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Numerical study of the statistical characteristics of the mixing processes in rivers.

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Abstract

A detailed analysis of statistical characteristics of the vertical mixing process in a horizontally homogeneous and stationary river flow is given. Stochastic models of Langevin type and random displacement models are developed to calculate the statistical characteristics of the vertical mixing. For validation, Langevin type models and random displacement models conventionally applied in this field are compared. All the methods show a good qualitative agreement. However the random diplacement model with constant coefficients is shown to perform with considerable deviations.

1 Introduction

The transport of species and impurities in turbulent flows is a problem which is treated via Lagrangian description. However in many problems, the mean concentration is often approximately evaluated through convection-diffusion equation under the Boussinesq hypothesis. This approximation is applicable if the time and space scales of the concentration field are sufficiently large compared to the relevant correlation scale of the velocity field (Monin and Yaglom, 1971). If, for instance, the concentrataion is calculated at the distances close to the source, or at small times, this method fails, and the Lagrangian stochastic models should be used. It should be noted that in many practically interesting cases the characteristic correlation scales are quite large, which implies that the limitations on applicability of the conventional convection-diffusion equation approach are not well defined. To study these limitations, the Lagrangian stochastic methods are well suited. These methods are used here to explore peculiarities of transport processes in rivers.

We will deal in this paper with transport processes in a river flow with depth much smaller than its width, representing the characteristic spatial scale of the Eulerian velocity. Therefore, when studying the vertical distribution of the concentration, the applicability of the convection-diffusion approximation is justified only after the period when the concentrtaion is slightly varying with distance from the river bed. However the time period when this kind of mixing over the depth happens is an interesting and complicated problem which cannot be solved in the framework of the convection-diffusion approximation. We will show that under some simplifications this problem can be effectively solved by the stochastic Lagrangian approach.

We develop here two different types of stochastic models: (1) Langevin type stochastic models, based on the statistics of the Eulerian velocity field, conventionally measured in practice (e.g., the mean velocity, the mean rate of the kinetic energy dissipation, and the Reynolds stress tensor), and (2) the random displacement models constructed for the solution of the convection-diffusion equation whose coefficients however are obtained in experiments quite approximately.

2 Formulation of the problem

Let us consider a river as an infinite domain $G = \{(x, y, z) : -\infty \le x \le \infty,\ 0 \le y < b,\ 0 \le z \le h\}$, where b is the width, and h is the depth of the river. We are interested in the case when $h << b$, therefore, along the centerline of the river the flow can be treated as horizontally homogeneous velocity field. We assume also that this field is stationary in time.

The particle is released at the point $(0, 0, z_s)$. The functions of interest are the following:
(1) The probability density function (pdf) of the ejection time t_{ej}, the time the particle first hits the layer $h - \Delta \le z \le h$. The height of the release point for this function is usually taken near the roughness height z_0 (from z_0 to $0.1\,h$ while the layer depth Δ varies from $0.05\,h$ to $0.2\,h$). In parallel, one usually evaluates the pdf of the ejection distance x_{ej}, the distance from the release point down the river stream the particle reaches, during the time t_{ej}.
(2) The pdf of the sweep time t_{sw}, the time the particle first hits the layer $z_0 \le z \le \Delta$, where z_0 is the roughness height. Evaluate also the pdf of the sweep length, x_{sw}. The layer depth Δ is the same as in p. (1), but z_s varies between $0.9\,h$ and h, in this case.
(3) The mean residence time, the mean of the random variable $res_t(z_1, z_2)$, the time the particle spends in the layer $z_1 \le z \le z_2$ during the time interval $(0, t)$.
(4) The distance from the source down the river stream at which the particles from a stationary line source are well mixed over the river depth.

3 Stochastic simulation models

We are interested in the case when $h << b$, therefore, along the centerline of the river the flow can be treated as horizontally homogeneous random velocity field. We assume also that this field is stationary in time. The characteristic Reynolds number in rivers is about 10^6, so the turbulence is pressumed to be fully developed, and the statistical Kolmogorov theory of the local isotropic turbulence is applcable at high wave numbers (e.g., see Sukhodolov et al., 1998).

3.1 Langevin type models

Let us suppose that a particle is conservative and follows the Lagrangian trajectories of the flow.

A Langevin type stochastic differential equation governing the motion of such a particle is given by (Thomson, 1987)

$$dX_i = (V_i' + \langle u_i \rangle)dt, \quad dV_i' = a_i\,dt + \sqrt{C_0 \bar{\varepsilon}}\,dB_i(t), \quad i = 1, 2, 3\ . \tag{1}$$

Here $V_i' + \langle u_i \rangle,\ i = 1, 2, 3$ are the components of the Lagrangian velocity vector, $\langle u_i \rangle$ are the components of the Eulerian mean velocity vector, $B_i(t)$ are three standard independent Wiener processes; a_i are functions of (X_3, V_1', V_2', V_3'), C_0 is the universal Kolmogorov's constant ($C_0 \simeq 5$, see Rodean, 1996), $\bar{\varepsilon}$ is the mean rate of the dissipation of the kinetic energy of turbulence which depends in our case on X_3,the distance from the river bed.

Simulating the Lagrangian trajectory $(X_1(t), X_2(t), X_3(t)) = (X(t), Y(t), Z(t))$ as the solution to the equation 1, one calculates the Lagrangian statistical characteristics of interest, like the mean ejection and sweep times and distances, as well as their probability densities.

It is worth to note that in this approach, the mean concentration is defined rigorously (without the Boussinesq assumption) by the formula (e.g., see Monin and Yaglom, 1971; Sawford, 1985)

$$c(x, y, z, t) = \int_{-\infty}^{t} dt_0 \int\int\int dx_0 dy_0 dz_0 Q(x_0, y_0, z_0, t_0)\, p(x, y, z, t; x_0, y_0, z_0, t_0) \qquad (2)$$

where $Q(x_0, y_0, z_0, t_0)$ is the source distribution, and $p(x, y, z, t; x_0, y_0, z_0, t_0)$ is the transition pdf of the Lagrangian trajectory starting at t_0 in the point (x_0, y_0, z_0).

Profiles of the statistical characteristics of the Eulerian velocity field.

