}$, respectively. To check the assembling of the stiffness matrix by recursion, we have set the smooth kernel $\mathbf{k}_0$ to zero. Correspondingly, to get the solution $u(x) = (1+x)^{2/3}$ we have deleted the third term of the right-hand side f in (7.4). Using the recursion from Section 6.3.2 to compute the stiffness matrix, we have obtained exactly the same results as with the quadrature approximated collocation. These results for the case $\mathbf{k}_0 \equiv 0$ are quite similar to those in Table 1. Whereas the calculations for $q = 1$ and $n = 1, 2, 4, 8, \dots, 1024$ requires 579 seconds with quadrature, the algorithm with recursion reduces the computing time to 10 seconds (pentium II, 233 MHz). The calculations for $q = 2$ and $n = 1, 2, 4, 8, \dots, 1024$ requires 1237 seconds with quadrature and 13 seconds with recursion. Though our quadrature code is not optimized, a reduction in computing time is obvious. Remember that, for each entry of the polynomial collocation, the quadrature is over the whole interval and not over the small support of a finite element function as for spline collocation. Using recursion, the "quadrature" over the whole interval is realized by a very small number of arithmetic operations.

Finally, we have applied the polynomial collocation with transformation parameter $q = 3$ to the convolution equation with $\mathbf{k}_0 \equiv 0$ and $\mathbf{k}$ from (7.1). Again the right-hand side f has been chosen such that $u(x) = (1+x)^{2/3}$ and $\tilde{u}(t) = (1+t)^2$. Now the proof of Theorem 5.1 implies that the collocation solution converges faster than any finite power of $1/n$. The numerical results in Table 2 confirm this fact as far as this is possible with the limited accuracy (double precision) of the computer.

8 The Proofs

Proof of Lemma 5.1. In this proof, without loss of generality, we shall assume that

$u(x) = (1+x)^{\kappa_0}[\log(1+x)]^{m_0}$. The other terms in the asymptotic expansion (3.1) and smooth remainder terms can be treated in a similar fashion. First, we assume the additional condition $[\kappa_0 + \sigma_-/2 - 1/4] \notin \mathbb{Z} \cap [0,\infty)$.

We denote the polynomial interpolation at the points x^{φ}_{kn}, $k = 1,\dots,n$ of a function v by $L_n(\varphi, v)$. Then we get $M_n u = \vartheta L_n(\varphi, [\vartheta^{-1}u])$ and

$$\begin{aligned}
\|u - M_n u\|_\sigma &= \left\| \Big[[\vartheta^{-1}u] - L_n(\varphi,[\vartheta^{-1}u])\Big]\vartheta\sqrt{\sigma}\right\|_{\mathbf{L}^2} \\
&= \left\| \Big[[\vartheta^{-1}u] - L_n(\varphi,[\vartheta^{-1}u])\Big]\sqrt{\varphi}\right\|_{\mathbf{L}^2}.
\end{aligned}$$

To estimate the error of interpolation applied to $[\vartheta^{-1}u]$ we utilize Theorem 3.1 of [26] (compare also [6] for estimates similar to the following). According to this we continue

$$\begin{aligned}
\|u - M_n u\|_\sigma &\leq \frac{C}{n^{1/2}} \int_0^{1/n} \frac{\Omega^k_\varphi([\vartheta^{-1}u],t)_{\sqrt{\varphi},2}}{t^{1+1/2}}\,\mathrm{d}t, \quad k \geq 1, \\
\Omega^k_\varphi([\vartheta^{-1}u],t)_{\sqrt{\varphi},2} &:= \sup_{0<h\leq t} \left\|(\triangle^k_{h\varphi}[\vartheta^{-1}u])\sqrt{\varphi}\right\|_{\mathbf{L}^2(I_{hk})}, \\
(\triangle^k_{h\varphi}[\vartheta^{-1}u])(x) &:= \sum_{i=0}^{k}(-1)^i \binom{k}{i} [\vartheta^{-1}u]\Big(x + \frac{kh}{2}\varphi(x) - ih\varphi(x)\Big), \\
I_{hk} &:= [-1 + 2h^2k^2, 1 - 2h^2k^2].
\end{aligned}$$

