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Wavelet Approximation Methods for Pseudodifferential Equations I: Stability and Convergence

W. Dahmen, S. Prössdorf and R. Schneider*

Abstract

This is the first part of two papers which are concerned with generalized Petrov-Galerkin schemes for elliptic periodic pseudodifferential equations in $\mathbb{R}^n$ covering classical Galerkin methods, collocation, and quasiinterpolation. These methods are based on a general setting of multiresolution analysis, i.e., of sequences of nested spaces which are generated by refinable functions. In this part we develop a general stability and convergence theory for such a framework which recovers and extends many previously studied special cases. The key to the analysis is a local principle due to the second author. Its applicability relies here on a sufficiently general version of a so called discrete commutator property. These results establish important prerequisites for developing and analysing in the second part methods for the fast solution of the resulting linear systems. These methods are based on compressing the stiffness matrices relative to wavelet bases for the given multiresolution analysis.

Key words: Refinable functions, wavelets, periodic pseudodifferential operators, generalized Galerkin-Petrov schemes, discrete commutator property, stability analysis, convergence estimates.

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1 Introduction

While initially typical applications of wavelets were concerned with signal and image analysis there have been recent attempts of applying wavelets to the solution of integral or differential equations (cf. [5, 3, 20, 22]). The objective of this paper is to analyse the potential of wavelet methods for the solution of pseudodifferential equations. At this

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stage we focus on the model case of periodic pseudodifferential equations to exploit the full advantages of Fourier transform techniques in connection with appropriate representations for the class of operators under consideration. However, we do consider variable symbols and it should be mentioned that this class covers all the classical examples such as Hörmander's class, in particular, those arising in connection with boundary element methods.

The present investigation draws upon two major sources of motivation. On one hand, a significant number of earlier papers (see, for instance, [38, 39, 37] and the literature cited there) treat Galerkin or collocation methods separately for special cases of operators and for various special choices of trial and test functions. Here we attempt to propose a general framework that allows us to develop a unified approach to all these cases and also to extend previous results. It seems that ascending sequences of nested spaces which are generated by the translates and dilates of a single *refinable* function provide a suitable setting for that purpose. Of course, spline spaces form a typical example which fits into this context. We will see that the essential conditions that will entail optimal convergence rates and stability of the methods can be conveniently formulated in terms of the Fourier transform of the refinable function.

On the other hand, such sequences of refinable spaces, often called *multiresolution analysis* (cf. [27, 30, 23]), offer convenient ways of constructing wavelet bases. Thus, one expects that the present setting should be able to take advantage of the recent interesting developments in this direction. In this regard, there are two issues which are of central importance for the present purposes, namely the preconditioning effect of wavelet bases [14], as well as the possibility of compressing stiffness matrices relative to wavelet bases. Such a compression technique has been proposed in [5] where, however, only operators of order zero where discussed.

This latter issue will be addressed in detail in a forthcoming second paper, while we concentrate here on convergence and stability analysis for generalized Petrov-Galerkin methods in the general framework of multiresolution analysis.

In Section 2 we collect some important facts on refinable functions that will be needed throughout the sequel. Specifically, we formulate appropriate periodized versions. Some prerequisits about the class of pseudodifferential operators which are to be investigated are presented in Section 3 where we also formulate the generalized Galerkin-Petrov schemes. In Section 4 we characterize stability of these methods first for the case of constant coefficient operators in terms of the so called *numerical symbol* which is a simple expression involving the symbol and the Fourier transform of the refinable function.

One possible approach to stability analysis for variable symbols is a reduction to the case of constant symbols by means of a certain *local principle* which could be viewed as a numerical counterpart to the well-known principle of *freezing coefficients.* Of course, the basic idea of localizing techniques has a long history in the theory as well as in the numerical analysis of partial differential equations. Here we focus only on those methods for pseudodifferential equations which can be formulated as projection methods. The first papers addressing this particular aspect seem to be [45, 46], where classical Galerkin schemes with trigonometric trial and test functions are analysed. In [36] collocation methods with piecewise linear trial functions for singular integral equations are investigated. The analysis in [36] already involved implicitly a certain *discrete commutator property* which also played then a crucial role in various subsequent papers treating one-dimensional problems (see e.g. [2, 33, 34, 35, 43]). An explicit abstract formulation of this principle was first given in [32, 33]. For an overview of the various univariate results see also [40]. First multivariate applications for tensor product spline spaces were obtained in [37, 38, 12].

In Section 5 we will prepare the ground for the application of this local principle in the present general multivariate setting. The main ingredient is a certain *super-approximation* result which should be of some independent interest. With these prerequisites at hand we characterize in Section 6 the stability of generalized Petrov-Galerkin schemes in terms of the ellipticity of the numerical symbol and establish corresponding optimal convergence rates for the approximate solutions.

2 Refinable Functions

Splines have been successfully employed for the numerical solution of operator equations. If an approximate solution in some fixed spline space, i.e., for a fixed set of knots, is to be updated subsequent knot insertion provides a powerful tool for increasing the flexibility and hence accuracy of the approximating spaces. The corresponding numerical manipulations become particularly simple when dealing with equidistant knots in which case each spline space is spanned by integer *shifts* of *dilates* of a *single* B-spline. This is a canonical example of what is usually referred to as *multiresolution approximation* framework as it typically arises in connection with the construction of wavelets (see e.g. [8, 17, 23, 27, 30, 41]).

In general, the main ingredient of such a multiresolution approximation is a so called *refinable function* $\varphi \in L_2(\mathbb{R}^n)$. By this we mean that φ satisfies a *refinement equation*

$$\varphi(x) = \sum_{k\in\mathbb{Z}^n} a_k\varphi(2x-k) \quad , \quad x \in \mathbb{R}^n \ , \tag{2.1}$$

where the *mask* $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}^n}$ is some fixed sequence in $\ell_1(\mathbb{Z}^n)$. To stress the dependence on $\mathbf{a}$, we will sometimes say φ is $\mathbf{a}$-refinable. We will be primarily interested in *compactly supported* refinable functions but in some cases it will be important to relax this assumption. It is shown in [23] that for many purposes it suffices to assume that φ belongs to the space

$$\mathcal{L}_2 := \{f \in L_2(\mathbb{R}^n) : \sum_{k\in\mathbb{Z}^n} |f(\cdot - k)| \in L_2([0,1]^n)\}.$$

It is clear that any function $\varphi \in L_2(\mathbb{R}^n)$ which has compact support or for which $\int_{k+[0,1]^n} |\varphi(x)|^2 dx$ decays exponentially, as $|k|$ tends to infinity, belongs to $\mathcal{L}_2$. Here $|x| := \langle x, x\rangle^{1/2}$ is the Euclidean distance and $\langle x, y\rangle := \sum_{j=1}^n x_j\overline{y}_j$ denotes the standard scalar product of $x, y \in \mathbb{C}^n$.

