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Random Walk on Spheres methods for iterative solution of elasticity problems

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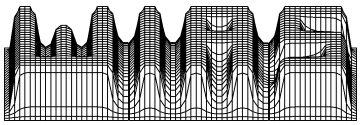

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Abstract

Random Walk on Spheres method for solving some 2D and 3D boundary value problems of elasticity theory are developed. The boundary value problems studied include the elastic thin plate problems with simply supported boundary, rigid fixing of the boundary, and general 2D and 3D problems for the Lamé equation. Unbiased estimators for some special classes of domains based on the generalized Mean Value Theorem which relates the solution at an arbitrary point inside the sphere with the integral of the solution over the sphere. We study a variance reduction technique based on the explicit simulation of the first passage of a sphere for the Wiener process starting at an arbitary point inside this sphere. Along with the conventional random walk methods we apply another type of iteration method, the Schwarz iterative procedure whose convergence for the Lamé equation was proved in 1936 by S.L. Sobolev. We construct also different types of iterative procedures which combine the probabilistic and conventional deterministic methods of solutions.

1 Introduction

It is well known that probabilistic representations of solutions to classical boundary value problems of parabolic and elliptic types in the form of expectations over diffusion stochastic processes can be used for a numerical solution by the Monte Carlo simulation. For the numerical solution of the relevant stochastic differential equation governing the diffusion process, one needs usually a simple finite-difference scheme, e.g., the Euler scheme, inside the domain, but considerable difficulties arise when approximating the random process near the boundary: one should take care that in each step, the process is inside the domain. This implies a rapid diminishing of the integration step when approaching the boundary, which in turn rapidly increases the computational cost.

There exists however another approach which appeared in 1956, the random Walk on Spheres Process (WSP), see [7]. The idea is quite simple. The probabilistic representation of the solution to the Laplace equation

$$\Delta u(x) = 0,\, x \in G, \quad u(y) = \varphi, \quad y \in \Gamma = \partial G,$$

has the form

$$u(x) = E_x\, \varphi(y_\Gamma),$$

where E_x denotes the expectation over all Wiener processes started at the point x and having the first passage on the boundary Γ at the exit point $y_\Gamma \in \Gamma$. Since only the random point $y_\Gamma \in \Gamma$ is involved in the probabilistic representation, it is natural to construct a process whose distribution of the exit point is close to that of the point y_Γ.

This Random Walk on Spheres process is constructed as follows.

First construct a sphere $S(x, d(x))$ centered at the point x whose radius $d(x)$ is equal to the distance from x to the boundary Γ, hence $S(x, d(x))$ is a maximal sphere which can be inscribed into the domain G provided it is centered at the point x. Then x_1 is chosen at random on the surface $S(x, d(x))$ with the uniform surface distribution (isotropic direction). Next take x_1 as the center of the second sphere $S(x_1, d(x_1))$ which is constructed analogously, and choose x_2 uniformly on $S(x_1, d(x_1))$, etc. Thus we come to a Markov chain $\{x, x_1, x_2, \ldots, \}$ started at x, and the state x_{k+1} has a uniform distribution on the sphere $S(x_k, d(x_k))$. Muller [7] has shown that this Markov chain converges to the boundary, and the distribution of the limit point x_∞ coincides with the distribution of the first passage point of the Wiener process on Γ.

With probability one, the WSP $\{x, x_1, x_2, \ldots, \}$ will not reach the boundary in finite number of steps, so in this form the process is of no use for numerical purposes. However there is an elegant cut-off approach: if instead of Γ we consider an ε-boundary

$$\Gamma_\varepsilon = \{x \in G : \, d(x) < \varepsilon\},$$

then the walk on spheres process $\{x, x_1, x_2, \ldots, x_{N_\varepsilon}\}$ hits the ε-boundary with probability one after N_ε steps. It is quite evident that, assuming the solution is at least continuous in ε-boundary, we can take in the probabilistic representation, as an approximation to $\varphi(y_\Gamma)$, the value of φ at a boundary point closest to the point x_{N_ε}.

Of course, two questions are here of primary interest: what a bias is caused by this cut-off procedure, and how large is N_ε. The answer encourages the further development of the method: the bias is at least of order $O(\varepsilon)$, and $\bar{N}_\varepsilon$, the mean number of steps, behaves like $const \cdot |\ln(\varepsilon)|$. So the cost is surprisingly low, moreover, the constant *const* is very slowly (approximately linear) dependent on the dimension of the problem. First proof of the logarithmic estimation was given in [6], and further extensions to general domains are given in [1], [2], [9].

The rigorous formulations and justifications of the walk on spheres method with applications to different kinds of equations can be found in [9], [2], [10]. It should be noted that the approach used in [9], and in subsequent publications is different: as a starting point not the probabilistic representation is used, but a reformulation of the original boundary value problem in an equivalent integral equation form. The integral formulation is written in the form of spherical mean value relation. If then the standard Monte Carlo Markov chain procedure is applied to solve this integral equation, we come exactly to the random walk on spheres method.

This approach is very convenient for the numerical purposes, and what is essential, it provides a technique for the construction of the random walk methods for broad classes of equations for which there are no probabilistic representations. As an example, we mention the random walk on boundary algorithms suggested first by K. Sabelfeld in [8] and described in the book [11]. In this method, the random walks leave on the boundary; the boundary integral equations of the potential theory are used to derive and justify the convergence of the method, to estimate the bias, the variance and the cost of the method. The random walk on boundary method works for all classical boundary value problems of the potential theory, including the exterior Dirichlet and Neumann problems.

Another example where the integral formulation works while there are no probabilistic representations is a class of systems of elliptic equations, see [10].

In this paper we consider some examples of domains where it is possible to avoid the ε-bias of the walk on spheres algorithm. For such domains we construct random walk on spheres algorithm for the bending of thin elastic plate governed by the biharmonic equation - in section 2. Section 3 deals with the classical Lamé equation. Here we work with two different approaches: one is the standard random walk on spheres based on the spherical mean value relation written for the center of the sphere. Second approach is based on the general mean value relation derived for an arbitrary point, not coincident with the center. We call the relevant random walk as Decentred Random Walk on Spheres (DRWS). Both 2D and 3D cases are considered. We study also some modifications of the standard walk on spheres process. For two overlapping circles we carried out a randomized evaluation of the Schwarz iterative procedure for the Lamé equation.

2 Random Walk on Spheres method for the biharmonic equation. Simply supported boundary.

2.1 Standard vector random walk on spheres estimator

Let us consider the following classical boundary value problem for the biharmonic equation governing the bending of a thin elastic plate G with a simply supported boundary Γ:

$$\Delta\Delta u(x) = 0, \quad x \in G \subset R^2, \quad u(y) = g_0(y), \quad \Delta u(y) = g_1(y), \quad y \in \Gamma. \tag{2.1}$$

The following integral relation for the solution to this boundary value problem can be found in [9] and [10].

Let $S(x, d(x))$ be an arbitrary circle in G centered at the point x, whose radius is $R = d(x)$.

Then the solution $u(x)$ satisfies the spherical mean value relation:

$$u(x) = \frac{1}{2\pi R} \int\limits_{S(x,R)} u(y) dS(y) - \frac{R^2}{4} \cdot \frac{1}{2\pi R} \int\limits_{S(x,R)} \Delta u(y) dS(y) \ , \tag{2.2}$$

$$\Delta u(x) = \frac{1}{2\pi R} \int\limits_{S(x,R)} \Delta u(y) dS(y) \ , \tag{2.3}$$

where $dS(y)$ is the surface element of $S(x, R)$ at the point y. The spherical mean value of a function $v(y)$ can be written also as

$$N_R\, v = \frac{1}{2\pi R} \int\limits_{S(x,R)} v(y) dS(y) = \frac{1}{2\pi} \int\limits_{\Omega} v(x + sR)\, d\Omega(s), \tag{2.4}$$

where Ω is the unit sphere $S(0, 1)$, and $d\Omega$ is its suface element.

The relations (2.2), (2.2) can be rewritten as a system of integral equations for the vector $(u, \Delta u)$ with a generalized kernel, see [9].

The random estimator for evaluation of the solution $u(x)$ has the following simple form:

$$\xi(x) = g_0(y^*_\Gamma) - \frac{g_1(y^*_\Gamma)}{4} \sum_{k=1}^{N_\varepsilon} d^2(x_k).$$

Here $\{x_1 = x, x_2, \ldots, x_{N_\varepsilon}\}$ is the walk on spheres process, y^*_Γ is a point at the boundary closest to the last state of the process absorbed in Γ_ε at the point y_{Γ_ε}; $d(x_k)$ are the radii of the spheres of the random walk on spheres process.

This estimator has a bias $\delta(\varepsilon)$, because of the ε cut-off procedure, hence

$$u(x) = E_x \xi(x) + \delta(\varepsilon)$$

where the form of $\delta(\varepsilon)$ depends on the continuity modulus of the functions $u(x)$ and $\Delta u(x)$ in Γ_ε (e.g., see [9], [2]).

What is important, the variance of the estimator $\xi(x)$ is uniformly bounded, as $\varepsilon \to 0$. This ensures that the accuracy can be increased by taking smaller values of ε, and the cost of the method behaves like $\sim |\ln(\varepsilon)|/\varepsilon^2$.

2.2 Two overlapping discs

In the walk on spheres method described the points x_k are the centers of the spheres, because we have used the spherical mean value relation which relates $u(x), \Delta u(x)$ with the integrals of these functions over the sphere $S(x, d(x))$.

Let us now take an arbitrary point x inside a disk $K(x_0, R)$ with the boundary $S(x_0, R)$. Then the following generalized mean value relation holds (see [10]; note that there was a mistake in this formula):

Theorem 1. *The regular solution to the biharmonic equation satisfies the following spherical mean value relation*

$$\begin{aligned} u(x) &= \frac{R^2 - r^2}{2\pi R} \int\limits_{S(x_0,R)} \frac{u(y) dS_y}{|x-y|^2} \\ &+ \frac{R^2 - r^2}{4\pi R} \int\limits_{S(x_0,R)} \Big\{ \frac{R \sin\alpha}{|x - x_0|} \mathrm{arctg} \Big\{ \frac{|x - x_0| \sin\alpha}{R - |x - x_0| \cos\alpha} \Big\} \\ &- \frac{1}{2} - \frac{R\cos\alpha}{2|x - x_0|} \ln \frac{|x-y|^2}{R^2} \Big\} \Delta u(y) dS_y \ , \end{aligned} \tag{2.5}$$

$$\Delta u(x) = \frac{R^2 - r^2}{2\pi R} \int\limits_{S(x_0,R)} \frac{\Delta u(y) dS_y}{|x-y|^2} \ . \tag{2.6}$$

Here $r = |x - x_0|$ is the distance from x to the circle's center x_0, and α is the angle between the vectors $x - x_0$ and $y - x_0$.

This relation can be used to construct a random estimator which is defined on a Markov chain $\{x = x_1, x_2, \ldots\}$ whose states are not the centers of the spheres, but are some random points inside these spheres.

Let us consider the simplest case when the domain G consists of two overlapping discs:

$$G = K(x_0^{(1)}, R_1) \cup K(x_0^{(2)}, R_2); \quad K(x_0^{(1)}, R_1) \cap K(x_0^{(2)}, R_2) \neq \emptyset \ , \tag{2.7}$$

and denote by γ_1 the part of the circle $S(x_0^{(1)}, R_1)$ which belongs to the second disc while Γ_1 is the part of the circle $S(x_0^{(1)}, R_1)$ not belonging to the second disc; analogously γ_2 and Γ_2 are defined. So the boundary of the domain G consists of Γ_1 and Γ_2.

Iteration with uniform directions.

Assume we are going to evaluate the solution to (2.1) in this domain, say, at a point $x \in K(x_0^{(1)}, R_1)$ centered at the point $x_0^{(1)} = (x_{01}, x_{02})$. The randomized iterative procedure suggests the following: sample a point x_1 uniformly on the circle $S(x_0^{(1)}, R_1)$. If x_1 is on Γ_1, the process stops, and the relevant score is calculated. If x_1 is inside the disc $K(x_0^{(2)}, R_2)$, then we are doing the same: sample a random point x_2 uniformly on the circle $S(x_0^{(2)}, R_2)$; again, if $x_2 \in \Gamma_2$, the process stops, and we calculate some score, if not, we are in the first disc, and the procedure goes further.

