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Mass-time-space scaling of a super-Brownian catalyst reactant pair

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MASS-TIME-SPACE SCALING OF A SUPER-BROWNIAN CATALYST REACTANT PAIR

KLAUS FLEISCHMANN AND JIE XIONG

Abstract. The one-dimensional super-Brownian reactant X^ϱ with a super-Brownian catalyst ϱ has a jointly continuous density field satisfying a stochastic partial differential equation. Consider any expectation preserving mass-time-space scaling of X^ϱ. Using the density field, one can pass to an fdd scaling limit of the measure-valued process, which degenerates also under the critical scaling of ϱ. For some of the scaling indexes, convergence on path spaces holds, too.

1. Introduction and main results

1.1. Background, motivation, and sketch of main result. We are dealing with the model of a super-Brownian reactant X^ϱ with a super-Brownian catalyst ϱ, introduced in [DF97a] and further studied in [DF97b, EF98, FK99, DF01, FK00]. X^ϱ is a continuous measure-valued process on R^d which exists non-trivially in dimensions $d \le 3$ (in higher dimensions it degenerates to the heat flow). Intuitively, the super-Brownian catalyst ϱ can be interpreted as a high density limit of an autonomous critical binary Brownian particle system with constant branching rate, whereas the super-Brownian reactant is also such a high density limit, but the branching rate varies in time and space and is given by the changing "density" of catalytic particles.

According to [FK99], off the catalyst ϱ, the reactant process X^ϱ has a smooth density field, which satisfies the heat equation. Since in dimensions $d \ge 2$ the catalyst ϱ lives on a time-space null set, this implies that in these dimensions the reactant X^ϱ has absolutely continuous measure states, in contrast to the catalyst ϱ.

On the other hand, in dimension $d = 1$, to which we restrict to from now on in this subsection, super-Brownian motions (SBM) in relatively general catalytic media have absolutely continuous states, at least for almost all time points; see [DFR91].

Our first aim is to verify that (in dimension one) X^ϱ has a jointly continuous density field on $(0, \infty) \times \mathsf{R}$, which satisfies a stochastic equation, just as ordinary SBM ϱ does ([KS88, Rei89]). By an abuse of notation, we often denote the density of a measure by the same symbol as the measure. The equations for the density

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fields are as follows:

$$\begin{aligned} \mathrm{d}\varrho_t(x) &= \frac{\sigma_{\mathrm{c}}^2}{2}\Delta\varrho_t(x)\,\mathrm{d}t + \sqrt{\gamma_{\mathrm{c}}\,\varrho_t(x)}\,\mathrm{d}W_t^{\mathrm{c}}(x), \\ \mathrm{d}X_t^{\varrho}(x) &= \frac{\sigma_{\mathrm{r}}^2}{2}\Delta X_t^{\varrho}(x)\,\mathrm{d}t + \sqrt{\gamma_{\mathrm{r}}\,\varrho_t(x)\,X_t^{\varrho}(x)}\,\mathrm{d}W_t^{\mathrm{r}}(x), \end{aligned} \tag{1}$$

$t>0,\ x\in\mathsf{R},$ where $\sigma_{\mathrm{c}}\,,\sigma_{\mathrm{r}}\,,\gamma_{\mathrm{c}}\,,\gamma_{\mathrm{r}}$ are (strictly) positive constants and $\mathrm{d}W^{\mathrm{c}},\mathrm{d}W^{\mathrm{r}}$ are independent (standard) time-space white noises (see Theorem 5 below).

Our *main purpose* is to deal with the large scale behavior of (ϱ, X^{ϱ}). To this aim we start with Lebesgue measures,

$$\varrho_0 \;=\; i_{\mathrm{c}}\ell, \quad X_0^{\varrho} \;=\; i_{\mathrm{r}}\ell, \tag{2}$$

with ℓ the (normed) Lebesgue measure on R, and positive constants i_{c} and i_{r}.

First we recall the *long-term behavior*. [DF97a] gives the stochastic convergence

$$(\varrho_t, X_t^{\varrho}) \underset{t\uparrow\infty}{\longrightarrow} (0, i_{\mathrm{r}}\ell) \tag{3}$$

(based on vague convergence of measures). In fact, according to a well-known result, the catalyst ϱ_t disappears locally as $t\uparrow\infty$, thus, at late times, locally the reactant X_t^{ϱ} will only be smeared out according to the heat flow, leading in law locally to the limiting Lebesgue measure, without losing any mass in the mean (persistence).

However, this point of view based on vague convergence does not expose what happens close to the huge, spatially escaping catalyst clumps. Intuitively, within a catalyst clump the reactant should die, whereas at the *boundary layer* of catalyst and reactant there are high fluctuations of the reactant, so one expects *hot spots* as seen in simulations for the two-dimensional case (see the figure in [FK99], for instance). Can one get some information on hot spots by a scaling procedure?

For this purpose, for a fixed constant $\eta\geq 0$, we *rescale* a measure-valued path $\mu=\{\mu_t : t\geq 0\}$ on R by

$${}^{K}\!\mu_t(B) \;:=\; K^{-\eta}\,\mu_{Kt}(K^{\eta}B), \qquad t\geq 0, \quad \text{Borel } B\subseteq\mathsf{R}, \quad K>0. \tag{4}$$

Note that by the critical branching, the expectations of our pair (ϱ, X^{ϱ}) of processes are just heat flows, which are invariant under this scaling by our choice (2) of initial states.

For SBM ϱ alone, such mass-time-space scaling under the *critical* parameter $\eta=1$ leads to a *non-trivial* limit in law as $K\uparrow\infty$, [DF88]. In fact, the limiting process ${}^{\infty}\varrho$ is the "boundary" SBM for which the diffusion constant equals zero (by the strong space scaling compared with the weaker Brownian spatial spread). This ${}^{\infty}\varrho$ is a measure-valued process starting from $i_{\mathrm{c}}\ell$ and, for each time $t>0$, describing a compound Poissonian field of mass points, where the mass in each point changes in time independently according to Feller's famous branching diffusion. Recall that Feller's branching diffusion (without drift) is the solution to the stochastic equation

$$\mathrm{d}\zeta_t^{\mathrm{c}} \;=\; \sqrt{\gamma_{\mathrm{c}}\,\zeta_t^{\mathrm{c}}}\,\mathrm{d}B_t^{\mathrm{c}}, \quad \zeta_0^{\mathrm{c}}\geq 0, \tag{5}$$

where B^{c} is a (standard) Brownian motion in R. In other words, in the scaling limit which describes the catalyst system from a macroscopic point of view, at time $t>0$ we have a homogeneous compound Poisson point field where the locations of the points describe the positions of the catalyst clumps (which are of order t apart), and their independent weights give the sizes of the clusters which are exponentially distributed with expectation of order t, and change in time independently according to Feller's branching diffusion as in (5). Clearly, under the weaker scaling $\eta < 1$ one expects the local extinction ${}^K\varrho_t \to 0$ in law as $K \uparrow \infty$ (for each $t > 0$), just as in the $\eta = 0$ case, whereas for $\eta > 1$ a law of large numbers should be true: ${}^K\varrho_t \to i_{\mathrm{c}}\ell$ as $K \uparrow \infty$.

Similarly, for the scaled reactant ${}^K X_t^\varrho$ one expects convergence in law to $i_{\mathrm{r}}\ell$ as $K \uparrow \infty$ under non-critical scaling $\eta \neq 1$. Indeed, under $\eta < 1$ the catalytic clumps will escape leading again in law to a locally uniform reactant, whereas under $\eta > 1$ an averaging holds. But what under the *critical scaling*? Does for the rescaled reactant a limit ${}^\infty X^\varrho$ exist, and is it random? For the first sight, it is not at all clear how to answer this question.

The *main result* of the present paper is, that the reactant is well-behaved also under the critical rescaling. But unfortunately, the fdd limit ${}^\infty X^\varrho$ is degenerate, as in the non-critical scaling cases. This means, the scaling we consider here is too rough to gain some information on the hot spots, at least as long as one considers only fdd convergence. In fact, the hot spots are close to the catalytic clumps, so to catch them one needs the strong space contraction as needed for the catalyst. This space contraction has to be compensated by a mass scaling (recall we restrict our consideration to expectation preserving scalings). But then the hot spots are too small on this scale, and they cannot be exposed this way. Nevertheless, we find it worth to give this unified answer on the scaling behavior of X^ϱ: For *all* scaling indexes $\eta \geq 0$, there is an *fdd* limiting reactant as $K \uparrow \infty$, which *degenerates* to ${}^\infty X_t^\varrho \equiv i_{\mathrm{r}}\ell$ (see Theorem 6(a) below).

For $\eta < 1$ or $\eta \geq 29/16$, the convergence can be sharpened to a *functional limit theorem* setting in law on spaces of continuous measure valued paths (see Theorem 6(b) below). Unfortunately, this leaves *open* whether tightness also holds for the intermediate range $1 \leq \eta < 29/16$. In particular, is perhaps tightness violated in the critical case $\eta = 1$, so that the reactant hot spots do have an influence?

We mention that this behavior of the catalytic branching system is quite different from the rescaling result in [DFM01] concerning the super-Brownian reactant in R with a stable time-independent catalyst. Opposed to our situation, that catalyst has infinite mean, but nevertheless some "subtle averaging" is going on leading to a continuous non-Markovian limiting reactant of infinite variance.

For recent surveys on catalytic branching models we refer to [DF00a, DF00b] or [Kle00].

1.2. **Preliminaries: Notation and spaces.** For $\lambda \in \mathsf{R}$, introduce the reference function

$$\phi_\lambda(x) := \mathrm{e}^{-\lambda|x|}, \qquad x \in \mathsf{R}^d. \tag{6}$$

For $f : \mathsf{R}^d \to \mathsf{R}$, put

$$|f|_\lambda := \|f/\phi_\lambda\|_\infty \tag{7}$$

where $\|\cdot\|_\infty$ refers to the supremum norm. For $\lambda \in \mathsf{R}$, denote by $\mathcal{C}_\lambda$ the space of all continuous functions $f : \mathsf{R}^d \to \mathsf{R}$ such that $|f|_\lambda$ is finite and that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \to \infty$. Introduce the spaces

$$\mathcal{C}_{\exp} = \mathcal{C}_{\exp}(\mathsf{R}^d) := \bigcup_{\lambda>0} \mathcal{C}_\lambda \quad \text{and} \quad \mathcal{C}_{\mathrm{tem}} = \mathcal{C}_{\mathrm{tem}}(\mathsf{R}^d) := \bigcap_{\lambda>0} \mathcal{C}_{-\lambda} \tag{8}$$

of *exponentially decreasing* and *tempered* continuous functions on R^d, respectively. (Roughly speaking, the functions in $\mathcal{C}_{\exp}$ decay exponentially, whereas the ones in $\mathcal{C}_{\text{tem}}$ are allowed to have a subexponential growth.) We also need the space $\mathcal{C}_{\text{com}} = \mathcal{C}_{\text{com}}(\mathsf{R}^d)$ of all continuous functions on R^d with compact support.

Write $\mathcal{C}_\lambda^{(m)} = \mathcal{C}_\lambda^{(m)}(\mathsf{R}^d)$, $\mathcal{C}_{\text{exp}}^{(m)} = \mathcal{C}_{\text{exp}}^{(m)}(\mathsf{R}^d)$, and $\mathcal{C}_{\text{com}}^{(m)} = \mathcal{C}_{\text{com}}^{(m)}(\mathsf{R}^d)$ if we additionally require that all partial derivatives up to the order $m \geq 1$ exist and belong to $\mathcal{C}_\lambda$, $\mathcal{C}_{\text{exp}}$, and $\mathcal{C}_{\text{com}}$, respectively.

For each $\lambda \in \mathsf{R}$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ is a separable Banach space. On the other hand, the space $\mathcal{C}_{\text{tem}}$ is topologized by the metric

$$({9}) \qquad {}^{\mathcal{C}}\mathrm{d}_{\text{tem}}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \left(|f-g|_{-1/n} \wedge 1 \right), \qquad f,g \in \mathcal{C}_{\text{tem}}\,,$$

making it to a Polish space. (For $\mathcal{C}_{\text{exp}}$ and $\mathcal{C}_{\text{com}}$ we do not need topologies.)

Let $\mathcal{M} = \mathcal{M}(\mathsf{R}^d)$ denote the set of all (non-negative) Radon measures μ on R^d and d_0 a complete metric on $\mathcal{M}$ which induces the vague topology. Mostly we consider the space $\mathcal{M}_{\text{tem}} = \mathcal{M}_{\text{tem}}(\mathsf{R}^d)$ of all measures μ in $\mathcal{M}$ such that $\langle \mu, \phi_\lambda \rangle < \infty$, for all $\lambda > 0$. We topologize this set $\mathcal{M}_{\text{tem}}$ of *tempered* measures by the metric

$$({10}) \qquad {}^{\mathcal{M}}\mathrm{d}_{\text{tem}}(\mu,\nu) \;:=\; \mathrm{d}_0(\mu,\nu) + \sum_{n=1}^{\infty} 2^{-n} \left(|\mu-\nu|_{1/n} \wedge 1 \right), \qquad \mu,\nu \in \mathcal{M}_{\text{tem}}\,.$$

Here $|\mu - \nu|_\lambda$ is an abbreviation for $\bigl|\langle \mu, \phi_\lambda \rangle - \langle \nu, \phi_\lambda \rangle\bigr|$. Note that $\bigl(\mathcal{M}_{\text{tem}}\,, {}^{\mathcal{M}}\mathrm{d}_{\text{tem}}\bigr)$ is a Polish space, and that $\mu_n \to \mu$ in $\mathcal{M}_{\text{tem}}$ if and only if

$$({11}) \qquad \langle \mu_n\,, \varphi \rangle \underset{n\uparrow\infty}{\longrightarrow} \langle \mu, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{C}_{\text{exp}}\,.$$

For each $m \geq 1$, write $\mathbf{C} := \mathcal{C}(\mathsf{R}_+\,, \mathcal{C}_{\text{tem}}^m)$ for the set of all continuous paths $t \mapsto \mathbf{f}_t$ in $\mathcal{C}_{\text{tem}}^m$, where $(\mathcal{C}_{\text{tem}}^m\,, {}^{\mathcal{C}}\mathrm{d}_{\text{tem}}^m)$ is defined as the m–fold Cartesian product of $\bigl(\mathcal{C}_{\text{tem}}\,, {}^{\mathcal{C}}\mathrm{d}_{\text{tem}}\bigr)$. When endowed with the metric

$$({12}) \qquad \mathrm{d}_{\mathbf{C}}(\mathbf{f}_\cdot\,, \tilde{\mathbf{f}}_\cdot) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \Bigl(\sup_{0\leq t\leq n} {}^{\mathcal{C}}\mathrm{d}_{\text{tem}}^m(\mathbf{f}_t\,, \tilde{\mathbf{f}}_t) \wedge 1 \Bigr), \qquad \mathbf{f}_\cdot\,, \tilde{\mathbf{f}}_\cdot \in \mathbf{C},$$

$\mathbf{C}$ is a Polish space. Let $\mathbf{P}$ denote the set of all probability measures on $\mathbf{C}$. Equipped with the Prohorov metric $\mathrm{d}_{\mathbf{P}}$, $\mathbf{P}$ is a Polish space, too ([EK86, Theorem 3.1.7]).

For each $m \geq 1$, write $\mathbb{C} := \mathcal{C}(\mathsf{R}_+\,, \mathcal{M}_{\text{tem}}^m)$ for the set of all continuous paths $t \mapsto \mu_t$ in $\mathcal{M}_{\text{tem}}^m$, where $(\mathcal{M}_{\text{tem}}^m\,, {}^{\mathcal{M}}\mathrm{d}_{\text{tem}}^m)$ is defined as the m–fold Cartesian product of $\bigl(\mathcal{M}_{\text{tem}}\,, {}^{\mathcal{M}}\mathrm{d}_{\text{tem}}\bigr)$. When endowed with the metric

$$({13}) \qquad \mathrm{d}_{\mathbb{C}}(\mu_\cdot\,, \tilde{\mu}_\cdot) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \Bigl(\sup_{0\leq t\leq n} {}^{\mathcal{M}}\mathrm{d}_{\text{tem}}^m(\mu_t\,, \tilde{\mu}_t) \wedge 1 \Bigr), \qquad \mu_\cdot\,, \tilde{\mu}_\cdot \in \mathbb{C},$$

$\mathbb{C}$ is a Polish space. Let $\mathbb{P}$ denote the set of all probability measures on $\mathbb{C}$. Equipped with the Prohorov metric $\mathrm{d}_{\mathbb{P}}$, $\mathbb{P}$ is a Polish space, too ([EK86, Theorem 3.1.7]).