To specify the Lagrangian stochastic models of Langevin type, one needs the following statistics: the mean velocity field, the Reynolds stress tensor, the mean rate of the dissipation of the kinetic energy of turbulence. Generally, such a detailed description for the whole cross-section of the river is very difficult. However in the central part of the river the flow can be considered as a horizontally homogeneous flow, hence all the statistics depend only on the distance from the river bed. Such measurement data was used in our calculations.

The mean velocity $\langle u \rangle(z)$ is directed along the x-axis, and it depends only on the distance from the river bed:

$$\langle u \rangle(z) = \frac{u_*}{\kappa} \ln(z/z_0)$$

where u_* is the shear velocity, $\kappa = 0.4$, and z_0 is the roughness height.

Let u', v', w' be the longitudinal, transversal, and vertical components of the velocity fluctuations, and let $\sigma_u^2 = \langle u'^2 \rangle$, $\sigma_v^2 = \langle v'^2 \rangle$, $\sigma_w^2 = \langle w'^2 \rangle$ be the relevant variances.

The variance profiles of the velocities are given by (Sukhadolov et al., 1998):

$$\begin{aligned} \sigma_u^2(z) &= (A_u\, u_*)^2\, exp(-2z/h), \\ \sigma_v^2(z) &= (A_v\, u_*)^2\, exp(-2z/h), \\ \sigma_w^2(z) &= (A_w\, u_*)^2\, exp(-2z/h), \end{aligned} \qquad (3)$$

where $A_u = 2.54, A_v = 1.85, A_w = 1.58$. The covariance $\overline{uw} = \langle u'w' \rangle$ is taken as

$$\overline{uw} = -u_*^2(1 - z/h),$$

and the profile of the mean rate of the dissipation of the kinetic energy is

$$\bar{\varepsilon}(z) = 13.4\, \frac{u_*^3}{h} \sqrt{\frac{h}{z}}\, exp(-3z/h)\ .$$

Thomson's model.

Thomson's model is given in the form (1) with the coefficients specified by (Thomson, 1987; see also Rodean, 1996):

$$a_i(z, u_1, u_2, u_3) = -\frac{C_0\bar{\varepsilon}}{2}\lambda_{ik}\, u_k + \frac{1}{2}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\lambda_{im}}{2}\langle u_k\rangle\frac{\partial \sigma_{jm}}{\partial x_k}\, u_j + \frac{\lambda_{im}}{2}\frac{\partial \sigma_{km}}{\partial x_j}\, u_j u_k, \quad i = 1, 2, 3. \quad (4)$$

Here

$$\sigma_{11} = \sigma_u^2,\ \sigma_{22} = \sigma_v^2,\ \sigma_{33} = \sigma_w^2,\ \sigma_{12} = \sigma_{21} = \sigma_{23} = \sigma_{32} = 0,\ \sigma_{13} = \sigma_{31} = \overline{uw}\ ,$$

and

$$\lambda_{11} = \sigma_w^2/\Delta,\ \lambda_{22} = 1/\sigma_v^2,\ \lambda_{33} = \sigma_u^2/\Delta,\ \lambda_{12} = \lambda_{21} = \lambda_{23} = \lambda_{32} = 0,\ \lambda_{13} = \lambda_{31} = -\overline{uw}/\Delta,$$

where $\Delta = \sigma_u^2\sigma_w^2 - (\overline{uw})^2$.

In (4) and below in (5), (6) and (7) we used the summation convention under repeated indeces.

The model of Kurbanmuradov and Sabelfeld (KS model).

In the stationary case the KS model is specified by (see Kurbanmuradov and Sabelfeld, 2000)

$$a_1(t, z, u, w) = -\frac{C_0\bar{\varepsilon}(1+\rho^2)}{2\sigma_{u/w}^2}(u - \rho w) + \frac{\rho\, C_0\bar{\varepsilon}}{2\sigma_w^2}w + \frac{\rho}{2}\frac{\partial \sigma_w^2}{\partial z}\left(\frac{w^2}{\sigma_w^2} + 1\right)$$
$$+\frac{\partial \rho}{\partial z}w^2 + \frac{1}{\sigma_{u/w}}\frac{\partial \sigma_{u/w}}{\partial z}(u - \rho w)w\ ,$$

$$a_2(t, z, u, v, w) = -\frac{C_0\bar{\varepsilon}}{2\sigma_v^2}v + \frac{1}{2}\frac{\partial \sigma_v^2}{\partial z}\frac{vw}{\sigma_v^2}\ ,$$

$$a_3(t, z, w) = -\frac{C_0\bar{\varepsilon}}{2\sigma_w^2}w + \frac{1}{2}\frac{\partial \sigma_w^2}{\partial z}\left(\frac{w^2}{\sigma_w^2} + 1\right).$$

where

$$\sigma_{u/w} = \frac{\Delta^{1/2}}{\sigma_w}, \quad \rho = \frac{\overline{uw}}{\sigma_w^2}, \quad \Delta = \sigma_u^2\sigma_w^2 - (\overline{uw})^2,$$

3.2 Random displacement models

Among the stochastic models described in section 3.1, there are different stochastic differential equations for the process of particles dispersion. The most often used is the random displacement model (e.g., see Rodean, 1996). In this model, the trajectory $(X_1(t), X_2(t), X_3(t)) = (X(t), Y(t), Z(t))$ is assumed to be a Markov process, governing by a stochastic differntial equation of the form

$$dX_i = a_i(X_1, X_2, X_3, t)dt + b_{i,j}(X_1, X_2, X_3, t)dB_j(t), \quad i = 1, 2, 3 \qquad (5)$$

where we turned to the notation $(X(t), Y(t), Z(t)) = (X_1(t), X_2(t), X_3(t))$.

The coefficients of this equation are related to the convection-diffusion equation for the mean concentration (e.g., see Rutherford, 1994):

$$\frac{\partial c}{\partial t} + u_i(x_1, x_2, x_3, t)\frac{\partial c}{\partial x_i} = \frac{\partial}{\partial x_i} k_{i,j}\frac{\partial c}{\partial x_j} + Q(x,t) \ , \tag{6}$$

through (see Monin and Yaglom, 1971)

$$k_{i,j} = \frac{1}{2} b_{i,l} b_{l,j}, \quad a_i = u_i + \frac{\partial k_{i,j}}{\partial x_j} \ . \tag{7}$$

Here (u_1, u_2, u_3), the mean velocity vector, is assumed to be incompressible ($\frac{\partial u_i}{\partial x_i} = 0$).