Two main features of such a random walk are: (1) there is no bias in the method since the scores are calculated exactly on the boundaries; (2) the process rapidly stops, since the stop probability is always very large, and in the worst case the probability to stop after 2 steps is not less than 0.25.

Two issues should be studied: first, the convergence of the method, and second, its cost.

But before doing that, let us modify the random walk procedure described.

Distribution of the first passage on $S(x_0^{(1)}, R)$ for a Wiener process starting at an arbitrary point $x \in K(x_0^{(1)}, R)$.

Note that the first integral in (2.5) is exactly the Poisson formula for the Laplace equation. It is not difficult to find out that the function

$$p(y; x) = \frac{R^2 - |x - x_0^{(1)}|^2}{2\pi R} \cdot \frac{1}{|x - y|^2} \tag{2.8}$$

is a probability density function of the variable $y \in S(x_0^{(1)}, R)$, for all $x \in K(x_0^{(1)}, R)$. This immediately follows from the representation of the solution $u = 1$ to the Dirichlet problem for the Laplace equation $\Delta u(x) = 0$, $u(y) = 1$ through the Poisson integral. From the probabilistic representation of the Dirichlet boundary value problem considered the density $p(y; x)$ coincides with the pdf of the first passage on $S(x_0^{(1)}, R)$ of a Wiener process starting at $x \in K(x_0^{(1)}, R)$.

To sample the point y on the circle $S(x_0^{(1)}, R)$ according to the density $p(y; x)$, it is convenient to choose new coordinates with the origin at x. Using simple transformations, in [4] the following simulation algorithm was suggested.

Simulation algorithm.

1. Sample a random direction in the "upper semisphere" $\omega = (\omega_1, \omega_2) = (\cos(\psi), \sin(\psi))$, where ψ is uniformly distributed between 0 and π.

2. Find $y = y_1$ on the circle $S(x_0^{(1)}, R)$ as the intersection point of $S(x_0^{(1)}, R)$ and the ray $x + \omega\,|x - y|$, and find also $y = y_2$ as the intersection point of $S(x_0^{(1)}, R)$ with the ray $x - \omega\,|x - y|$.

3. Let $a_1 = |x - y_1|$, and $a_2 = |x - y_2|$. Then with probability $a_1/(a_1 + a_2)$ take $y = y_2$, and with probability $a_2/(a_1 + a_2)$ take $y = y_1$.

The algorithm has two remarkable properties: (1) the simulation algorithm is the same in arbitrary dimensions, and (2) the closer the point x to the boundary $S(x_0^{(1)}, R)$, the larger is the probability that the sampled point y is the point closest to x. Note that in [2], a slightly different simulation algorithm is presented.

Decentred Random Walk on Spheres (DRWS).

Let us denote by $\{Y_1 = x, Y_2, \ldots, Y_m\}$ the Markov chain constructed for our overlapping discs as described above in the *Iteration with uniform directions* but with the difference that at each step, the point Y_k on the consequent circle is sampled not uniformly, but according to $p(y; Y_{k-1})$, starting from $Y_1 = x$; m is the number of steps until the process stops on Γ_1 or Γ_2. Not that Y_k, for $k > 1$ is on the first circle for k even, and it is on the second circle, if k is odd.

Then the random estimators for $u(x)$ and $\Delta u(x)$, according to the spherical mean value relation, can be written in the form:

$$\xi_1(x) = g_0(Y_m) - g_1(Y_m) \sum_{k=2}^{m} Q(Y_{k-1}, Y_k)$$

and

$$\xi_2(x) = g_0(Y_m)\ .$$

Here

$$Q(Y_{k-1}, Y_k) = \frac{|Y_{k-1} - Y_k|^2}{2} \left[\frac{R_{k-1}\sin(\alpha_k)}{|Y_{k-1} - x_0^{(*)}|}\, arctg \left\{ \frac{|Y_{k-1} - x_0^{(*)}|\sin(\alpha_k)}{R_{k-1} - |Y_{k-1} - x_0^{(*)}|\cos(\alpha_k)} \right\} - \frac{1}{2} \right.$$
$$\left. - \frac{R_{k-1}\cos(\alpha_k)}{2|Y_{k-1} - x_0^{(*)}|} \ln \frac{|Y_{k-1} - Y_k|^2}{R_{k-1}^2} \right],$$

α_k is the angle between the vectors $Y_{k-1} - x_0^{(*)}$ and $Y_k - x_0^{(*)}$. We use here the notation $x_0^{(*)}$ for the centers of the both circles: it is the center of the first disc if k is odd, and for k even it is the center of the second disc.

Theorem 2. *For any overlapping discs, the estimators ξ_1 and ξ_2 are unbiased,*

$$u(x) = E_x \xi_1(x), \quad \Delta u(x) = E_x\, \xi_2(x),$$

and have finite variance.

Proof. Two different approaches can be used to prove this statement. One is just a direct estimation of the expectation and second moments of ξ_1 and ξ_2. Another approach

is to estimate the spectral radius of the system of integral equations generated by the spherical mean value relation. Since g_0 and g_1 are bounded functions, the statement for ξ_2 is evident. As to the estimate ξ_1, the finiteness of the expectation and the variance follows from the convergence of the Neumann series for the integral operator. Indeed, since the integrands in (2.5) are all bounded, we find that the Nemann series is estimated by

$$const \cdot D^2 \sum_{k=1}^{\infty} k\, q^k,$$

where $D = max\{R_1, R_2\}$, $q = max\{q_1, q_2\}$, and q_1, q_2 are defined by

$$q_1 = \sup_{x \in \Gamma_1} \int_{\Gamma_2} p(y;x) dS(y), \;\; q_2 = \sup_{x \in \Gamma_2} \int_{\Gamma_1} p(y;x) dS(y) \; .$$

Since $p(y;x)$ is the probability density function for each x, we conclude that $q_1 < 1$ and $q_2 < 1$ which ensures the convergence. Note that the same series is to be estimated if we evaluate directly the expectation or the variance of the estimator $\xi_1(x)$.

Remark 1.

For simplicity, we presented here the random walk method for two overlapping discs. It is not difficult to find out that this method converges and has a finite variance for any bounded domain which is a unification of, say, m discs each of them having an overlapping at least with one disc. For brevity, we call this type of domains by Km-domins.

This suggests a new biased method which is different from the conventional walk on spheres method with the ε-bias. Indeed, approximate the given domain by a Km-domain, and make an extrapolation of the boundary conditions to the boundary of the domain Km. Of course, this extrapolation will cause a bias, but for a broad classes of domains this bias can be made very small. Our calculations have shown that the modified walk on spheres method presented is very effective.

2.3 Rigid fixing of the boundary

In this section we aim at generalising the method to the case when the plate has a rigid fixing of the boundary, which means that we have to find a solution to the biharmonic equation with the following boundary:

$$u\Big|_{\Gamma} = g_0, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma} = g_1. \qquad (2.9\;)$$

Here n is the exterior normal vector to the boundary Γ.

It should be mentioned that this problem is quite difficult for the numerical solution, including the Monte Carlo methods. The authors know some attempts to construct a stochastic solution to this problem but all the attempts fail.

In this section we suggest a stochastic method which works at least for some particular cases.

The biharmonic function satisfies at an arbitrary point $x \in K(x_0, R)$ the following spherical mean value relation, [10]:

$$u(x) = -\frac{(R^2-|x-x_0|^2)^2}{2R^2 2\pi}\int\limits_{S(x_0,R)}\frac{g_1(y)dS_y}{|x-y|^2} + \frac{(R^2-|x-x_0|^2)^2}{2R^3 2\pi}\int\limits_{S(x_0,R)}\frac{(2R^2-2R|x-x_0|\cos\alpha)g_0(y)dS_y}{|x-y|^4}\ , \qquad (2.10\ )$$

where α is the angle between the vectors $x-x_0$ and $y-x_0$; $x_0=(x_{01},x_{02})$.

To derive a system of integral equations relating the function $u(x)$ and its normal derivative at the point $x=(x_1,x_2)$ we could differentiate the relation (2.10) with respect to x_1 and x_2, and then use the relation at the point $y=(y_1,y_2)$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1}\cdot\frac{y_1-x_{01}}{R} + \frac{\partial u}{\partial x_2}\cdot\frac{y_2-x_{02}}{R} \qquad (2.11\ )$$

where R equals R_1, if the derivative is taken at $x\in\gamma_1$, or R_2 at γ_2. However it is more convenient to derive the system of integral equations for the vector which includes the solution and its derivatives, i.e., the vector $v=(v_1,v_2,v_3)\equiv(u,u_{x_1},u_{x_2})$. We differentiate (2.10) and use the Poisson kernel $p(y;x)$ defined in (2.8) (here we use the notation $r_1=|x-x_0^{(1)}|$:

$$u(x) = \int\limits_{S(x_0,R_1)} p(y;x)\left\{\frac{R_1^2-r_1^2}{|x-y|^2}(1-\frac{r_1}{R_1}\cos(\alpha_1))\,u(y) - \frac{R_1^2-r_1^2}{2R_1}\frac{y_1-x_{01}}{R_1}\frac{\partial u}{\partial x_1}(y) - \frac{R_1^2-r_1^2}{2R_1}\frac{y_2-x_{02}}{R_1}\frac{\partial u}{\partial x_2}(y)\right\}dS(y)$$

$$\frac{\partial u}{\partial x_1} = \int\limits_{S(x_0,R_1)} p(y;x)\left\{\left[\frac{-4(x_1-x_{01})(1-\frac{r_1}{R_1}\cos(\alpha_1))}{|x-y|^2} - \frac{(R_1^2-r_1^2)(y_1-x_{01})}{R_1^2\,|x-y|^2} - \frac{4(R_1^2-r_1^2)(x_1-y_1)}{|x-y|^4}(1-\frac{r_1}{R_1}\cos(\alpha_1))\right]u(y) + \left[\frac{2(x_1-x_{01}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_1-y_1}{R_1}\right]\frac{y_1-x_{01}}{R_1}\frac{\partial u}{\partial x_1}(y) + \left[\frac{2(x_1-x_{01}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_1-y_1}{R_1}\right]\frac{y_2-x_{02}}{R_1}\frac{\partial u}{\partial x_2}(y)\right\}dS(y)$$

$$\frac{\partial u}{\partial x_2} = \int\limits_{S(x_0,R_1)} p(y;x)\left\{\left[\frac{-4(x_2-x_{02})(1-\frac{r_1}{R_1}\cos(\alpha_1)}{|x-y|^2} - \frac{(R_1^2-r_1^2)(y_2-x_{02})}{R_1^2\,|x-y|^2} - \frac{4(R_1^2-r_1^2)(x_2-y_2)}{|x-y|^4}(1-\frac{r_1}{R_1}\cos(\alpha_1))\right]u(y) + \left[\frac{2(x_2-x_{02}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_2-y_2}{R_1}\right]\frac{y_1-x_{01}}{R_1}\frac{\partial u}{\partial x_1}(y) + \left[\frac{2(x_2-x_{02}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_2-y_2}{R_1}\right]\frac{y_2-x_{02}}{R_1}\frac{\partial u}{\partial x_2}(y)\right\}dS(y)$$

which can be conveniently rewritten in a matrix form as follows:

$$v(x) = \int\limits_{S(x_0,R_1)} p(y;x)Q(x,y)v(y)\,dS(y), \qquad (2.12\ )$$

where $Q(x,y)$ is a 3×3-matrix with the entries

$$q_{11}(x,y) = \frac{R_1^2-r_1^2}{|x-y|^2}(1-\frac{r_1}{R_1}\cos(\alpha_1)),\ q_{12} = \frac{R_1^2-r_1^2}{2R_1}\frac{y_1-x_{01}}{R_1}\,, q_{13} = \frac{R_1^2-r_1^2}{2R_1}\frac{y_2-x_{02}}{R_1}\,,$$

$$\begin{aligned} q_{21} = {} & \frac{-4(x_1-x_{01})(1-\frac{r_1}{R_1}\cos(\alpha_1))}{|x-y|^2} - \frac{(R_1^2-r_1^2)(y_1-x_{01})}{R_1^2\,|x-y|^2} \\ & -\frac{4(R_1^2-r_1^2)(x_1-y_1)}{|x-y|^4}(1-\frac{r_1}{R_1}\cos(\alpha_1))\ , \end{aligned}$$

$$q_{22} = \left[\frac{2(x_1-x_{01}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_1-y_1}{R_1}\right]\frac{y_1-x_{01}}{R_1}\ ,$$

$$q_{23} = \left[\frac{2(x_1-x_{01}}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_1-y_1}{R_1}\right]\frac{y_2-x_{02}}{R_1}\ ,$$

$$\begin{aligned} q_{31} = {} & \frac{-4(x_2-x_{02})(1-\frac{r_1}{R_1}\cos(\alpha_1))}{|x-y|^2} - \frac{(R_1^2-r_1^2)(y_2-x_{02})}{R_1^2\,|x-y|^2} \\ & -\frac{4(R_1^2-r_1^2)(x_2-y_2)}{|x-y|^4}(1-\frac{r_1}{R_1}\cos(\alpha_1))\ , \end{aligned}$$

$$q_{32} = \left[\frac{2(x_2-x_{02})}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_2-y_2}{R_1}\right]\frac{y_1-x_{01}}{R_1}\ ,$$

$$q_{33} = \left[\frac{2(x_2-x_{02})}{R_1}+\frac{R_1^2-r_1^2}{|x-y|^2}\frac{x_2-y_2}{R_1}\right]\frac{y_2-x_{02}}{R_1}\ .$$