Occasionally, instead of $\mathcal{C}(\mathsf{R}_+\,, \mathcal{M}_{\text{tem}}^m)$ we consider also $\mathcal{C}\bigl((0,\infty), \mathcal{M}^m\bigr)$. Then in (13) the supremum has to be taken only over $n^{-1} \leq t \leq n$, and ${}^{\mathcal{M}}\mathrm{d}_{\text{tem}}^m$ has to be replaced by d_0^m.

Random objects are always thought of as defined over a large enough stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$ satisfying the usual hypotheses. If $Y = \{Y_t : t \geq 0\}$ is a random process, then as a rule the law of Y is denoted by P^Y. We use $\mathcal{F}_t^Y$ to denote the completion of the σ–field $\bigcap_{\varepsilon>0} \sigma\{Y_s : s \leq t+\varepsilon\}$, $t \geq 0$. Sometimes we write $\mathcal{L}(Y)$ and $\mathcal{L}(Y\,|\,\cdot)$ for the law and conditional law of Y, respectively.

For a constant $\sigma > 0$ let $\mathrm{p} = \mathrm{p}^\sigma$ denote the heat kernel in R^d related to $\frac{\sigma^2}{2}\Delta$:

$$\text{(14)} \qquad \mathrm{p}_t(x) \;=\; \mathrm{p}_t^\sigma(x) \;:=\; (2\pi\sigma^2 t)^{-d/2} \exp\Big[-\frac{|x|^2}{2\sigma^2 t}\Big], \qquad t>0, \quad x \in \mathsf{R}^d.$$

Write $\xi = (\xi, \Pi_{r,x})$ for the related Brownian motion in R^d, with $\Pi_{r,x}$ denoting the law of ξ if it starts at time r at $\xi_r = x \in \mathsf{R}^d$ (using for convenience this time-inhomogeneous writing for the time-homogeneous process).

With $c = c(q)$ we always denote a positive constant which (in the present case) might depend on a quantity q and might also change from place to place. Moreover, an index on c as $c_{(\#)}$ or $c_\#$ will indicate that this constant first occurred in formula line (#) or Lemma #, respectively, for instance.

1.3. **Modelling.** First we recall the notion of collision local time ([BEP91]).

Definition 1 (**Collision local time**)**.** Let $\mathbf{Y} = \big(Y^1, Y^2\big)$ be an $\mathcal{M}_{\mathrm{tem}}^2$–valued continuous process. The *collision local time* of $\mathbf{Y}$ (if it exists) is a continuous non-decreasing $\mathcal{M}_{\mathrm{tem}}$–valued stochastic process $t \mapsto L_{\mathbf{Y}}(t) = L_{\mathbf{Y}}(t, \cdot\,)$ such that

$$\text{(15)} \qquad \big\langle L_{\mathbf{Y}}^\varepsilon(t), \varphi\big\rangle \;\longrightarrow\; \big\langle L_{\mathbf{Y}}(t), \varphi\big\rangle \quad \text{as} \quad \varepsilon \downarrow 0 \quad \mathcal{P}\text{–in probability},$$

for all $t > 0$ and $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^d)$. Here the approximating collision local times $L_{\mathbf{Y}}^\varepsilon$ are defined by

$$\text{(16)} \qquad \big\langle L_{\mathbf{Y}}^\varepsilon(t), \varphi\big\rangle \;:=\; \int_0^t \mathrm{d}s \int_{\mathsf{R}^d} Y_s^1(\mathrm{d}x) \int_{\mathsf{R}^d} Y_s^2(\mathrm{d}y)\, \mathrm{p}_\varepsilon(x-y)\, \varphi\big(\tfrac{x+y}{2}\big).$$

The collision local time $L_{\mathbf{Y}}$ will also be considered as a (locally finite) measure $L_{\mathbf{Y}}\big(\mathrm{d}(s,x)\big)$ on $\mathsf{R}_+ \times \mathsf{R}^d$. ◇

Here is now the basic model we start from.

Definition 2 (**SB reactant with SB catalyst**)**.** A random element (ϱ, X) in $\mathcal{C}\big(\mathsf{R}_+, \mathcal{M}_{\mathrm{tem}}^2(\mathsf{R}^d)\big)$ is said to be a *catalyst reactant pair* with diffusion constants $\sigma_{\mathrm{c}}, \sigma_{\mathrm{r}} > 0$ and branching rates $\gamma_{\mathrm{c}}, \gamma_{\mathrm{r}} > 0$, if for each $\varphi^{\mathrm{c}}, \varphi^{\mathrm{r}} \in \mathcal{C}_{\exp}^{(2)}(\mathsf{R}^d)$,

$$\text{(17)} \qquad M_t^{\mathrm{c}}(\varphi^{\mathrm{c}}) \;:=\; \langle \varrho_t, \varphi^{\mathrm{c}}\rangle - \langle \varrho_0, \varphi^{\mathrm{c}}\rangle - \int_0^t \mathrm{d}s\, \Big\langle \varrho_s, \frac{\sigma_{\mathrm{c}}^2}{2}\Delta\varphi^{\mathrm{c}}\Big\rangle, \qquad t \ge 0,$$

is a square-integrable continuous $\mathcal{F}_\cdot^\varrho$–martingale with square function

$$\text{(18)} \qquad \big\langle\!\big\langle M^{\mathrm{c}}(\varphi^{\mathrm{c}})\big\rangle\!\big\rangle_t \;=\; \gamma_{\mathrm{c}} \int_0^t \mathrm{d}s\, \big\langle \varrho_s, (\varphi^{\mathrm{c}})^2\big\rangle, \qquad t \ge 0,$$

and

$$\text{(19)} \qquad M_t^{\mathrm{r}}(\varphi^{\mathrm{r}}) \;:=\; \langle X_t, \varphi^{\mathrm{r}}\rangle - \langle X_0, \varphi^{\mathrm{r}}\rangle - \int_0^t \mathrm{d}s\, \Big\langle X_s, \frac{\sigma_{\mathrm{r}}^2}{2}\Delta\varphi^{\mathrm{r}}\Big\rangle, \qquad t \ge 0,$$

is a square-integrable continuous $\mathcal{G}_\cdot$–martingale with square function

$$\text{(20)} \qquad \big\langle\!\big\langle M^{\mathrm{r}}(\varphi^{\mathrm{r}})\big\rangle\!\big\rangle_t \;=\; \gamma_{\mathrm{r}} \big\langle L_{(\varrho,X)}(t), (\varphi^{\mathrm{r}})^2\big\rangle, \qquad t \ge 0,$$

where $\mathcal{G}_t := \mathcal{F}_\infty^\varrho \vee \mathcal{F}_t^X$, $t \ge 0$, and $L_{(\varrho,X)}$ is the collision local time between ϱ and X. Here for the collision local time $L_{(\varrho,X)}$ we additionally assume that the convergence (15) even holds $\mathcal{P}$–almost surely. ◇

Essentially from the literature we can get the following result, see Section 2 below.

Lemma 3 (**Unique existence**). *Fix constants* $\sigma_{\rm c}, \sigma_{\rm r}, \gamma_{\rm c}, \gamma_{\rm r}, i_{\rm c} > 0$ *and a measure* $\mu_{\rm r} \in \mathcal{M}_{\rm tem}(\mathsf{R}^d)$. *In dimensions* $d \leq 3$, *there is a unique (in law) catalyst reactant pair* (ϱ, X) *with diffusion constants* $\sigma_{\rm c}, \sigma_{\rm r}$, *branching rates* $\gamma_{\rm c}, \gamma_{\rm r}$, *and initial states* $\varrho_0 = i_{\rm c}\ell$, $X_0 = \mu_{\rm r}$. *The following expectation formulas*

$$\mathcal{P}\varrho_t(\mathrm{d}x) = \varrho_0 * \mathrm{p}_t^{\sigma_{\rm c}}(x)\,\mathrm{d}x \equiv i_{\rm c}\ell, \quad \mathcal{P}\big\{X_t(\mathrm{d}x) \,\big|\, \varrho\big\} = \mu_{\rm r} * \mathrm{p}_t^{\sigma_{\rm r}}(x)\,\mathrm{d}x, \tag{21}$$

and covariance formulas

$$\begin{aligned} &\mathcal{C}\mathrm{ov}\left[\langle \varrho_{t_1}, \varphi_1\rangle, \langle \varrho_{t_2}, \varphi_2\rangle\right] \\ &\quad = i_{\rm c}\,\gamma_{\rm c} \int_0^{t_1\wedge t_2} \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}z_1\; \varphi_1(z_1) \int_{\mathsf{R}^d} \mathrm{d}z_2\; \varphi_2(z_2)\, \mathrm{p}^{\sigma_{\rm c}}_{t_1+t_2-2s}(z_1 - z_2) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\mathcal{C}\mathrm{ov}\left\{\langle X_{t_1}, \varphi_1\rangle, \langle X_{t_2}, \varphi_2\rangle \,\big|\, \varrho\right\} \\ &\quad = \gamma_{\rm r} \int_{\mathsf{R}^d} \mu_{\rm r}(\mathrm{d}z) \int_0^{t_1\wedge t_2} \mathrm{d}s \int_{\mathsf{R}^d} \varrho_s(\mathrm{d}y)\; \mathrm{p}_s^{\sigma_{\rm r}}(y - z)\, \mathrm{p}^{\sigma_{\rm r}}_{t_1-s} * \varphi_1(y)\, \mathrm{p}^{\sigma_{\rm r}}_{t_2-s} * \varphi_2(y) \end{aligned} \tag{23}$$

hold, $t_1, t_2 \geq 0$, $\varphi_1, \varphi_2 \in \mathcal{C}_{\rm exp}(\mathsf{R}^d)$.

Remark 4 (**One-dimensional case**). In the case $d = 1$ the previous lemma is also true if the catalyst process ϱ starts in any initial measure $\mu_{\rm c}$ in $\mathcal{M}_{\rm tem}(\mathsf{R})$. In the other dimensions, one needs to impose an additional assumption on ϱ_0 in order to make sense of the model, see [FK99, Proposition 5]; for simplicity, here we worked with a Lebesgue initial measure $i_{\rm c}\ell$. ◇

1.4. **The jointly continuous density fields.** From now on we restrict our attention to the one-dimensional case $d = 1$, and we come back to the equation system as in (1):

Theorem 5 (**Jointly continuous density fields**). *Fix a pair* $\mathbf{f} = (f_{\rm c}, f_{\rm r})$ *of non-negative functions in* $\mathcal{C}_{\rm tem}(\mathsf{R})$. *Then the system*

$$\begin{aligned} \mathrm{d}\varrho_t(x) &= \frac{\sigma_{\rm c}^2}{2}\Delta\varrho_t(x)\,\mathrm{d}t + \sqrt{\gamma_{\rm c}\,\varrho_t(x)}\,\mathrm{d}W_t^{\rm c}(x), \\ \mathrm{d}X_t^{\varrho}(x) &= \frac{\sigma_{\rm r}^2}{2}\Delta X_t(x)\,\mathrm{d}t + \sqrt{\gamma_{\rm r}\,\varrho_t(x)\,X_t(x)}\,\mathrm{d}W_t^{\rm r}(x), \end{aligned} \tag{24}$$

$t > 0$, $x \in \mathsf{R}$, *where* $\mathrm{d}W^{\rm c}, \mathrm{d}W^{\rm r}$ *are independent time-space white noises, has a unique (in law) non-negative solution* (ϱ, X) *in* $\mathcal{C}\big(\mathsf{R}_+, \mathcal{C}^2_{\rm tem}\big)$ *satisfying* $(\varrho_0, X_0) = \mathbf{f}$. *This solution are the jointly continuous density fields of the (one-dimensional) catalyst reactant pair of Lemma 3 and Remark 4 in the case of the initial states* $\varrho_0(\mathrm{d}x) = f_{\rm c}(x)\,\mathrm{d}x$ *and* $X_0(\mathrm{d}x) = f_{\rm r}(x)\,\mathrm{d}x$. *The following expectation formulas*

$$\mathcal{P}\varrho_t(x) = \mathrm{p}_t^{\sigma_{\rm c}} * f_{\rm c}(x), \quad \mathcal{P}\left\{X_t(x) \,\big|\, \varrho\right\} = \mathrm{p}_t^{\sigma_{\rm r}} * f_{\rm r}(x), \tag{25}$$

and covariance formulas

$$\begin{aligned} &\mathcal{C}\mathrm{ov}\left[\varrho_{t_1}(x_1), \varrho_{t_2}(x_2)\right] \\ &\quad = i_{\rm c}\,\gamma_{\rm c} \int_{\mathsf{R}} \mathrm{d}z\; f_{\rm c}(z) \int_0^{t_1\wedge t_2} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y\; \mathrm{p}_s^{\sigma_{\rm c}}(y - z)\, \mathrm{p}^{\sigma_{\rm c}}_{t_1-s}(y - x_1)\, \mathrm{p}^{\sigma_{\rm c}}_{t_2-s}(y - x_2) \end{aligned} \tag{26}$$

and

$$(27)\quad \mathcal{C}\mathrm{ov}\left\{X_{t_1}(x_1), X_{t_2}(x_2) \,\middle|\, \varrho\right\}$$
$$= \gamma_{\mathrm{r}} \int_{\mathsf{R}} \mathrm{d}z \; f_{\mathrm{r}}(z) \int_0^{t_1 \wedge t_2} \mathrm{d}s \int_{\mathsf{R}} \varrho_s(\mathrm{d}y) \; \mathrm{p}_s^{\sigma_{\mathrm{r}}}(y-z) \, \mathrm{p}_{t_1-s}^{\sigma_{\mathrm{r}}}(y-x_1) \, \mathrm{p}_{t_2-s}^{\sigma_{\mathrm{r}}}(y-x_2)$$

hold, $t_1, t_2 \geq 0,\ x_1, x_2 \in \mathsf{R}$.

This theorem will be proved in Subsection 3.2 below.

1.5. **Scaling limits.** For the moment, we fix a scaling index $\eta \geq 0$ and consider the pair $({}^K\!\varrho, {}^K\!X)$ of rescaled processes as in (4). Our *main result* is as follows. Recall that $d = 1$.

Theorem 6 (**Scaling limit theorem**). *Start the catalyst reactant pair* (ϱ, X) *with Lebesgue measures as in* (2). *Consider the path space* $\mathcal{C}\big(\mathsf{R}_+, \mathcal{M}^2_{\mathrm{tem}}\big)$ *if* $\eta \geq 1$ *and* $\mathcal{C}\big((0,\infty), \mathcal{M}^2\big)$ *if* $\eta < 1$.

- **(a) (Fdd convergence):** *In terms of convergence of finite-dimensional distributions, the pair* $({}^K\!\varrho, {}^K\!X)$ *of rescaled processes converges as* $K \uparrow \infty$ *to a limit denoted by* $({}^\infty\!\varrho, {}^\infty\!X)$. *The limiting reactant* ${}^\infty\!X$ *is always degenerate:* ${}^\infty\!X_t \equiv i_{\mathrm{r}}\ell$. *On the other hand, for the limiting catalyst* ${}^\infty\!\varrho$ *we have the following three cases.*
$$ {}^\infty\!\varrho_t = i_{\mathrm{c}}\ell, \quad t \geq 0, \quad \text{if} \quad \eta > 1,$$
$$ {}^\infty\!\varrho_t = 0, \quad t > 0, \quad \text{if} \quad \eta < 1;$$
however, under the critical scaling $\eta = 1$, *for each* $\varepsilon > 0$ *the limiting catalyst* ${}^\infty\!\varrho$ *has the representation in law*
$$ {}^\infty\!\varrho_t \;=\; \int_{\mathsf{R}} \pi_\varepsilon(\mathrm{d}x) \; \zeta_t^{\mathrm{c}}(x) \, \delta_x \,, \qquad t \geq \varepsilon > 0.$$
Here π_ε *is a Poisson point field on* R *with intensity measure* $i_{\mathrm{c}}\,\varepsilon^{-1}\ell$, *and, independently of this field,* $\zeta^{\mathrm{c}} = \{\zeta^{\mathrm{c}}(x) : x \in \mathsf{R}\}$ *is a family of independent Feller's diffusions as in* (5) *starting from independent identically exponentially distributed variables* $\{\zeta_\varepsilon^{\mathrm{c}}(x) : x \in \mathsf{R}\}$ *with mean* ε.
- **(b) (Functional limit theorem):** *For the catalyst, convergence on path space holds for all* $\eta \geq 0$. *On the other hand, for the reactant, convergence on path space is true provided that* $\eta < 1$ *or* $\eta \geq 29/16$.