We define the Fourier transform of $f \in L_1(\mathbb{R}^n)$ by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,y\rangle}dx \ .$$

It is known that $\mathcal{L}_2 \subseteq L_1(\mathbb{R}^n)$ (see [23]) so that $\hat{\varphi}$ is continuous and (2.1) implies that for any $l \in \mathbb{N}$

$$\hat{\varphi}(y) = \left(\prod_{j=1}^{l} 2^{-n}\mathbf{a}(2^{-j}y)\right) \hat{\varphi}(2^{-l}y), \tag{2.2}$$

where

$$\mathbf{a}(y) := \sum_{k\in\mathbb{Z}^n} a_k e^{-i2\pi\langle y,k\rangle}.$$

Thus $\hat{\varphi}$ and hence φ would vanish identically if $\mathbf{a}(0) < 2^n$. It is also easy to see that, when

$$2^n = \mathbf{a}(0) = \sum_{k\in\mathbb{Z}^n} a_k, \tag{2.3}$$

the products $\prod_{j=1}^{l} 2^{-n}\mathbf{a}(2^{-j}y)$ converge uniformly on compact sets in $\mathbb{C}^n$ so that any nontrivial solution to (2.1) must satisfy $\hat{\varphi}(0) \neq 0$. Thus we may assume in the following without loss of generality that (2.3) as well as

$$\hat{\varphi}(0) = 1 \tag{2.4}$$

hold.

It is remarkable that refinability combined with additional smoothness properties, has a number of important far reaching consequences which we will record for later use. To this end, we will employ standard multiindex notation, i.e., $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Proposition 2.1 *Suppose φ is* $\mathbf{a}$*-refinable where we assume (2.3) and (2.4). Then the following properties hold.*

(i) If φ belongs to $C^d(\mathbb{R}^n)$ and $x^\nu\varphi(x)$ is in $L_1(\mathbb{R}^n)$ for $\nu \in \mathbb{N}_0$, $|\nu| \leq d$, then φ satisfies Fix-Strang *conditions of order d, i.e.,*

$$(\partial^\alpha\hat{\varphi})(k) = 0 \quad , \quad |\alpha| \leq d \; , \; k \in \mathbb{Z}^n\backslash\{0\} \; . \tag{2.5}$$

Thus, in particular, when $\varphi \in C^d(\mathbb{R}^n)$ has compact support, (2.5) is valid.

(ii) If $\varphi \in C^d(\mathbb{R}^n)$ has compact support, then there exists a positive real number $\rho = \rho(\mathbf{a}) < 1$ and a constant c depending only on the mask $\mathbf{a}$ *such that*

$$|\partial^\alpha\varphi(x) - \partial^\alpha\varphi(y)| \leq c|x-y|^\rho \; , \qquad x, y \in \mathbb{R}^n \; , \qquad |\alpha| = d. \tag{2.6}$$

(iii) If φ has compact support and belongs to $C^d(\mathbb{R}^n)$, then for every $P \in \Pi_d(\mathbb{R}^n)$, the space of polynomials of degree at most d on $\mathbb{R}^n$, there exists a unique polynomial $Q \in \Pi_d(\mathbb{R}^n)$ such that

$$P(x) = \sum_{k\in\mathbb{Z}^n} Q(k)\varphi(x-k), \qquad x \in \mathbb{R}^n, \tag{2.7}$$

and

$$P - Q \in \Pi_{d-1}(\mathbb{R}^n).$$

(i) is proved in [7], Theorem 8.4. Property (ii) is also implicitly contained in Section 7 of [7] and is proved explicitly in [14]. Theorem 9.1 in [7] covers (iii).

For sufficient conditions on the mask **a** to ensure dth order differentiability of the corresponding **a**-refinable function φ the reader is referred again to [7].

The second important property which we will require is that the integer translates of φ are *stable* in the sense that

$$\|\lambda\|_{\ell_2(\mathbb{Z}^n)} \sim \|\lambda *' \varphi\|_{L_2(\mathbb{R}^n)}, \tag{2.8}$$

where $A \sim B$ means that there exist two positive constants c_1, c_2 such that $c_1 A \leq B \leq c_2 A$ holds uniformly with respect to all parameters the quantities A, B may depend on. Here $\|\lambda\|^2_{\ell_2(\mathbb{Z}^n)} = \sum_{k\in\mathbb{Z}^n} |\lambda_k|^2$, $\|\cdot\|_{L_2(\mathbb{R}^n)}$ denotes the usual L_2-norm on $\mathbb{R}^n$, and the *semi-discrete convolution* $\lambda *' \varphi$ is defined by

$$\lambda *' \varphi = \sum_{k\in\mathbb{Z}^n} \lambda_k \varphi(\cdot - k).$$

Again cardinal B-splines or, more generally, certain cube-splines satisfy both (2.1) and (2.8) (cf. e.g. [7, 23]).

In terms of the Fourier transform of φ the stability of φ is well-known to be equivalent to

$$[\hat{\varphi}\hat{\varphi}](\omega) = \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\omega + k)|^2 > 0 \quad \text{for all} \quad \omega \in [0,1]^n, \tag{2.9}$$

where for

$$(f,g) := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx,$$

we define in general

$$[\hat{f}\hat{g}](\omega) := \sum_{k\in\mathbb{Z}^n} \hat{f}(\omega + k)\overline{\hat{g}(\omega + k)} = \sum_{k\in\mathbb{Z}^n} (f, g(\cdot - k))e^{2\pi i\langle\omega,k\rangle}. \tag{2.10}$$

Note that the trigonometric series on the right hand side of (2.10) is known to converge absolutely whenever $f, g \in \mathcal{L}_2$ (see Theorem 3.1 in [23]).