The same relation is of course true for any point $x\in K(x_0^{(2)},R_2)$:

$$v(x) = \int\limits_{S(x_0^{(2)},R_2)} p(y;x)Q(x,y)v(y)\,dS(y)\ . \qquad (2.13\ )$$

Thus the random estimator $\xi(x) = (\xi_1,\xi_2,\xi_3)$ for the solution $v(x) = (v_1(x),v_2(x),v_3(x))$ is obtained by iterating the relevant integral relations

$$v(x) = E\,\xi(x), \quad \xi(x) = \left\{\prod_{k=2}^{m} Q(Y_{k-1},Y_k)\right\} g(Y_m) \qquad (2.14\ )$$

where m is the random number of steps, and $g(y) = (u(y),u_{x_1}(y),u_{x_2}(y))$ are the boundary values. Here $Y_1 = x, Y_2,\ldots,Y_m$ is the DRWS process.

Numerical calculations show however that the variance of the estimator is very large. The reason is that after the first iteration, the point Y_1 may be sampled very close to the

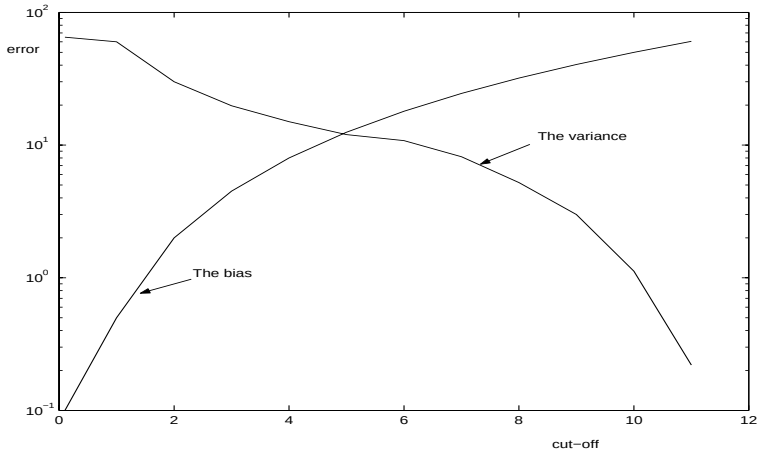

Figure 1: To the choice of the cut-off procedure

boundary of the second disc which leads in turn to the evaluation of the spherical mean at a point which is too close to the boundary. We have made a cut-off of the integral at the point of intersections of the two circles. The calculations show that this drastically influences the estimator: the variance is getting small, while the bias caused by the cut-off can be estimated and controlled. To find a reasonable strategy of the cut-off procedure, let us show schematically the bias and the statistical error as functions of the dimensionless cut-off size.

From the Figure 1 we see that it is reasonable to take the value of the cut-off at the point of intersection of the two curves. This strategy has given very good results for two arbitrary overlapping discs.

Calculations for three overlapping discs show that the variance is increasing too fast to provide a stable numerical solution. So the convergence of the method in this case is open.

3 Lamé equation

It was found in [10] that the straightforward extension of the walk on spheres method to the first boundary value problem for the Lamé equation fails. It is interesting to note that for any fixed number of steps the method gives an estimator with a finite variance. The problem is that the variance is exponentially increasing with the number of steps which implies that the cost would tend to infinity as the ε, the size of the ε-boundary Γ_ε tends to zero.

Several attempts to find a reasonable modification of the walk on spheres method were made in [10], let us mention some of them. The issues listed are simultaneously the problems we are dealing with in this section.

1. First, the following property was used in [9]: if the number of steps of the ε-process is not large, the variance remains relatively small. This suggests (see [10]) a modification where a combined strategy is used: for relatively large ε, the rigorous walk on spheres iteration is carried out in the domain $G_\varepsilon = G \setminus \Gamma_\varepsilon$, while closer to the boundary, a

special approximative iterative procedure is used. In this section we will give a further improvement of this approach.

2. Second, the special structure of the kernel of the k-iteration of the matrix integral operator was used: it turns out that the k-iteration is essentially a product of cosines-functions which can be included into the probability density function for the distribution of the random walker on the sphere. This leads to a non-isotropic random walk on spheres process with a variance which increases considerably slower.

3. Third, we have studied in [10] the method which exploits the general spherical mean value property written for an arbitrary point inside the sphere. This is especially well suited for domains we have defined above as Km-domains. The relevant estimator has no bias, and the number of steps is very small. However we have not given there a practical numerical example.

4. The Schwarz iterative procedure. The general analysis of this classical procedure for the Lamé equation was given by S.L. Sobolev in [15]. Randomized evaluation of the iterations involve some features which we will present in this section.

3.1 Spherical mean value relation

Suppose a homogeneous isotropic medium $G \subset \mathbb{R}^n$ with a boundary Γ is given, whose state in the absence of body forces is governed by the classical static equation, the Lamé equation, see, e.g., [5], [10]:

$$\Delta \mathbf{u}(x) + \alpha \operatorname{grad} \operatorname{div} \mathbf{u}(x) = 0, \quad x \in G, \tag{3.1}$$

where $\mathbf{u}(x) = (u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n))$ is a vector of displacements, whose components are real-valued regular functions. The elastic constant α

$$\alpha = \frac{\lambda + \mu}{\mu}$$

is expressed through the Lamé constants of elasticity λ and μ. Here we will employ the summation convention, for example,

$$u_{i,jj} = \sum_{j=1}^{n} \frac{\partial^2 u_i}{\partial x_j^2} \ .$$

Hence, the equation (3.1) can be written in the form

$$u_{i,jj} + \alpha \, u_{j,ji} = 0, \quad i, j = 1, \ldots, n \quad . \tag{3.2}$$

The first boundary value problem for the Lamé equation consists in finding a vector function $\mathbf{u} \in C^2(G) \cap C(\bar{G})$ satisfying the boundary condition

$$\mathbf{u}(y) = \mathbf{g}(y), \quad y \in \Gamma \ , \tag{3.3}$$

where $\mathbf{g} \in C(\Gamma)$ is a given vector-function.

Spherical Mean Value Relation

Let us first present the spherical mean value relation for the general n-dimensional case.

The regular solutions to the system (3.2) satisfy the following spherical mean value relation in $S(x,r)$

$$u_i(x) = \frac{n}{2(n+\alpha)\omega_n} \int\limits_{\Omega} \Big((2-\alpha)\delta_{ij} + \alpha\,(n+2)s_i s_j\Big) u_j(x+rs)\; d\Omega(s), \qquad (3.4\ )$$

$i = 1, \ldots, n$, δ_{ij} is the Kronecker symbol, s_i are the cosine directions of the unit vector s, and $\omega_n = 2\pi^{n/2}\Gamma(n/2)$ is the area of the surface of Ω, the unit sphere in $\mathbf{R}^n$.

We rewrite this mean value relation in the vector form

$$\mathbf{u} = p\mathcal{N}_r\mathbf{u} + q\mathcal{S}_r\mathbf{u}\ , \qquad (3.5\ )$$

where

$$p = \frac{n(2-\alpha)}{2(n+\alpha)} \geq 0\ , \quad q = \frac{\alpha(n+2)}{2(n+\alpha)} \geq 0\ ,$$

hence $p+q=1$, and $\mathcal{N}_r$ is the matrix-integral operator

$$\mathcal{N}_r\mathbf{u} = \begin{pmatrix} N_r u_1 & 0 & \cdots & 0 \\ 0 & N_r u_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & N_r u_n \end{pmatrix}$$

where the diagonal operators N_r are the spherical means defined in (2.4), and

$$(\mathcal{S}_r\mathbf{u})_l(x) = \frac{n}{\omega_n} \int_{\Omega} s_l s_j u_j(x+Rs) d\Omega(s)\ , \quad l,j = 1,2,\ldots,n\ . \qquad (3.6\ )$$

A naive vector Monte Carlo estimator can be constructed as a generalization of the standard isotropic Random Walk on Spheres method. Indeed, introduce the unbiased vector estimator

$$\xi_\varepsilon = \prod_{i=1}^{N_\varepsilon} (pI + qS_i)\mathbf{u}(X_{N_\varepsilon})\ , \qquad (3.7\ )$$

and the ε-biased random estimator

$$\bar{\xi} = \prod_{i=1}^{N_\varepsilon} (pI + qS_i)\mathbf{g}(\bar{X}_{N_\varepsilon})\ , \qquad (3.8\ )$$

where I is the $n \times n$ identity matrix, and S_i is the matrix-kernel of the operator (3.6) in i-th sphere:

$$S_i = n \begin{pmatrix} s_1^2 & s_1 s_2 & \cdots & s_1 s_n \\ s_2 s_1 & s_2^2 & \cdots & s_2 s_n \\ \vdots & & \ddots & \vdots \\ s_n s_1 & \cdots & & s_n^2 \end{pmatrix}\ . \qquad (3.9\ )$$

Thus $s_1, s_2, \ldots, s_n$ are the components of the unit isotropic vector in i-th sphere of the Random Walk on Spheres process.

From the integral formulation of the Lamé equation given in Chap. 5 of [10] it follows that

$$\mathbf{u}(x) = \mathbb{E}_x \xi_\varepsilon \ . \tag{3.10}$$

However, the estimators (3.7),(3.8) cannot be used if ε is small enough. Indeed, the variance of $\bar{\xi}(x)$ is exponentially increasing: $\mathbf{E}(\xi_\varepsilon)^2 \cong n^{N_\varepsilon}$, where N_ε is the number of steps of the ε-spherical process.

3.2 A modification of the algorithm

In [12], [10] the following modification was described.

It is known that if ε is small, then after a certain number of steps the trajectory is concentrated near the boundary, r_i, the radii of the spheres being of order of ε (see, e.g., [9] and [7]). This gives the motivation to construct the following modification of the algorithm.

Let us consider the sum of the non-diagonal terms of the operator (3.5)

$$m_i = \frac{1}{\omega_n} \int_\Omega s_i \ s_j \ u_j(s) \ d\Omega(s) \ , \quad (j \neq i) \ .$$

Let us keep the details for the 2D case, and rewrite (3.5) in the form :

$$u_i(x) = p \, (\mathcal{N}_r \mathbf{u})_i(x) + \frac{2q}{2\pi} \int_\Omega s_i^2 \ u_i(x + rs) \ d\Omega(s) + m_i \ , \quad i = 1, 2 \ .$$

Thus, in Γ_ε we can approximate the local integral equation (3.5) by

$$\hat{u}_i(x) = p \, (\mathcal{N}_r \hat{\mathbf{u}})_i(x) + \frac{2q}{2\pi} \int_\Omega s_i^2 \ \hat{u}_i(x + rs) d\Omega(s) \ , \quad i = 1, 2 \ . \tag{3.11}$$

Define a diagonal matrix-operator $\mathcal{C}$ by

$$(\mathcal{C}\mathbf{v})_i(x) = \frac{2}{2\pi} \int_\Omega s_i^2 \ v_i(x + rs) \ d\Omega(s) \ ,$$

(no summation over i).

Since $p + q = 1$, we obtain that the Neumann series for (3.11) absolutely converges for an arbitrary small ε because the norm of the second iteration of the integral operator in (3.11) has the following estimate (see [10]) :

$$\| (p\mathcal{N}_r + q\mathcal{C})^2 \|_{L^\infty(G)} < 1 - \nu(\varepsilon) < 1 \ ,$$

where $\nu(\varepsilon)$ is a small positive number.