Recall that the main point in the theorem is the fdd degeneration of ${}^\infty\!X$ under the critical scaling $\eta = 1$. As already mentioned, we do not know whether tightness holds for the reactant in the *intermediate region* $1 \leq \eta < 29/16$, covering the critical case $\eta = 1$.

Remark 7 (**Partial discontinuity at** $t = 0$)**.** The main reason why we excluded the starting time $t = 0$ in the convergence statement under $\eta < 1$ is that the limiting catalyst process ${}^\infty\!\varrho$ is here sample-discontinuous at $t = 0$ (note that ${}^K\!\varrho_0 \equiv i_{\mathrm{c}}\,\ell$ holds, implying ${}^\infty\!\varrho_0 = i_{\mathrm{c}}\,\ell$, whereas ${}^\infty\!\varrho_t \equiv 0$ otherwise). ◇

1.6. **Outline.** The structure of the paper is as follows. In the next section we recall the existence and uniqueness of the catalyst reactant pair in the martingale problem of Definition 2 and add several discussions (as Lemma 8 and Corollary 10). The joint continuous density fields of the catalyst reactant pair are constructed in Section 3 using [Shi94].

The main part of the paper concerns the proof of the scaling limit theorem. Here different scaling regimes will require different methods. The proof of Theorem 6 is finally completed in Subsection 4.7 by putting together all pieces.

2. The catalyst reactant pair

The purpose of this section is to prove the unique existence Lemma 3, and to discuss some properties of the catalyst reactant pair (ϱ, X).

2.1. **Existence of a catalyst reactant pair.** For all $\mu_c \in \mathcal{M}_{\text{tem}}(\mathsf{R}^d)$, $d \geq 1$, the unique existence of the catalyst ϱ with $\varrho_0 = \mu_c$, satisfying the first martingale problem in Definition 2, is a well-known fact. Moreover, as already mentioned in Subsection 1.1, in dimension $d = 1$ there exists a jointly continuous density field $\{\varrho_t(x) : t > 0,\ x \in \mathsf{R}\}$ ([KS88, Rei89]).

Given ϱ with $\varrho_0 = i_c \ell$, the reactant X^ϱ was constructed in [DF97a, Definition 5, p.261] as a non-degenerate continuous (time-inhomogeneous) Markov process in dimensions $d \leq 3$ via the log-Laplace approach [with the slight difference using a reference function with potential decay instead of the ϕ_λ from (6)]. If additionally $d = 1$, any initial measure $\varrho_0 \in \mathcal{M}_{\text{tem}}$ can be allowed for this construction ([DF97a, Remark 6, p.261]). As already mentioned, if $d = 2, 3$, for almost all ϱ, the reactant X^ϱ has a smooth density field $\{X_t^\varrho(x) : t > 0,\ x \notin \operatorname{supp}\varrho_t\}$ off the time-space support of the catalyst, satisfying the heat equation ([FK99]).

It is easy to see, that in Definition 1 of collision local time one can give up the symmetry by replacing $\frac{x+y}{2}$ in (16) by x, say, (see, for instance, [EP94, Proof of Lemma 3.4]). Using the jointly continuous density field of ϱ in $d = 1$ and of X^ϱ in $d = 2, 3$, this implies that the collision local time $L_{(\varrho, X^\varrho)}$ exists in any dimension $d \leq 3$. This remains true, if the convergence in probability in (15), based on (16) or the just mentioned asymmetric expressions, is replaced by almost sure convergence as needed for Definition 2. Moreover, as mentioned in [DF01], X^ϱ satisfies the second martingale problem in Definition 2.

Altogether, using this pair (ϱ, X^ϱ), we *proved the existence claim in Lemma* 3.

We finish this subsection with a simple observation.

Lemma 8 (**Quenched approach**). *If* (ϱ, X) *is a catalyst reactant pair according to Definition* 2 *and* $P^{X|\varrho}$ *denotes the regular conditional probability of* X *given* ϱ, *then for* P^ϱ*–almost all* ϱ *and all* $\varphi \in \mathcal{C}_{\exp}^{(2)}$, *the process* $t \mapsto M_t^{\mathrm{r}}(\varphi)$ *defined as in* (19) *is a square-integrable continuous* $P^{X|\varrho}$*–martingale with respect to* $\mathcal{F}^X$ *with square function as in* (20), *now with* $L_{(\varrho,X)}$ *the collision local time of* (ϱ, X) *given* ϱ.

Proof. Consider a catalyst reactant pair (ϱ, X). Let $0 \leq s < t$, $\varphi \in \mathcal{C}_{\exp}^{(2)}$, $A_c \in \mathcal{F}_\infty^\varrho$, and $A_r \in \mathcal{F}_s^X$. Then, by the definition of the conditional probability $P^{X|\varrho}$ and the second martingale property in Definition 2,

$$
\begin{aligned}
(28) \qquad P^\varrho\, 1_{A_c}(\varrho)\, P^{X|\varrho}\, 1_{A_r}(X)\, M_t^{\mathrm{r}}(\varphi) \;&=\; \mathcal{P} 1_{A_c}(\varrho)\, 1_{A_r}(X)\, M_t^{\mathrm{r}}(\varphi) \\
&=\; \mathcal{P} 1_{A_c}(\varrho)\, 1_{A_r}(X)\, M_s^{\mathrm{r}}(\varphi) \;=\; P^\varrho\, 1_{A_c}(\varrho)\, P^{X|\varrho}\, 1_{A_r}(X)\, M_s^{\mathrm{r}}(\varphi).
\end{aligned}
$$

Hence,

$$
(29) \qquad P^{X|\varrho}\, 1_{A_r}(X)\, M_t^{\mathrm{r}}(\varphi) \;=\; P^{X|\varrho}\, 1_{A_r}(X)\, M_s^{\mathrm{r}}(\varphi), \quad \text{for } P^\varrho\text{–almost all } \varrho.
$$

Therefore, $t\mapsto M_t^{\mathrm{r}}(\varphi)$ is a $P^{X|\varrho}$–martingale with respect to $\mathcal{F}^X$, for P^{ϱ}–almost all ϱ. Similarly, $t\mapsto \bigl(M_t^{\mathrm{r}}(\varphi)\bigr)^2-\gamma_{\mathrm{r}}\bigl\langle L_{(\varrho,X)}(t),\varphi^2\bigr\rangle$ is a $P^{X|\varrho}$–martingale with respect to $\mathcal{F}^X$, for P^{ϱ}–almost all ϱ. Here $L_{(\varrho,X)}$ is the collision local time with respect to $\delta_\varrho\times P^{X|\varrho}$, $P^{X|\varrho}$–a.s. This finishes the proof. □

2.2. **Uniqueness of the catalyst reactant pair.** In order to prove the uniqueness claim in Lemma 3, we will exploit Mytnik's [Myt98] method of approximate dual processes. Consider any catalyst reactant pair (ϱ,X). We proceed with the family $\bigl\{M^{\mathrm{r}}(\varphi):\ \varphi\in\mathcal{C}^{(2)}_{\exp}\bigr\}$ of $P^{X|\varrho}$–martingales from Lemma 8, given ϱ. By standard arguments this family (given ϱ) extends ([Wal86]) to a square-integrable *martingale measure* $M^{\mathrm{r}}\bigl(\mathrm{d}(s,x)\bigr)$ and to the usual class of predictable integrands. In particular, for $\psi\in\mathcal{C}^{(1,2)}_{T,\exp}$, $T>0$,

$$(30)\qquad \langle M^{\mathrm{r}},\psi\rangle_t \;:=\; \int_{[0,t]\times\mathsf{R}^d} M^{\mathrm{r}}\bigl(\mathrm{d}(s,x)\bigr)\,\psi_s(x) \;=\; \langle X_t\,,\psi_t\rangle-\langle X_0\,,\psi_0\rangle-\int_0^t\mathrm{d}s\;\Bigl\langle X_s\,,\,\frac{\partial}{\partial s}\psi_s+\frac{\sigma_{\mathrm{r}}^2}{2}\Delta\psi_s\Bigr\rangle$$

with square function

$$(31)\qquad \langle\!\langle M^{\mathrm{r}},\psi\rangle\!\rangle_t \;=\; \gamma_{\mathrm{r}}\int_{[0,t]\times\mathsf{R}^d} L_{(\varrho,X)}\bigl(\mathrm{d}(s,x)\bigr)\,\psi_s^2(x) \;=:\; \gamma_{\mathrm{r}}\,\bigl\langle L_{(\varrho,X)},\psi^2\bigr\rangle_t\,,$$

$0\le t\le T$. Here $\mathcal{C}^{(1,2)}_{T,\exp}:=\bigcup_{\lambda>0}\mathcal{C}^{(1,2)}_{T,\lambda}$ with $\mathcal{C}^{(1,2)}_{T,\lambda}=\mathcal{C}^{(1,2)}_{T,\lambda}\bigl([0,T]\times\mathsf{R}^d\bigr)$ the set of all (real-valued) functions ψ defined on $[0,T]\times\mathsf{R}^d$ such that $t\mapsto\psi(t,\cdot)$, $t\mapsto\frac{\partial}{\partial t}\psi(t,\cdot)$, and $t\mapsto\Delta\psi(t,\cdot)$ are continuous $\mathcal{C}_\lambda$–valued functions.

Next we recall from [DF97a] a basic fact on the *log-Laplace equation* related to X^ϱ. For this purpose, let $d\le 3$, and assume $\varrho_0=\mu_{\mathrm{c}}\in\mathcal{M}_{\mathrm{tem}}$ if $d=1$, otherwise $\varrho_0=i_{\mathrm{c}}\ell$. For $T>0$ and $\varphi\in\mathcal{C}^{+}_{\exp}(\mathsf{R}^d)$, given ϱ, the equation

$$(32)\qquad v_r(x) \;=\; \mathrm{p}_{T-r}*\varphi\,(x)-\frac{\gamma_{\mathrm{r}}}{2}\int_r^T\mathrm{d}s\int_{\mathsf{R}^d}\varrho_s(\mathrm{d}y)\;\mathrm{p}_{s-r}(x-y)\,v_s^2(y),$$

$0\le r\le T$, $x\in\mathsf{R}^d$, with the heat kernel $\mathrm{p}=\mathrm{p}^{\sigma_{\mathrm{r}}}$, has a unique non-negative solution $v=v^{\varrho}[T;\varphi]=\bigl\{v_r(x):\,0\le r\le T,\ x\in\mathsf{R}^d\bigr\}$.

For $\varepsilon\in(0,1]$ introduce the smoothed catalyst ${}^{\varepsilon}\!\varrho_s:=\mathrm{p}_\varepsilon*\varrho_s$, $s\in[0,T]$ and denote by ${}^{\varepsilon}v\ge 0$ the unique solution to (32) if $\varrho_s(\mathrm{d}y)$ is replaced by ${}^{\varepsilon}\!\varrho_s(y)\,\mathrm{d}y$. Assuming additionally $\varphi\in\mathcal{C}^{(2)}_{\exp}$, then ${}^{\varepsilon}v$ belongs to $\mathcal{C}^{(1,2)}_{T,\exp}$ and is the unique non-negative solution to the partial differential equation related to (32), that is to

$$(33)\qquad -\frac{\partial}{\partial s}\,{}^{\varepsilon}v_s(x) \;=\; \frac{\sigma_{\mathrm{r}}^2}{2}\Delta\,{}^{\varepsilon}v_s(x)-\frac{\gamma_{\mathrm{r}}}{2}\,{}^{\varepsilon}\!\varrho_s(x)\,{}^{\varepsilon}v_s^2(x),\qquad 0\le s\le T,\quad x\in\mathsf{R}^d,$$

with terminal condition ${}^{\varepsilon}v_T=\varphi$. Trivially, we have the uniform domination

$$(34)\qquad 0\;\le\;{}^{\varepsilon}v_s(x)\;\le\;\mathrm{p}_{T-s}*\varphi\,(x),\qquad 0\le s\le T,\quad x\in\mathsf{R}^d.$$

Moreover,

$$(35)\qquad {}^{\varepsilon}v_s(x)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;v_s(s),\qquad 0\le s\le T,\quad x\in\mathsf{R}^d.$$

Entering ${}^{\varepsilon}v$ into (30) and (31) in place of ψ, by using (33) gives

$$(36)\qquad \mathrm{d}\langle M^{\mathrm{r}},{}^{\varepsilon}v\rangle_s \;=\; \mathrm{d}\langle X_s\,,{}^{\varepsilon}v_s\rangle-\frac{\gamma_{\mathrm{r}}}{2}\,\bigl\langle X_s\,,\,{}^{\varepsilon}\!\varrho_s\,{}^{\varepsilon}v_s^2\bigr\rangle\,\mathrm{d}s$$

with square function

$$\mathrm{d}\langle\!\langle M^{\mathrm{r}}, {}^{\varepsilon}v\rangle\!\rangle_s \;=\; \gamma_{\mathrm{r}}\,\mathrm{d}\big\langle L_{(\varrho,X)}, {}^{\varepsilon}v^2\big\rangle_s\,. \tag{37}$$

By Itô's formula, this implies

$$\mathrm{d}\mathrm{e}^{-\langle X_s\,,{}^{\varepsilon}v_s\rangle} \;=\; \mathrm{e}^{-\langle X_s\,,{}^{\varepsilon}v_s\rangle}\Big[-\,\mathrm{d}\langle M^{\mathrm{r}}, {}^{\varepsilon}v\rangle_s - \frac{\gamma_{\mathrm{r}}}{2}\langle X_s\,, {}^{\varepsilon}\varrho_s\, {}^{\varepsilon}v_s^2\rangle \mathrm{d}s + \frac{1}{2}\,\gamma_{\mathrm{r}}\,\mathrm{d}\big\langle L_{(\varrho,X)}, {}^{\varepsilon}v^2\big\rangle_s\Big].$$

Hence, for each $0<\varepsilon\le 1$,

$$\begin{aligned} P^{X|\varrho}\,\mathrm{e}^{-\langle X_T\,,\varphi\rangle} \;=\;& P^{X|\varrho}\,\mathrm{e}^{-\langle X_0\,,{}^{\varepsilon}v_0\rangle} - \frac{\gamma_{\mathrm{r}}}{2}\,P^{X|\varrho}\int_0^T \mathrm{d}s\;\mathrm{e}^{-\langle X_s\,,{}^{\varepsilon}v_s\rangle}\big\langle X_s\,, {}^{\varepsilon}\varrho_s\, {}^{\varepsilon}v_s^2\big\rangle \\ &+\; \frac{\gamma_{\mathrm{r}}}{2}\,P^{X|\varrho}\int_{[0,T]\times\mathsf{R}^d} L_{(\varrho,X)}\big(\mathrm{d}(s,x)\big)\;{}^{\varepsilon}v_s^2(x)\,\mathrm{e}^{-\langle X_s\,,{}^{\varepsilon}v_s\rangle}. \end{aligned} \tag{38}$$

By the asymmetric version of definition of collision local time as mentioned in the beginning of Subsection 2.1, and by the assumed almost sure property in approaching $L_{(\varrho,X)}$, for each $f\in\mathcal{C}_{\exp}$ we have

$$\int_0^T \mathrm{d}s\;\big\langle X_s\,, {}^{\varepsilon}\varrho_s\, f\big\rangle \;\underset{\varepsilon\downarrow 0}{\longrightarrow}\; \big\langle L_{(\varrho,X)}(T), f\big\rangle, \quad \mathcal{P}\text{–almost surely,} \tag{39}$$

hence $P^{X|\varrho}$–almost surely, for P^{ϱ}–almost all ϱ. Thus, by the pointwise convergence of approximate solutions as in (35) and domination (34), the second and third term at the right hand side of (38) cancel each other as $\varepsilon\downarrow 0$. Therefore

$$P^{X|\varrho}\,\mathrm{e}^{-\langle X_T\,,\varphi\rangle} \;=\; \lim_{\varepsilon\downarrow 0} P^{X|\varrho}\,\mathrm{e}^{-\langle X_0\,,{}^{\varepsilon}v_0\rangle}, \qquad 0\le\varphi\in\mathcal{C}_{\exp}^{(2)}\,, \tag{40}$$

(which is in fact $P^{X|\varrho}\,\mathrm{e}^{-\langle X_0\,,v_0\rangle}$).