Model with parabolic profile of the diffusivity:

$$\langle u\rangle \frac{\partial c}{\partial x} = \frac{\partial}{\partial z}\nu(z)\frac{\partial c}{\partial z}, \quad \langle u\rangle(z) = \frac{u_*}{\kappa}\ln{(z/z_0)}, \quad \nu(z) = \kappa u_* z(1 - z/h) \ ,$$

$$dX = \langle u\rangle(Z)\ dt, \quad dZ = \frac{d\nu}{dz}(Z)\, dt + \sqrt{2\nu(Z)}\ dB(t) \ .$$

The model with constant coefficients:

$$\bar{u}\frac{\partial c}{\partial x} = \frac{\partial}{\partial z}\bar{\nu}\frac{\partial c}{\partial z}, \quad \bar{u} = \frac{u_*}{\kappa}[\ln{(h/z_0)} - 1], \quad \bar{\nu} = 0.067\ u_* h \ ,$$

$$dX = \bar{u}\ dt, \quad dZ = \sqrt{2\bar{\nu}}\ dB(t) \ .$$

With the random displacement model, it is also possible to evaluate different statistical characteristics, in particular, the mean concentrtaion is calculated by (2).

4 Simulation results

In this section we present the results of simulation for the following statistical characteristics: the pdf's of the ejection and sweep lengths and times, the vertical mixing distance, and the vertical distribution of the mean residence time obtained by Thomson's model.

4.1 Detales of numerical schemes

We have chosen a simple semi-explicit numerical scheme for the equation (1):

$$\begin{aligned} X_i(t+\Delta t) &= X_i(t) + (\langle u_i\rangle(X(t)) + V_i')\,\Delta t, \quad i = 1,2,3; \\ V_i'(t+\Delta t) &= V_i'(t) + a_i(X(t+\Delta t), V'(t))\,\Delta t + \sqrt{2C_0\bar{\varepsilon}(X(t+\Delta t))\Delta t}\,\eta_t, \end{aligned}$$

where $\Delta t = 0.02\tau_L$, $\tau_L = 2\sigma_w^2/(C_0\bar{\varepsilon})$, η_t is a standard Gaussian random variable.

In the RDM with parabolic profile we used the scheme:

$$\begin{aligned} X(t+\Delta t) &= X(t) + \langle u\rangle(Z(t))\Delta t, \\ Z(t+\Delta t) &= Z(t) + \sqrt{2\nu(Z(t))\Delta t}\,\eta_t\,. \end{aligned}$$

To ensure that in one time step, the value of $\nu(z)$ is changing slowly (less than 10 %), we have taken $\Delta t = (\nu + 0.01 u_* h)/(32 u_*^2)$. The calculations with this choice of the integration step were numerically stable.

The RDM with constant coefficients was solved by the same scheme, where the mean velocity changed with the depth averaged mean velocity $\bar{u}$, and ν - with $\bar{\nu}$, the depth averaged value of ν. A numerically stable process was reached at $\Delta t = 0.00025 h^2/\bar{\nu}$.

At the upper and lower boundaries ($z = h$ and $z = z_0$, respectively) the perfect reflection condition was used (Thomson and Montgomery, 1994).

Recall that our concentration field is two-dimensional since the line source is chosen transverse to the mean flow direction. The mean concentration $c_* = c(x_*, z_*)$ from stationary line source was approximately evaluated by the formula:

$$c_* = \lim_{|V^*|\to 0} \frac{1}{|V^*|} \langle \int_0^\infty \#_{V^*}(X(t), Z(t))\, dt \rangle, \tag{8}$$

where V^* is a surface element which inludes the point (x_*, z_*), $|V^*|$ is its area, and $\#_{V^*}$ is the indicator function of V^*.

This formula follows from (2) (for more details, see, Kurbanmuradov et al., 2001).

In our numerical simulations we take h and u_* as the characteristic scales of length and velocity. Therefore all the statistical characteristics under interest depend on z_0/h and z_s/h, where z_s is the source height.

4.2 Evaluation of the ejection and sweep statistics

Ejection statistics

Let us present the function $p_{t,ej}(t)$, the pdf of the ejection time. By the definition, this function depends also on z_s/h, Δ/h, and z_0/h. Calculations have shown that the dependence on z_s/h is very weak so we can ignore it. For the ejections calculations we have taken $z_s/h = 0.1$. The mean values and the root-mean-square (rms) value of t_{ej} are given in Table 1.

The curves presented in the Figure 1 (left picture) suggest that the dependence of the pdf $p_{t,ej}(t)$ on the parameter z_0/h is weak. This follows from the fact that the vertical component of the model is governed by an equation whose coefficients do not depend on z_0/h. However when the value of z_0/h approaches $5 \cdot 10^{-2}$, this dependence cannot be ignored, as seen from the Table 1. This effect comes from the reflection at the lower roughness boundary $z = z_0$.

The dependence on the parameter Δ/h is seen in Figure 1, right picture. The curves, as well as the results presented in the Table 1 show that this dependence is quite strong. Thus, $p_{t,ej}(t)$ depends mainly on two parameters, the time t and Δ/h.