It is known [23] that stability (2.8) is implied by the somewhat stronger notion of *algebraic linear independence* of the shifts of φ, which means that the mapping

$$\lambda \mapsto \lambda *' \varphi \tag{2.11}$$

is injective on the space of *all* complex-valued sequences λ defined on $\mathbb{Z}^n$. Again cardinal B-splines or, more generally, certain cube splines have this property.

In subsequent sections we will frequently have to work with a function φ which satisfies the following requirements:

- φ is refinable, has compact support and belongs to $C^d(\mathbb{R}^n)$.
- The integer shifts of φ are algebraically linearly independent.

In this case we will briefly say that φ satisfies condition $\mathbf{C}_0^{\mathbf{d}}$.

It is also known [4] that linear independence of integer translates of a compactly supported function implies their *local* linear independence, which means that

$$(\lambda *' \varphi)(x) = 0 \quad \text{for} \quad x \in \Omega,$$

for any open domain $\Omega \subseteq \mathbb{R}^n$ implies $\lambda_k = 0$ for which $\operatorname{supp} \varphi(\cdot - k) \cap \Omega \neq \emptyset$. It is shown in [18] that for a given compactly supported function φ with locally linearly independent translates there exists a functional F such that

$$F(\varphi(\cdot - k)) = \delta_{0,k}, \qquad k \in \mathbb{Z}^n, \tag{2.12}$$

and

$$|F(g)| \leq c \left(\int_{[0,1]^n} |g(x)|^2 dx \right)^{1/2}, \quad g \in L_2(\mathbb{R}^n), \tag{2.13}$$

holds for some constant c independent of g.

We now turn to an appropriate *periodic setting* based on the above notions. Identifying one-periodic functions, i.e., functions f satisfying

$$f(x+k) = f(x), \quad \text{for all} \quad k \in \mathbb{Z}^n,$$

with functions on the n-dimensional torus

$$\mathcal{T}^n = \mathbb{R}^n / \mathbb{Z}^n,$$

the periodization operator

$$[f](x) := \sum_{k \in \mathbb{Z}^n} f(x+k) \tag{2.14}$$

maps $L_2(\mathbb{R}^n)$ into $L_2(\mathcal{T}^n)$. Likewise we will identify for notational convenience the cosets $[x] := x + \mathbb{Z}^n$, $x \in \mathbb{R}^n$, with its representer $x \in [0,1]^n$. For any function $\phi \in \mathcal{L}_2$ we define now

$$\phi_k^j := 2^{\frac{jn}{2}} [\phi(2^j \cdot - k)], \qquad k \in \mathbb{Z}^n. \tag{2.15}$$

Specifically, for any refinable function $\varphi \in \mathcal{L}_2$ we define the finite dimensional spaces

$$V^j := \operatorname{span} \{\varphi_k^j \ : \ k \in \mathbb{Z}^{n,j}\}, \tag{2.16}$$

where

$$\mathbb{Z}^{n,j} := \mathbb{Z}^n / (2^j \mathbb{Z}^n).$$

Since by (2.1) and (2.15)

$$\varphi_k^j = 2^{nj/2} \sum_{\ell \in \mathbb{Z}^n} a_\ell [\varphi(2^{j+1} \cdot -2k - \ell)] = 2^{-n/2} \sum_{m \in \mathbb{Z}^n} a_{m-2k} \varphi_m^{j+1}$$

we conclude

$$V^0 \subset V^1 \subset \ldots \subset V^j \subset V^{j+1} \subset \ldots \subset L_2(\mathcal{T}^n) \ . \tag{2.17}$$

One easily confirms now from corresponding results on the non-periodic case [23] that, under the above assumptions,

$$\overline{\bigcup_{j\in \mathbb{N}_0} V^j} = L_2(\mathcal{T}^n). \tag{2.18}$$

We will refer to the sequence $\{V^j\}_{j\in\mathbb{Z}}$ as the periodic multiresolution analysis generated by φ.

Let for any two one-periodic functions u, v

$$(u,v)_0 := \int_{[0,1]^n} u(x)\overline{v(x)}dx.$$

Next, note that for any $g \in \mathcal{L}_2$, $u \in L_2(\mathcal{T}^n)$

$$([g], u)_0 = (g, u) \tag{2.19}$$

so that, for any $f, g \in \mathcal{L}_2$

$$([f],[g])_0 = (f,[g]) = ([f],g). \tag{2.20}$$

One easily derives from these facts the following observation.

Remark 2.1 *Let $f, g \in \mathcal{L}_2$ satisfy*

$$(f, g(\cdot - \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n.$$

Then

$$(f_k^j, g_l^j)_0 = \delta_{k,l}, \quad k, l \in \mathbb{Z}^{n,j},\ j \in \mathbb{N}_0.$$

More generally, if η is any functional of compact support satisfying

$$\eta(g(\cdot + \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n,$$

then

$$\eta_k^j(v) := 2^{-nj/2}\eta(v(2^{-j}(\cdot + k))) \tag{2.21}$$

satisfies

$$\eta_k^j(g_l^j) = \delta_{k,l}, \quad k, l \in \mathbb{Z}^{n,j},\ j \in \mathbb{N}_0.$$

The following facts are now immediate consequences of (2.12), (2.13) and Remark 2.1.

Lemma 2.1 *If φ satisfies $\mathbf{C}_0^{\mathbf{d}}$ then one has for $F_k^j(v) := 2^{-nj/2}F(v(2^{-j}(\cdot + k)))$, where F satisfies (2.12) and (2.13),*

$$F_k^j(\varphi_\ell^j) = \delta_{k,\ell}, \quad k, \ell \in \mathbb{Z}^{n,j}, \tag{2.22}$$

and

$$|F_k^j(u^j)| \le c \left(\int_{2^{-j}(k+[0,1]^n)} |u^j(x)|^2 dx \right)^{1/2}, \tag{2.23}$$

where c is independent of j, k and $u^j \in V^j$.

3 Periodic Pseudodifferential Operators and Numerical Approximation Methods

In this section we shall introduce a class of periodic pseudodifferential equations which will be studied throughout the remainder of this paper. Pseudodifferential operators on smooth manifolds are usually defined in terms of local representations and partitions of unity (see e.g [21, 48, 44]).