Summarizing, the Laplace functional of X_T with respect to $P^{X|\varrho}$ applied to all non-negative $\varphi\in\mathcal{C}_{\exp}^{(2)}$ is uniquely determined, hence the law of X_T, consequently the law of X with respect to $P^{X|\varrho}$ is uniquely determined ([EK86, 4.4.2]). Thus, (ϱ,X) coincides in law with the catalyst reactant pair (ϱ,X^{ϱ}) of the previous subsection. This finishes the proof of Lemma 3 altogether, including Remark 4. □

Remark 9 (**No dependence on the future catalyst**)**.** From (40) and (33) we get

$$P^{X|\varrho}(A) \;=\; P^{X|\varrho^t}(A), \qquad A\in\mathcal{F}_t^X, \quad t\ge 0, \tag{41}$$

where $\{\varrho_s^t := \varrho_{s\wedge t}:\ s\ge 0\}$ is the catalyst process stopped at time $t\ge 0$. ◇

By the latter remark, we can redefine our basic martingale problem in Definition 2 by using the more natural filtration $\mathcal{F}_{\cdot}^{\varrho}$ instead of the σ–field $\mathcal{F}_{\infty}^{\varrho}$:

Corollary 10 (**Orthogonality**)**.** *If (ϱ,X) is the catalyst reactant pair of Lemma 3, then the martingales $M_t^{\mathrm{c}}(\varphi^{\mathrm{c}})$ and $M_t^{\mathrm{r}}(\varphi^{\mathrm{r}})$ from (17) and (19) are orthogonal with respect to the filtration $\mathcal{F}_{\cdot}$ defined by $\mathcal{F}_t := \mathcal{F}_t^{\varrho}\vee\mathcal{F}_t^X$, $t\ge 0$.*

Proof. It is clear that $M^{\mathrm{r}}(\varphi^{\mathrm{r}})$ is an $\mathcal{F}_{\cdot}$–martingale since $\mathcal{F}_t\subseteq\mathcal{G}_t$, $t\ge 0$. Let $0\le s<t$, $A_{\mathrm{c}}\in\mathcal{F}_s^{\varrho}$, and $A_{\mathrm{r}}\in\mathcal{F}_s^X$. Then, with the notation ϱ^t of the stopped catalyst process from Remark 9,

$$\begin{aligned} &\mathcal{P}\,1_{A_{\mathrm{c}}\times A_{\mathrm{r}}}(\varrho,X)\,M_t^{\mathrm{c}}(\varphi^{\mathrm{c}}) \;=\; P^{\varrho}\,1_{A_{\mathrm{c}}}(\varrho)\,M_t^{\mathrm{c}}(\varphi^{\mathrm{c}})\,P^{X|\varrho}(A_{\mathrm{r}}) \\ &\quad=\; P^{\varrho}\,1_{A_{\mathrm{c}}}(\varrho)\,P^{X|\varrho^s}(A_{\mathrm{r}})\,M_t^{\mathrm{c}}(\varphi^{\mathrm{c}}) \;=\; P^{\varrho}\,1_{A_{\mathrm{c}}}(\varrho)\,P^{X|\varrho^s}(A_{\mathrm{r}})\,P^{\varrho}\big\{M_t^{\mathrm{c}}(\varphi^{\mathrm{c}})\;\big|\;\mathcal{F}_s^{\varrho}\big\} \\ &\quad=\; P^{\varrho}\,1_{A_{\mathrm{c}}}(\varrho)\,P^{X|\varrho^s}(A_{\mathrm{r}})\,M_s^{\mathrm{c}}(\varphi^{\mathrm{c}}) \;=\; \mathcal{P}\,1_{A_{\mathrm{c}}\times A_{\mathrm{r}}}(\varrho,X)\,M_s^{\mathrm{c}}(\varphi^{\mathrm{c}}). \end{aligned} \tag{42}$$

Therefore,

$$P^{\varrho}\big\{M_t^{\rm c}(\varphi^{\rm c})\;\big|\;\mathcal{F}_s^{\varrho}\big\} \;=\; M_s^{\rm c}(\varphi^{\rm c}). \tag{43}$$

Moreover,

$$\begin{aligned}
&\mathcal{P}\,1_{A_{\rm c}\times A_{\rm r}}(\varrho,X)\,M_t^{\rm c}(\varphi^{\rm c})\,M_t^{\rm r}(\varphi^{\rm r}) \\
&= P^{\varrho}\,1_{A_{\rm c}}(\varrho)\,M_t^{\rm c}(\varphi^{\rm c})\,P^{X|\varrho}\,1_{A_{\rm r}}(X)\,M_t^{\rm r}(\varphi^{\rm r}) \\
&= P^{\varrho}\,1_{A_{\rm c}}(\varrho)\,M_t^{\rm c}(\varphi^{\rm c})\,P^{X|\varrho}\,1_{A_{\rm r}}(X)\,M_s^{\rm r}(\varphi^{\rm r}) \\
&= P^{\varrho}\,1_{A_{\rm c}}(\varrho)\,M_t^{\rm c}(\varphi^{\rm c})\,P^{X|\varrho^s}\,1_{A_{\rm r}}(X)\,M_s^{\rm r}(\varphi^{\rm r}) \\
&= P^{\varrho}\,1_{A_{\rm c}}(\varrho)\,M_s^{\rm c}(\varphi^{\rm c})\,P^{X|\varrho^s}\,1_{A_{\rm r}}(X)\,M_s^{\rm r}(\varphi^{\rm r}) \\
&= \mathcal{P}\,1_{A_{\rm c}\times A_{\rm r}}(\varrho,X)\,M_s^{\rm c}(\varphi^{\rm c})\,M_s^{\rm r}(\varphi^{\rm r}).
\end{aligned} \tag{44}$$

Hence, $M^{\rm c}(\varphi^{\rm c})\,M^{\rm r}(\varphi^{\rm r})$ is an $\mathcal{F}_{\cdot}$–martingale, thus, $\big\langle\!\big\langle M^{\rm c}(\varphi^{\rm c})\,M^{\rm r}(\varphi^{\rm r})\big\rangle\!\big\rangle_t \equiv 0$, finishing the proof. □

3. Jointly continuous density field

The purpose of this section is to prove Theorem 5.

3.1. **An SPDE result of Shiga.** For the moment, fix $d,k\geq 1$. Let $\mathcal{S}^{d,k}$ denote the collection of all (real-valued) $d\times k$ matrices. We apply the Euclidean norm $|\cdot|$ also to elements in $\mathcal{S}^{d,k}$. A slight generalization of Theorems 2.2 and 2.3 of [Shi94] is as follows. (For the first part of the following proposition, see also [Kot92].)

Proposition 11 (**A well-posed SPDE**). *Suppose that* $\mathbf{a}:\mathsf{R}_+\times\mathsf{R}\times\mathsf{R}^d\times\Omega\to\mathcal{S}^{d,k}$ *and that* $\mathbf{b}:\mathsf{R}_+\times\mathsf{R}\times\mathsf{R}^d\times\Omega\to\mathsf{R}^d$ *are predictable maps satisfying the following two conditions. For each* $T>0$ *there is a constant* $c_{11}=c_{11}(T)>0$ *such that*

$$\big|\mathbf{a}(t,x,u,\omega)\big| + \big|\mathbf{b}(t,x,u,\omega)\big| \;\leq\; c_{11}\,\big(1+|u|\big) \tag{45}$$

for $0\leq t\leq T$, $(x,u)\in\mathsf{R}\times\mathsf{R}^d$, *and* $\mathcal{P}$*–almost all* $\omega\in\Omega$, *and*

$$\begin{aligned}\big|\mathbf{a}(t,x,u',\omega)-\mathbf{a}(t,x,u'',\omega)\big| + \big|\mathbf{b}(t,x,u',\omega)-\mathbf{b}(t,x,u'',\omega)\big| \\ \leq\; c_{11}|u'-u''|\end{aligned} \tag{46}$$

for $0\leq t\leq T$, $(x,u',u'')\in\mathsf{R}\times\mathsf{R}^d\times\mathsf{R}^d$, *and* $\mathcal{P}$*–almost all* $\omega\in\Omega$. *Let* $\mathbf{W}=(W^1,\ldots,W^k)^{\top}$ *denote a (column) vector of independent (standard) Brownian sheets. Fix constants* $\sigma_1,\ldots,\sigma_d>0$. *Then for each* $\mathbf{f}\in\mathcal{C}^d_{\rm tem}(\mathsf{R})$, *the system of stochastic partial differential equations*

$$\mathrm{d}u_i(t,x) \;=\; \frac{\sigma_i^2}{2}\Delta u_i(t,x)\,\mathrm{d}t \;+\; b_i\big(t,x,\mathbf{u}(t,x)\big) + \sum_{j=1}^{k} a_{i,j}\big(t,x,\mathbf{u}(t,x)\big)\,\mathrm{d}W_t^j(x),$$

$(t,x)\in(0,\infty)\times\mathsf{R}$, $1\leq i\leq d$, *with initial condition* $\mathbf{u}(0,\cdot\,)=\mathbf{f}$ *has a (pathwise) unique* $\mathcal{C}^d_{\rm tem}(\mathsf{R})$*–valued solution* $\mathbf{u}=(u_1,\ldots,u_d)$.

Suppose additionally that for each $t\geq 0$ *and* $\omega\in\Omega$,

$$(x,u)\mapsto\big(\mathbf{a}(t,x,u,\omega),\mathbf{b}(t,x,u,\omega)\big)\quad \textit{is continuous,} \tag{48}$$

$$\mathbf{a}(t,x,0,\omega)=0\quad \textit{and}\quad \mathbf{b}(t,x,0,\omega)\geq 0,\qquad x\in\mathsf{R}, \tag{49}$$

and

$$\mathbf{f}\geq 0. \tag{50}$$

Then

$$\mathcal{P}(\mathbf{u} \geq 0) \ = \ 1. \tag{51}$$

Proof. The existence of the unique solution follows as in [Shi94]. To prove the non-negativity, fix $u_2, \ldots, u_d$ and reread the equation for u_1 as follows:

$$\mathrm{d}u_1(t,x) \ = \ \frac{\sigma_1^2}{2}\Delta u_1(t,x)\,\mathrm{d}t + \tilde{b}\big(t,x,u_1(t,x)\big) + \tilde{a}\big(t,x,u_1(t,x)\big)\,\mathrm{d}\tilde{B}_t(x) \tag{52}$$

where

$$\tilde{b}\big(t,x,u_1(t,x)\big) \ := \ b_1\big(t,x,\mathbf{u}(t,x)\big), \tag{53}$$

$$\tilde{a}\big(t,x,u_1(t,x)\big) \ := \ \Big[\sum_{j=1}^{k}\big[a_{1,j}\big(t,x,\mathbf{u}(t,x)\big)\big]^2\Big]^{1/2}, \tag{54}$$

and the Brownian sheet $\tilde{B}$ is defined by

$$\begin{aligned}\mathrm{d}\tilde{B}_t(x) \ := \ & \sum_{j=1}^{k} 1_{\{\tilde{a}(t,x,u_1(t,x))\neq 0\}}\,\frac{a_{1,j}\big(t,x,\mathbf{u}(t,x)\big)}{\tilde{a}\big(t,x,u_1(t,x)\big)}\,\mathrm{d}W_t^j(x)\\ & + \ 1_{\{\tilde{a}(t,x,u_1(t,x))=0\}}\,\mathrm{d}\tilde{W}_t(x),\end{aligned} \tag{55}$$

where $\tilde{W}$ is a Brownian sheet independent of $\mathbf{W}$. Obviously, Theorem 2.3 of [Shi94] is applicable to (52), hence we get the non-negativity of u_1. Proceed then with the other components in the same way to finish the proof. □

3.2. **Proof of Theorem 5.** In order to apply Proposition 11, set $d = 2 = k$, $\sigma_1 := \sigma_{\mathrm{c}}$, $\sigma_2 := \sigma_{\mathrm{r}}$, $\mathbf{W} = (W^{\mathrm{c}}, W^{\mathrm{r}})^{\top}$, $\mathbf{u} = (\varrho, X)$. For the moment, fix $n \geq 1$, and introduce the function $\psi^n : \mathsf{R} \to \mathsf{R}_+$ by

$$\psi^n(u) \ := \ \begin{cases} |u| & \text{if} \quad |u| \geq n^{-1}, \\ n^{-1} & \text{if} \quad |u| < n^{-1}, \end{cases} \tag{56}$$

and put

$$\mathbf{a}^n(t,x,u,\omega) \ := \ \begin{pmatrix} \sqrt{\gamma_{\mathrm{c}}\,\psi^n(u_1)} & 0 \\ 0 & \sqrt{\gamma_{\mathrm{r}}\,\psi^n(u_1)\,\psi^n(u_2)} \end{pmatrix}, \tag{57}$$

$(t,x,u,\omega) \in \mathsf{R}_+ \times \mathsf{R} \times \mathsf{R}^2 \times \Omega$, and $\mathbf{b}^n = \mathbf{0}$. Note that $\mathbf{a}^n$ and $\mathbf{b}^n$ satisfy the conditions (45) and (46). Then by Proposition 11 with these replacements, there is a unique solution $\mathbf{u}^n = (\varrho^n, X^n)$ to the system of equations

$$\begin{aligned}\mathrm{d}\varrho_t^n(x) \ &= \ \frac{\sigma_{\mathrm{c}}^2}{2}\Delta\varrho_t^n(x)\,\mathrm{d}t + \sqrt{\gamma_{\mathrm{c}}\,\psi^n\big(\varrho_t^n(x)\big)}\,\mathrm{d}W_t^{\mathrm{c}}(x),\\ \mathrm{d}X_t^n(x) \ &= \ \frac{\sigma_{\mathrm{r}}^2}{2}\Delta X_t^n(x)\,\mathrm{d}t + \sqrt{\gamma_{\mathrm{r}}\,\psi^n\big(\varrho_t^n(x)\big)\ \psi^n\big(X_t^n(x)\big)}\,\mathrm{d}W_t^{\mathrm{r}}(x),\end{aligned} \tag{58}$$

$t > 0$, $x \in \mathsf{R}$, with non-negative initial condition $(\varrho_0, X_0) = \mathbf{f} \in \mathcal{C}_{\mathrm{tem}}^2(\mathsf{R})$. Since to each $T > 0$ there is a constant $c_{(59)} = c_{(59)}(T) > 0$ such that

$$\sup_{n\geq 1,\ 0\leq t\leq T,\ x\in\mathsf{R},\ \omega\in\Omega} \mathbf{a}^n(t,x,u,\omega) \ \leq \ c_{(59)}\big(1+|u|\big), \tag{59}$$

by standard methods (see [FX01, Lemmas 12 and 13]) we get the following statements: For $T, p, q, \lambda > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 5$ we have

$$\sup_{n\geq 1,\ 0\leq t\leq T} \int_{\mathsf{R}} \mathrm{d}x\ \phi_\lambda(x)\,\mathcal{P}\Big(\big|\varrho_t^n(x)\big|^{2q} + \big|X_t^n(x)\big|^{2q}\Big) \ <\ \infty \tag{60}$$

and, with a constant $c_{(61)} = c_{(61)}(T, p, \lambda)$,

$$\begin{aligned} \sup_{n\geq 1} \mathcal{P}\Big(&|Y_t^{\mathrm{c},n}(x) - Y_{t'}^{\mathrm{c},n}(x')|^{2q} + |Y_t^{\mathrm{r},n}(x) - Y_{t'}^{\mathrm{r},n}(x')|^{2q}\Big) \\ &\leq\ c_{(61)}\,\big(\,|t-t'|^{1/2} + |x-x'|\,\big)^{q/p}\,\phi_{-\lambda}(x) \end{aligned} \tag{61}$$

whenever $t, t' \in (0, T],\ x, x' \in \mathsf{R},\ |x - x'| \leq 1,$ where

$$\begin{aligned} Y_t^{\mathrm{c},n}(x) \ &:=\ \int_{[0,t)\times\mathsf{R}} \mathrm{d}W_s^{\mathrm{c}}(y)\ \mathrm{p}_{t-s}^{\sigma_{\mathrm{c}}}(y-x)\,\sqrt{\gamma_{\mathrm{c}}\,\psi^n\big(\varrho_t^n(x)\big)}\,, \\ Y_t^{\mathrm{r},n}(x) \ &:=\ \int_{[0,t)\times\mathsf{R}} \mathrm{d}W_s^{\mathrm{r}}(y)\ \mathrm{p}_{t-s}^{\sigma_{\mathrm{r}}}(y-x)\,\sqrt{\gamma_{\mathrm{r}}\,\psi^n\big(\varrho_t^n(x)\big)\ \psi^n\big(X_t^n(x)\big)}\,. \end{aligned} \tag{62}$$

Then, as in the proof of Proposition 9 in [FX01], the family of laws of the processes $\big\{(\varrho^n, X^n) : n \geq 1\big\}$ is tight in the set of all laws on $\mathcal{C}(\mathsf{R}_+, \mathcal{C}_{\mathrm{tem}}^2)$. Let (ϱ, X) be distributed according to any of its limit points. It is easy to see that the related $\mathcal{M}_{\mathrm{tem}}^2$–valued process (ϱ, X) solves the martingale problems in Definition 2. Moreover, by Corollary 10 and the martingale representation theorem, we see that the density fields (ϱ, X) solve the system of equations (24).