However, if we choose k sufficiently large and if we estimate the finite difference operator $\triangle^k_{h\varphi}$ by the derivative, then we arrive at

$$\begin{aligned}
\left|(\triangle^k_{h\varphi}[\vartheta^{-1}u])(x)\right| &\leq C\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}[\vartheta^{-1}u]\right)(x)\,[h\varphi(x)]^k \\
&\leq Ch^k(1+x)^{\Re\mathrm{e}\,\kappa_0 - 1/4 + \sigma_-/2 - k/2}[\log(1+x)]^{m_0}, \qquad (8.1) \\
\left\|(\triangle^k_{h\varphi}[\vartheta^{-1}u])\sqrt{\varphi}\right\|^2_{\mathbf{L}^2(I_{hk})} &\leq C\,h^{2k}\int_{-1+h^2/C}^{1-h^2/C} \left|(1+x)^{\Re\mathrm{e}\,\kappa_0 + \sigma_-/2 - k/2}[\log(1+x)]^{m_0}\right|^2 \mathrm{d}x \\
&\leq C\,h^{2[2\Re\mathrm{e}\,\kappa_0 + \sigma_- + 1]}[\log h^{-1}]^{2m_0}. \qquad (8.2)
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\Omega^k_\varphi([\vartheta^{-1}u],t)_{\sqrt{\varphi},2} &\leq C\,t^{2[\Re\mathrm{e}\,\kappa_0 + (\sigma_- + 1)/2]}[\log t^{-1}]^{m_0}, \\
\frac{C}{n^{1/2}}\int_0^{1/n} \frac{\Omega^k_\varphi([\vartheta^{-1}u],t)_{\sqrt{\varphi},2}}{t^{1+1/2}}\,\mathrm{d}t &\leq \frac{C}{n^{1/2}}\int_0^{1/n} t^{2[\Re\mathrm{e}\,\kappa_0 + (\sigma_- + 1)/2] - 3/2}[\log t^{-1}]^{m_0}\,\mathrm{d}t \\
\|u - M_n u\|_\sigma &\leq \frac{C}{n^{2[\Re\mathrm{e}\,\kappa_0 + (\sigma_- + 1)/2]}}[\log n]^{m_0},
\end{aligned}$$

which is the desired error bound.

Analogously, we get the error bound for the alternative cases. The only difference in the treatment of the case $[\kappa_0 + \sigma_-/2 - 1/4] \in \mathbb{Z} \cap [0,\infty)$ and $m_0 > 0$ is that one of the logarithmic factors drops out in the estimate for the derivative of $[\vartheta^{-1}u]$ (cf. (8.1) and (8.2)). The difference in handling the case $[\kappa_0 + \sigma_-/2 - 1/4] \in \mathbb{Z} \cap [0,\infty)$ and $m_0 = 0$ is that the derivatives of $[\vartheta^{-1}u]$ in the corresponding estimates (8.1) and (8.2) stay bounded such that the factor h^{2k} leads to arbitrarily high orders of convergence. ∎

Proof of Lemma 5.2. The integral operator K_0 with the smooth kernel function $\mathbf{k}_0$ is well known to map any kind of weighted Lebesgue space into any Banach space of differentiable functions even without weight. In particular, it maps $\mathbf{L}^2_\sigma$ compactly into $\mathbf{C}_\nu$. Hence, we only have to prove the assertion for the Mellin operator K. Clearly, the norm of this mapping from $\mathbf{L}^2_\sigma$ to $\mathbf{C}_\nu$ can be estimated as