However, for our purposes it is more convenient to represent them by means of Fourier transforms. It will be seen that the resulting class of operators covers all classical pseudodifferential operators so that confining our discussions to this class will not impose any essential restrictions. The approximation methods that will be used for the solution of corresponding pseudodifferential equations will be formulated as generalized *Galerkin-Petrov* schemes. In particular, this framework covers also collocation methods.

The Fourier transform on $\mathcal{T}^n$, often referred as *discrete Fourier transform*, of a function $f \in L_1(\mathcal{T}^n)$ is defined by

$$\tilde{f}(\xi) = \mathcal{F}_{\mathcal{T}^n} f(\xi) = \int_{\mathcal{T}^n} e^{-2\pi i \langle \xi, x \rangle} f(x) dx \quad , \quad \xi \in \mathbb{Z}^n \ , \tag{3.1}$$

and, conversely, due to Fourier's inversion formula, f can be recovered under suitable smoothness assumptions by the Fourier series

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) e^{2\pi i \langle \xi, x \rangle} \quad , \quad x \in \mathcal{T}^n \ . \tag{3.2}$$

Let $e^j = (\delta_{j,\ell})_{\ell=1}^n$ denote the jth coordinate vector and define

$$\Delta := (\tau_1 - 1, \ldots, \tau_n - 1)^T$$

where

$$(\tau_j f)(x) := f(x + e^j).$$

We shall use the subscript ξ in $\Delta_{(\xi)}^\alpha$ to indicate that the multinomial Δ^α acts on the variable ξ.

We are now ready to introduce the notion of a global symbol. For $r \in \mathbb{R}$, let $S^r(\mathcal{T}^n)$ denote the set of all functions $\sigma \in C^\infty(\mathcal{T}^n \times \mathbb{Z}^n)$ satisfying

$$|\partial_x^\beta \Delta_{(\xi)}^\alpha \sigma(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{r - |\alpha|} \text{ for all } x \in \mathcal{T}^n \ , \ \xi \in \mathbb{Z}^n \ . \tag{3.3}$$

The function $\sigma \in S^r(\mathcal{T}^n)$ is called a *global symbol* of order r on $\mathcal{T}^n$.

For a given symbol $\sigma \in S^r(\mathcal{T}^n)$ we define the global pseudodifferential operator $\sigma(x, D)$ by

$$\sigma(x, D) u(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i \langle \xi, x \rangle} \sigma(x, \xi) \tilde{u}(\xi) \ , \ u \in C^\infty(\mathcal{T}^n) \ . \tag{3.4}$$

The symbol of a global pseudodifferential operator on $\mathcal{T}^n$ is uniquely defined up to a smooth function $t \in S^{-\infty}(\mathcal{T}^n) = \bigcap_{r \in \mathbb{R}} S^r(\mathcal{T}^n)$ (see [1, 28]).

In what follows we restrict ourselves to the following subclass of $S^r(\mathcal{T}^n)$. For any $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu = r$ we denote by $\Sigma^\mu(\mathcal{T}^n)$ the class of all symbols $\sigma \in S^r(\mathcal{T}^n)$ which admit a decomposition $\sigma = \sigma_0 + \sigma_1$, where $\sigma_1 \in S^{r_1}(\mathcal{T}^n)$ with some $r_1 < r$ and $\sigma_0 \in C^\infty(\mathcal{T}^n \times \mathbb{R}^n \backslash \{0\})$. Here we will assume that

$$\sigma_0(x, 0) = 1$$

and that σ_0 is positively homogeneous of degree μ, i.e.,

$$\sigma_0(x, \lambda\xi) = \lambda^\mu \sigma_0(x, \xi) \quad \text{for} \quad \lambda > 0 \;,\; \xi \neq 0 \;. \tag{3.5}$$

Since $\operatorname{Re}\mu = r$, σ_0 is automatically contained in $S^r(\mathcal{T}^n)$.

The main result in [28], [29] (see also [1]) and a corresponding version for operators on the torus [38] asserts that all the classical pseudodifferential operators defined in [25], including those appearing in boundary element methods (see [51, 11, 13, 50]), can be described by a global symbol and vice versa. Accordingly, we will denote by $\Psi^r(\mathcal{T}^n)$ ($\Phi^\mu(\mathcal{T}^n)$) the class of pseudodifferential operators of the form $A = \sigma(x, D) + K$ where $\sigma \in S^r(\mathcal{T}^n)$ ($\Sigma^\mu(\mathcal{T}^n)$) and K given by $Ku(x) = \int_{\mathcal{T}^n} k(x,y)u(y)dy$ with $k \in C^\infty(\mathcal{T}^n \times \mathcal{T}^n)$, is a smoothing operator.

Finally, for $\sigma \in S^r(\mathcal{T}^n)$ the pseudodifferential operator $\sigma(x, D)$ is said to be *elliptic* if there exist $C, R > 0$ such that

$$|\sigma(x, \xi)| \geq C|\xi|^r \quad \text{for} \quad |\xi| > R \;,\; x \in \mathcal{T}^n \;. \tag{3.6}$$

A simple and useful instance of an elliptic operator is induced by the function $\xi \mapsto \langle\xi\rangle$, $\xi \in \mathbb{Z}^n$, where $\langle\xi\rangle = |\xi|$ if $\xi \neq 0$. Further we set $\langle\xi\rangle = 1$ if $\xi = 0$. It is well-known that for $s \in \mathbb{R}$, one can define the Sobolev spaces $H^s(\mathcal{T}^n)$ by

$$H^s(\mathcal{T}^n) = \{u \in \mathcal{D}'(\mathcal{T}^n) : \langle D\rangle^s u \in L^2(\mathcal{T}^n)\} \tag{3.7}$$

equipped with the norm

$$\|u\|_s = \Big(\sum_{\xi\in\mathbb{Z}^n} \langle\xi\rangle^{2s} |\tilde{u}(\xi)|^2 \Big)^{\frac{1}{2}} \tag{3.8}$$

and the inner product

$$(u, v)_s = \sum_{\xi\in\mathbb{Z}^n} \langle\xi\rangle^{2s} \tilde{u}(\xi)\overline{\tilde{v}(\xi)} \;. \tag{3.9}$$

Notice that $H^{-s}(\mathcal{T}^n)$ can be identified with the dual space of $H^s(\mathcal{T}^n)$ with respect to the sesquilinear form $(f, u)_0$.