Thus, the modified algorithm reads as follows.

First, choose $\delta = t\varepsilon$ where $t > 1$ is a constant. Evaluate (3.7) over the standard Walk on spheres process X_k until $X_m \in \Gamma_\delta$. After that use the operator (3.11). The estimator is then

$$\tilde{\xi}_\varepsilon = \left\{ \prod_{i=0}^{N_\delta} (pI + qS_i) \right\} \left\{ \prod_{k=0}^{N_\varepsilon} (pI + qC_k) \right\} \mathbf{g}(\bar{X}_{N_\varepsilon}) \ .$$

Here C_k are diagonal matrices whose entries c_{ii} are $2s_i^2$, $i = 1, 2$.

Of course, this procedure is reasonable only if the diagonal system approximates well the solution in the δ-boundary.

In the general case, it is not possible to improve the algorithm by this approach. However there is one possibility to make an essential improvement by using another approximation near the boundary.

3.3 A modification near the boundary

Let us approximate the solution near the boundary as follows: take the first equation as it is, while omit the last integral in the second equation:

$$u_1(x) = \frac{1}{2\pi} \int\limits_{\Omega} \left[(p + q\, 2\, s_1^2)\, u_1(x + Rs) + q\, 2 s_1 s_2\, u_2(x + Rs) \right]\, d\Omega(s),$$
$$u_2(x) = \frac{1}{2\pi} \int\limits_{\Omega} (p + q\, s_2^2) u_2(x + Rs)\, \, d\Omega(s)\ .$$

The main advantage of this approximation is that the walk on spheres estimator generated by this equation converges. This follows from the structure of this integral equation: it is triangular, having probabilistic kernels on the diagonals. This immediately leads to the behaviour of the Neumann series analogous to that of Theorem 1, since all the entries of the matrix kernel are bounded.

3.4 Decentred walk on Spheres Process for the Lame equation

The Decentered Walk on Spheres process introduced in section 2.2 has remarkable properties which drastically changes the behaviour of the random process near the boundary: it turns out that for this process not only the mean number of steps is considerably smaller, but also, it's tale of distribution of the number of steps is essentially shorter than that for the standard ε-spherical process. This gives us a motivation to construct the algorithm for the Lamé equation based on the decentred walk on spheres process. To do this, we need to derive the spherical mean value relation for the Lamé equation at an arbitrary point inside the sphere. In 3D and 2D, the derivations are given in [13], [14] and in the book [10], however, in the 2D case, the formulae are presented there with a technical mistake. So we give here the derivation of the correct mean value relation in 2D. The 3D case is also presented: we have derived the mean value relation in a simpler form which includes only two functions to be additionally numerically calculated in advance.

3.4.1 The Decentred Mean Value Relation for the Lamé equation: 2D case

Consider an arbitrary point x with polar coordinates (ρ, φ') inside a disk $K(x_0, R)$. The point y situated on the circle $S(x_0, R)$ has the coordinates (R, θ), where $\theta = \varphi' + \alpha$, and z is defined by $z = y - x$, with the absolute value $Z = |z|$; note that α is the angle between

the vectors x and y; ψ is the angle between x and z.
Let us introduce the notation

$$\hat{s}_1 = \cos\varphi, \quad \hat{s}_2 = \sin\varphi, \quad \varphi = \psi + \varphi',$$

where $\hat{s}_i$ are the direction cosines of the vector z,

$$k = \rho/R, \quad J_k = \sqrt{1 - k^2 \sin^2\psi}\,.$$

The next relations run out from the above definitions immediately

$$Z^2 = R^2 + \rho^2 - 2R\rho\cos\alpha,$$

$$Z = -\rho\cos\psi + \sqrt{R^2 - \rho^2\sin^2\psi},$$
$$Z_k = Z/R = -ks_1 + J_k,$$

$$\begin{aligned} \cos\alpha &= Z/R\cos\psi + \rho/R = Z_k s_1 + k\ , \\ \sin\alpha &= Z/R\sin\psi = Z_k s_2, \end{aligned}$$

where $s_1 = \cos\ \psi$, $s_2 = \sin\ \psi$, and

$$\begin{aligned} \hat{s}_1 &= (R\cos\theta - \rho\cos\varphi')/Z = (\cos\theta - k\cos\varphi')/Z_k\ , \\ \hat{s}_2 &= (R\sin\theta - \rho\sin\varphi')/Z = (\sin\theta - k\sin\varphi')/Z_k\ . \end{aligned} \tag{3.12}$$

Theorem 2.

The solution to the equation (3.1) satisfies the following mean value relation, x being an arbitrary point in $K(x_0, R)$:

$$u_i(x) = \frac{R^2 - |x - x_0|^2}{2\pi R} \sum_{j=1}^{2} \int\limits_{S(x_0,R)} \frac{b_{ij}u_j(y)}{|x-y|^2}\, dS_y\ , \quad i = 1, 2, \tag{3.13}$$

where b_{ij} are functions of x, y, conveniently represented as the entries of the following matrix

$$B = \frac{1}{\sigma'}\begin{pmatrix} (\sigma' - 1) + 2\cos^2\varphi + \frac{|x-y|}{R}\cos(\theta+\varphi) & 2\cos\varphi\sin\varphi + \frac{|x-y|}{R}\sin(\theta+\varphi) \\ 2\cos\varphi\sin\varphi + \frac{|x-y|}{R}\sin(\theta+\varphi) & (\sigma'-1) + 2\sin^2\varphi - \frac{|x-y|}{R}\cos(\theta+\varphi) \end{pmatrix},$$

$$\sigma' = 1 + \frac{2\mu}{\lambda + \mu}\ .$$

In the notation of $p(y;x)$ introduced in (2.8), the relation (3.13) reads in the matrix form:

$$\mathbf{u}(x) = \int\limits_{S(x_0,R)} p(y;x)\, B\mathbf{u}(y)\, dS(y)\ . \tag{3.14}$$

Proof. Let $i = 1$, (the case $i = 2$ is deduced in a similar manner). We introduce the notation

$$u_i(y) = g_i(y), \quad \Delta u_i(y) = g_i^1(y), \quad i = 1, 2, \quad y \in S(x_0, R).$$

It is known [16] that any biharmonic function can be represented through two arbitrary harmonic functions as

$$u_i(x) = (R^2 - \rho^2)v_i(x) + w_i(x), \quad i = 1, 2\ . \tag{3.15}$$

It is convenient to choose the Dirichlet conditions for the functions w_i,

$$\begin{cases} \Delta w_i(x) = 0, & x \in K(x_0, R) \\ w_i(y) = g_i(y), & y \in S(x_0, R) \end{cases}\ , \tag{3.16}$$

then the functions v_i should solve the problem

$$\begin{cases} \Delta v_i(x) = 0, & x \in K(x_0, R) \\ \left\{ \frac{\partial v_i}{\partial r} + \frac{1}{r} v_i \right\}\Big|_{r=R} = -\frac{g_i^1(y)}{4R}, & y \in S(x_0, R) \end{cases}\ . \tag{3.17}$$

The Poisson formula yields the solution to the problem (3.16):

$$w_i = \frac{(1-k^2)R}{2\pi} \int\limits_{S(x_0,R)} \frac{g_i(y)}{Z^2}\ dS. \tag{3.18}$$

As to the problem (3.17), by separation of variables we come to [10]

$$\begin{aligned} v_1(\rho, \varphi') \ = \ & \frac{2}{\pi k \sigma' R^2} \int\limits_0^k (1-t^2) \int\limits_0^{2\pi} \Big(\frac{(\hat{s}_1^2 - 0.5)}{Z_t^4} u_1 + \frac{\hat{s}_1 \hat{s}_2}{Z_t^4} u_2 \Big)\ d\theta\ dt \\ & - \frac{1}{\pi k \sigma' R^2} \int\limits_0^k t \int\limits_0^{2\pi} \frac{1}{Z_t^4} \Big(\cos(\theta + \varphi')\, u_1 + \sin(\theta + \varphi')\, u_2 \Big)\ d\theta\ dt \\ & + \frac{1}{\pi k \sigma' R^2} \int\limits_0^k t^2 \int\limits_0^{2\pi} \frac{1}{Z_t^4} \Big(\cos 2\varphi'\, u_1 + \sin 2\varphi'\, u_2 \Big)\ d\theta\ dt. \end{aligned}$$

Here

$$Z_t^2 = 1 - 2t\cos\alpha + t^2,$$

and the functions $\hat{s}_i$ are given by the formulae (3.12) where k should be replaced with t. Changing the order of integration we rewrite the last equation to

$$\begin{aligned} v_1(\rho, \varphi') \ = \ & \frac{2}{\pi k \sigma' R^2} \int\limits_0^{2\pi} \int\limits_0^k \frac{1-t^2}{Z_t^6} \Big((\cos\theta - t\cos\varphi')^2 - 0.5 Z_t^2 \Big) u_1 \\ & + \frac{(1-t^2)}{Z_t^6} (\cos\theta - t\cos\varphi')(\sin\theta - t\sin\varphi')\ u_2\ d\theta\ dt \\ & - \frac{1}{\pi k \sigma' R^2} \int\limits_0^{2\pi} \int\limits_0^k \frac{t}{Z_t^4} \Big(\cos(\theta + \varphi')\, u_1 + \sin(\theta + \varphi')\, u_2 \Big)\ d\theta\ dt \\ & + \frac{1}{\pi k \sigma' R^2} \int\limits_0^{2\pi} \int\limits_0^k \frac{t^2}{Z_t^4} \Big(\cos 2\varphi'\, u_1 + \sin 2\varphi'\, u_2 \Big)\ d\theta\ dt. \end{aligned}$$

Now, using [3], we evaluate all the integrals over t, which results in

$$\begin{aligned} v_1(\rho,\varphi')\pi k\sigma' &= -\int_0^{2\pi} f_0(k)u_1\ d\theta + \int_0^{2\pi} 2f_1(k)\left[\cos^2\theta u_1 + \cos\theta\sin\theta u_2\right]\ d\theta \\ &\quad -\int_0^{2\pi} 2f_2(k)[2\cos\theta\cos\varphi' u_1 + \sin(\theta+\varphi')u_2]\ d\theta \\ &\quad +\int_0^{2\pi} 2f_3(k)[cos^2\varphi' u_1 + \sin\varphi'\cos\varphi' u_2]\ d\theta \\ &\quad -\int_0^{2\pi} f_4(k)[\cos(\theta+\varphi')u_1 + \sin(\theta+\varphi')u_2]\ d\theta \\ &\quad +\int_0^{2\pi} f_5(k)[\cos 2\varphi' u_1 + \sin 2\varphi' u_2]\ d\theta\ , \end{aligned}$$

where

$$f_0(k) = \int_0^k \frac{1-t^2}{Z_t^4}\ dt = \left.\frac{t}{Z_t^2}\right|_{t=0}^{t=k} = \frac{k}{Z_k^2}\ ,$$

$$f_1(k) = \int_0^k \frac{1-t^2}{Z_t^6}\ dt = \left\{-\frac{\cos\alpha - t}{4\sin^2\alpha\, Z_t^2} + \frac{t}{2Z_t^4}\right\}\Bigg|_{t=0}^{t=k} + \frac{A(k)}{4\sin^2\alpha}\ ,$$

$$f_2(k) = \int_0^k \frac{(1-t^2)t}{Z_t^6}\ dt = \left.\frac{t^2}{2Z_t^4}\right|_{t=0}^{t=k}\ ,$$

$$f_3(k) = \int_0^k \frac{(1-t^2)t^2}{Z_t^6}\ dt = \left\{\frac{t(Z_t^2+t^2)}{2Z_t^4} + \frac{\cos\alpha - t}{4\sin^2\alpha\, Z_t^2}\right\}\Bigg|_{t=0}^{t=k} - \frac{A(k)}{4\sin^2\alpha}\ ,$$

$$f_4(k) = \int_0^k \frac{t}{Z_t^4}\ dt = \left.\frac{t\cos\alpha - 1}{2\sin^2\alpha\, Z_t^2}\right|_{t=0}^{t=k} + \frac{A(k)\cos\alpha}{2\sin^2\alpha}\ ,$$

$$f_5(k) = \int_0^k \frac{t^2}{Z_t^4}\ dt = \left.\frac{2\cos^2\alpha\, t - t - \cos\alpha}{2\sin^2\alpha\, Z_t^2}\right|_{t=0}^{t=k} + \frac{A(k)}{2\sin^2\alpha}\ ,$$

and

$$A(k) = \int_0^k \frac{1}{Z_t^2}\ dt\ .$$

Since

$$\frac{A(k)\,u_1}{2\sin^2(\theta-\varphi')}\Big(\cos^2\theta - \cos^2\varphi' - \cos(\theta-\varphi')\cos(\theta+\varphi') + \cos 2\varphi'\Big) = 0,$$

and

$$\frac{A(k)\,u_2}{4\sin^2(\theta-\varphi')}\Big(\sin 2\theta - \sin^2 2\varphi' + 2\sin 2\varphi' - 2\sin(\theta-\varphi')\sin(\theta+\varphi')\Big) = 0\ .$$