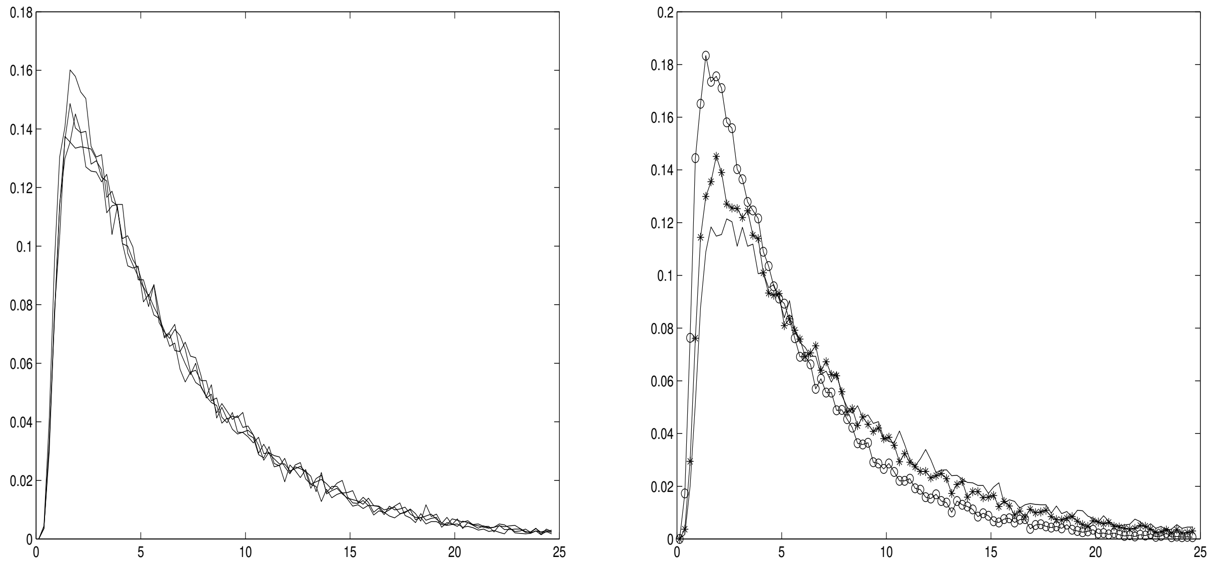

Figure 1: Left picture: the dimensionless probability density function $p_{\tau,ej}(\tau) = \frac{h}{u_*} p_{t,ej}(h\tau/u_*)$ versus the dimensionless time $\tau = u_*\, t/h$ for different values of z_0/h ($= 10^{-5}$, 10^{-3}, 10^{-2}, and $5 \cdot 10^{-2}$) for $\Delta/h = 0.1$. Right picture: the same as on the right picture, but for different values of Δ/h ($= 0.05$ - low curve, 0.1 - the mid curve, and 0.2 - upper curve), for $z_0/h = 10^{-3}$.

Table 1. The normalized mean value $m = u_* \langle t_{ej} \rangle / h$ and σ, the rms of $u_* t_{ej}/h$, for different values of z_0/h and Δ/h.

	$\Delta/h = 0.05$		$\Delta/h = 0.1$		$\Delta/h = 0.2$	
z_0/h	m	σ	m	σ	m	σ
10^{-5}	7.52	6.34	6.64	5.60	5.14	4.27
10^{-4}	7.52	6.35	6.66	5.64	5.13	4.29
10^{-3}	7.47	6.28	6.61	5.55	5.12	4.29
$5 \cdot 10^{-3}$	7.42	6.26	6.57	5.54	5.12	4.27
10^{-2}	7.37	6.24	6.50	5.47	5.06	4.24
$5 \cdot 10^{-2}$	6.92	5.80	6.08	5.09	4.65	3.87

Let us turn to the pdf of the ejection length x_{ej}, $p_{x,ej}(x)$. As calculations show, this function also slightly depends on z_s/h, but strongly depends on the distance x, the parameter z_0/h and on Δ/h. The dependence on z_0/h is clear: this parameter affects the mean velocity which in turn affects the length x_{ej}.

>From calculations we have found a simple approximation:

$$p_{x,ej}(x) \simeq \frac{1}{\bar{v}} p_{t,ej}(x/\bar{v}) \ . \tag{9}$$

Here $\bar{v}$ has a velocity dimension, and is given by

$$\bar{v} = k_{ej} \bar{u},$$

where

$$\bar{u} = \frac{1}{h} \int_{z_0}^{h} \langle u(z) \rangle \, dz = \frac{u_*}{\kappa} \left(\ln \frac{h}{z_0} - 1 \right) \ ,$$

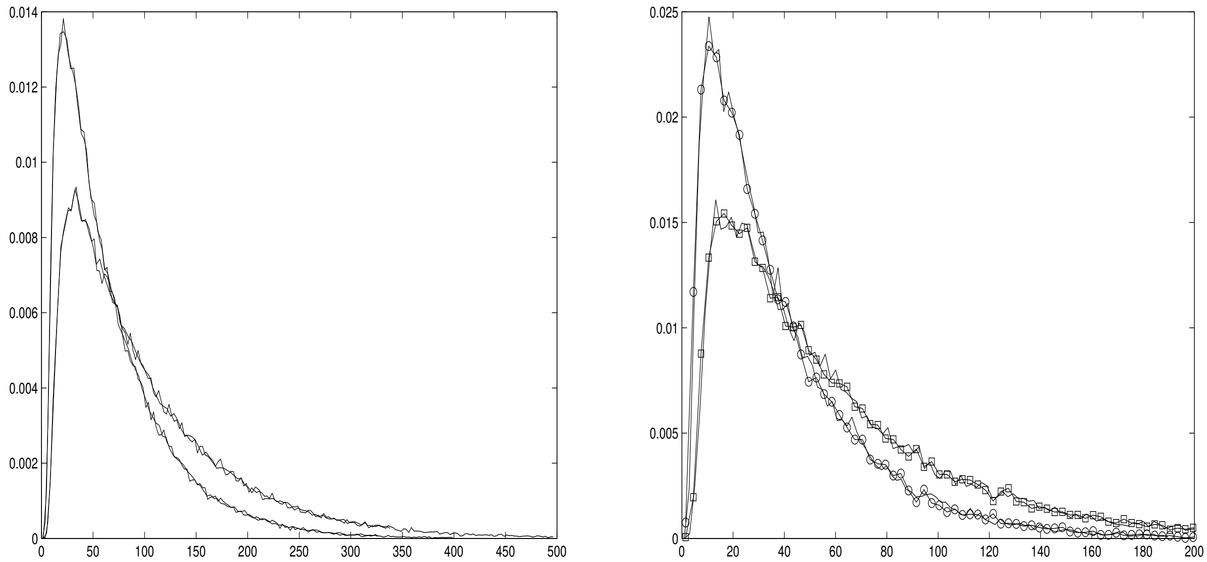

Figure 2: Illustration to the approximation (9): Left picture: the dimensionless probability density function $p_{l,ej}(l) = h\ p_{x,ej}(lh)$ versus the dimensionless distance $l = x/h$, for $\Delta/h = 0.05$ (the lower curve) and $\Delta/h = 0.2$ (the upper curve) for $z_0/h = 10^{-3}$. The approximations to these pdf's given by (9) which practically coincide with the relevant functions are also plotted. The right picture presents the same curves for $z_0/h = 10^{-2}$.

is the depth averaged mean velocity, and k_{ej} is an ajusting parameter depending on $z_0/h, \Delta/h$. This parameter is weakly varying in z_0/h and Δ/h , and we give the values at some points in Table 3.