Recall that a pseudodifferential operator $A \in \Psi^r(\mathcal{T}^n)$ maps

$$A : H^s(\mathcal{T}^n) \to H^{s-r}(\mathcal{T}^n) \quad , \quad s \in \mathbb{R} \;, \tag{3.10}$$

boundedly.

Our central objective is to solve the pseudodifferential equation

$$Au = f \tag{3.11}$$

on $\mathcal{T}^n$ for $u \in H^s(\mathcal{T}^n)$, where $A \in \Phi^\mu(\mathcal{T}^n)$, $\operatorname{Re}\mu = r$, and $f \in H^{s-r}(\mathcal{T}^n)$.

Let the spaces V^j be defined by (2.16) with respect to some fixed refinable function φ. We will study a rather general class of numerical schemes for the solution of (3.11) based on a fixed compactly supported distribution

$$\eta \in H^{-s'}(\Gamma), \tag{3.12}$$

where $s' \geq 0$ satisfies $AV^j \subset H^{s'}(\mathcal{T}^n)$, and where $\Gamma \subset \mathbb{R}^n$ is some fixed reference domain such as a hypercube. Thus, defining for $g \in H^{s'}(\mathbb{R}^n)$

$$\eta_k^j(g) := 2^{-nj/2}\eta(g(2^{-j}(\cdot + k))), \tag{3.13}$$

the space

$$X^j := \operatorname{span}\{\eta_k^j \ : \ k \in \mathbb{Z}^{n,j}\} \tag{3.14}$$

is contained in $(AV^j)'$, the dual of AV^j. The corresponding Galerkin-Petrov scheme is then given by

$$\eta_k^j(Au^j) = \eta_k^j(f), \quad k \in \mathbb{Z}^{n,j}. \tag{3.15}$$

Specifically, the choice $\eta = \delta(\cdot - \omega_0)$, i.e.,

$$\eta(g) := g(\omega_0), \tag{3.16}$$

gives rise to the type of collocation schemes studied in [42] for $n = 1$ and in [38, 12] for arbitrary spatial dimension and tensor product spline spaces, while

$$\eta(g) = (g, \overline{\varphi}) \tag{3.17}$$

corresponds to the standard Galerkin scheme. Of course, one has to assert that (3.15) admits a unique solution. Our approach to this question as well as to the corresponding convergence analysis is based on an appropriate notion of *stability* which will be formulated in the following section.

4 Stability for Constant Coefficient Operators

In this section we will confine the discussion to the case of constant coefficient operators. The general case will be reduced later to this situation by means of principle of locally freezing coefficients established in [33].

Thus we consider here the equation

$$\sigma(D)u = f, \tag{4.1}$$

where

$$\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\}), \qquad \sigma(0) = 1,$$

is positively homogeneous of degree $\mu \in \mathbb{C}$, $\operatorname{Re}\mu = r$, i.e., $\sigma\,|_{\mathbb{Z}^n} \in \Sigma^\mu(\mathcal{T}^n)$. Throughout this section φ will be a fixed refinable and stable function in $H^s(\mathbb{R}^n)$ whose additional properties will be specified when there is any need. Writing

$$\mathbf{u}^j := (u_k^j \ : \ k \in \mathbb{Z}^{n,j}), \quad \mathbf{u}^j *^\circ \varphi^j := \sum_{k \in \mathbb{Z}^{n,j}} u_k^j \varphi_k^j \in V^j$$

(3.15) gives rise to a linear sytem involving the *circulant* matrix

$$\mathbf{A}^j := (a^j_{k,m})_{k,m\in\mathbb{Z}^{n,j}} = (a^j_{[k-m]})_{k,m\in\mathbb{Z}^{n,j}}, \tag{4.2}$$

where

$$a^j_k := \eta^j_k(\sigma(D)\varphi^j_0), \quad k\in\mathbb{Z}^{n,j}, \tag{4.3}$$

and $[k-m] := (k-m) \bmod 2^j$. It is well-known that the unitary matrix

$$\mathbf{F} := \left(2^{-jn/2}e^{-2\pi i 2^{-j}\langle k,m\rangle}\right)_{k,m\in\mathbb{Z}^{n,j}} \tag{4.4}$$

diagonalizes $\mathbf{A}^j$, i.e.,

$$\mathbf{F}^{-1}\mathbf{A}^j\mathbf{F} = (\alpha_k\delta_{k,m})_{k,m\in\mathbb{Z}^{n,j}}, \tag{4.5}$$

where

$$\alpha_k := \sum_{m\in\mathbb{Z}^{n,j}} a^j_m e^{-2\pi i 2^{-j}\langle k,m\rangle}. \tag{4.6}$$

In the following let $\hat{\eta}$ denote the Fourier transform of η in the distributional sense, i.e., $\hat{\eta}(\phi) = \eta(\hat{\phi})$ holds for any $\phi\in\mathcal{S}$.

Theorem 4.1 *Suppose the series*

$$\alpha(\omega) := [\sigma\hat{\varphi}\overline{\hat{\eta}}](\omega) = \sum_{m\in\mathbb{Z}^n}\sigma(\omega+m)\hat{\varphi}(\omega+m)\overline{\hat{\eta}(\omega+m)} \tag{4.7}$$

converges absolutely for $\omega\in\mathcal{T}^n$. Then the eigenvalues α_k (4.6) arising from the Galerkin-Petrov scheme (3.15) are given by

$$\alpha_0 = 1, \qquad \alpha_k = 2^{j\mu}\alpha(2^{-j}k), \quad k\in\mathbb{Z}^{n,j},\ j\in\mathbb{N}_0. \tag{4.8}$$

Proof: By (2.19) and (2.15) we have

$$\tilde{\varphi}^j_0(\xi) = 2^{-nj/2}\hat{\varphi}(2^{-j}\xi), \quad \xi\in\mathbb{Z}^n, \tag{4.9}$$

so that, in view of (3.4),

$$(A\varphi^j_0)(x) = 2^{-nj/2}\sum_{\xi\in\mathbb{Z}^n}\sigma(\xi)\hat{\varphi}(2^{-j}\xi)e^{2\pi i\langle x,\xi\rangle}. \tag{4.10}$$