Therefore,

$$
\begin{aligned}
v_1(\rho,\varphi')\pi k\sigma' \;=\; & -\int_0^{2\pi}\frac{k}{Z_k^2}u_1\,d\theta \\
& +\int_0^{2\pi}\Big(\frac{k}{Z^4}-\frac{\cos\alpha-k}{2\sin^2\alpha\,Z_k^2}+\frac{\cos\alpha}{2\sin^2\alpha}\Big)\Big(\cos^2\theta\,u_1+\cos\theta\sin\theta\;u_2\Big)\,d\theta \\
& -\int_0^{2\pi}\frac{t^2}{Z_k^4}\Big(2\cos\theta\cos\varphi' u_1+\sin(\theta+\varphi')u_2\Big)\,d\theta \\
& +\int_0^{2\pi}\Big(\frac{(Z_k^2+k^2)k}{Z^4}+\frac{\cos\alpha-k}{2\sin^2\alpha\,Z_k^2}-\frac{\cos\alpha}{2\sin^2\alpha}\Big)\Big(\cos\varphi'^2 u_1+\sin\varphi'\cos\varphi' u_2\Big)\,d\theta \\
& +\int_0^{2\pi}\Big(\frac{1-k\cos\alpha}{2\sin^2\alpha\,Z_k^2}-\frac{1}{2\sin^2\alpha}\Big)(\cos(\theta+\varphi')u_1+\sin(\theta+\varphi')u_2)\,d\theta \\
& +\int_0^{2\pi}(\frac{2\cos^2\alpha\,k-k-\cos\alpha}{2\sin^2\alpha\,Z_k^2}+\frac{\cos\alpha}{2\sin^2\alpha})(\cos 2\varphi' u_1+\sin 2\varphi' u_2)\,d\theta\ .
\end{aligned}
$$

We conclude by substituting (3.18) and the expression just derived into (3.15). By a full analogy, the mean value relation for the second component u_2 is obtained. Theorem is proved.

3.4.2 The decentred Mean Value Relation: 3D case

We turn now to the $3D$ case. The general scheme of derivation follows [5] and [14], however here we represent the kernel matrix in a simpler form.

Let $B(0,R)$ be a ball of radius R centered at the origin of coordinates, and let x be an arbitrary point inside this ball whose spherical coordinates are (ρ,θ,φ). We denote by y a point on the sphere $S=\partial B(0,R)$.

Let W be the distance $|x-y|$, and let $\hat{s}=(\hat{s}_1,\hat{s}_2,\hat{s}_3)$ be the direction cosines of the vector $y-x$. We need also a triple of coordinate axes $(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)$ centered at the point x and oriented with respect to the axes $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ so that $\mathbf{n}_1$ coincides with the vector x.

In these new coordinates, the direction cosines of the vector $y-x$ are $s=(s_1,s_2,s_3)$, and those of the vector x are $p=(p_1,p_2,p_3)$.

Simple geometrical considerations lead to

$$W^2=R^2+\rho^2-2R\rho p_1\ .$$

Analogous to the 2D case, we introduce the notations:

$$k=\rho/R,\quad W_t^2=1+t^2-2tp_1,\ 0\le t\le k,\quad W_k^2=1+k^2-2kp_1\ ,$$

then

$$W_t^2=1+t^2-kt+(W_k^2-1)t/k\ .$$

The vectors s and $\hat{s}$ are related by

$$s = A^T \hat{s},$$

where

$$A = \{a_{ij}\} = \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} ,$$

and

$$s_1 = W^{-1}(Rp_1 - \rho), \quad s_2 = W^{-1}Rp_2, \quad s_3 = W^{-1}Rp_3.$$

For $0 \le t \le k$ we define $\hat{s}_i^t$ and s_1^t as

$$\hat{s}_i^t = \frac{W_k \hat{s}_i + a_{i1}(k-t)}{W_t}, \quad s_1^t = s_1 \frac{W_k}{W_t} + \frac{k-t}{W_t} .$$

Theorem 3.

The components of the displacement vector $\mathbf{u}$, the solution to the Lamé equation satisfy the following Mean Value Relation:

$$u_j(x) = \frac{1}{4\pi R^2} \int_S k_{ji}(x,y) u_i(y)\, dS(y) , \qquad (3.19\ )$$

$$k_{ji}(x,y) = \frac{\delta_{ij} R(R^2-\rho^2)}{W^3} + \frac{R(R^2-\rho^2)\rho^{(1-3\sigma')/(2\sigma')}}{\sigma'} \int_0^\rho \frac{\Phi_1(t) + \Phi_2(t) + \Phi_3(t)}{t^{(1-\sigma')/(2\sigma')}}\, dt ,$$

where

$$\Phi_1(t) = \frac{\delta_{ij}}{2W_t^3} ,$$

$$\Phi_2(t) = -3t \frac{a_{i1}\hat{s}_j^t + a_{j1}\hat{s}_i^t - \delta_{ij} s_1^t}{W_t^4} ,$$

and

$$\Phi_3(t) = \frac{3(5\hat{s}_i^t \hat{s}_j^t - 2\delta_{ij})(1-t^2)}{2W_t^5} .$$

The integral over t can be represented through the following two integrals:

$$I_1(k) = \int_0^k \frac{t^{(\sigma'-1)/(2\sigma')}}{W_t^3} dt , \qquad I_2(k) = \int_0^k \frac{t^{(\sigma'-1)/(2\sigma')}}{W_t^5} dt .$$

Proof. The general expression (3.19) can be found in [10]. The integral of the function $\frac{\Phi_1(t)}{t^{(1-\sigma')/(2\sigma')}}$ is, by definition, given by $I_1(k)$, so let us evaluate the integrals

$$\int_0^k \frac{\Phi_2(t)}{t^{(1-\sigma')/(2\sigma')}}\, dt = -3W_k(a_{i1}\hat{s}_j + a_{j1}\hat{s}_i - \delta_{ij}s_1^t)F_1(k) - 3(2a_{j1}a_{i1}W_k - \delta_{ij})F_2(k) ,$$

and

$$\int_0^k \frac{\Phi_3(t)}{t^{(1-\sigma')/(2\sigma')}}\,dt = \frac{15}{2}\Big((\hat{s}_i\hat{s}_j - a_{j1}a_{i1})W_k^2 F_3(k) + (a_{j1}a_{i1} - 0.4\delta_{ij})F_4(k) + (a_{i1}\hat{s}_j + a_{j1}\hat{s}_i - 2a_{j1}a_{i1}s_1^t)W_k F_5(k)\Big) .$$

The functions $F_i(k)$, $i = 1, \ldots, 5$ can be evaluated:

$$F_1(k) = \int_0^k \frac{t^{(3\sigma'-1)/(2\sigma')}}{W_t^5}dt = -\frac{3\sigma'+1}{6\sigma' p_1}I_1(k) + \frac{1}{p_1}I_2(k) - \frac{k^{(3\sigma'-1)/(2\sigma')}}{3p_1 W_k^3} ,$$

$$\begin{aligned} F_2(k) &= \int_0^k \frac{(k-t)t^{(3\sigma'-1)/(2\sigma')}}{W_t^5}dt \\ &= \Big(\frac{1}{3\sigma'} - \frac{(3\sigma'+1)k}{6\sigma' p_1}\Big)I_1(k) + (\frac{k}{p_1} - 1)I_2(k) + (2 - \frac{k}{p_1})\frac{k^{(3\sigma'-1)/(2\sigma')}}{3W_k^3} \\ &= kF_1(k) + F_4(k) - I_2(k) , \end{aligned}$$

$$F_3(k) = \int_0^k \frac{(1-t^2)t^{(\sigma'-1)/(2\sigma')}}{W_t^7}dt = \frac{2\sigma'+1}{5\sigma'}I_2(k) + \frac{2k^{(3\sigma'-1)/(2\sigma')}}{5W_k^5} ,$$

$$F_4(k) = \int_0^k \frac{(1-t^2)t^{(\sigma'-1)/(2\sigma')}}{W_t^5}dt = \frac{1}{3\sigma'}I_1(k) + \frac{2k^{(3\sigma'-1)/(2\sigma')}}{3W_k^3} ,$$

$$\begin{aligned} F_5(k) &= \int_0^k \frac{(k-t)(1-t^2)t^{(\sigma'-1)/(2\sigma')}}{W_t^5}dt = \frac{3\sigma'+1}{30\sigma'^2 p_1}I_1(k) \\ &+ \Big(-\frac{1}{5p_1\sigma'} + \frac{k(1+1\sigma')}{5\sigma'}\Big)I_2(k) + \frac{k^{(3\sigma'-1)/(2\sigma')}}{15p_1\sigma' W_k^3} . \end{aligned}$$

3.5 Some properties of the matrix kernel

We proceed by analysing the 2D case. Note that the matrix B in the representation (3.14) can be rewritten in a different form, which is more convenient for evaluation of the iterations of the integral operator. Let us give two such convenient forms.

Simple trigonometry transformations yield

$$B = B^{(1)} = \frac{1}{\sigma'}\left\{(\sigma' - 1 - \frac{|x-y|}{R})I + 2\hat{S} + 2\frac{|x-y|}{R}Q\right\} ,$$

where I is the identity matrix, and

$$\hat{S} = \begin{pmatrix} \hat{s}_1^2 & \hat{s}_1\hat{s}_2 \\ \hat{s}_2\hat{s}_1 & \hat{s}_2^2 \end{pmatrix} , \qquad Q = \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 \\ \beta_2\beta_1 & \beta_2^2 \end{pmatrix} .$$

By definition, $\hat{s}$ is the unit vector at the point x, and

$$\beta_1 = cos((\varphi+\theta)/2), \quad \beta_2 = sin((\varphi+\theta)/2).$$

The following property of matrices $\hat{S}$ and Q is useful:

$$\prod_{i=1}^{m} \hat{S}_i = 2^m (\hat{s}^{(1)}, \hat{s}^{(2)}) \dots (\hat{s}^{(m-1)}, \hat{s}^{(m)}) \, [\hat{s}^{(1)} \otimes \hat{s}^{(m)}],$$

where

$$[\hat{s}^{(1)} \otimes \hat{s}^{(m)}] = \begin{pmatrix} \hat{s}_1^{(1)}\hat{s}_1^{(m)} & \hat{s}_1^{(1)}\hat{s}_2^{(m)} \\ \hat{s}_2^{(1)}\hat{s}_1^{(m)} & \hat{s}_2^{(1)}\hat{s}_2^{(m)} \end{pmatrix},$$

and $(\hat{s}^{(i)}, \hat{s}^{(i+1)})$ is the scalar product of two unit vectors $\hat{s}^{(i)}$ and $\hat{s}^{(i+1)}$, i.e., the cosine between the vectors y_{i+1} and y_i, $cos(\psi_{i+1} - \psi_i)$.