On the other hand, if (ϱ, X) is a solution to (24), then the pair of related measure-valued processes solves the martingale problems, hence its distribution is uniquely determined. This finishes the proof. □

4. Scaling limits

Here we want to prove Theorem 6. After adapting the martingale problems to the scaled processes, with Lemma 12 we prove tightness of the ${}^K\!\varrho$ under $\eta \geq 1$. With Corollary 14 we get the extinction of ${}^K\!\varrho$ under $\eta < 1$ in a functional limit setting. The convergence in law ${}^K\!X_t \to {}^\infty\!X_t$ for fixed t and all $\eta \geq 0$ will be shown in Subsection 4.4 below by a modification of the proof in the $\eta = 0$ case from [DF97a]. Tightness questions of the ${}^K\!X$ for $\eta < 1$ are dealt with in Lemma 18. Finally, tightness of the ${}^K\!X$ in the case $\eta \geq 29/16$ is provided with Lemma 20. Subsection 4.7 then summarizes the proof of Theorem 6.

4.1. **Preparation: Scaled martingale problems.** Recall that we are dealing with the pair $({}^K\!\varrho, {}^K\!X) \in \mathcal{C}\big(\mathsf{R}_+, \mathcal{M}_{\mathrm{tem}}^2\big)$ of rescaled processes with scaling index $\eta \geq 0$, corresponding to the one-dimensional catalyst reactant pair $(\varrho, X) = (\varrho, X^\varrho)$ starting with Lebesgue measures as in (2). Recall also that (ϱ, X) has jointly continuous density fields denoted by the same symbol and which solves the equation system (24). Then for $\eta \geq 0$ and $K \geq 1$ fixed, the *scaled density fields* $({}^K\!\varrho,\ {}^K\!X)$ solve the system of equations

$$\begin{aligned} \mathrm{d}{}^K\!\varrho_t(x) \ &=\ K^{1-2\eta}\,\frac{\sigma_{\mathrm{c}}^2}{2}\,\Delta\,{}^K\!\varrho_t(x)\,\mathrm{d}t \,+\, \sqrt{K^{1-\eta}\,\gamma_{\mathrm{c}}\,{}^K\!\varrho_t(x)}\,\mathrm{d}{}^K\!W_t^{\mathrm{c}}(x), \\ \mathrm{d}{}^K\!X_t(x) \ &=\ K^{1-2\eta}\,\frac{\sigma_{\mathrm{r}}^2}{2}\,\Delta\,{}^K\!X_t(x)\,\mathrm{d}t \,+\, \sqrt{K^{1-\eta}\,\gamma_{\mathrm{r}}\,{}^K\!\varrho_t(x)\,{}^K\!X_t(x)}\,\mathrm{d}{}^K\!W_t^{\mathrm{r}}(x), \end{aligned} \tag{63}$$

$t > 0,\ x \in \mathsf{R},$ [with constant initial condition $(i_{\rm c}\,, i_{\rm r})$], where $\mathrm{d}^K W^{\rm c}, \mathrm{d}^K W^{\rm r}$ are the following independent (standard) time-space white noises:

$$\mathrm{d}^K W_s^{\cdot}(x) \ := \ K^{-\frac{1+\eta}{2}}\, \mathrm{d}W_{Ks}^{\cdot}(K^\eta x) \tag{64}$$

(with "$\cdot$" referring to the index c and r, respectively).

Let ${}^K\mathrm{p} = {}^K\mathrm{p}^{\sigma_\cdot}$ denote the heat kernel related to our scaling, that is, related to the operator $K^{1-2\eta}\,\frac{\sigma_\cdot^2}{2}\Delta$ (where the dot refers either to c or to r). In a standard way, equation system (63) can be turned in its *convolution form:*

$$\left\{\begin{aligned} {}^K\varrho_t(x) \ &= \ {}^K\mathrm{p}_t^{\sigma_{\rm c}} * {}^K\varrho_0\,(x) \\ &\quad + \int_{[0,t)\times\mathsf{R}} \mathrm{d}^K W_s^{\rm c}(y)\ {}^K\mathrm{p}_{t-s}^{\sigma_{\rm c}}(y-x)\sqrt{K^{1-\eta}\,\gamma_{\rm c}\,{}^K\varrho_s(y)}, \\ {}^K X_t(x) \ &= \ {}^K\mathrm{p}_t^{\sigma_{\rm r}} * {}^K X_0\,(x) \\ &\quad + \int_{[0,t)\times\mathsf{R}} \mathrm{d}^K W_s^{\rm r}(y)\ {}^K\mathrm{p}_{t-s}^{\sigma_{\rm r}}(y-x)\sqrt{K^{1-\eta}\,\gamma_{\rm r}\,{}^K\varrho_s(y)\,{}^K X_s(y)}, \end{aligned}\right. \tag{65}$$

$t > 0,\ x \in \mathsf{R},\ K \ge 1.$

On the other hand, equation system (63) leads also to the following *scaled martingale problems* instead of the ones in Definition 2:

For each $\varphi^{\rm c}, \varphi^{\rm r} \in \mathcal{C}_{\exp}^{(2)}(\mathsf{R}^d)$,

$$M_t^{{\rm c},K}(\varphi^{\rm c}) \ := \ \big\langle {}^K\varrho_t\,, \varphi^{\rm c}\big\rangle - \big\langle {}^K\varrho_0\,, \varphi^{\rm c}\big\rangle - K^{1-2\eta}\int_0^t \mathrm{d}s\,\Big\langle {}^K\varrho_s\,,\,\frac{\sigma_{\rm c}^2}{2}\Delta\varphi^{\rm c}\Big\rangle, \qquad t \ge 0, \tag{66}$$

is a square-integrable continuous $\mathcal{F}_\cdot^{{}^K\!\varrho}$–martingale with square function

$$\big\langle\!\big\langle M^{{\rm c},K}(\varphi^{\rm c})\big\rangle\!\big\rangle_t \ = \ K^{1-\eta}\,\gamma_{\rm c}\int_0^t \mathrm{d}s\ \big\langle {}^K\varrho_s\,, (\varphi^{\rm c})^2\big\rangle\,, \qquad t \ge 0, \tag{67}$$

and, similarly,

$$M_t^{{\rm r},K}(\varphi^{\rm r}) \ := \ \big\langle {}^K X_t\,, \varphi^{\rm r}\big\rangle - \big\langle {}^K X_0\,, \varphi^{\rm r}\big\rangle - K^{1-2\eta}\int_0^t \mathrm{d}s\,\Big\langle {}^K X_s\,,\,\frac{\sigma_{\rm r}^2}{2}\Delta\varphi^{\rm r}\Big\rangle, \tag{68}$$

$t \ge 0,$ is a square-integrable continuous $\mathcal{G}_\cdot^K$–martingale with square function

$$\big\langle\!\big\langle M^{{\rm r},K}(\varphi^{\rm r})\big\rangle\!\big\rangle_t \ = \ K^{1-\eta}\,\gamma_{\rm r}\,\big\langle L_{({}^K\varrho,{}^K X)}(t), (\varphi^{\rm r})^2\big\rangle, \qquad t \ge 0, \tag{69}$$

where $\mathcal{G}_t^K := \mathcal{F}_\infty^{{}^K\!\varrho} \vee \mathcal{F}_t^{{}^K\!X},\ t \ge 0.$

Roughly speaking, the difference to the martingale problems in Definition 2 is that the diffusion constants $\sigma_\cdot^2$ get the factor $K^{1-2\eta}$, and the branching rates $\gamma_\cdot$ the factor $K^{1-\eta}$.

Of course, the definition of $\mathcal{G}_t^K$ can be simplified by using $\mathcal{F}_\infty^{{}^K\!\varrho} \equiv \mathcal{F}_\infty^{\varrho}$.

Clearly, if $\eta > 1$, then the square functions in (67) and (69) will disappear as $K \uparrow \infty$, as well as the diffusion terms in (66) and (68). Hence, under this supercritical scaling the claim in Theorem 6 follows once we showed tightness, which will be provided in Subsections 4.2 and 4.6, respectively, (for $\eta \ge 29/16$ in the latter case).

We finish this subsection by recalling that there is a smoothed version $\tilde{\phi}_\lambda$ of ϕ_λ such that to each $\lambda \in \mathsf{R}$ and $m \ge 0$ there are positive constants $\underline{c}_{(70)} = \underline{c}_{(70)}(\lambda, m)$

and $\overline{c}_{(70)} = \overline{c}_{(70)}(\lambda, m)$ with the property

$$\underline{c}_{(70)}\,\phi_\lambda(x) \;\le\; \left|\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,\tilde{\phi}_\lambda\,(x)\right| \;\le\; \overline{c}_{(70)}\,\phi_\lambda(x), \qquad x \in \mathsf{R}, \tag{70}$$

(see, for instance, [FX01, formula (7)]).

4.2. **Tightness of the $^K\varrho$ in the case $\eta \ge 1$.** Recall that $^K\varrho_0 \equiv i_{\mathrm{c}}\ell$. As usual, we say that a family of random processes is tight, if their laws form a tight family.

Lemma 12 (**Tightness of the $^K\varrho$ under $\eta \ge 1$**). *Under $\eta \ge 1$, the processes $\{^K\varrho :\, K \ge 1\}$ are tight in $\mathcal{C}(\mathsf{R}_+\,,\,\mathcal{M}_{\mathrm{tem}})$.*

Proof. For the moment, fix $T, \lambda > 0$ and $q > 2$. Consider

$$f_K(t) \;:=\; \mathcal{P} \sup_{0\le s\le t} \Big[1 + \big\langle {}^K\varrho_s\,,\,\tilde{\phi}_\lambda\big\rangle\Big]^q, \qquad K \ge 1, \quad 0 \le t \le T. \tag{71}$$

Using the martingale (66), the estimate (70) in the case $m = 2$, and assuming $\eta \ge 1$, we have

$$f_K(t) \;\le\; c_{(72)} \left[f_K(0) + \mathcal{P}\Big(\int_0^t \mathrm{d}s\;\big\langle {}^K\varrho_s\,,\,\tilde{\phi}_\lambda\big\rangle\Big)^q + \mathcal{P}\sup_{0\le s\le t}\big|M_s^{\mathrm{c},K}(\tilde{\phi}_\lambda)\big|^q\right], \tag{72}$$

$0 \le t \le T$, $K \ge 1$, with a constant $c_{(72)} = c_{(72)}(\lambda, q)$. Here and in the further procedure, as a rule we do not take care on dependencies of estimation constants on model parameters as $i_{\mathrm{c}}\,, \sigma_{\mathrm{c}}\,, \gamma_{\mathrm{c}}\,.$ By Burkholder's inequality,

$$\mathcal{P}\sup_{0\le s\le t}\big|M_s^{\mathrm{c},K}(\tilde{\phi}_\lambda)\big|^q \;\le\; c(q)\,\mathcal{P}\Big(\gamma_{\mathrm{c}}\int_0^t \mathrm{d}s\,\big\langle {}^K\varrho_s\,,(\tilde{\phi}_\lambda)^2\big\rangle\Big)^{q/2}. \tag{73}$$

Using $\tilde{\phi}_\lambda \le c(\lambda)$ and the simple inequality $|a| \le 2^{-1/2}\,(1 + a^2)$, $a \in \mathsf{R}$, we may continue with

$$\le\; c_{(72)}\,f_K(0) \,+\, c_{(74)}\int_0^t \mathrm{d}s\;f_K(s) \tag{74}$$

$0 \le t \le T$, $K \ge 1$, with a constant $c_{(74)} = c_{(74)}(\lambda, q, T)$. Then Gronwall's inequality gives

$$\sup_{K\ge 1}\,\mathcal{P}\sup_{0\le s\le T}\big\langle {}^K\varrho_s\,,\,\tilde{\phi}_\lambda\big\rangle^q \;\le\; c_{(75)} \tag{75}$$

with a constant $c_{(75)} = c_{(75)}(\lambda, q, T)$. Again from the martingale (66), for $\varphi \in \mathcal{C}_\lambda^{(2)}$, $\lambda > 0$, and $0 \le t' \le t \le T$,

$$\begin{aligned}\mathcal{P}\,\big|\langle {}^K\varrho_t - {}^K\varrho_{t'}\,,\,\varphi\rangle\big|^q \;\le\; & c_{(76)}\,\mathcal{P}\Big(\int_{t'}^t \mathrm{d}s\;\big\langle {}^K\varrho_s\,,\,\tilde{\phi}_\lambda\big\rangle\Big)^q \\ &+\, c_{(76)}\,\mathcal{P}\Big(\gamma_{\mathrm{c}}\int_{t'}^t \mathrm{d}s\,\big\langle {}^K\varrho_s\,,(\tilde{\phi}_\lambda)^2\big\rangle\Big)^{q/2}\end{aligned} \tag{76}$$

with $c_{(76)} = c_{(76)}(q, \lambda)$. By (75) we may continue with

$$\sup_{K\ge 1}\,\mathcal{P}\,\big|\langle {}^K\varrho_t - {}^K\varrho_{t'}\,,\,\varphi\rangle\big|^q \;\le\; c_{(77)}\,|t - t'|^{q/2}, \qquad 0 \le t, t' \le T, \tag{77}$$

with $c_{(77)} = c_{(77)}(\lambda, q, T)$.

To finish the tightness proof, we want to exploit [EK86, Theorem 3.9.1]. To this end, we use the relatively compact subsets

$$K\big((c_n)_{n\geq 1}\big) \;:=\; \Big\{\mu\in\mathcal{M}_{\mathrm{tem}}:\ \langle\mu,\tilde{\phi}_{1/n}\rangle\leq c_n\,,\ n\geq 1\Big\}\;\subseteq\;\mathcal{M}_{\mathrm{tem}} \tag{78}$$

with $(c_n)_{n\geq 1}$ a sequence of positive numbers. Given $0<\varepsilon\leq 1$, using (75), we can find $(c_n)_{n\geq 1}$ such that

$$\mathcal{P}\Big({}^{K}\!\varrho_t\in K\big((c_n)_{n\geq 1}\big)\ \text{ for all }\ t\in[0,T]\Big)\;\geq\;1-\varepsilon. \tag{79}$$

Then by (77), for $\varphi\in\mathcal{C}^{(2)}_{\exp}$, the families of random processes $t\mapsto\langle{}^{K}\!\varrho_t\,,\varphi\rangle$ restricted to $[0,T]$ are tight in $\mathcal{C}\big([0,T],\mathsf{R}\big)$. Then by [EK86, Theorem 3.9.1] the tightness claim follows. (Note that all of our processes are continuous, thus the tightness in Skorohod space implies the tightness in our $\mathcal{C}$–space.) This finishes the proof. □

4.3. **Extinction of ${}^{K}\!\varrho$ under $\eta<1$.** This extinction will follow from the following strong local extinction property of one-dimensional super-Brownian motion ϱ starting from a Lebesgue measure, which we expose as a lemma, since we did not find it directly in the literature.

Lemma 13 (**Almost sure local finite time extinction of ϱ**). *For all bounded Borel sets B in R, and $0\leq\eta<1$, almost surely,*

$$\varrho_T(T^{\eta}B)\;=\;0,\quad \textit{for all sufficiently large } T. \tag{80}$$

Proof. We adapt a method occurring in [DF97a, Subsection 6.2]. We may assume that B is a centered "ball" of radius $r\geq 1$, say. Using the branching property, we decompose $\varrho=\sum_{i\in\mathsf{Z}}\varrho^i$ in independent copies ϱ^i of ϱ, but where ϱ^i starts from $\varrho^i_0=\ell\left([i,i+1)\cap(\,\cdot\,)\right)$, $i\in\mathsf{Z}$.

For the moment, fix i such that $|i|\geq 2r$. Consider the event

$$\varrho^i_t(t^{\eta}B)\;>\;0,\quad \text{for some}\quad t\geq 1, \tag{81}$$

which we denote by E^i. Under E^i there are two cases: Such a t satisfies $t\leq t_i:=\big(|i|/2r\big)^{1/\eta}$, or $t>t_i$. If $t\leq t_i$, then ϱ^i gives mass to the centered ball with radius $|i|/2$ at some time after 1 (note that $|i|/2=t_i^{\eta}r$). Call this event E^i_1. On the other hand, if $t>t_i$, then ϱ^i has to survive by time t_i. Call this event E^i_2. Consequently, $E^i\subseteq E^i_1\cup E^i_2$.