The relation (9) can be used to evaluate statistical characteristics of the ejection length via the statistics of the ejection time, for instance, $\langle x_{ej}\rangle \simeq \bar{v}\langle t_{ej}\rangle$, or $\langle x_{ej}^2\rangle \simeq \bar{v}^2\langle t_{ej}^2\rangle$.

To illustrate the approximation (9), we show in Figure 2 the two functions which stand in the left and right hand side of (9), for different values of z_0/h and Δ/h.

Sweep statistics

Here we present the function $p_{t,sw}(t)$, the pdf of the sweep time. By the definition, this function depends on $z_s/h, \Delta/h$, but not on z_0/h. Calculations have shown that the dependence on z_s/h is also very weak, so we ignore it. In calculations, we have taken $z_s/h = 0.9$. In Table 2, the dependence on Δ/h is presented.

Table 2. The mean value $m = u_*\langle t_{sw}\rangle/h$ and σ, the rms of u_*t_{sw}/h, for different values of Δ/h.

	$\Delta/h = 0.05$	$\Delta/h = 0.1$	$\Delta/h = 0.2$
m	6.34	5.52	4.38
σ	5.2	4.46	3.52

Calculations show that the pdf of the sweep length x_{sw}, $p_{x,sw}(x)$, also slightly depends on z_s/h, so we ignore it, as usually. The dependence on the parameter z_0/h and Δ/h is considerable. Agian, the dependence on z_0/h is explained by the fact that this parameter affects the mean velocity which in turn affects the length x_{sw}.

As in the case of ejections, we have found a simple approximation:

$$p_{x,sw}(x) \simeq \frac{1}{\bar{v}} p_{t,sw}(x/\bar{v}) \ . \tag{10}$$

Here $\bar{v}$ has a velocity dimension, and is given by $\bar{v} = k_{sw}\bar{u}$,, and k_{sw} is an ajusting parameter depending on $z_0/h, \Delta/h$. This parameter is slightly varying with $z_0/h, \Delta/h$, and we give the values at some points in Table 3.

Table 3. The values of k_{ej} and k_{sw}, for different values of z_0/h and Δ/h.

	$\Delta/h = 0.05$		$\Delta/h = 0.1$		$\Delta/h = 0.2$	
z_0/h	k_{ej}	k_{sw}	k_{ej}	k_{sw}	k_{ej}	k_{sw}
10^{-5}	0.96	1.05	0.95	1.05	0.94	1.06
10^{-4}	0.95	1.06	0.94	1.07	0.92	1.08
10^{-3}	0.93	1.09	0.92	1.10	0.90	1.11
$5 \cdot 10^{-3}$	0.91	1.12	0.89	1.14	0.87	1.16
10^{-2}	0.90	1.15	0.88	1.16	0.85	1.19
$5 \cdot 10^{-2}$	0.90	1.27	0.88	1.30	0.83	1.34

Table 4. The value of dimensionless times τ_{ej}^{η} and τ_{sw}^{η}, for different values of Δ/h and η.

	$\Delta/h = 0.05$		$\Delta/h = 0.1$		$\Delta/h = 0.2$	
η	τ_{ej}^{η}	τ_{sw}^{η}	τ_{ej}^{η}	τ_{sw}^{η}	τ_{ej}^{η}	τ_{sw}^{η}
0.1	1.7	1.5	1.5	1.4	1.1	1.
0.25	2.9	2.6	2.6	2.2	2.	1.7
0.5	5.4	4.7	4.8	4.1	3.7	3.3
0.75	9.8	8.4	8.6	7.3	6.7	5.6
0.9	15.6	13.	13.6	11.4	10.6	9.1

It should be noted that the random variables t_{ej} and t_{sw} are only roughly characterized by their means and variances. For more detailed qualitative characterisation, the distributions can be used. We show these in Table 4, for different values of Δ/h. Here τ_{ej}^{η} is the dimensionless time such that the probability that the particle reaches the upper layer $h - \Delta < z < h$ to within the time interval $(0, h\ \tau_{ej}^{\eta}/u_*)$ is equal to the value η: $Prob(t_{ej} < h\ \tau_{ej}^{\eta}/u_*) = \eta$. The dimensionless time τ_{sw}^{η} is defined analogously.

To construct similar distributions for the ejection and sweep distances, the following relations can be used:

$$Prob(x_{ej} < h\ \bar{v}\, \tau_{ej}^{\eta}/u_*) = \eta, \quad Prob(x_{sw} < h\ \bar{v}\, \tau_{sw}^{\eta}/u_*) = \eta \ .$$

4.3 The vertical mixing distance

The vertical mixing distance L_z is defined as the distance from a linear stationary source situated transverse to the mean flow, at the given height z_s, down the river flow at which the vertical distribution of the concentration is sufficiently uniform.

In the literature (e.g., see Rutherford, 1994) one can find that "sufficiently uniform" may imply that the ratio of the minimal value of the mean concentration to the maximal one lies in the interval $0.8 - 0.98$. In our calculations, we take this value equal to 0.9.

We recall that in our considerations, the river width is assumed to be much larger than the depth. In this case, the vertical mixing happens much faster than the particles released at the central part of the river reach the river banks. Therefore, we may consider the concentration $c(x, z)$ in the central part of the river as a function not depending on the transverse coordinate y.

In Table 5 we present the values of the vertical mixing distance L_z normalized by $h\bar{u}/u_*$.