Second representation of the matrix B is also easily derived:

$$B = B^{(2)} = \frac{1}{\sigma'} \left\{ (\sigma' I + P_{2\varphi_i} + \frac{|x-y|}{R} P_{\varphi_i+\theta_i} \right\} ,$$

where I is the identity matrix, $P_{\varphi+\theta}$ is:

$$P_{2\varphi} = \begin{pmatrix} cos(2\varphi) & sin(2\varphi) \\ sin(2\varphi) & -cos(2\varphi) \end{pmatrix}$$

and

$$P_{\varphi+\theta} = \begin{pmatrix} cos(\varphi+\theta) & sin(\varphi+\theta) \\ sin(\varphi+\theta) & -cos(\varphi+\theta) \end{pmatrix} .$$

The matrices $P_{2\varphi}$ and $P_{\varphi+\theta}$ are orthogonal, and their product is also an orthogonal matrix:

$$P_{\varphi_1} P_{\varphi_2} = \hat{P}_{\varphi_2-\varphi_1} ,$$

where

$$\hat{P}_{\varphi_2-\varphi_1} = \begin{pmatrix} cos(\varphi_2-\varphi_1) & sin(\varphi_2-\varphi_1) \\ -sin(\varphi_2-\varphi_1) & cos(\varphi_2-\varphi_1) \end{pmatrix} .$$

For a randomized calculation of the iterations of the matrix-integral operator in (3.14) we need the evaluation of the products of the matrices B. Using the derived properties, this can be done successively:

$$\begin{aligned} \prod_{i=1}^{N} (\sigma' I + P_{2\varphi_i} + Z_i P_{\varphi_i+\theta_i}) = {\sigma'}^N I \;&+\; {\sigma'}^{N-1} Z_1 P_{\varphi_1+\theta_1} + {\sigma'}^{N-1} Z_2 P_{\varphi_2+\theta_2} + \dots \\ &+\; {\sigma'}^{N-1} P_{2\varphi_1} + {\sigma'}^{N-1} P_{2\varphi_2} + \dots \\ &+\; {\sigma'}^{N-2} Z_1 P_{2\varphi_2-\varphi_1-\theta_1} + {\sigma'}^{N-2} Z_1 P_{2\varphi_3-\varphi_1-\theta_1} + \dots \; . \end{aligned}$$

3.5.1 Properties of the decentred walk on spheres process

Let us first consider some properties of the decentred walk on spheres process (DWSP). For simplicity, we consider a special case when the domain is a layer $G = \{(x_1, x_2) \in G : -L \le x_2 \le L, -\infty < x_1 < \infty\}$ (which however reflects the main properties of DWSP), and the ε-boundary consists of the upper ε-boundary $\{L - \varepsilon \le x_2 \le L\}$, and the lower ε-boundary $\{-L \le x_2 \le -L + \varepsilon\}$. In general, we could taken the depth of the upper ε-boundary not equal to that of the lower ones. Since we are interested in small ε-boundaries, this is not important.

The Decentred Walk on Spheres process $\{Y_1 = x, Y_2, \ldots, Y_{N_\varepsilon}\}$ starting in a point $x = (x_{01}, x_{02})$ is defined in this case as follows. On the circle of radius $R = L$, centered at the point $(x_{01}, 0)$ sample a random point $Y_2 = (y_1^1, y_2^1)$ according to the distribution (2.8). The simulation algorithm is given in section 2.2. If y_2^1 is in the upper or lower ε-boundary, then the process is stopped, otherwise sample a point $Y_3 = (y_1^2, y_2^2)$ on the circle of radius $R = L$ centered at the point $(y_1^1, 0)$, etc., the random point $Y_{k+1} = (y_1^k, y_2^k)$ is sampled on the surface of radius $R = L$ centered at the point $(y_1^{k-1}, 0)$.

Let us consider the main features of this random walk process.

Mean number of steps, as a function of the distance from the boundary to the starting point.

The mean number of steps of the standard walk on spheres process is well studied, and behaves like $ln(d(x)/\varepsilon)$, where $d(x)$ is the distance from the starting point x to the boundary. This implies that the process "feels" the boundary only in a narrow strip along the boundary.

Let us estimate the mean number of steps of the DWSP. To do this, we first estimate the probability that the process makes one step, i.e., the probability of absorption in the ε-boundary after the first step. By the definition, this probability equals the integral:

$$Prob(k = 1) = I_{S_\varepsilon} = \int\limits_{S_\varepsilon} p(y; x)\, dS(y).$$

Here S_ε is the part of the circle lying in Γ_ε.

Let us denote by α the angle at which the part of the circle belonging to, say, upper boundary, is seen from the center of the circle. We evaluate this integral over the part of circle lying in the upper boundary. From the equality

$$I_{S_\varepsilon} = \frac{2}{2\pi} arctg \left\{ \frac{R + r}{R - r}\, tg(\alpha/2) \right\} \Bigg|_{\alpha_1}^{\alpha_2},$$

where α_1 and α_2 are the limiting directions of the angle α, and r is the distance from the point x to the center of the circle. We choose $\alpha_1 = 0$, and due to symmetry, we get

$$Prob(k = 1) = \frac{2}{\pi} arctg \left\{ \frac{R + x}{R - x} \sqrt{\frac{\varepsilon}{2R - \varepsilon}} \right\} .$$

where $-R \le x \le R - \varepsilon$. Since x may be positive and negative, it is more convenient to rewrite this formula, using the introduced notation for the distance $d(x)$:

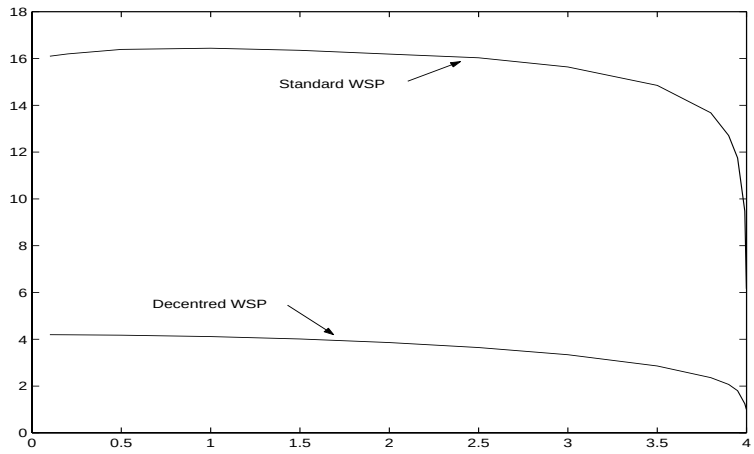

Figure 2: The mean number of steps, as a function of the distance from the starting point to the boundary of the layer. Here $\varepsilon = 10^{-4}$, and $x_2 = 0$, the origin of coordinates is taken in the center of the layer, while $x_2 = 4$ is the upper, and $x_2 = -4$ is the lower boundary of the layer.

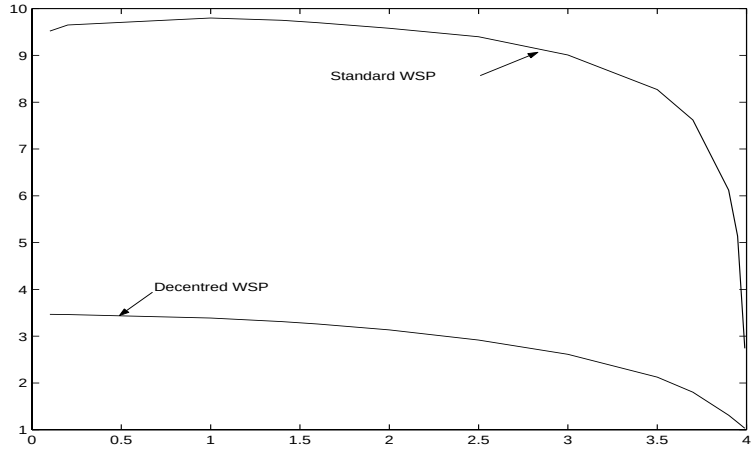

Figure 3: The same as in Figure 1, but for $\varepsilon = 0.01$

$$Prob(k=1) = \frac{2}{\pi} arctg \left\{ \frac{2R - d(x)}{d(x)} \sqrt{\frac{\varepsilon}{2R-\varepsilon}} \right\} .$$

This function monotonically increases from

$$\frac{2}{\pi} arctg \left(\frac{\varepsilon}{2R-\varepsilon} \right)^{3/2} \quad \text{to} \quad \frac{2}{\pi} arctg \sqrt{\frac{2R-\varepsilon}{\varepsilon}}$$

as x increases from $-R+\varepsilon$ to $R-\varepsilon$. This is the probability that the walker starting at the point x hits after the first step the upper Γ_ε-boundary. The probability to hit the lower Γ_ε-boundary at the first step behaves conversely: it monotonically decreases in this region, and is obtained from $Prob$ by a symmetry reflection relative to the axes $x_2 = 0$. Consequently, the probability to reach the lower or upper Γ_ε-boundary is the

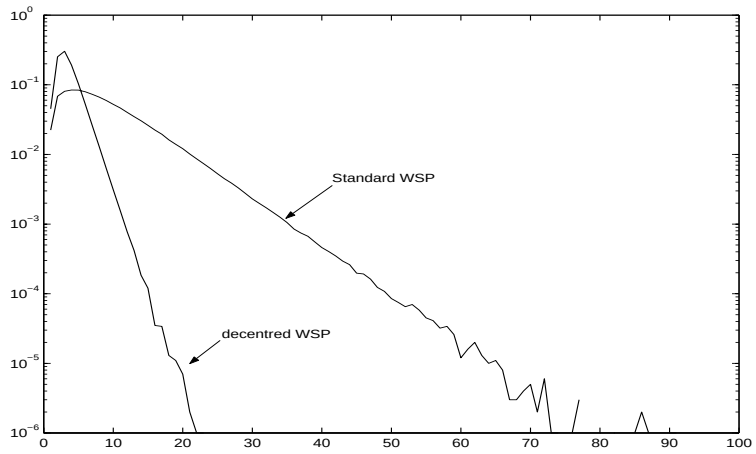

Figure 4: The distribution of the number os steps, for Decentred WSP and Standard WSP, for $\varepsilon = 0.01$. The processes started at the center of the layer.

sum of these functions, which reaches its minimum at the point $x_2 = 0$, and is equal to $g = \frac{4}{\pi} arctg \sqrt{\frac{\varepsilon}{2R-\varepsilon}}$.

We can estimate the mean number of steps in the Bernoulli scheme (at each step, we have a positive probability, not less than g, that the process stops) as $1/g$. From this we can approximately write for small (compared to R) values of ε:

$$\bar{N}_\varepsilon < \frac{\pi}{2\sqrt{2}} \cdot \sqrt{\frac{R}{\varepsilon}} \ .$$

However this estimation is crude. In fact, the asymptotic behaviour of the mean number of steps is also logarithmic in $d(x)/\varepsilon$, i.e., it behaves like $c \cdot ln(d(x)/\varepsilon)$, as in the Standard WSP (e.g., see [2]), however the factor c in this behaviour is many times less.

Let us compare the mean number of steps for the Decentred and Standard WSP for our layer G. In Figs. 1 - 2. we present the results for $\varepsilon = 10^{-4}$, and $\varepsilon = 10^{-2}$, respectively.

More pronounced are the differences in the distribution of the number of steps. In Figures 4-7, we show the distributions of the number of steps, for different starting points, and for different values of ε.

From these results we conclude that for all cases, both for starting points in the center of the layer and close to the boundary, and for all considered values of ε, the distributions for the DRWS are much narrower than that for the standard WSP.

The pictures 8-9 show that the Decentred WSP leaves closer to the boundary, even if it starts far from the boundary, while the Standard WSP is distributed much more uniformly over the layer depth.

The distributions of the number of steps of the two processes being for small values of ε both exponentially decaying, are different: for the the Decentred WSP it is quite compact, while for the Standard WSP, the right tale is very long. This property is a crucial point for the algorithm we develop for the Lamé equation: the main contribution to the variance increase comes from the tale. It implies, that the cut-off procedure in the Decentred WSP is much less affecting the bias than in the case of the Standard WSP.

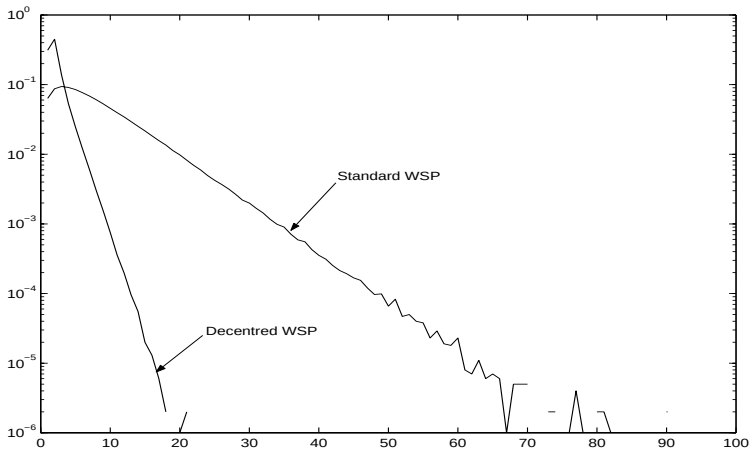

Figure 5: The same as in Figure 4, but the trajectories started at a point situated at the distance 0.5 from the upper boundary.