Now the event E^i_1 has a probability bounded by $c\,|i|^{-2}$, see [Isc88, Theorem 1]. On the other hand, E^i_2 has probability bounded by $c\,t_i^{-1}=c\,|i|^{-1/\eta}$, since the total mass process of ϱ^i is Feller's branching diffusion (see, for instance, [DFM00, formula (73)]).

Consequently, E^i has probability bounded by $c\left(|i|^{-2}+|i|^{-1/\eta}\right)$ which is summable in the considered i with $|i|\geq 2r$. By Borel-Cantelli, E^i occurs only for finitely many $i\in\mathsf{Z}$. But these finitely many ϱ^i die in finite (random) time, that is, for them we have $\varrho^i_t=0$ for all sufficiently large t, a.s. This gives

$$\varrho_t(t^{\eta}B)\;=\;\sum_{i\in\mathsf{Z}}\varrho^i_t(t^{\eta}B)\;=\;0,\quad \text{for all sufficiently large } t,\quad \text{a.s.}, \tag{82}$$

that is the claim (80). □

Corollary 14 (**Almost sure local finite time extinction of** ${}^K\varrho$ **under** $\eta<1$). *In the case* $\eta<1$, *for each* $\varepsilon>0$, *bounded Borel set* $B\subset\mathsf{R}$, *and* $\delta\geq 0$ *satisfying* $\eta+\delta<1$,

$$ {}^K\varrho_t(K^\delta B)=0,\quad t\geq\varepsilon,\quad \textit{for all sufficiently large } K,\quad \mathcal{P}^{\varrho}\text{–}a.s. \tag{83}$$

In particular, as $K\uparrow\infty$, *the processes* ${}^K\varrho$ *converge in law to* 0 *in path space* $\mathcal{C}\big((0,\infty),\mathcal{M}\big)$.

Proof. Fix $\eta,\varepsilon,B,\delta$ as in the corollary. Set $\tilde\eta:=\eta+\delta$. For $t\geq\varepsilon$ and $K\geq 1$,

$$ {}^K\varrho_t(K^\delta B)\;=\;K^{-\eta}\,\varrho_{Kt}\big((Kt)^{\tilde\eta}\,t^{-\tilde\eta}B\big). \tag{84}$$

But there is a bounded Borel set $\tilde B\subset\mathsf{R}$, such that $t^{-\tilde\eta}B\subseteq\tilde B$, for all $t\geq\varepsilon$. Therefore (83) follows from Lemma 13 with η,B replaced by $\tilde\eta,\tilde B$. Taking $\delta=0$, by the definition of the topology in $\mathcal{C}\big((0,\infty),\mathcal{M}\big)$ this implies the convergence claim, since ε and B had been arbitrary. This finishes the proof. ◻

Remark 15 (**Amost sure local finite time extinction of densities**). The almost sure local finite time extinction properties in (80) and (83) can be restated in terms of the jointly continuous density fields of ϱ and ${}^K\varrho$, respectively. ◇

4.4. **Fdd convergence of** ${}^KX$. One would be seduced to try a variance calculation in order to prove convergence of one-dimensional distributions to a degenerate limit. But if $\eta\leq 1$, the variances do not go to zero as $K\uparrow\infty$. In fact, for any $\eta\geq 0$ and $\varphi\in\mathcal{C}^{(2)}_{\exp}$, from the scaled martingale problem in Subsection 4.1 and the conditional expectation formula in (21),

$$ \mathcal{V}\mathrm{ar}\,\mathcal{P}\left\{\langle {}^KX_t\,,\varphi\rangle\;\middle|\;{}^K\varrho\right\}\;=\;0, \tag{85}$$

hence from covariance formula (23),

$$\begin{aligned} \mathcal{V}\mathrm{ar}\,\big\langle {}^KX_t\,,\varphi\big\rangle\;&=\;\mathcal{P}\mathcal{V}\mathrm{ar}\left\{\langle {}^KX_t\,,\varphi\rangle\;\middle|\;{}^K\varrho\right\}\\ &=\;i_{\mathrm{r}}\,K^{1-\eta}\,\gamma_{\mathrm{r}}\int_0^t\mathrm{d}s\int_{\mathsf{R}}\mathcal{P}\,{}^K\varrho_s(\mathrm{d}y)\,\big[\mathrm{p}^{\sigma_{\mathrm{r}}}_{K^{1-2\eta}(t-s)}*\varphi\big]^2(y)\\ &=\;i_{\mathrm{c}}\,i_{\mathrm{r}}\,K^{1-\eta}\,\gamma_{\mathrm{r}}\int_0^t\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;\big[\mathrm{p}^{\sigma_{\mathrm{r}}}_{K^{1-2\eta}s}*\varphi\big]^2(y). \end{aligned} \tag{86}$$

Therefore, if $\eta=1$,

$$ \mathcal{V}\mathrm{ar}\,\big\langle {}^KX_t\,,\varphi\big\rangle\;\underset{K\uparrow\infty}{\longrightarrow}\;i_{\mathrm{c}}\,i_{\mathrm{r}}\,\gamma_{\mathrm{r}}\int_0^t\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;\varphi^2(y)\;\neq\;0, \tag{87}$$

provided that $t>0$ and $\varphi\neq 0$, whereas for $\eta<1$,

$$ \mathcal{V}\mathrm{ar}\,\big\langle {}^KX_t\,,\varphi\big\rangle\;=\;i_{\mathrm{c}}\,i_{\mathrm{r}}\,\gamma_{\mathrm{r}}\int_0^{K^{1-\eta}t}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;\big[\mathrm{p}^{\sigma_{\mathrm{r}}}_{K^{-\eta}s}*\varphi\big]^2(y)\;\underset{K\uparrow\infty}{\longrightarrow}\;\infty, \tag{88}$$

despite ${}^\infty X_t\equiv i_{\mathrm{r}}\ell$ according to our claim. Roughly speaking, calculating the variance in the annealed model means to pass to the variance in the constant medium case, which will not disappear under non-supercritical scaling in this subcritical dimension, despite $\mathcal{V}\mathrm{ar}\,\langle {}^\infty X_t\,,\varphi\rangle\equiv 0$ by the (claimed) degeneration of ${}^\infty X$.

So we will need some more subtle method. Actually, we will prove convergence of one-dimensional distributions simultaneously for all $\eta\geq 0$ by a modification of the proof in the $\eta=0$ case from [DF97a]. In fact, replace the a.s. statement concerning the catalyst in Theorem 6, p.273, there by convergence in probability, then the proof goes through, we will give the details.

Lemma 16 (**Fdd convergence**). *For all finite sequences* $0 \le t_1 \le \cdots \le t_m$ *and* $\varphi_1, \ldots, \varphi_m \in \mathcal{C}^+_{\exp}(\mathsf{R}),\ m \ge 1,$ *as well as any* $\eta \ge 0,$

$$\mathcal{P} \exp\Big[-\sum_{i=1}^m \big\langle {}^K\!X_{t_i}, \varphi_i\big\rangle\Big] \underset{K\uparrow\infty}{\longrightarrow} \exp\Big[-i_{\mathrm{r}} \sum_{i=1}^m \langle \ell, \varphi_i\rangle\Big]. \tag{89}$$

Proof. We start with recalling the *log-Laplace representation* of X^ϱ, given ϱ. For $0 \le r \le T$ and $\varphi \in \mathcal{C}^+_{\exp}$,

$$-\log \mathcal{P}\Big\{\exp\langle X_T, -\varphi\rangle \,\Big|\, \varrho, X_r\Big\} = \big\langle X_r, v_r^\varrho[T;\varphi]\big\rangle \tag{90}$$

with $v^\varrho[T;\varphi]$ the unique non-negative solution to the log-Laplace equation (32). For any $\eta \ge 0$, abbreviating

$$\varphi^K := K^{-\eta}\varphi(K^{-\eta}\,\cdot), \qquad K \ge 1, \tag{91}$$

identity (90) implies

$$-\log \mathcal{P}\Big\{\exp\big\langle {}^K\!X_T, -\varphi\big\rangle \,\Big|\, \varrho, {}^K\!X_r\Big\} = \big\langle X_{Kr}, v_{Kr}^\varrho[KT;\varphi^K]\big\rangle. \tag{92}$$

For the proof of (89), it suffices to assume that $m=1$. Fix $t>0$, $\varphi \in \mathcal{C}^+_{\exp}$, and $\eta \ge 0$. By (92) it suffices to show that

$$\big\|v_0^\varrho[Kt;\varphi^K]\big\|_1 \underset{K\uparrow\infty}{\longrightarrow} \|\varphi\|_1 \quad \text{in } P^\varrho\text{–law}, \tag{93}$$

with $v^\varrho[T;\varphi]$ the unique non-negative solution to the log-Laplace equation (32), and $\|\cdot\|_1$ the L^1–norm. From domination as in (34),

$$v_s^\varrho[Kt;\varphi^K](x) \le \mathrm{p}^{\sigma_{\mathrm{r}}}_{Kt-s} * \varphi^K(x), \qquad 0 \le s \le Kt, \quad x \in \mathsf{R}^d, \tag{94}$$

and trivially,

$$\|\mathrm{p}^{\sigma_{\mathrm{r}}}_{Kt-s} * \varphi^K\|_1 = \|\varphi^K\|_1 = \|\varphi\|_1. \tag{95}$$

Therefore,

$$\big\|v_0^\varrho[Kt;\varphi^K]\big\|_1 \le \|\varphi\|_1, \tag{96}$$

and instead of (93) it suffices to verify that

$$\liminf_{K\uparrow\infty} \big\|v_0^\varrho[Kt;\varphi^K]\big\|_1 = \|\varphi\|_1 \quad \text{in } P^\varrho\text{–law}. \tag{97}$$

From the Feynman-Kac form of the log-Laplace equation (32),

$$\big\|v_0^\varrho[Kt;\varphi^K]\big\|_1 = \int_{\mathsf{R}} \mathrm{d}x\ \Pi_{0,x}\, \varphi^K(W_{Kt}) \exp\Big[-\int_0^{Kt} \mathrm{d}s\ \varrho_s(W_s)\, v_s^\varrho[Kt;\varphi^K](W_s)\Big],$$

see [DF97a, formula (6.4)]. But from domination (94),

$$v_s^\varrho[Kt;\varphi^K](W_s) \le \int_{\mathsf{R}} \mathrm{d}y\ \mathrm{p}^{\sigma_{\mathrm{r}}}_{Kt-s}(y-W_s)\, \varphi^K(y) \le c_{(98)}\,(Kt-s)^{-1/2} \tag{98}$$

with a constant $c_{(98)} = c_{(98)}(\|\varphi\|_1)$. Hence,

$$\begin{aligned}
&\big\|v_0^\varrho[Kt;\varphi^K]\big\|_1 \\
&\ge \int_{\mathsf{R}} \mathrm{d}x\ \Pi_{0,x}\, \varphi^K(W_{Kt}) \exp\Big[-c_{(98)}\int_0^{Kt} \mathrm{d}s\ (Kt-s)^{-1/2}\, \varrho_s(W_s)\Big] \\
&= \Pi_{0,0} \int_{\mathsf{R}} \mathrm{d}x\ \varphi(x) \exp\Big[-c_{(98)}\int_0^{Kt} \mathrm{d}s\ (Kt-s)^{-1/2}\, \varrho_s(K^\eta x - W_{Kt} + W_s)\Big].
\end{aligned}$$

We want to show that the latter P^{ϱ}–random $\Pi_{0,0}$–expectation expression converges in law to $\|\varphi\|_1$. But it is bounded (by $\|\varphi\|_1$), so we may show instead the convergence to $\|\varphi\|_1$ of its P^{ϱ}–expectation:

$$\begin{aligned}
&\Pi_{0,0}\int_{\mathsf{R}}\mathrm{d}x\;\varphi(x)\,P^{\varrho}\exp\Big[-c_{(98)}\int_0^{Kt}\mathrm{d}s\;(Kt-s)^{-1/2}\,\varrho_s(K^{\eta}x-W_{Kt}+W_s)\Big]\\
&=\;\Pi_{0,0}\int_{\mathsf{R}}\mathrm{d}x\;\varphi(x)\,P^{\varrho}\exp\Big[-c_{(98)}\int_0^{Kt}\mathrm{d}s\;(Kt-s)^{-1/2}\,\varrho_s(W_s)\Big]\\
&=\;\|\varphi\|_1\,\Pi_{0,0}\,P^{\varrho}\exp\Big[-c_{(98)}\int_0^{Kt}\mathrm{d}s\;(Kt-s)^{-1/2}\,\varrho_s(W_s)\Big].
\end{aligned}$$

Here in the last but one step we used that the density field ϱ with constant initial state i_{c} is invariant in P^{ϱ}–law with respect to the spatial shift by $K^{\eta}x-W_{Kt}$.

For the further proof we may set $t=1$. We still need to show that

$$\int_0^K\mathrm{d}s\;(K-s)^{-1/2}\,\varrho_s(W_s)\;\underset{K\uparrow\infty}{\longrightarrow}\;0,\quad \Pi_{0,0}\times P^{\varrho}\text{–a.s.}\tag{99}$$

But there is a finite time of interference, say $\tau=\tau(\varrho,W)$, of ϱ and W, that is

$$\varrho_s(W_s)\;=\;0,\quad\text{for }\;s\geq\tau,\quad \Pi_{0,0}\times P^{\varrho}\text{–a.s.},\tag{100}$$

see [DF97a, Proposition 7, p.264]. Thus,

$$\begin{aligned}
&\int_0^K\mathrm{d}s\;(K-s)^{-1/2}\,\varrho_s(W_s)\;=\;\int_0^{K\wedge\tau}\mathrm{d}s\;(K-s)^{-1/2}\,\varrho_s(W_s)\\
&\leq\;(K-K\wedge\tau)^{-1/2}\int_0^{\tau}\mathrm{d}s\;\varrho_s(W_s)\;\underset{K\uparrow\infty}{\longrightarrow}\;0,\quad\Pi_{0,0}\times P^{\varrho}\text{–a.s.},
\end{aligned}\tag{101}$$

where the total collision local time $\int_0^{\tau}\mathrm{d}s\,\varrho_s(W_s)$ of ϱ and W is finite by the joint continuity of the density field ϱ (see also [DF97a, Proposition 8, p.265]). This finishes the proof. □

4.5. **Tightness formulations for the ${}^K\!X$ under $\eta<1$.** In order to deal with tightness of the ${}^K\!X$ in the case $\eta<1$, we will decompose them into two parts which will be handled separately. For this, we impose the following assumption:

Assumption 17 (**Choice of parameters**). Fix $0\leq\eta<1$, $0<2\varepsilon<T$, as well as a non-vanishing $\varphi\geq 0$ in $\mathcal{C}^{(2)}_{\mathrm{com}}$. Choose $\delta\geq 0$ such that $1/2-\eta<\delta<1-\eta$. Let $B\subset\mathsf{R}$ denote a centered "ball" covering the support of φ. ◇

For the moment, fix also $K\geq 1$. From the convolution form (65), with $\mathrm{d}W:=\mathrm{d}^K W^{\mathrm{r}}$, given $\mathcal{G}^K_{\varepsilon}:=\mathcal{F}^{\varrho}_{\infty}\vee\mathcal{F}^{{}^K\!X}_{\varepsilon}$,

$${}^K\!X_t(x)\;=\;{}^K\!I_t(x)\;+\;{}^K\!J^{\varepsilon,t}_t(x),\qquad t>\varepsilon,\quad x\in\mathsf{R},\tag{102}$$

where

$$\begin{aligned}
{}^K\!I_t(x)\;:=\;&{}^K\mathrm{p}_{t-\varepsilon}\ast{}^K\!X_{\varepsilon}\,(x)\\
&+\int_{[\varepsilon,t)\times K^{\delta}B}\mathrm{d}W_s(y)\,{}^K\mathrm{p}_{t-s}(y-x)\sqrt{K^{1-\eta}\,\gamma_{\mathrm{r}}\,{}^K\varrho_s(y)\,{}^K\!X_s(y)}\,,
\end{aligned}\tag{103}$$

and, for $\varepsilon<t\leq\tau$,

$${}^K\!J^{\varepsilon,\tau}_t(x)\;:=\;\int_{[\varepsilon,t)\times K^{\delta}B^{\mathrm{c}}}\mathrm{d}W_s(y)\,{}^K\mathrm{p}_{\tau-s}(y-x)\sqrt{K^{1-\eta}\,\gamma_{\mathrm{r}}\,{}^K\varrho_s(y)\,{}^K\!X_s(y)}\,.\tag{104}$$

Then we have the following *decomposition:*

$$\big\langle {}^K\!X_t\,,\varphi\big\rangle \;=\; \big\langle {}^K\!I_t\,,\varphi\big\rangle \;+\; \big\langle {}^K\!J_t^{\varepsilon,t},\varphi\big\rangle, \qquad t\geq\varepsilon, \tag{105}$$

understanding the pairings in the obvious way. Our purpose is to deal with tightness of the two terms at the right hand side separately, and, in fact, in the case of the first one, conditioned on ϱ.