Upon combining (4.3), (4.10), and (3.13), we obtain

$$\begin{aligned} a^j_k &= 2^{-nj/2}\eta\left(A\varphi^j_0(2^{-j}(\cdot+k))\right) \\ &= 2^{-nj}\sum_{\xi\in\mathbb{Z}^n}\sigma(\xi)\hat{\varphi}(2^{-j}\xi)e^{2\pi i 2^{-j}\langle k,\xi\rangle}\eta(e^{2\pi i 2^{-j}\langle\cdot,\xi\rangle}) \\ &= 2^{-nj}\sum_{\xi\in\mathbb{Z}^n}\sigma(\xi)\hat{\varphi}(2^{-j}\xi)\overline{\hat{\eta}(2^{-j}\xi)}e^{2\pi i 2^{-j}\langle k,\xi\rangle}. \end{aligned} \tag{4.11}$$

Using the homogeneity of $\sigma\in\Sigma^\mu(\mathcal{T}^n)$, (4.11) and (4.6) yield

$$\alpha_k = 2^{-nj}\sum_{m\in\mathbb{Z}^{n,j}}\left(\sum_{\xi\in\mathbb{Z}^n}\sigma(\xi)\hat{\varphi}(2^{-j}\xi)\overline{\hat{\eta}(2^{-j}\xi)}e^{2\pi i 2^{-j}\langle m,\xi\rangle}\right)e^{-2\pi i 2^{-j}\langle m,k\rangle}.$$

By our hypothesis, we may interchange the summation in the above double sum so that the orthogonality relation

$$\sum_{m\in\mathbb{Z}^{n,j}} e^{-2\pi i 2^{-j}\langle m,k-\xi\rangle} = \begin{cases} 2^{nj} & \text{if } \xi = k + 2^j\zeta \text{ with } \zeta\in\mathbb{Z}^n, \\ 0 & \text{otherwise,} \end{cases}$$

provides

$$\alpha_k = 2^{j\mu}\sum_{\xi\in\mathbb{Z}^n}\sigma(\xi+2^{-j}k)\hat{\varphi}(\xi+2^{-j}k)\overline{\hat{\eta}(\xi+2^{-j}k)},$$

proving the assertion. □

As in [38] we will refer to the periodic function α defined by (4.7) as the *numerical symbol* of the approximation method (3.15). Our next step toward defining our notion of stability is to introduce *discrete Sobolev norms* and *discrete Bessel potential operators.* To this end, let $\theta(\omega)$ denote the symbol of the forward difference operator Δ, i.e., the ℓ-th component of $\theta(\omega)$ is given by

$$\theta_\ell(\omega) = e^{2\pi i\omega_\ell} - 1, \tag{4.12}$$

(see e.g. [39]), and define

$$\lambda^s(\omega) = \begin{cases} 1 & \text{if } \omega = 0, \\ |\theta(\omega)|^s & \text{if } \omega\in\mathcal{T}^n\setminus\{0\}. \end{cases} \tag{4.13}$$

Now let

$$\mathbf{\Lambda}^{s,j} := \mathbf{F}^{-1}\left(\operatorname{diag}\left(2^{sj}\lambda^s(2^{-j}\nu)\right)\right)\mathbf{F}, \tag{4.14}$$

i.e.,

$$(\mathbf{\Lambda}^{s,j})_{k,m} = 2^{-nj}\sum_{\nu\in\mathbb{Z}^{n,j}} 2^{js}\lambda^s(2^{-j}\nu)e^{2\pi i 2^{-j}\langle k-m,\nu\rangle},$$

and define the norm

$$\|\mathbf{u}^j\|_{s,j} := \|\mathbf{\Lambda}^{s,j}\mathbf{u}^j\|_{\ell_2(\mathbb{Z}^{n,j})}, \quad \mathbf{u}^j\in\mathbb{C}^{2^{jn}}. \tag{4.15}$$

The space $\mathbb{C}^{2^{jn}}$ equipped with the norm $\|\cdot\|_{s,j}$ is denoted by $h^s(\mathbb{Z}^{n,j})$ and we refer to $h^s(\mathbb{Z}^{n,j})$ as a discrete Sobolev space with discrete Sobolev norm $\|\cdot\|_{s,j}$.

We are now in a position to formulate what we mean by stability. Writing briefly

$$\eta^j(g) := \{\eta^j_k(g)\}_{k\in\mathbb{Z}^{n,j}},$$

we note that

$$\eta^j(A(\mathbf{u}^j *^\circ \varphi^j)) = \mathbf{A}^j\mathbf{u}^j.$$

The scheme (3.15) is called (s,r)-*stable* if there exists some constant $c>0$ such that

$$\|\mathbf{A}^j\mathbf{u}^j\|_{s-r,j} \geq c\|\mathbf{u}^j\|_{s,j} \tag{4.16}$$

holds for all $\mathbf{u}^j\in h^s(\mathbb{Z}^{n,j})$, uniformly in $j\in\mathbb{N}_0$.

So far we have hardly made any assumptions on η and φ relative to σ. A convenient hypothesis for proving stability in the above sense may be stated as follows. The triple (σ,φ,η) is called *admissible* if the following properties hold:

(i) The series (4.7) converges absolutely, and

(ii) if

$$|\hat{\varphi}(\zeta+\omega)||\hat{\eta}(\zeta+\omega)| = o(|\omega|^r), \quad \omega \to 0, \tag{4.17}$$

uniformly in $\zeta \in \mathbb{Z}^n \setminus \{0\}$, where r is the order of σ.

Remark 4.1 *If the convolution $\eta * \varphi$ belongs to $C^{d'}(\mathbb{R}^n)$, is also refinable, and $|x|^{d'}(\eta * \varphi)(x) \in L_1(\mathbb{R}^n)$, then by Proposition 2.1 $\eta * \varphi$ satisfies Fix-Strang conditions of order d'. Therefore, (4.17) holds for $r < d'$. Moreover, if φ satisfies condition $\mathrm{C}_0^{\mathrm{d}}$ (cf. Section 2) so that $\varphi \in H^s(\mathbb{R}^n)$ for $s < d + \rho(\mathrm{a})$, one easily concludes from (2.5) and the fact that $\hat{\varphi}$ is entire, that the triple $(\langle\cdot\rangle^{2s}, \varphi, \varphi)$ is admissible.*

The following observation relates the discrete Sobolev norms to the continuous ones and justifies (4.16).