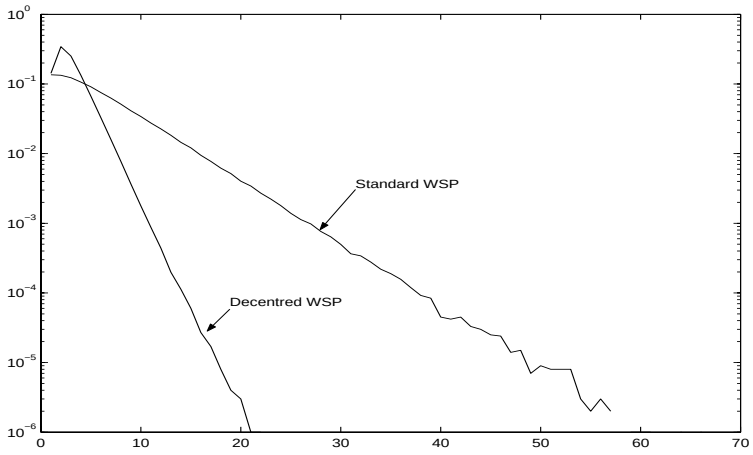

Figure 6: The same as Figure 4, but for $\varepsilon = 0.1$.

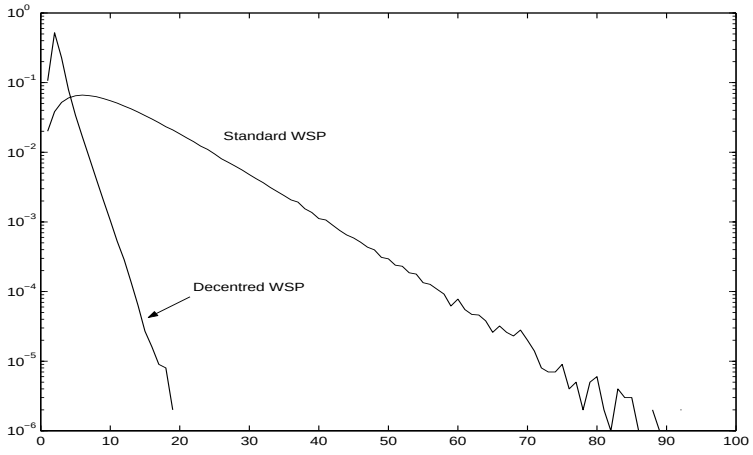

Figure 7: The same as in the Figure 5, but for $\varepsilon = 0.001$.

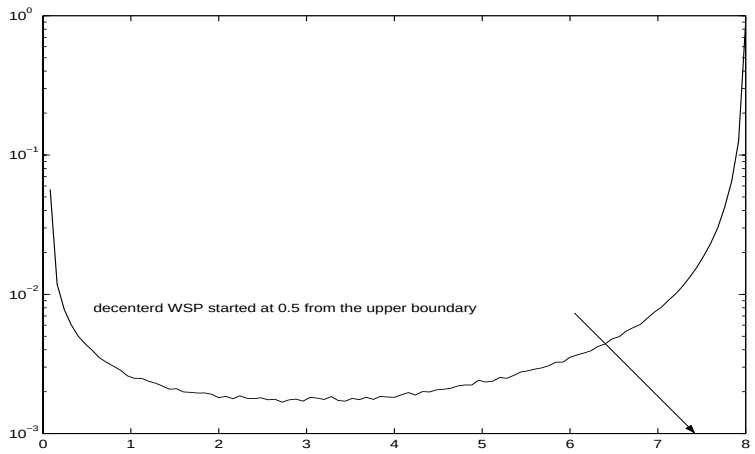

Figure 8: The x_2-position distribution over the layer's depth, for the Decentred WSP, $\varepsilon = 0.1$, the process started at 0.5 distance from the upper boundary.

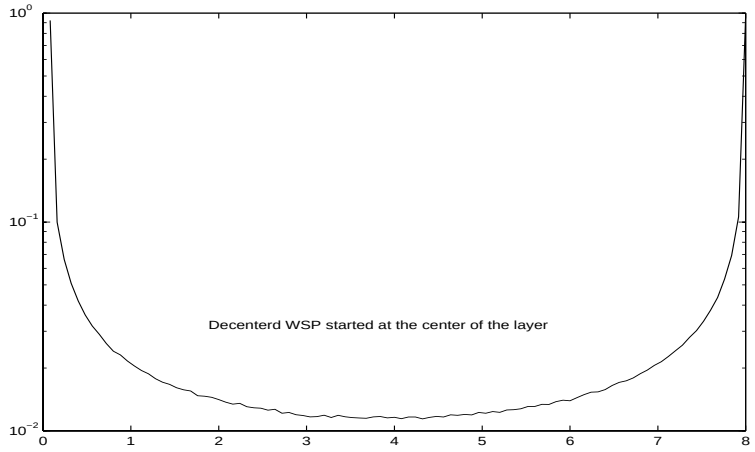

Figure 9: The same as in the previous Figure, but the process started at the center of the layer.

3.6 Numerical simulations

In the layer described in the previous section, we solved the following model problem:

$$\Delta \mathbf{u}(x) + \alpha \operatorname{grad} \operatorname{div} \mathbf{u}(x) = 0, \quad x \in G, \tag{3.20}$$

with the Dirichlet boundary conditions $\mathbf{u}(y) = \mathbf{g}(y)$, for $y \in \partial G$. We have chosen the case with the exact solution $u_i(x_1, x_2) = 1 + x_i^2 - 3.5 x_1 x_2$, $i = 1, 2$ which solves the Lamé equation for $\lambda = 3\mu$.

We solved this problem by the Decentred WSP. The Standard WSP was unable to solve this problem: the variance was rapidly increasing with ε. The Decentred WSP gave satisfactory results, moreover, we have improved the results by a quite natural modification by introducing a cut-off procedure: in the averaging we have taken into account only the trajectories whose length was not larger than the mean number of steps multiplied by some integer $mcut$, in our case, we have taken $mcut = 3$. This considerably improves the results: the curve obtained by this method is almost coincident (see Figure 10) with the exact solution. The upper curve was obtained by the Decentred WSP where all the trajectories were taken into account. It is clearly seen that the original method (without the cut-off) works well near the boundary, already beginning from the distance which are about 25% the diameter of the domain.

This suggests a new modification, which we call *ε-boundary propagation method*: first calculate the solution in a δ_1-boundary where $\delta_1 = m_1 \varepsilon$, m_1 being a parameter, e.g., equal to $2-4$ depending on the behaviour of solution's gradient, after that calculate the solution in a δ_2-boundary, with $\delta_2 = m_2 \delta_1$, and using the solution calculated in δ_1-boundary as the known boundary conditions. So in few steps (4-5 steps, actually, was sufficient in our case) we obtain the solution by the Decentred WSP whose average number of steps is very small in each step, with small variance. It is easy to estimate the number of steps required: since $S_n \equiv 1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$, and assuming that $L/\varepsilon = K$, we get, e.g., for $K \approx 100$, that the number of steps is about 6 even if we take in each step $m_i = 2$.

One problem here is to be treated carefully: we have to use a clever interpolation procedure when taking the boundary conditions in the last step, after stopping in a point where the solution is not known. It is not difficult to provide a simple procedure which guarantees the accuracy of order ε.

3.7 Two overlapping circles

In this section we solve the boundary value problem (3.20) for the domain consisting of two overlapping discs:

$$G = K_1(x^{(1)}, R_1) \cup K_2(x^{(2)}, R_2) \ .$$

Let us choose $R_1 = 2$, $R_2 = 2$, take the distance between the centers equal to $shift = 3$, and place $x^{(1)}$ in the origin of coordinates. Then $x^{(2)}$ is the point $(shift, 0)$. As a test problem, we take

$$u_1(x) = x_1^2 - 3.5 x_1 x_2 + x_2 + 1, \quad u_2(x) = x_2^2 - 3.5 x_1 x_2 + x_1 + 1$$

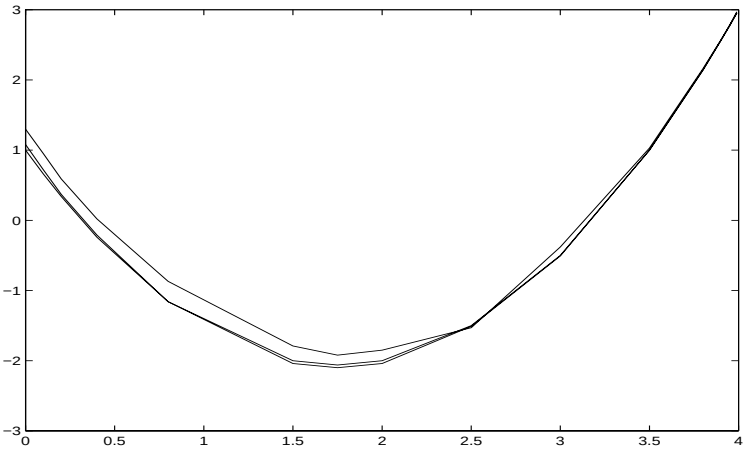

Figure 10: The second component of the solution, as a function of x_2, the layer depth: $\varepsilon = 10^{-3}$, the number of trajectories $N = 10^6$. The two lower (almost coincident) curves are the exact solution and the numerical results obtained by the cut-off procedure where all the trajectories whose length is larger than $3\cdot$mean number of steps do not give a contribution to the score. The upper curve was obtained by averaging over all trajectories.

which solves the Lamé problem for $\lambda = \mu/3$, $\mu = 0.5$.

Let us calculate the solution at the point $(1, 0)$ by the Schwarz iterative procedure and DRWS method.

1. The Schwarz iterative procedure.

Let us consider the two overlapping discs introduced in (2.7). In the generalized Schwarz method whose convergence was proved in [15] a sequence of functions $\mathbf{u}^{(i)}$, $i = 1, 2, \ldots$, is constructed as follows. The functions $\mathbf{u}^{(2k-1)}$ are regular in K_2 and $K_1 \setminus K_2$, satisfy in these domains the Lamé equation, they are continuous in G and

$$\mathbf{u}^{(2k-1)}|_{\Gamma_1} = \mathbf{g}^{(1)},\ \mathbf{u}^{(2k-1)}|_{\gamma_1} = \begin{cases} \mathbf{g}^{(0)} & \text{if } k = 1\ , \\ \mathbf{u}^{(2k)} & \text{if } k > 1\ . \end{cases}$$

Here $\mathbf{g}^{(0)}$ is an arbitrary sufficiently smooth vector function.

All the functions $\mathbf{u}^{(2k)}$ are continuous in G, regular in K_1 and $K_2 \setminus K_1$, satisfy the Lamé equation, and

$$\mathbf{u}^{(2k)}|_{\Gamma_2} = \mathbf{g}^{(2)},\ \mathbf{u}^{(2k)}|_{\gamma_2} = \mathbf{u}^{(2k-1)}|_{\gamma_2},\quad k \geq 0.$$

The proof of convergence of $\mathbf{u}^{(k)}$ is based (see [15]) on the estimations

$$E(\mathbf{u}^{2k}) \leq E(\mathbf{u}^{2k-1})$$

and

$$E(\mathbf{u}^{2k+1}) \leq E(\mathbf{u}^{2k})\ .$$

Here $E(\mathbf{v})$ is the energy of deformation for any displacement vector $\mathbf{v}$ as

$$E(\mathbf{v}) = \int\limits_G \left[\lambda(\mathrm{div}\mathbf{v})^2 + \frac{\mu}{2}(\nabla^\alpha v_\beta + \nabla_\beta v^\alpha)^2\right] dV\ . \qquad (3.21\ )$$

The boundary value problem (3.20) is equivalent to the problem of finding a function $\mathbf{v}$ such that $\mathbf{v}|_\Gamma = \mathbf{g}$ and which minimizes the energy integral (3.21). Then, representing $\mathbf{u}^{(k)}$ through the Green function and taking the limit in the Green formula one finds that $\lim_{k\to\infty} \mathbf{u}^{(k)}$ solves the original problem.