Lemma 18 (**Tightness concerning** ${}^K\!X$ **under** $\eta<1$)**.** *Impose Assumption* 17.

(a) (First term given ϱ**):** *Conditioned on* ϱ, *the processes*

$$\big\{t\mapsto\langle{}^K\!I_t\,,\varphi\rangle:\ K\geq1\big\}$$

are tight in $\mathcal{C}\big([2\varepsilon,T],\mathsf{R}_+\big)$.

(b) (Second term): *The processes*

$$\big\{t\mapsto\langle{}^K\!J_t^{\varepsilon,t},\varphi\rangle:\ K\geq1\big\}$$

are tight in $\mathcal{C}\big([\varepsilon,T],\mathsf{R}_+\big)$.

Proof of Lemma 18(a). Since $\eta+\delta<1$, by the extinction Corollary 14, $\mathcal{P}^\varrho$–almost surely,

$${}^K\!\varrho_s(y) \;=\; 0,\quad s\geq\varepsilon,\quad y\in K^\delta B,\quad K\geq K_0=K_0(\varrho,\eta,\varepsilon,\varphi,\delta),\ \text{say.} \tag{106}$$

Hence, for these K, the integral term in (103) vanishes. Thus,

$$\big\langle{}^K\!I_t\,,\varphi\big\rangle \;=\; \big\langle{}^K\!X_\varepsilon\,,\,{}^K\mathrm{p}_{t-\varepsilon}*\varphi\big\rangle \;=:\; {}^K Y_{t-\varepsilon}\,, \qquad \varepsilon<t\leq T. \tag{107}$$

Given ϱ, introduce the events

$${}^K\!E_N \;=\; {}^K\!E_N(\varrho,\varepsilon,T,\varphi) \;:=\; \Big\{\sup_{\varepsilon\leq t\leq 2T}{}^K Y_t\leq N\Big\}, \qquad N\geq1. \tag{108}$$

By Markov's inequality, for the complement ${}^K\!E_N^{\mathrm{c}}$ of ${}^K\!E_N$,

$$\mathcal{P}\left\{{}^K\!E_N^{\mathrm{c}}\;\middle|\;\varrho\right\} \;\leq\; N^{-1}\,\mathcal{P}\Big\{\sup_{\varepsilon\leq t\leq 2T}\big\langle{}^K\!X_\varepsilon\,,\,{}^K\mathrm{p}_t*\varphi\big\rangle\;\Big|\;\varrho\Big\}. \tag{109}$$

But with a constant $c_{(110)}=c_{(110)}(\varepsilon,T)$,

$${}^K\mathrm{p}_t \;\leq\; (2T/\varepsilon)^{1/2}\,{}^K\mathrm{p}_{2T} \;=\; c_{(110)}\,{}^K\mathrm{p}_{2T}, \qquad \varepsilon\leq t\leq 2T. \tag{110}$$

Hence, by the conditional expectation formula in (21) and our Lebesgue initial states, inequality (109) can be continued with

$$\leq\; N^{-1}\,c_{(110)}\,\mathcal{P}\Big\{\big\langle{}^K\!X_\varepsilon\,,\,{}^K\mathrm{p}_{2T}*\varphi\big\rangle\;\Big|\;\varrho\Big\} \;=\; N^{-1}\,c_{(110)}\,c_{\mathrm{r}}\,\|\varphi\|_1\,. \tag{111}$$

Thus, for each $\delta>0$ we find an $N_0=N_0(\delta,\varepsilon,T,\varphi)$ such that

$$\sup_{K\geq K_0}\mathcal{P}\left\{{}^K\!E_N^{\mathrm{c}}\;\middle|\;\varrho\right\} \;\leq\; \delta, \qquad N\geq N_0\,. \tag{112}$$

On the other hand, on ${}^K\!E_N$, for $\varepsilon\leq s\leq t\leq T$,

$$\big|{}^K Y_t-{}^K Y_s\big| \;\leq\; \big\langle{}^K\!X_\varepsilon\,,\,|{}^K\mathrm{p}_t-{}^K\mathrm{p}_s|*\varphi\big\rangle. \tag{113}$$

However, for any diffusion constant $\sigma>0$,

$$\Big|\frac{\partial}{\partial r}\,\mathrm{p}_r(x)\Big| \;\leq\; c\,\frac{1}{r}\,\mathrm{p}_{2r}(x), \qquad r>0,\quad x\in\mathsf{R}. \tag{114}$$

Therefore,

$$\left|{}^K\mathrm{p}_t(x) - {}^K\mathrm{p}_s(x)\right| \;\le\; \int_s^t \mathrm{d}r\,\left|\frac{\partial}{\partial r}\,{}^K\mathrm{p}_r(x)\right| \;\le\; c\int_s^t \mathrm{d}r\;\mathrm{p}_{2r}(x), \tag{115}$$

$\varepsilon \le s \le t \le T,\ x \in \mathsf{R}$. Inserting into (113), on ${}^KE_N$,

$$\left|{}^KY_t - {}^KY_s\right| \;\le\; c\int_s^t \mathrm{d}r\;\bigl\langle {}^KX_\varepsilon\,, {}^K\mathrm{p}_{2r}{*}\varphi\bigr\rangle \;=\; c\int_s^t \mathrm{d}r\;{}^KY_{2r} \;\le\; c\,N\,|t-s|, \tag{116}$$

$\varepsilon \le s \le t \le T$. Consequently, given ϱ and on ${}^KE_N$, the processes

$$\bigl\{{}^KY_t:\ \varepsilon \le t \le T\bigr\},\quad K \ge K_0\,, \tag{117}$$

are equi-continuous, hence, the processes

$$\Bigl\{\bigl\langle {}^KX_\varepsilon\,, {}^K\mathrm{p}_{t-\varepsilon}{*}\varphi\bigr\rangle:\ 2\varepsilon \le t \le T+\varepsilon\Bigr\},\quad K \ge K_0\,, \tag{118}$$

are also equi-continuous on ${}^KE_N$, given ϱ. But then the processes

$$\Bigl\{\bigl\langle {}^KI_t\,,\varphi\bigr\rangle:\ 2\varepsilon \le t \le T\Bigr\},\quad K \ge K_0\,, \tag{119}$$

are also equi-continuous on ${}^KE_N$, given ϱ. This then gives tightness of the family

$$\Bigl\{\bigl\langle {}^KI_t\,,\varphi\bigr\rangle:\ 2\varepsilon \le t \le T\Bigr\},\quad K \ge 1, \tag{120}$$

of processes, given ϱ, finishing the proof. □

Proof of Lemma 18(b). Here we proceed without conditioning to ϱ. It suffices to show that there is a constant $c_{(121)} = c_{(121)}(\varepsilon,\varphi,B,T,\eta,\delta)$ such that

$$\sup_{K\ge 1}\,\mathcal{P}\left|\langle {}^KJ_t^{\varepsilon,t},\varphi\rangle - \langle {}^KJ_r^{\varepsilon,r},\varphi\rangle\right|^2 \;\le\; c_{(121)}\,|t-r|^2, \qquad r,t\in[\varepsilon,T]. \tag{121}$$

For this we may assume that $r<t$. Recalling definition (104) of ${}^KJ_t^{\varepsilon,\tau}$, we decompose

$$\langle {}^KJ_t^{\varepsilon,t},\varphi\rangle - \langle {}^KJ_r^{\varepsilon,r},\varphi\rangle \;=\; \langle {}^KJ_t^{r,t},\varphi\rangle + \langle {}^KJ^{\varepsilon,r,t},\varphi\rangle, \tag{122}$$

where

$${}^KJ^{\varepsilon,r,t} \;:=\; \int_{[\varepsilon,r)\times K^\delta B^{\mathrm{c}}} \mathrm{d}W_s(y)\,\bigl[{}^K\mathrm{p}_{t-s}(y-x) - {}^K\mathrm{p}_{r-s}(y-x)\bigr]\sqrt{K^{1-\eta}\,{}^K\varrho_s(y)\,{}^KX_s(y)},$$

and deal with both decomposition terms separately in order to prove (121).

1° (*First term in the decomposition* (122)). Actually, there is a constant $c_{(123)} = c_{(123)}(\varepsilon,\varphi,B,T,\eta,\delta)$ such that

$$\sup_{K\ge 1}\,\mathcal{P}\bigl|\langle {}^KJ_t^{r,t},\varphi\rangle\bigr|^2 \;\le\; c_{(123)}\,(t-r)^2. \tag{123}$$

Indeed, note that

$$t \mapsto \langle {}^KJ_t^{r,\tau},\varphi\rangle, \qquad \varepsilon \le t \le \tau \le T, \tag{124}$$

is a $\mathcal{G}^K_\cdot = \mathcal{F}^\varrho_\infty \vee \mathcal{F}^{{}^KX}_\cdot$–martingale. Then, by Burkholder-Davis-Gundy's inequality,

$$\begin{aligned}\mathcal{P}\bigl|\langle {}^KJ_t^{r,t},\varphi\rangle\bigr|^2 \\ \le\; c\,\mathcal{P}\int_r^t \mathrm{d}s\int_{K^\delta B^{\mathrm{c}}}\mathrm{d}y\,\Bigl(\int_B \mathrm{d}x\;\varphi(x)\,{}^K\mathrm{p}_{t-s}(y-x)\Bigr)^2 K^{1-\eta}\,{}^K\varrho_s(y)\,{}^KX_s(y).\end{aligned} \tag{125}$$

But

$$\mathcal{P}\,{}^K\!\varrho_s(y)\,{}^K\!X_s(y) \;\equiv\; i_{\rm c}\,i_{\rm r}\,, \tag{126}$$

since ${}^K\!\varrho$ and ${}^K\!X$ are uncorrelated, and

$$\mathcal{P}\,{}^K\!\varrho_t(x) \;\equiv\; i_{\rm c} \quad \text{and} \quad \mathcal{P}\,{}^K\!X_t(x) \;\equiv\; i_{\rm r} \tag{127}$$

by the expectation formulas in (25) and our uniform initial states. Moreover,

$$|y-x| \;\geq\; |y|/2, \qquad x\in B, \quad y\in K^\delta B^{\rm c}, \quad K\geq K_0=K_0(\delta,B), \text{ say.} \tag{128}$$

Hence, for $K\geq K_0$,

$$\begin{aligned} \mathcal{P}\big|\langle{}^K\!J_t^{r,t},\varphi\rangle\big|^2 \;&\leq\; c\,\|\varphi\|_\infty^2\,K^{1-\eta}\int_r^t\!\mathrm{d}s\int_{K^\delta B^{\rm c}}\mathrm{d}y\;{}^K\mathrm{p}_{t-s}^2(y/2) \\ &=\; c\,K^{1-\eta}\int_0^{t-r}\frac{\mathrm{d}s}{s}\int_{|y|\,\geq\,c\,K^\delta}\mathrm{d}y\;{}^K\!\sigma^{-2}\,\exp\Big[-\frac{y^2}{4\,{}^K\!\sigma^2\,s}\Big], \end{aligned} \tag{129}$$

with the diffusion constant

$${}^K\!\sigma \;:=\; K^{1/2-\eta}\sigma_{\rm r} \tag{130}$$

of the heat kernel ${}^K\mathrm{p}$. Substituting $s\mapsto s\,(t-r)$, the latter double integral equals

$${}^K\!\sigma^{-2}\int_0^1\frac{\mathrm{d}s}{s}\int_{|y|\,\geq\,c\,K^\delta}\mathrm{d}y\;\exp\Big[-\frac{y^2}{4\,{}^K\!\sigma^2\,s\,(t-r)}\Big], \tag{131}$$

and $y\mapsto\big(4\,{}^K\!\sigma^2\,s\,(t-r)\big)^{1/2}y$ now gives

$${}^K\!\sigma^{-2}\int_0^1\frac{\mathrm{d}s}{s}\int_{({}^K\!\sigma^2\,s\,(t-r))^{1/2}\,|y|\,\geq\,c\,K^\delta}\mathrm{d}y\;\big(4\,{}^K\!\sigma^2\,s\,(t-r)\big)^{1/2}\,\mathrm{e}^{-y^2}. \tag{132}$$

Passing in the latter integration bound from s to 1, the ds–integral $\int_0^1\mathrm{d}s\;s^{-1/2}=c$ can be separated. Then, by the concrete form (130) of the diffusion constant ${}^K\!\sigma$, for the double integral in (129) we found the bound

$$c\,K^{-1/2+\eta}\,(t-r)^{1/2}\int_{|y|\,\geq\,c\,K^{\eta+\delta-1/2}\,(t-r)^{-1/2}}\mathrm{d}y\;\mathrm{e}^{-y^2}. \tag{133}$$

But

$$0\;<\;\eta+\delta-\frac{1}{2}\;=:\;\frac{\tilde\eta}{2}\;<\;\frac{1}{2} \tag{134}$$

by our Assumption 17. Fix a number

$$\varsigma\;:=\;\varsigma(\eta,\delta)\;>\;1/\tilde\eta\vee 3. \tag{135}$$

Clearly, there is a constant $c_{(136)}=c_{(136)}(\varsigma)$ such that

$$\int_{|y|\,\geq\,L}\mathrm{d}y\;\mathrm{e}^{-y^2}\;\leq\;\frac{2}{L}\,\mathrm{e}^{-L^2}\;\leq\;c_{(136)}\,L^{-\varsigma}, \quad \text{for all } K \text{ sufficiently large.} \tag{136}$$

Hence, for the integral in (133) we may use the bound

$$c\,K^{-\varsigma\,\tilde\eta/2}\,(t-r)^{\varsigma/2}, \quad \text{for all } K \text{ sufficiently large.} \tag{137}$$

Inserting into (133) and (129) we get

$$\begin{aligned} \mathcal{P}\big|\langle{}^K\!J_t^{r,t},\varphi\rangle\big|^2 \;&\leq\; c\,K^{1-\eta}\,K^{-1/2+\eta}\,(t-r)^{(1+\varsigma)/2}\,K^{-\varsigma\,\tilde\eta/2} \\ &\leq\; c\,K^{(1-\varsigma\tilde\eta)/2}\,(t-r)^{(1+\varsigma)/2} \end{aligned} \tag{138}$$

for all sufficienlty large K. Since $1-\varsigma\tilde{\eta}<0$, this gives (123).