With this scaling, it is natural to expect that the dependence on the parameter z_0/h will be weak. Indeed, considering the time t_{wm} at which the vertical distribution of a particle is getting uniform, it is plausible that $L_z \simeq \bar{u}\, t_{wm}$. This implies that the ratio $L_z/\bar{u}$ is weakly dependent on z_0/h, since t_{wm} is slightly dependent on z_0/h. The last fact is related with the governing equation in which the coefficients responsible for the vertical motion do not depend on z_0/h. Weak dependence may appear only due to the reflection at the roughness boundary $z = z_0$. This is confirmed by the results given in Table 5.

Table 5. The ratio $L_z u_*/(h\bar{u})$ for different value of z_0/h and z_s/h.

z_0/h	$z_s/h = 0.1$	$z_s/h = 0.25$	$z_s/h = 0.5$	$z_s/h = 0.75$	$z_s/h = 0.9$
10^{-5}	4.37	4.18	2.28	3.35	3.82
10^{-4}	4.38	4.29	2.26	3.41	3.8
10^{-3}	4.33	4.13	2.4	3.35	3.8
$5 \cdot 10^{-3}$	4.23	3.95	2.5	3.4	3.82
10^{-2}	3.8	3.94	2.74	3.35	3.7
$5 \cdot 10^{-2}$	3.7	3.9	3.05	3.03	3.45

In Figure 3, $q(x) = c_{min}/c_{max}$, the ratio of the minimum to maximum concentrtaion values at a fixed distance x is shown as a function of the dimensionless distance $l = xu_*/(h\bar{u})$. The curves are shown for different values of z_0/h (left picture), and for different values of z_s/h (right picture). It is seen from the left picture, that for the values of q around 0.9 all the curves converge which confirms our observation that $L_z/\bar{u}$ slowly depends on z_0/h. Note that for smaller values of q the curves are considerably different. For instance, for $q = 0.8$, the value of $L_z/\bar{u}$ at $z_0/h = 5 \cdot 10^{-2}$ is almost 2 times larger than that at $z_0/h = 10^{-5}$.

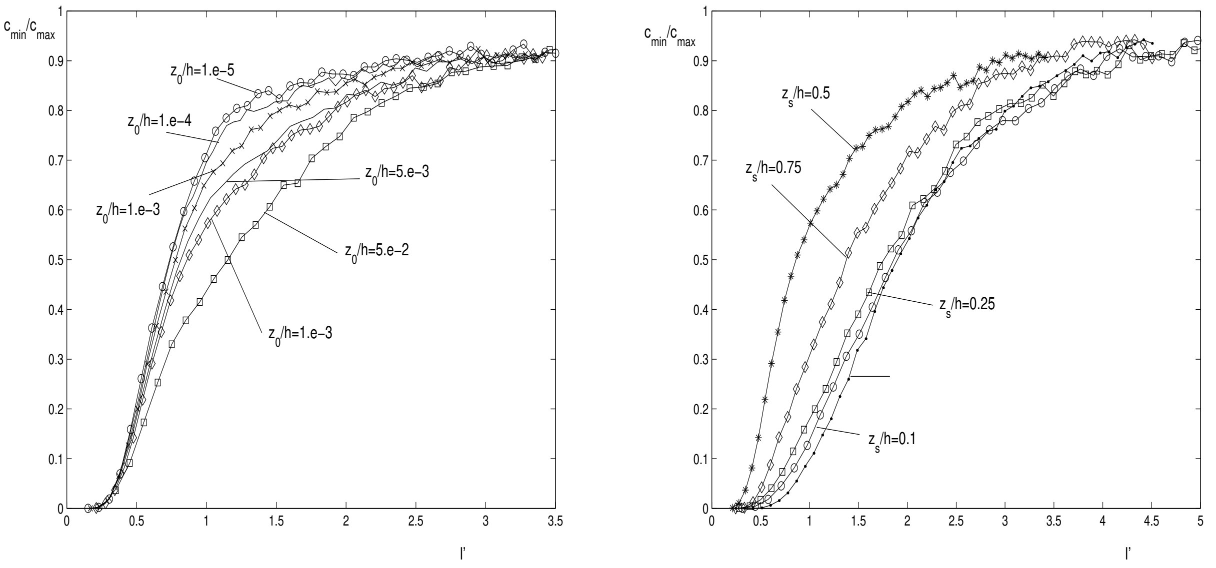

Figure 3: The ratio c_{min}/c_{max} versus the dimensionless distance $l' = xu_*/(h\bar{u})$, for different values of z_0/h at $z_s/h = 0.5$ (left picture), and for different values of z_s/h at $z_0/h = 0.01$ (right picture).

4.4 Mean residence time

The mean time the particle spends in a certain layer of the river is important in different applied problems. For instance, the growth of organic organisms like plankton and nekton is much influenced by the solar radiation absorbed by the layer where these organisms live.

Let us indroduce the vertical distribution density function of the mean residence time:

$$p_{res}(z;t) = \frac{1}{t} \lim_{\Delta z \to 0} \frac{1}{\Delta z} \langle res_t(z, z + \Delta z) \rangle \ .$$

This function is useful when evaluating the mean time a fluid particle spends in a layer during the time interval $(0, t)$:

$$\langle res_t(z_1, z_2) \rangle = t \int_{z_1}^{z_2} p_{res}(z;t) dz.$$

The calculations have shown that p_{res} slowly depends on z_0/h, and in the interval $z_0/h < 10^{-2}$ the dependence is neglegible, see Table 6, where we use the notation $p_{res,min}/p_{res,max}$ for the ratio of the minimal value of p_{res} to the maximal value over the height.

Obviously, the dependence on z_s/h is strong only for short times, while with the time increase, the particles "forget" their starting position, and we come to the uniform distribution over the depth, see Table 7. Of course, the time needed to reach the uniformity depends on the starting position z_s, see the curves in Figure 4. For instance, the particles started at the centerline, $z_s/h = 0.5$, were mixed almost 2 times faster than the particles released at $z_s/h = 0.1$.