The sequence of functions $\mathbf{u}^{(i)}$ from the Generalized Poisson formula (see Theorem 2 in $2D$ and Theorem 3 in $3D$) for an arbitrary point $x \in K_1$ can be written in the matrix form as

$$\mathbf{u}^{(1)}(x) = \int\limits_{\Gamma_1} C(x,y)\mathbf{g}^{(1)}(y)dS(y) + \int\limits_{\gamma_1} C(x,y)\mathbf{g}^{(0)}(y)dS(y),$$

or

$$\mathbf{u}^{(1)}(x) = K_1\mathbf{g}^{(1)} + F_1\mathbf{g}^{(0)},$$

where

$$K_1\mathbf{g}^{(1)}(x) = \int\limits_{\Gamma_1} C(x,y)\mathbf{g}^{(1)}(y)dS(y),$$

$$F_1\mathbf{g}^{(0)}(x) = \int\limits_{\gamma_1} C(x,y)\mathbf{g}^{(0)}(y)dS(y),$$

and the matrix $C(x,y)$ is defined in the above mentioned theorems. For $\mathbf{u}^{(2)}(x)$:

$$\mathbf{u}^{(2)}(x) = \int\limits_{\Gamma_2} C(x,y)\mathbf{g}^{(2)}(y)dS(y) + \int\limits_{\gamma_2} C(x,y)\mathbf{u}^{(1)}(y)dS(y),$$

$$\mathbf{u}^{(2)}(x) = K_2\mathbf{g}^{(2)}(x) + F_2\mathbf{u}^{(1)}(x),$$

where

$$K_2\mathbf{u}(y) = \int\limits_{\Gamma_2} C(y,y')\mathbf{g}^{(2)}(y')dS(y'),$$

$$F_2\mathbf{u}^{(1)}(y) = \int\limits_{\gamma_2} C(y,y')\mathbf{u}^{(1)}(y')dS(y').$$

Hence,

$$\begin{aligned}
\mathbf{u}^{(2)}(x) &= \int\limits_{\Gamma_2} C(x,y)\mathbf{g}^{(2)}(y)dS(y) + \int\limits_{\gamma_2}\int\limits_{\Gamma_1} C(x,y_1)C(y_1,y_2)\mathbf{g}^{(1)}(y_2)dS(y_1)dS(y_2) \\
&+ \int\limits_{\gamma_2}\int\limits_{\gamma_1} C(x,y_1)C(y_1,y_2)\mathbf{g}^{(0)}(y_2)dS(y_1)dS(y_2) \\
&= K_2\mathbf{g}^{(2)} + F_2K_1\mathbf{g}^{(1)} + F_2F_1\mathbf{g}^{(0)}.
\end{aligned}$$

For $\mathbf{u}^{(2n+1)}$:

$$\mathbf{u}^{(2n+1)}(x) = K_1\mathbf{g}^{(1)} + F_1\mathbf{u}^{(2n)} = K_1\mathbf{g}^{(1)} + F_1K_2\mathbf{g}^{(2)} + F_1F_2K_1\mathbf{g}^{(1)} + \ldots + F_1F_2F_1\ldots F_1\mathbf{g}^{(0)},$$

which can formally be represented as

$$\mathbf{u}^{(2n+1)}(x) = \sum_{j=0}^{n} G^jK_l\mathbf{g}^{(l)} + G^{2n+1}\mathbf{g}^{(0)}, \qquad (3.22)$$

where

$$G^0 = I, \quad G^j = \underbrace{F_1F_2F_1F_2\ldots F_1F_2\ldots F_p}_{j \text{ times}},$$

$$l = \begin{cases} 2, & \text{if } j \text{ is even,} \\ 1 & \text{if } j \text{ is odd or } 0, \end{cases}$$

while

$$p = \begin{cases} 2, & \text{if } j \text{ is odd,} \\ 1 & \text{if } j \text{ is even.} \end{cases}$$

Now we consider the iterations of our decentred mean value relation. For an arbitrary point $x \in K_1$ we can write

$$\mathbf{u}(x) = (K_1 + F_1)\mathbf{u},$$

where K_1 and F_1 are defined above. The same relation is true at $y \in K_2$:

$$\mathbf{u}(y) = (K_2 + F_2)\mathbf{u}.$$

Then, formally,

$$\mathbf{u}(x) = K_1\mathbf{g}^{(1)} + F_1K_2\mathbf{g}^{(2)} + F_1F_2K_1\mathbf{g}^{(1)} + F_1F_2F_1K_2\mathbf{g}^{(2)} + F_1F_2F_1F_2K_1\mathbf{g}^{(1)} + \ldots .$$

Let

$$S_k = \sum_{j=0}^{k} G^j K_l \mathbf{g}^{(l)}. \tag{3.23}$$

From (3.22) and (3.23) we conclude

$$S_m \;=\; \mathbf{u}^{(2m+1)}(x) - \underbrace{F_1F_2F_1F_2\ldots F_1}_{2m+1 \text{ times}}\mathbf{g}^{(0)} = \mathbf{u}^{(2m+1)}(x) - G^{2m+1}\mathbf{g}^{(0)}.$$

We choose the arbitrary initial function as $\mathbf{g}^{(0)} = 0$; in this case the partial sums S_m of the Neumann series coincide with the iteraions $\mathbf{u}^{(2m+1)}$ for all m which implies that the Neumann series converges.

It should be noted however that this leaves the problem of the variance finiteness of the Monte Carlo procedure open. However the numerical simulations show that the variance behaves very stable, and the DWS method works very fast, with high accuracy.

In Figure 11 we present the numerical results obtained by this algorithm. The exact value of the first component of the solution is 2. The upper curve was obtained by the Schwarz iterative procedure (first twenty iteartions are shown), where at each iteration, $N = 10^4$ random points uniformly distributed over the surfaces of the spheres (sampled once, and used for all iterations) were used to calculate the relevant integrals over the circles by the Monte Carlo method. The lower curve was obtained by averaging over $N = 10^5$ samples. It should be stressed once more: first store the points sampled on the both circles, and then use these points in each iteration. This not only saves the computer time, but also provides higher accuracy, as the dependent sampling usually does.

In Figure 12, the relevant statistical error ($rms/\sqrt{N}$) is shown as a function of iterations. In this picture, the error in the first iterations was large, but then it stabilizes. The

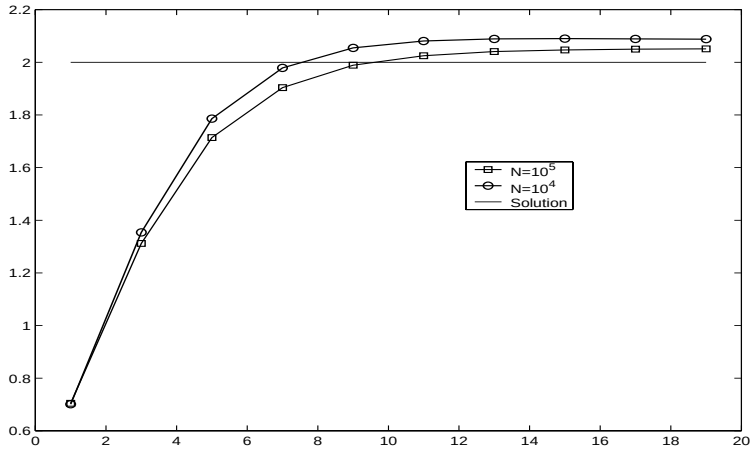

Figure 11: The first component of the solution to the Lamé equation obtained by the Schwarz iterative procedure. The exact solution is $(2.0; 2.0)$, Computing time: if for $N = 10^4$ it takes about 1 unit, then for $N = 10^5$ it was about $t = 60$ units.

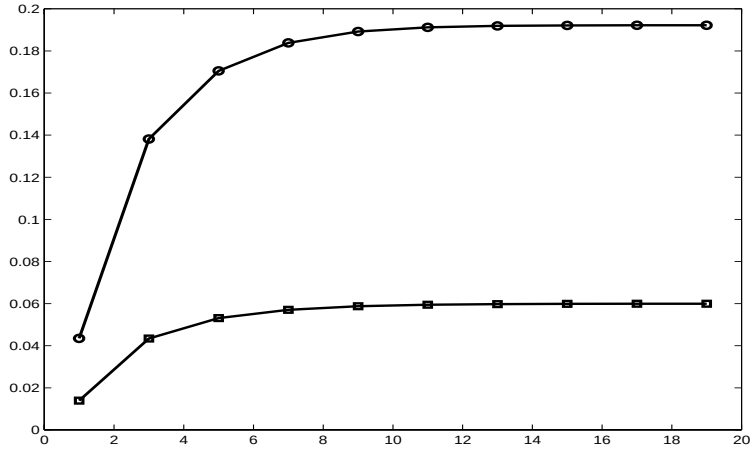

Figure 12: Statistical error for $N = 10^4$ (upper curve), and $N = 10^5$ (lower curve).

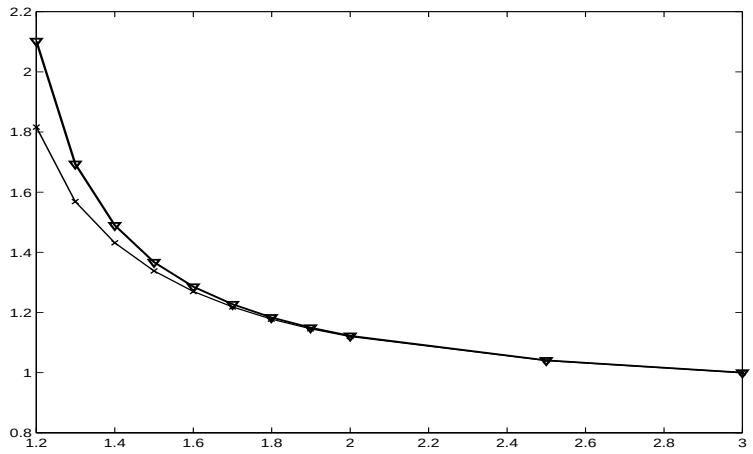

Figure 13: The upper curve - the solution obtained by DRWS method with the number of trajectories $N = 10^7$, the lower curve - the exact solution, versus the parameter σ'.

dependence of the initial iteration is not so simple: if the initial error was too large, we observed a numerical divergence.

1. Random Walk Algorithm

Here we use the Decentred Walk on Spheres process $\{x, y_1, y_2, \ldots\}$. We start from the point x, $x \in K(x^{(1)}, R_1)$. Then according to the density $p(y_1; x)$ we simulate the point y_1 as described in Section 2.2. If the sampled point y_1 is on Γ_1, the trajectory is stopped; if $y_1 \in \gamma_1$ the process continues. Now, if $y_1 \in K(x^{(2)}, R_2)$ we simulate the next point y_2 according to the density $p(y_2; y_1)$, and so on.

The results of calculations are shown in Table 1.

Table 1: Monte Carlo results: N is the number of trajectories, the exact solution is $(2.0, 2.0)$.

N	u_1	$3\,rms/\sqrt{N}$	u_2	$3\,rms/\sqrt{N}$
10^6	2.0525	0.0718	1.9892	0.0397
10^7	2.0089	0.0230	2.0019	0.0284

Note that the error of the algorithm can be sensitive to the factor $\frac{1}{\sigma'}$ in the matrix B. In Figure 13 we show the solution as a function of σ', in Figure 14 - its statistical error. Note that in the region of practically important case $\sigma' \approx 2$ the error is very small.

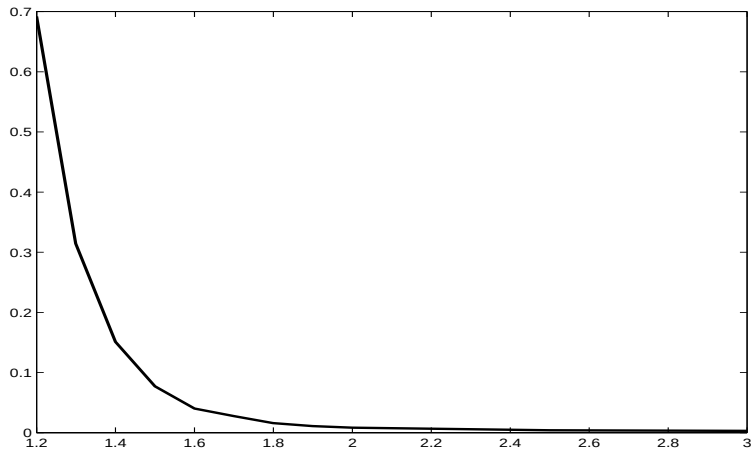

Figure 14: Statistical error, as a function of σ', for $N = 10^5$.

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