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The time-varying stabilization of linear discrete control systems

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Abstract

The Brockett stabilization problem for linear discrete control systems is considered. The method of synthesis of time-varying feedback for stabilization is described.

1 Introduction

In [1] R. Brockett has stated the time-varying stabilization problem for continuous linear systems. We consider the analogue of this problem for discrete systems.

There are given three constant matrices A, B, C. Under what conditions does there exist a time-dependent matrix $K(t)$ such that the system

$$x(t+1) = Ax(t) + BK(t)Cx(t), \qquad x \in \mathbb{R}^n, \ t \in \mathcal{N} \tag{1}$$

is asymptotically stable?

Here $\mathcal{N} = \{0, 1, 2, \ldots\}$ is set of nonnegative integer numbers.

In this paper we apply methods developed for continous systems (see [2, 3]) to discrete control systems.

2 The stabilization criteria

Suppose there exist matrices K_1 and K_2 such that for $j = 1, 2$ the system

$$x(t+1) = (A + BK_jC)x(t) \tag{2}$$

has a stable invariant linear manifold L_j and an invariant linear manifold M_j. We assume for $j = 1, 2$

(i)
$$M_j \cap L_j = \{0\}, \ \dim M_j + \dim L_j = n.$$

(ii) There are positive numbers λ_j, κ_j, α_j, β_j, $j = 1, 2$, such that for $t \in \mathcal{N}$, $j = 1, 2$, the inequalities hold

$$|x(t)| \leq \alpha_j e^{-\lambda_j t}|x(0)| \qquad \text{for} \qquad x(0) \in L_j, \tag{3}$$

$$|x(t)| \leq \beta_j e^{\kappa_j t}|x(0)| \qquad \text{for} \qquad x(0) \in M_j. \tag{4}$$

Assume also that to any $t \in \mathcal{N}$ there exists a matrix $U(t)$ and that there is an integer $\tau > 0$ such that for the system

$$y(t+1) = (A + BU(t)C)y(t) \tag{5}$$

the inclusion

$$Y(\tau)M_1 \subset L_2 \tag{6}$$

is valid, where

$$Y(t+1) = \prod_{j=0}^{t} (A + BU(j)C), \qquad Y(0) = I.$$

Theorem 1. *If the inequality*

$$\lambda_1 \lambda_2 > \kappa_1 \kappa_2 \tag{7}$$

holds, then there exists a periodic matrix $K(t)$ such that system (1) is asymptotically stable.

Lemma 1. *Suppose the inequality (7) is satisfied. Then for any $T > 0$ there exist integers $t_1 > 0$ and $t_2 > 0$ such that*

$$\begin{aligned} -\lambda_1 t_1 + \kappa_2 t_2 < -T, \\ -\lambda_2 t_2 + \kappa_1 t_1 < -T. \end{aligned} \tag{8}$$

Proof. Condition (7) implies the validity of the inequalities

$$\frac{T}{\lambda_1} + \frac{\kappa_2}{\lambda_1} t_2 < t_1 < -\frac{T}{\kappa_1} + \frac{\lambda_2}{\kappa_1} t_2 \tag{9}$$

for sufficiently large integer $t_2 > 0$. Here t_1 is some positive integer. The inequalities (9) are equivalent to the inequalities (8).

The following lemma is obvious.

Lemma 2. *Let D_i, $i = 1, \ldots, 4$, be real matrices. From*

$$\begin{pmatrix} D_2 w \\ 0 \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} 0 \\ w \end{pmatrix} \qquad \forall\, w \in \mathbb{R}^l$$

we get $D_4 = 0$.

Proof of Theorem 1. Let T be an arbitrarilly positive number. Under the condition (7) there are positive integers t_1 and t_2 satisfying the inequalities (8) (see Lemma 1).

We now define the periodic matrix $K(t)$ in the following way

$$\begin{aligned} K(t) &= K_1, && \text{for} \quad t \in [0, t_1), \\ K(t) &= U(t - t_1), && \text{for} \quad t \in [t_1, t_1 + \tau], \\ K(t) &= K_2, && \text{for} \quad t \in (t_1 + \tau, t_1 + t_2 + \tau). \end{aligned} \tag{10}$$

The minimal period of the matrix $K(t)$ is $t_1 + t_2 + \tau$. We shall prove that for sufficiently large T system (1) with the matrix $K(t)$ defined in (10) is asymptotically stable.

Let S_j $(j = 1, 2)$ be a nonsingular matrix. Then by (2) we have

$$S_j x_{n+1} = S_j(A + BK_jC)x_n = S_j(A + BK_jC)S_j^{-1} S_j x_n. \tag{11}$$

We assume that S_j is a matrix such that

(i) $S_j(A + BK_jC)S_j^{-1} = \begin{pmatrix} Q_j & 0 \\ 0 & P_j \end{pmatrix}$.

(ii) $Q_j : L_j \to L_j,\ P_j : M_j \to M_j$.