Proposition 4.1 *Suppose that for $\varphi \in H^s(\mathbb{R}^n)$ the triple $(\langle\xi\rangle^{2s}, \varphi, \varphi)$ is admissible and that φ is stable. Then*

$$\|\mathbf{u}^j *^\circ \varphi^j\|_{H^s(\mathbb{R}^n)} \sim \|\mathbf{u}^j\|_{s,j}, \quad \mathbf{u}^j \in \mathbb{C}^{2^{jn}}, \tag{4.18}$$

uniformly in $j \in \mathbb{N}_0$.

Proof: Choosing $\sigma(D) = \langle D\rangle^{2s}$ and $\eta = \varphi$, i.e., $X^j = V^j$, we obtain, in view of (3.8),

$$\|\mathbf{u}^j\|_s^2 = (\langle D\rangle^{2s} u^j, u^j)_0 = \mathbf{u}^j \mathbf{G}^j \mathbf{u}^j,$$

where $\mathbf{G}^j$ denotes the stiffness matrix for $\langle D\rangle^{2s}$. By Theorem 4.1 (4.8) and (4.7), its eigenvalues are given by

$$\alpha_k^G = 2^{2js} \sum_{\xi \in \mathbb{Z}^n} |\xi + 2^{-j}k|^{2s} |\hat{\varphi}(\xi + 2^{-j}k)|^2, \quad k \in \mathbb{Z}^{n,j} \setminus \{0\}.$$

Admissibility ensures that this series converges absolutely. On the other hand, the eigenvalues $\lambda_k^{s,j}$ of the matrix $(\mathbf{\Lambda}^{s,j})^* \mathbf{\Lambda}^{s,j}$ are

$$\lambda_k^{s,j} = 2^{2js} |\theta(2^{-j}k)|^{2s}, \quad k \in \mathbb{Z}^{n,j} \setminus \{0\}.$$

Observe next that outside a fixed neighborhood of zero in $\mathbb{T}^n$ one has

$$|\omega + \zeta|^{2s} / |\theta(\omega)|^{2s} > c_1 > 0, \quad \text{for all } \zeta \in \mathbb{Z}^n.$$

Moreover, by the boundedness of the sum $\sum_{\zeta \in \mathbb{Z}^n} |\omega + \zeta|^{2s} |\hat{\varphi}(\omega + \zeta)|^2$ on $\mathbb{T}^n$ and since $|\theta(\omega)|^{2s} > c_2$ outside a neighborhood of zero, we conclude that there

$$0 \le c_1' |\theta(\omega)|^{2s} \le \sum_{\zeta \in \mathbb{Z}^n} |\omega + \zeta|^{2s} |\hat{\varphi}(\omega + \zeta)|^2 \le c_2' |\theta(\omega)|^{2s}. \tag{4.19}$$

But since $\hat{\varphi}(0) = 1$ (see Proposition 2.1), condition (4.17) guarantees that $|\theta(\omega)|^{2s}$ and $\sum_{\zeta \in \mathbb{Z}^n} |\omega + \zeta|^{2s} |\hat{\varphi}(\omega + \zeta)|^2$ have a zero of the same order at $\omega = 0$. Hence (4.19) holds everywhere on $\mathbb{T}^n$. Therefore there exist constants $c, c' > 0$ such that

$$c\lambda_k^{s,j} \le \alpha_k^G \le c'\lambda_k^{s,j}, \quad k \in \mathbb{Z}^{n,j} \setminus \{0\}, \ j \in \mathbb{N}_0. \tag{4.20}$$

This finishes the proof. □

We mention in passing that the analogous equivalence

$$2^{-jn/2}\|\mathbf{v}^j\|_{s,j} \sim \|\mathbf{v}^j *^\circ C^j\|_s, \tag{4.21}$$

where $C_k^j := [C(2^j\cdot -k)]$ and $C(x)$ is the fundamental cardinal spline function of some coordinate degrees $d_1,\ldots,d_n$, $s < d_i + 1/2$, $i = 1,\ldots,n$, was established in [38].

Adhering to the terminology of [38], we call the numerical symbol α of a homogeneous pseudodifferential operator of order r *elliptic* if there exists a constant $c > 0$ such that

$$\inf_{\omega\in\mathcal{T}^n\setminus\{0\}} \frac{|\alpha(\omega)|}{|\theta(\omega)|^r} \geq c. \tag{4.22}$$

These notions are related as follows.

Theorem 4.2 *If $\varphi \in H^s(\mathcal{T}^n)$ and (σ,φ,η) is admissible, then the corresponding Galerkin-Petrov scheme (3.15) is (s,r)-stable if and only if the corresponding numerical symbol is elliptic.*

Proof: By (4.15) we have

$$\|\mathbf{A}^j\mathbf{u}^j\|_{s-r,j} = \|\mathbf{\Lambda}^{s-r,j}\mathbf{A}^j\mathbf{\Lambda}^{-s,j}\mathbf{v}^j\|_{\ell_2(\mathbb{Z}^{n,j})},$$

where we have set $\mathbf{v}^j := \mathbf{\Lambda}^{s,j}\mathbf{u}^j$. Note next that the eigenvalues of the matrix

$$(\mathbf{\Lambda}^{s-r,j}\mathbf{A}^j\mathbf{\Lambda}^{-s,j})^*(\mathbf{\Lambda}^{s-r,j}\mathbf{A}^j\mathbf{\Lambda}^{-s,j})$$

are given by $\beta_0 = 1$ and

$$\beta_k = |2^{jr}\alpha(2^{-j}k)\lambda^{s-r}(2^{-j}k)\lambda^{-s}(2^{-j}k)2^{-jr}|^2, \qquad k\in\mathbb{Z}^{n,j}\setminus\{0\}. \tag{4.23}$$

It is easy to see that the eigenvalues (4.23) are uniformly bounded away from below by a positive constant if and only if the numerical symbol $\alpha(\omega)$ is elliptic. Hence

$$\|\mathbf{\Lambda}^{s-r,j}\mathbf{A}^j\mathbf{\Lambda}^{-s,j}\mathbf{v}^j\|_{\ell_2(\mathbb{Z}^{n,j})} \geq c\|\mathbf{v}^j\|_{\ell_2(\mathbb{Z}^{n,j})}, \tag{4.24}$$

proving that the ellipticity of the numerical symbol implies (s,r)-stability.