2° (*Second term in the decomposition* (122)). It is also true that there is a constant $c_{(139)}=c_{(139)}(\varepsilon,\varphi,B,T,\eta,\delta)$ such that

$$\sup_{K\ge 1}\mathcal{P}\left|\langle{}^K\!J^{\varepsilon,r,t},\varphi\rangle\right|^2 \;\le\; c_{(139)}\,|t-r|^2\,. \tag{139}$$

In fact, as in (125),

$$\begin{aligned}\mathcal{P}\left|\langle{}^K\!J^{\varepsilon,r,t},\varphi\rangle\right|^2 \;\le\; c\,K^{1-\eta}\,\mathcal{P}\int_\varepsilon^r \mathrm{d}s\int_{K^\delta B^c}\mathrm{d}y \\ \times\,\Big(\int_B \mathrm{d}x\,\left|{}^K\mathrm{p}_{t-s}(y-x)-{}^K\mathrm{p}_{r-s}(y-x)\right|\Big)^2\,{}^K\!\varrho_s(y)\,{}^K\!X_s(y).\end{aligned} \tag{140}$$

The $\mathrm{d}x$–integral can be handled as follows:

$$\begin{aligned}\int_B \mathrm{d}x\,\left|{}^K\mathrm{p}_{t-s}(y-x)-{}^K\mathrm{p}_{r-s}(y-x)\right| \;&\le\; \int_B\mathrm{d}x\int_{r-s}^{t-s}\mathrm{d}\theta\,\Big|\frac{\partial}{\partial\theta}\,{}^K\mathrm{p}_\theta(y-x)\Big| \\ &\le\; c\int_{r-s}^{t-s}\mathrm{d}\theta\,\frac{1}{\theta}\,{}^K\mathrm{p}_{2\theta}(y/2),\end{aligned} \tag{141}$$

where we used (114) and (128). Inserting into the $\mathrm{d}y$–integral of (140), and using identity (126) as well as Cauchy-Schwarz gives

$$c\int_{|y|\,\ge\,c\,K^\delta}\mathrm{d}y\;|t-r|\int_{r-s}^{t-s}\mathrm{d}\theta\;\frac{1}{\theta^2}\,{}^K\mathrm{p}^2_{2\theta}(y/2). \tag{142}$$

Interchanging the order of integration and substituting $y\mapsto(8\,\theta\,{}^K\!\sigma^2)^{1/2}y$ results into

$$c\,|t-r|\int_{r-s}^{t-s}\mathrm{d}\theta\;\frac{1}{\theta^2}\,\theta^{1/2}\,K^{-1/2+\eta}\int_{|y|\,\ge\,c\,K^{\eta+\delta-1/2}\,\theta^{-1/2}}\mathrm{d}y\;\mathrm{e}^{-y^2}. \tag{143}$$

Using $\tilde{\eta}$ as in (134), we may fix a number

$$\varsigma\;:=\;\varsigma(\eta,\delta)\;>\;1/\tilde{\eta}\vee 5 \tag{144}$$

and exploit (136) in order to get for the latter integral the bound

$$c\,K^{-\varsigma\,\tilde{\eta}/2}\,\theta^{\varsigma/2},\quad \text{for all } K\ge K_0\,,\ \text{say}. \tag{145}$$

Thus, for (143) we get the upper estimate

$$c\,|t-r|\int_{r-s}^{t-s}\mathrm{d}\theta\,\theta^{-5/2}\,K^{-1/2+\eta}\,K^{-\varsigma\,\tilde{\eta}/2}\,\theta^{\varsigma/2}\;\le\;c\,|t-r|^2\,K^{-1/2+\eta-\varsigma\,\tilde{\eta}/2}, \tag{146}$$

for $K\ge K_0$, since $\varsigma-5>0$ and $0\le\theta\le T$. Inserting this into (140), we obtain

$$\mathcal{P}\left|\langle{}^K\!J^{\varepsilon,r,t},\varphi\rangle\right|^2\;\le\;c\,K^{(1-\varsigma\tilde{\eta})/2}\,|t-r|^2\;\le\;c\,|t-r|^2\,,\qquad K\ge K_0\,, \tag{147}$$

since $1-\varsigma\tilde{\eta}<0$. This gives (139).

3° (*Conclusion*). Combining (123) and (139), by decomposition (122) claim (121) follows. This finishes the proof. □

4.6. **Tightness of the $^K X$ under $\eta \geq 29/16$.** The key for this tightness will be a moment estimate concerning the collision measures behind the collision local times $L_{({}^K\varrho, {}^K X)}$ entering into the square function (69). Recall that $^K\varrho$ and $^K X$ start with Lebesgue measures as in (2).

Lemma 19 (2^{nd} moment of collision measure). *Fix $\lambda > 0$ and $T > 0$. Under $\eta \geq 29/16$,*

$$\sup_{0 \leq t \leq T,\ K \geq 1} \mathcal{P}\left| K^{1-\eta} \int_{\mathsf{R}} \mathrm{d}x \ {}^K\varrho_t(x)\, {}^K X_t(x)\, \phi_\lambda(x) \right|^2 \ < \ \infty. \tag{148}$$

Proof. Fix λ, T, η, K as in the lemma.

1° (*A second moment bound for $^K\varrho$*). Recalling (127), from covariance formula (26),

$$\mathcal{P}\, {}^K\varrho_t^2(x) \ \leq \ c + c\, K^{1-\eta} \int_0^t \mathrm{d}s \ {}^K\mathrm{p}_s^{\sigma_\mathrm{c}}(0) \tag{149}$$

with $^K\mathrm{p}^{\sigma_\mathrm{c}} = {}^K\mathrm{p}$ introduced before the convolution form (65) of our stochastic equations. But

$$\int_0^t \mathrm{d}s \ {}^K\mathrm{p}_s^{\sigma_\mathrm{c}}(0) \ \leq \ c\, K^{-1/2+\eta}\, t^{1/2}, \tag{150}$$

since the heat kernel $^K\mathrm{p}^{\sigma_\mathrm{c}}$ has diffusion constant

$$^K\sigma_\mathrm{c} \ := \ K^{1/2-\eta} \sigma_\mathrm{c}\,. \tag{151}$$

Consequently, there is a constant $c_{(152)} = c_{(152)}(T)$ such that

$$\mathcal{P}\, {}^K\varrho_t^2(x) \ \leq \ c_{(152)}\, K^{1/2}, \qquad 0 \leq t \leq T, \quad x \in \mathsf{R}. \tag{152}$$

2° (*A third moment estimate for $^K\varrho$*). The second moment estimate (152) implies

$$\mathcal{P}\, {}^K\varrho_t^3(x) \ \leq \ c_{(153)}\, K^{9/8}, \qquad 0 \leq t \leq T, \quad x \in \mathsf{R}, \tag{153}$$

with a constant $c_{(153)} = c_{(153)}(T)$. In fact, from the convolution form (65),

$$^K\varrho_t(x) \ = \ i_\mathrm{c} \ + \int_{[0,t)\times \mathsf{R}} \mathrm{d}W_s(y) \ {}^K\mathrm{p}_{t-s}^{\sigma_\mathrm{c}}(y-x) \sqrt{K^{1-\eta}\, \gamma_\mathrm{c}\, {}^K\varrho_s(y)}. \tag{154}$$

Using the martingale

$$t \mapsto \int_{[0,t)\times \mathsf{R}} \mathrm{d}W_s(y) \ {}^K\mathrm{p}_{\tau-s}^{\sigma_\mathrm{c}}(y-x) \sqrt{K^{1-\eta}\, \gamma_\mathrm{c}\, {}^K\varrho_s(y)}, \qquad 0 \leq t \leq \tau, \tag{155}$$

from Burkholder-Davis-Gundy's inequality,

$$\begin{aligned} \mathcal{P}\, {}^K\varrho_t^3(x) \ &\leq \ c \ + \ c\, \mathcal{P}\Big(\int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y \ {}^K\mathrm{p}_{t-s}^2(y-x)\, K^{1-\eta}\, {}^K\varrho_s(y) \Big)^{3/2} \\ &\leq \ c \ + \ c\, K^{3(1-\eta)/2} \bigg(\mathcal{P}\Big[\int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y \ {}^K\mathrm{p}_{t-s}^2(y-x)\, {}^K\varrho_s(y) \Big]^2 \bigg)^{3/4}. \end{aligned} \tag{156}$$

If we write twice the double integral using different integration variables, and interchange all the integrations with the expectation, we can use

$$\mathcal{P}\, {}^K\varrho_s(y)\, {}^K\varrho_{s'}(y') \ \leq \ c_{(152)}\, K^{1/2}, \qquad s, s' \in [0,T] \tag{157}$$

by (152). Therefore, exploiting again (150),

$$\begin{aligned}\mathcal{P}\,{}^K\!\varrho_t^3(x) \;&\le\; c \,+\, c\,K^{3(1-\eta)/2}\,c_{(152)}^{3/4}\,K^{3/8}\Big(\Big(\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;{}^K\mathrm{p}_s^2(y-x)\Big)^2\Big)^{3/4}\\ &\le\; c \,+\, c\,K^{3(1-\eta)/2}\,K^{3/8}\Big(\int_0^t \mathrm{d}s\;{}^K\mathrm{p}_s(0)\Big)^{3/2}\\ &\le\; c \,+\, c\,K^{3(1-\eta)/2}\,K^{3/8}\,K^{-3/4+3\eta/2} \;\le\; c\,K^{9/8}, \qquad (158)\end{aligned}$$

and we arrived at (153).

3° (*A second moment for collision*). Next we will verify

$$\mathcal{P}\big[{}^K\!\varrho_t(x)\,{}^K\!X_t(x)\big]^2 \;\le\; c_{(159)}\,K^{13/8}, \qquad 0\le t\le T, \quad x\in\mathsf{R}, \tag{159}$$

for a constant $c_{(159)}=c_{(159)}(T)$. Indeed, the left hand side can be written as

$$\mathcal{P}\,{}^K\!\varrho_t^2(x)\,\mathcal{P}\big\{{}^K\!X_t^2(x)\,\big|\,{}^K\!\varrho\big\}. \tag{160}$$

But

$$\mathcal{P}\big\{{}^K\!X_t^2(x)\,\big|\,{}^K\!\varrho\big\} \;\le\; c+cK^{1-\eta}\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;{}^K\!\varrho_s(y)\,{}^K\mathrm{p}_{t-s}^2(y-x), \tag{161}$$

by covariance formula (27) and (127). Therefore, the left hand side of (159) is bounded by

$$\mathcal{P}\,{}^K\!\varrho_t^2(x)\,c\,\Big(1+K^{1-\eta}\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\;{}^K\!\varrho_s(y)\,{}^K\mathrm{p}_{t-s}^2(y-x)\Big). \tag{162}$$

In view of (152), (153), and again (150), we obtain

$$\mathcal{P}\big[{}^K\!\varrho_t(x)\,{}^K\!X_t(x)\big]^2 \;\le\; c\,K^{1/2} \,+\, c\,K^{1-\eta}\,K^{9/8}\,K^{-1/2+\eta} \;\le\; c\,K^{13/8}, \tag{163}$$

that is, (159).

4° (*Conclusion*). By Jensen's inequality, there is a constant $c_{(164)}=c_{(164)}(\lambda)$ such that

$$\begin{aligned}&\mathcal{P}\Big|K^{1-\eta}\int_{\mathsf{R}}\mathrm{d}x\;{}^K\!\varrho_t(x)\,{}^K\!X_t(x)\,\phi_\lambda(x)\Big|^2\\ &\quad\le\; c_{(164)}\,K^{2(1-\eta)}\int_{\mathsf{R}}\mathrm{d}x\;\phi_\lambda(x)\,\mathcal{P}\big[{}^K\!\varrho_t(x)\,{}^K\!X_t(x)\big]^2. \qquad (164)\end{aligned}$$

Insert (159) and use $2(1-\eta)+13/8\le 0$ by our assumption on η in order to finish the proof. □

As a consequence, we obtain the following result.

Lemma 20 (**Tightness of the** ${}^K\!X$ **under** $\eta\ge 29/16$). *If* $\eta\ge 1$, *the processes* $\{{}^K\!X : K\ge 1\}$ *are tight in* $\mathcal{C}(\mathsf{R}_+,\mathcal{M}_{\text{tem}})$.

Proof. For $\varphi \in \mathcal{C}_\lambda^{(2)}$, $\lambda > 0$, and $0 \le t' \le t \le T$, for a constant $c_{(165)} = c_{(165)}(\varphi)$,

$$\begin{aligned}\mathcal{P}\left\langle L_{({}^K\!\varrho,{}^K\!X)}(t) - L_{({}^K\!\varrho,{}^K\!X)}(t'),\varphi\right\rangle^2 \\ = \mathcal{P}\Big(\int_{t'}^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}x \; {}^K\!\varrho_s(x)\, {}^K\!X_s(x)\, |\varphi|(x)\Big)^2 \\ \le c_{(165)}\, (t-t') \int_{t'}^t \mathrm{d}s\; \mathcal{P}\Big|\int_{\mathsf{R}} \mathrm{d}x\; {}^K\!\varrho_t(x)\, {}^K\!X_t(x)\, \phi_\lambda(x)\Big|^2.\end{aligned} \tag{165}$$

Therefore, by Lemma 19, for a constant $c_{(166)} = c_{(166)}(\varphi)$,

$$\mathcal{P}\left\langle L_{({}^K\!\varrho,{}^K\!X)}(t) - L_{({}^K\!\varrho,{}^K\!X)}(t'),\varphi\right\rangle^2 \le c_{(166)}\,(t-t')^2. \tag{166}$$

Hence, from the martingale (68) with square function (69), and Burkholder's inequality,

$$\begin{aligned}\mathcal{P}\left|M_t^{\mathrm{r},K}(\varphi) - M_{t'}^{\mathrm{r},K}(\varphi)\right|^4 \\ \le c\,\mathcal{P}\left\langle L_{({}^K\!\varrho,{}^K\!X)}(t) - L_{({}^K\!\varrho,{}^K\!X)}(t'),\varphi^2\right\rangle^2 \le c\,(t-t')^2.\end{aligned} \tag{167}$$

Thus, as we concluded in the proof of Lemma 12, the family of processes $M^{\mathrm{r},K}(\varphi)$ is tight, and the same holds for the ${}^K\!X$. This finishes the proof. □

4.7. **Completion of the proof of Theorem 6.** For all $\eta \ge 0$, the convergence of finite-dimensional distributions of the ${}^K\!X$ was provided with Lemma 16. Since the fdd limit ${}^\infty\!X$ is deterministic, in order to complete the proof it suffices to consider the ${}^K\!\varrho$ and ${}^K\!X$ separately.

Under $\eta \ge 1$, by Lemma 12 the processes ${}^K\!\varrho$, $K \ge 1$, are tight in $\mathcal{C}(\mathsf{R}_+, \mathcal{M}_{\mathrm{tem}})$. Let ${}^\infty\!\varrho$ denote any of its limit points. In Subsection 4.1, we identified already the limit ${}^\infty\!\varrho$ under $\eta > 1$. For $\eta = 1$, the convergence to and the identification of the limit ${}^\infty\!\varrho$ of the ${}^K\!\varrho$ was provided in [DF88] (with a slightly different reference function and using a Skorohod space, but note that all of our processes are continuous). The extinction of ${}^K\!\varrho$ under $\eta < 1$ on the path space $\mathcal{C}\big((0,\infty), \mathcal{M}\big)$ was verified in Corollary 14.

It suffices to deal with the ${}^K\!X$. By Lemma 20, the ${}^K\!X$ are tight if $\eta \ge 29/16$. It remains to argue concerning the convergence in law ${}^K\!X \to {}^\infty\!X$ as $K \uparrow \infty$ on function space $\mathcal{C}\big([2\varepsilon, T], \mathcal{M}\big)$ in the case $\eta < 1$, for any choice of $0 < 2\varepsilon < T$. For this purpose, for fixed $\varphi \in \mathcal{C}_{\mathrm{com}}^{(2)}$, we can decompose as in (105):

$$\left\langle {}^K\!X_t, \varphi\right\rangle = \left\langle {}^K\!I_t, \varphi\right\rangle + \left\langle {}^K\!J_t^{\varepsilon,t}, \varphi\right\rangle, \qquad t \ge 2\varepsilon, \tag{168}$$

By Lemma 18(b), the second part forms a tight family of processes in $\mathcal{C}\big([2\varepsilon, T], \mathsf{R}\big)$. Moreover, by (138), for fixed t,

$$\mathcal{P}\left|\langle {}^K\!J_t^{\varepsilon,t}, \varphi\rangle\right|^2 \le c\, K^{(1-\varsigma\tilde{\eta})/2} \underset{K\uparrow\infty}{\longrightarrow} 0. \tag{169}$$

Therefore,

$$\left\langle {}^K\!J_\cdot^{\varepsilon,t}, \varphi\right\rangle \underset{K\uparrow\infty}{\longrightarrow} 0 \quad \text{on function space.} \tag{170}$$

On the other hand, for fixed t, the term at the left hand side of (168) convergence in law to the required deterministic limit $\left\langle {}^\infty\!X_t, \varphi\right\rangle$. Therefore also the first term at the right hand side of (168) converges fdd to that limit. Hence, the P^ϱ–random finite dimensional distributions of the processes $t \mapsto \langle {}^K I_t, \varphi \rangle$ conditioned on ϱ converge in law to the ones of $\delta_{\langle {}^\infty X_\cdot, \varphi\rangle}$. Then by the conditioned tightness in Lemma 18(a), the P^ϱ–random distributions of the processes $\langle {}^K I_\cdot, \varphi \rangle$ converge in law to $\delta_{\langle {}^\infty X_\cdot, \varphi\rangle}$. Integrating out ϱ, the processes $\langle {}^K I_\cdot, \varphi \rangle$ converge in law to $\langle {}^\infty X_\cdot, \varphi \rangle$. Putting this together with (170), by the decomposition (122) the processes $t \mapsto \langle {}^K X_t, \varphi \rangle$ converge in law to $\langle {}^\infty X_\cdot, \varphi \rangle$ on function space $\mathcal{C}([2\varepsilon, T], \mathsf{R})$. Since φ was arbitrary, the proof of Theorem 6 is finished altogether. □

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