Thus, S_j defines by (11) the decomposition

$$S_j x = \begin{pmatrix} z_j \\ w_j \end{pmatrix}, \tag{12}$$

and (2) is equivalent to

$$\begin{aligned} z_j(t+1) &= Q_j z_j(t), \qquad && \dim z_j = \dim L_j, \\ w_j(t+1) &= P_j w_j(t), \qquad && \dim w_j = \dim M_j, \end{aligned} \tag{13}$$

where without loss of generality we may assume that for $t \in \mathcal{N}$

$$\begin{aligned} |z_j(t)| &\le \alpha_j e^{-\lambda_j t}|z_j(0)|, \\ |w_j(t)| &\le \beta_j e^{\kappa_j t}|w_j(0)|. \end{aligned} \tag{14}$$

From the relations (13) and (14) it follows that

$$\begin{pmatrix} z_2(t_1+\tau) \\ w_2(t_1+\tau) \end{pmatrix} = S_2 Y(\tau) S_1^{-1} \begin{pmatrix} z_1(t_1) \\ w_2(t_1) \end{pmatrix}.$$

Inclusion (6) implies that the matrix $S_2Y(\tau)S_1^{-1}$ has the form (see Lemma 2)

$$S_2 Y(\tau) S_1^{-1} = \begin{pmatrix} R_{11}(\tau) & R_{12}(\tau) \\ R_{21}(\tau) & 0 \end{pmatrix}.$$

Therefore (8), (13) and (14) result in the estimates

$$|z_2(t_1+t_2+\tau)| \le \alpha_1\alpha_2 |R_{11}(\tau)|\, e^{-2T}|z(0)| + \alpha_2\beta_1|R_{12}(\tau)|\, e^{-T}|w_1(0)|,$$

$$|w_2(t_1+t_2+\tau)| \le \alpha_1\beta_2|R_{21}(\tau)|\, e^{-T}|z_1(0)|.$$

Hence, to any $x(0)$ with $|x(0)| \le 1$ there is a sufficiently large T such that the solution of (1) satisfies

$$|x(t_1+t_2+\tau, 0, x(0)| \le \frac{1}{2}.$$

This relation and the periodicity of the matrix $K(t)$ imply the asymptotic stability of system (1).

Theorem 2. *Suppose $B \in \mathbb{R}^n$, $C^* \in \mathbb{R}^n$, (A, B) is controllable, (A, C^*) is observable, $M_1 = M_2$, $L_1 = L_2$, $\dim M_1 = 1$, $\dim L_1 = n-1$. Then there exists a matrix $U_0 \equiv U(t)$ such that*

$$Y(1)M_1 \subset L_2.$$

Proof. Consider vectors $h \in \mathbb{R}^n$, $q \in \mathbb{R}^n$ such that

$$L_1 = \{h^* x = 0\}, \qquad q \in M_1, \ q \neq 0.$$

From the controllability of (A, B) and from the observability of (A, C^*) it follows the controllability of $(A + BK_1C, B)$ and the observability of $(A + BK_1C, C^*)$.

Suppose that $h^*B = 0$. In this case the invariance of L_1 implies the relations

$$h^*B = 0, \qquad h^*(A + BK_1C)B = 0, \ldots, h^*(A + BK_1C)^{n-1}B = 0.$$

From this relations and from the controllability of $(A + BK_1C, B)$ it follows $h = 0$. Hence, the assumption $h^*B = 0$ is incorrect and we have $h^*B \neq 0$.

From the relation $Cq = 0$ and from the invariance of M_1 it follows

$$C(A + BK_1C)q = 0, \ldots, C(A + BK_1C)^{n-1}q = 0.$$

Therefore, the observability of $(A+BK_1C, C^*)$ implies $q = 0$. Hence, the assumption $Cq = 0$ is incorrect and we have $Cq \neq 0$.

Let us consider system (5) with $y(0) = q$. From

$$h^*y(1) = h^*Aq + U(0)h^*BCq$$

and from the inequalities $h^*B \neq 0$, $Cq \neq 0$ it follows by (6) for $\tau = 1$

$$U(0) = -h^*Aq \Big/ (h^*BCq).$$

From Theorem 1 and Theorem 2 the following result can be obtained.

Theorem 3. *Suppose $B \in \mathbb{R}^n$, $C^* \in \mathbb{R}^n$, (A, B) is controllable, (A, C^*) is observable, $M_1 = M_2$, $L_1 = L_2$, $\lambda_1 = \lambda_2$, $\kappa_1 = \kappa_2$, $\dim M_1 = 1$, $\dim L_2 = n-1$ and $\lambda_1 > \kappa_1$. Then there exists a periodic function $K(t)$ such that system (1) is asymptotically stable.*

Theorem 3 implies the following result.

Theorem 4. *Suppose $B \in \mathbb{R}^n$, $C^* \in \mathbb{R}^n$, (A, B) is controllable, (A, C^*) is observable. There is some number K_0 such that the eigenvalues μ_j $(j = 1, \ldots, n)$ of the matrix $A + K_0BC$ satisfy the conditions*

$$|\mu_j| < 1, \qquad \textit{for } \ j = 1, \ldots, n-1,$$
$$|\mu_n \mu_j| < 1, \qquad \textit{for } \ j = 1, \ldots, n-1.$$

Then there exists a periodic function $K(t)$ such that system (1) is asymptotically stable.

3 Two-dimensional linear systems

Let us consider system (1) with $B \in \mathbb{R}^2$, $C^* = \mathbb{R}^2$, $n = 2$ and with the transfer function

$$W(p) = C(A - pI)^{-1}B = \frac{\nu p + \gamma}{p^2 + \alpha p + \beta}.$$

We assume that α, β, γ, ν are numbers such that

$$\gamma^2 - \alpha\nu + \beta\nu^2 \neq 0.$$

This inequality is a necessary and sufficient condition for the controllability and observability of system (1) in case $n = 2$.

The eigenvalues μ_1, μ_2 of the matrix $A + K_0BC$ are the zeroes of the polynomial

$$p^2 + (\alpha + K_0\nu)p + \beta + K_0\gamma.$$

Therefore, it holds

$$|\mu_1\mu_2| = |\beta + K_0\gamma|.$$

Hence, all conditions of Theorem 4 are fulfiled if $\gamma \neq 0$ or $|\beta| < 1$ and

$$\gamma^2 - \alpha\nu + \beta\nu^2 \neq 0.$$

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