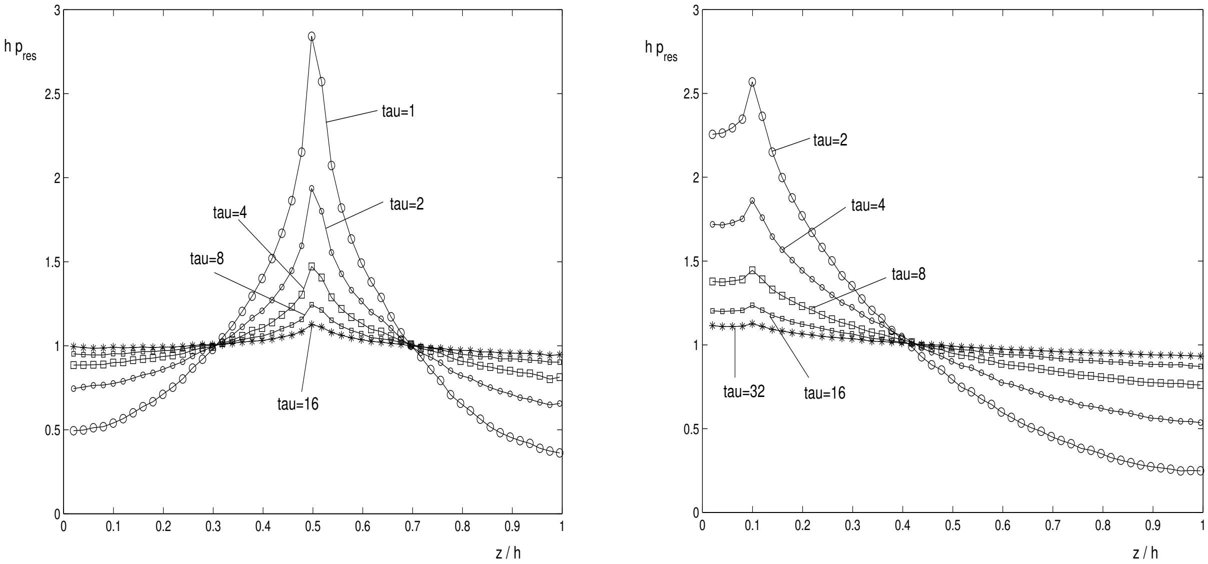

Figure 4: The dimensionless pdf hp_{res} versus the dimensionless height z/h for different values of the dimensionless time $\tau = tu_*/h$ at $z_s/h = 0.5$ (left picture) and at $z_s/h = 0.1$ (right picture). Both pictures are obtained at $z_0/h = 5 \cdot 10^{-3}$.

Table 6. The value of the ratio $p_{res,min}/p_{res,max}$, for different roughness heights and times, at $z_s/h = 0.5$.

z_0/h	$t = 4h/u_*$	$t = 8h/u_*$	$t = 16h/u_*$	$t = 32h/u_*$	$t = 64h/u_*$
10^{-5}	0.58	0.75	0.85	0.91	0.94
$5 \cdot 10^{-3}$	0.55	0.74	0.85	0.91	0.94
10^{-2}	0.54	0.73	0.83	0.9	0.94
$5 \cdot 10^{-2}$	0.52	0.68	0.81	0.9	0.94

Table 7. The value of the ratio $p_{res,min}/p_{res,max}$, for different initial heights and times, at $z_0/h = 5 \cdot 10^{-3}$.

z_s/h	$t = 4h/u_*$	$t = 8h/u_*$	$t = 16h/u_*$	$t = 32h/u_*$	$t = 64h/u_*$
0.1	0.28	0.52	0.69	0.82	0.89
0.25	0.4	0.6	0.76	0.86	0.91
0.5	0.55	0.74	0.85	0.91	0.94
0.75	0.38	0.61	0.79	0.9	0.95
0.9	0.29	0.53	0.72	0.85	0.92

4.5 Comparison against other stochastic models

In this section we present numerical results obtained by the following methods: (1) Thomson's model; (2) KS model; (3) random displacement model with parabolic profile of the diffusivity coefficient; (4) random displacement model with constant coefficients.

In Figure 5, the dimensionless pdf's for ejection time, $p_{\tau,ej}(\tau) = \frac{h}{u_*} p_{t,ej}(h\tau/u_*)$ (left picture) and ejection distance, $p_{l,ej}(l) = hp_{x,ej}(hl)$ (right picture) calculated by the methods (1-4) are presented. In Figure 6 the same pdf's for the sweep time and sweep distance are shown.

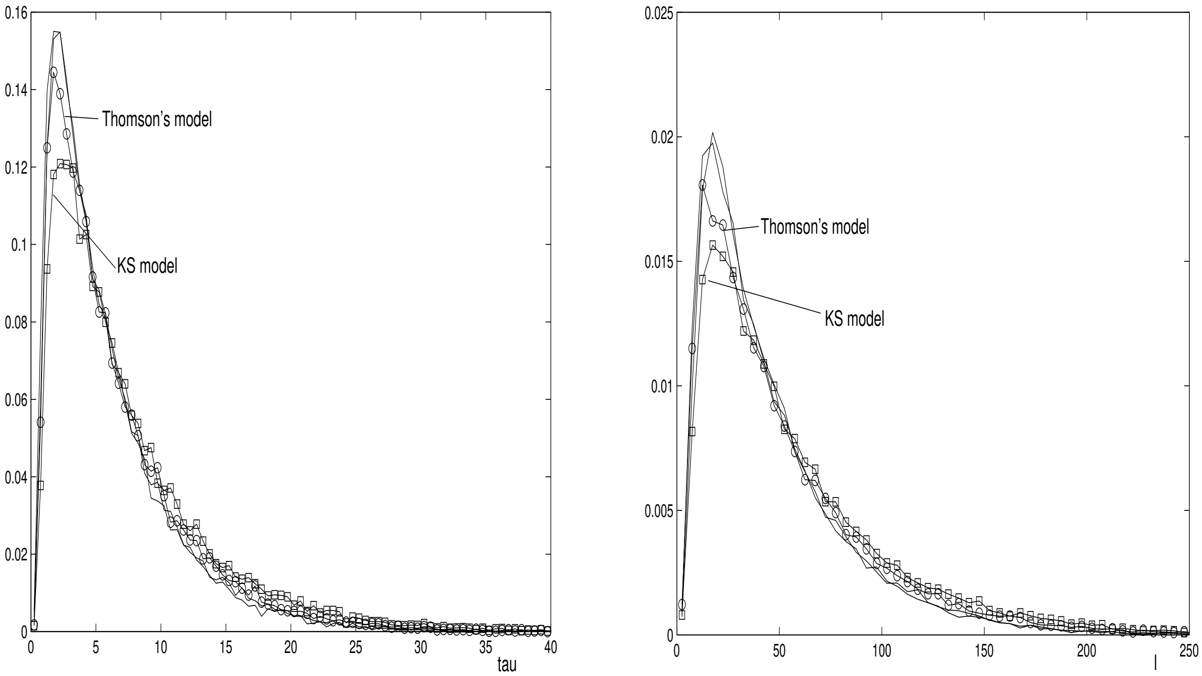

Figure 5: The dimensionless pdf's of the ejection time (left picture) and the ejection distance (right picture) obtained by different models for $z_0/h = 10^{-2}$, $\Delta/h = 0.1$. The two upper curves which are not labelled are obtained by the random displacement models.