$$\begin{aligned} \|K\|^2 \;&\le\; \sup_{x:\,-1<x<1} \nu(x)^2 \int_{-1}^{1} \left| \mathbf{k}\left(\frac{1+x}{1+y}\right) \frac{1}{1+y} \sigma(y)^{-1/2} \right|^2 \mathrm{d}y \\ &\le\; C \sup_{x:\,-1<x<1} \int_{-1}^{0} \left| \mathbf{k}\left(\frac{1+x}{1+y}\right) \right|^2 \left[\frac{1+x}{1+y}\right]^{1+\sigma_-} \frac{1}{1+y} \mathrm{d}y \;+\; C \\ &\le\; C \sup_{x:\,0<x<\infty} \int_{0}^{\infty} \left| \mathbf{k}\left(\frac{x}{y}\right) \right|^2 \left[\frac{x}{y}\right]^{1+\sigma_-} \frac{1}{y} \mathrm{d}y \;+\; C. \end{aligned}$$

Substituting $z = x/y$ and $dz/z = -dy/y$, we continue

$$\|K\|^2 \;\le\; C \int_0^\infty |\mathbf{k}(z)|^2 z^{1+\sigma_-} \frac{1}{z} \mathrm{d}z \;+\; C \;\le\; C \int_{\Re e\, z=(1+\sigma_-)/2} \left| \widehat{\mathbf{k}}(z) \right|^2 \mathrm{d}z \;+\; C,$$

where in the last step we have used the fact that the Mellin transform maps $\mathbf{L}^2_{\sigma_-/2}$ isometrically onto the $\mathbf{L}^2$ space over the line $\{z :\ \Re e\, z = (1+\sigma_-)/2\}$. This fact is a simple reformulation of Plancherel's theorem according to which the classical Fourier transform is unitary. In view of the last estimate, of $\alpha < (\sigma_- + 1)/2 < \beta$ and of (2.2), we obtain the desired boundedness property. ■

Proof of Lemma 5.3. Since the nth order Gauss rule $\int_{-1}^1 f\varphi \sim \sum_{k=1}^n \lambda^\varphi_{kn} f(x^\varphi_{kn})$ with $\lambda^\varphi_{kn} := \pi[\varphi(x^\varphi_{kn})]^2/(n+1)$ corresponding to the Chebyshev weight function φ and the Chebyshev nodes of the second kind x^φ_{kn} is exact for polynomials of degree less than $2n$, we obtain (compare also the Marcinkiewicz inequality in [26])

$$\|M_n v\|_\sigma \;=\; \sqrt{\sum_{k=1}^n \omega_{kn}^2 \left| v\left(x^\varphi_{kn}\right) \right|^2}, \quad \omega_{kn} \;:=\; \sqrt{\frac{\pi}{n+1} \varphi\left(x^\varphi_{kn}\right) \sigma\left(x^\varphi_{kn}\right)}. \tag{8.3}$$

Consequently, we get

$$\|M_n v\|_\sigma \;\le\; \sup_{k=1,\dots,n} \left| v\left(x^\varphi_{kn}\right) \nu\left(x^\varphi_{kn}\right) \right| \sqrt{\sum_{k=1}^n \omega_{kn}^2 \left| \nu\left(x^\varphi_{kn}\right)^{-1} \right|^2}.$$

It remains to prove that the last square root is less than a constant times $\sqrt{\log n}$. However, due to $2(1/4 + \sigma_\pm/2 - \nu_\pm) = -1/2$, the square of the last square root can be estimated by

$$\begin{aligned} &\frac{\pi}{n+1} \sum_{k=1}^n \left(1 - \cos\left(\frac{\pi k}{n+1}\right)\right)^{-1/2} \left(1 + \cos\left(\frac{\pi k}{n+1}\right)\right)^{-1/2} \\ &\le\; C \frac{1}{n} \sum_{k=1}^{n/2-1} \left(\frac{\pi k}{n+1}\right)^{-1} + C \frac{1}{n} \sum_{k=n/2}^{n} \left(\frac{\pi [n+1-k]}{n+1}\right)^{-1} \\ &\le\; C \frac{1}{n} \sum_{k=1}^{n/2-1} \left(\frac{k}{n}\right)^{-1} + C \frac{1}{n} \sum_{k=1}^{n/2} \left(\frac{k}{n}\right)^{-1} \;\le\; C \log n\,. \end{aligned}$$

■

Proof of the Instability of (6.7) and the Stability of the Modification.