Table 8. Comparison of statistical characteristics of ejection and sweep for different stochastical models ($\Delta/h = 0.1, z_0/h = 0.01$).

Characte-ristic	Thomson's model	KS model	RDM with parab. prof. dif.	RDM with const. coeff.
$\langle\tau_{ej}\rangle$	6.5	7.4	5.8	6.
$\langle\tau_{sw}\rangle$	5.5	6.4	5.6	6.1
$\langle l_{ej}\rangle$	51.7	57.6	45.8	47.
$\langle l_{sw}\rangle$	58.	68.4	58.	63.6
$\sigma_{\tau,ej}$	5.5	6.2	4.8	4.9
$\sigma_{\tau,sw}$	4.5	5.3	4.6	5.
$\sigma_{l,ej}$	44.	48.	37.2	38.
$\sigma_{l,sw}$	47.	56.	48.6	53.

Table 9. Comparison of the value $u_* L_z/(h\bar{u})$ for different stochastic models at $z_0/h = 0.005$.

z_s/h	Thomson's model	KS model	RDM with parab. prof. dif.	RDM with const. coeff.
0.1	4.2	4.8	4.8	5.6
0.25	4.	5.2	4.	5.2
0.5	2.5	3.6	2.9	1.56
0.75	3.4	3.8	3.9	5.2
0.9	3.8	4.6	4.8	5.6

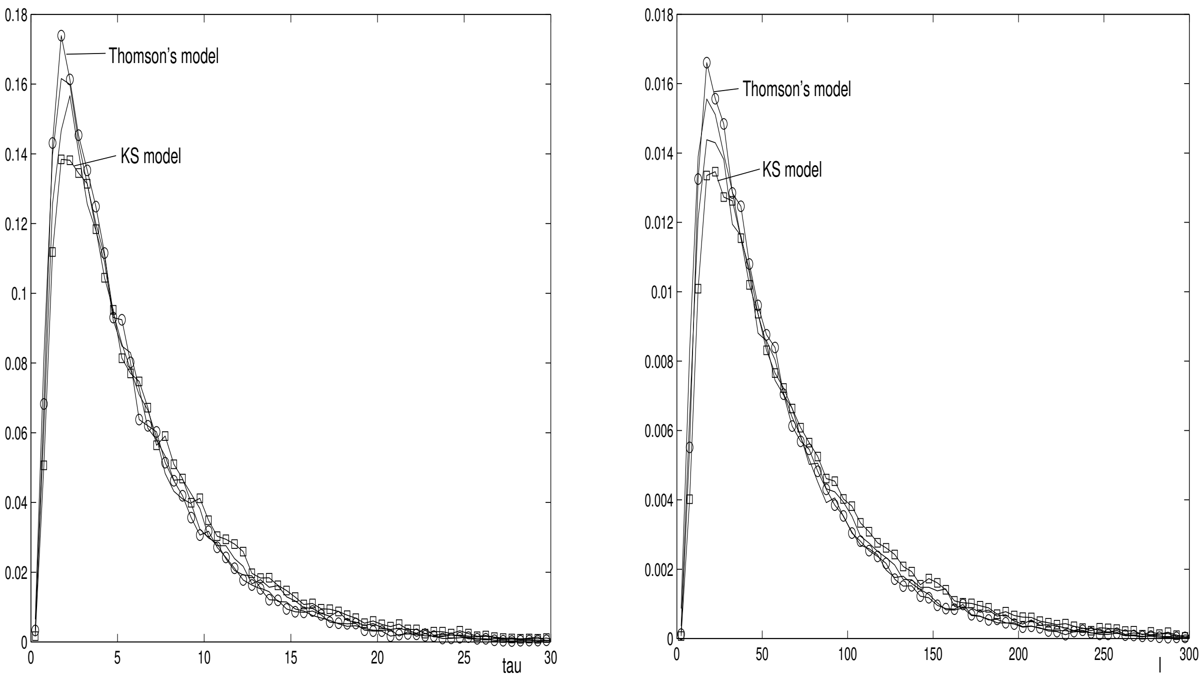

Figure 6: The same as in Figure 5, but for the sweep statistics

Generally, the results obtained by all the methods show a good agreement. The values are slightly different only around the maximum value. The detailed comparison can be made from the Table 8 where the mean values and the rms' of the following quantities are given:

$$\tau_{ej} = u_* t_{ej}/h, \quad l_{ej} = x_{ej}/h, \quad \tau_{sw} = u_* t_{sw}/h, \quad l_{sw} = x_{sw}/h.$$

Finally, the models were compared by the calculation of the vertical mixing distance (see Table 9). Here Thomson's model, KS model and the RDM with parabolic profile of the diffusivity agree good enough while the RDM with the constant coefficients shows considerable deviations.

5 Conclusions

A detailed numerical analysis of statistical characteristics of the vertical mixing process in a horizontally homogeneous and stationary river flow is given on the basis of stochastic simulation models. For validation, we compared Langevin type models with the random dicplacement models conventionally applied in this field. All the methods show a qualitatively agreement. Quantitatively, the random diplacement model with constant coefficients produced some bias.

The results will be used for the development of predictive methods of vertical mixing in river flows. Further detailed field experiments are necessary for direct verification of models. The experimental design should provide possibilities for measurements of fluctuating flows of mass sinchronously with fluctuations of velocity field. Furthemore, the results form a basis for the development of an optimal field measurements strategy.

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