First we consider the linear system of equations $T_m(g)\xi = \eta$ (cf. Section 6.3.2). The solution ξ and the right-hand side η are finite sections of two infinite vectors containing the Fourier coefficients of two integrable functions with respect to the orthogonal polynomials U_m. Hence, stability may be defined to mean that the condition number of the linear system $T_m(g)\xi = \eta$ is bounded uniformly with respect to m (cf. e.g. [17, Chapters 7 and 8]). However, the $T_m(g)$ are finite sections of a Toeplitz matrix with the analytic generating function $g(t) := 1 - 2zt^{-1} + t^{-2}$. The uniform boundedness of the condition numbers of the $T_m(g)$ is equivalent to the condition $g(t) \neq 0$ for any $t \in \mathbb{C}$ with $|t| \geq 1$ (cf. [14] and compare the characteristic equation (34) in [13, Section V.4] and (8.7-4) in [1, Section 8.7]). Thus the recursion is stable if and only if the two roots $t_\pm^{-1}$ of the polynomial $g(t)$ have a modulus less than one, i.e. if

$$|t_\pm| \;>\; 1, \quad t_\pm \;:=\; z \pm \sqrt{z^2 - 1}. \tag{8.4}$$

Unfortunately, we have $|t_\pm| = 1$ for real $z \in [-1, 1]$. For any other z, only one of the $t_\pm$ has a modulus greater than one, and the other modulus is strictly less than one. Thus (8.4) is violated and the recursion is unstable.

To get a better feeling for the kind of instability, we look at the recursion as a form of an iteration. The latter reads as $\xi_{m+1} = E\xi_m + \eta_m$ with $\xi_m := [M_l^{m,l'}(x_{jn}^\varphi), M_l^{m-1,l'}(x_{jn}^\varphi)]^T$ and $\eta_m := [M_l^{m,l'-1}(x_{jn}^\varphi), 0]^T$. The matrix E is given by

$$E \;:=\; \begin{bmatrix} 2z & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.5}$$

If the spectral radius of E is larger than one, then the errors will grow exponentially during the iterative process. The spectral radius of E is $\max\{|t_-|, |t_+|\}$ with $t_\pm$ from (8.4). In other words, we arrive at the same criterion of stability. However, if $\max\{|t_-|, |t_+|\}$ is equal or slightly larger than one and if the maximal number m of recursion steps is not too big, then the errors grow slowly and the recursion computes acceptable values. In particular, for x_{jn}^φ close to -1, the value z is close to -1 resulting in a mild degree of instability.

Next we consider $T_{m-1}(\tilde{g})\tilde{\xi} = \tilde{\eta}$ (cf. Section 6.3.2). The generating function $\tilde{g}$ of the Toeplitz matrix takes the form $\tilde{g}(t) := t - 2z + t^{-1}$. The uniform boundedness of the condition numbers of the $T_{m-1}(\tilde{g})$ is equivalent to the conditions $\tilde{g}(t) \neq 0$, $|t| = 1$ and $\operatorname{wind}\tilde{g} = 0$ (cf. [14]). Here $\operatorname{wind}\tilde{g}$ is the winding number of the closed smooth curve $\{\tilde{g}(e^{\mathbf{i}s}) : 0 \leq s \leq 2\pi\}$. Observing $\tilde{g}(e^{\mathbf{i}s}) = 2[\cos s - z]$, we easily get $\operatorname{wind}\tilde{g} = 0$ for $\Im\mathrm{m}\, z \neq 0$. For real z, we get $\operatorname{wind}\tilde{g} = 0$ from $|z| > 1$.

Concerning the recursion (6.10) we remark the following. For large m the recurrence relation is almost of the form $M_l^{m+1,0} = M_l^{m-1,0}$. This however can be analyzed as above. The matrix E takes the form (compare (8.5))

$$E \;:=\; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalues have modulus one, which means a slight instability, only.

■

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