Conversely, suppose α were not elliptic. Then there exists a sequence $\{k_j\}_{j\in\mathbb{N}_0}$, $k_j\in\mathbb{Z}^{n,j}$, such that

$$\alpha(2^{-j}k_j)|2^{-j}k_j|^{-r} \to 0, \quad j\to\infty. \tag{4.25}$$

Note that

$$\mathbf{v}^j := 2^{-jn/2}\left(e^{2\pi i\langle k_j,2^{-j}k\rangle}\right)_{k\in\mathbb{Z}^{n,j}},$$

is the normalized eigenvector to the eigenvalue $\sqrt{\beta_{k_j}}$ given by (4.23) and (4.25). Setting

$$\mathbf{u}^j := \mathbf{\Lambda}^{-s,j}\mathbf{v}^j,$$

we obtain

$$\begin{aligned} 1 &= \|\mathbf{u}^j\|_{j,s} = \|\mathbf{v}^j\|_{\ell_2(\mathbb{Z}^{n,j})} \\ &\leq c\|\mathbf{\Lambda}^{s-r,j}\mathbf{A}^j\mathbf{\Lambda}^{-s,j}\mathbf{v}^j\|_{\ell_2(\mathbb{Z}^{n,j})} = c\sqrt{\beta_{k_j}}. \end{aligned} \tag{4.26}$$

But by (4.25) we know that $\beta_{k_j} \to 0,\ j \to \infty$, contradicting (4.26). This completes the proof. □

Since admissibility ensures that the asymptotic behavior of σ and of the numerical symbol coincide near zero the same arguments as used in [38] allow to establish the following fact.

Proposition 4.2 *If (σ, φ, η) is admissible and if the numerical symbol of the homogeneous pseudodifferential operator $\sigma(D)$ is elliptic, then $\sigma(D)$ is elliptic.*

As pointed out in [38, 39] the ellipticity of the pseudodifferential operator does in general not imply ellipticity of the numerical symbol.

To analyse stability and convergence properties of the schemes (3.15) for variable symbols it is convenient to rephrase these schemes as projection methods. To describe this, let, for some $s' \geq 0$, $\eta \in H^{-s'}(\Gamma)$ be a given functional as described in (3.12) above.

We will then seek for an appropriate 'dual function' $\phi \in H^s(\mathbb{R}^n) \cap \mathcal{L}_2$ satisfying

$$\eta(\phi(\cdot - \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n. \tag{4.27}$$

The mappings

$$Q_j u := \sum_{k \in \mathbb{Z}^{n,j}} \eta_k^j(u) \phi_k^j \tag{4.28}$$

project $H^s(\mathcal{T}^n)$ onto the spaces

$$Y^j := \operatorname{span}\{\phi_k^j : k \in \mathbb{Z}^{n,j}\}. \tag{4.29}$$

It is then clear that solving (3.15) is equivalent to finding $u^j \in V^j$ such that

$$Q_j A u^j = Q_j f. \tag{4.30}$$

Let us briefly comment on the construction of ϕ. One easily verifies that (4.27) is equivalent to

$$[\hat{\phi}\bar{\hat{\eta}}](\omega) = 1, \quad \omega \in \mathcal{T}^n. \tag{4.31}$$

Usually ϕ can be found by first looking for a compactly supported function $\gamma \in H^s(\mathbb{R}^n)$ satisfying

$$[\hat{\gamma}\bar{\hat{\eta}}](\omega) \neq 0, \quad \omega \in \mathcal{T}^n. \tag{4.32}$$

In fact, since η and γ have compact support

$$[\hat{\gamma}\bar{\hat{\eta}}](\omega) = \sum_{\xi \in \mathbb{Z}^n} \eta(\gamma(\cdot + \xi)) e^{2\pi i \langle \xi, \omega \rangle},$$

is a trigonometric polynomial which, on account of (4.32), does not vanish on $\mathcal{T}^n$. Hence

$$\frac{1}{[\hat{\gamma}\bar{\hat{\eta}}](\omega)} = \sum_{\xi \in \mathbb{Z}^n} g_\xi e^{2\pi i \langle \xi, \omega \rangle}$$

is a well-defined trigonometric series with exponentially decaying coefficients g_ξ. By construction, the Fourier transform of the function

$$\phi := g *' \gamma \tag{4.33}$$

is given by

$$\hat{\phi}(\omega) = \frac{\hat{\gamma}(\omega)}{[\hat{\gamma}\bar{\hat{\eta}}](\omega)}. \tag{4.34}$$

Thus ϕ satisfies (4.31) and therefore (4.27).

Remark 4.2 *Suppose γ satisfies $\mathrm{C}_0^{\mathrm{d}}$ and let $\rho \in (0,1)$ denote the Hölder coefficient of the dth order derivatives of γ (see Proposition 2.1). Let ϕ be given by (4.34). Then $(\langle\cdot\rangle^{2s}, \phi, \phi)$ is admissible for $s < d + \rho$.*

Proof: By (4.34) one has

$$\sum_{\xi \in \mathbb{Z}^n} \langle \omega + \xi \rangle^{2s} |\hat{\phi}(\omega + \xi)|^2 = |[\hat{\gamma}\bar{\hat{\eta}}](\omega)|^{-2} \sum_{\xi \in \mathbb{Z}^n} \langle \omega + \xi \rangle^{2s} |\hat{\gamma}(\omega + \xi)|^2,$$

whence, in view of (4.32), the assertion follows. □

We immediately conclude now from Proposition 4.1 and Remark 4.1

Remark 4.3 *Let γ satisfy $\mathrm{C}_0^{\mathrm{d}}$ and let the projectors Q_j be defined by (4.28) and (4.33). Then the scheme (3.15) is (s,r)-stable if and only if there exists some constant c such that*

$$\|Q_j A u^j\|_{s-r} \geq c\, \|u^j\|_s, \quad \textit{for all } \; u^j \in V^j,\ j \in \mathbb{N}. \tag{4.35}$$

The above framework covers the following cases of particular interest.

(I) Classical Galerkin scheme:

$$\eta(g) = (g, \varphi), \quad \gamma = \varphi, \quad s = 0, \quad X^j = V^j = Y^j.$$

In fact, the stability of φ (2.8) assures, in view of (2.9), that (4.32) holds, so that in this case

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{[\hat{\varphi}\bar{\hat{\varphi}}](\omega)} \tag{4.36}$$

is well-defined.

(II) Biorthogonal Galerkin-Petrov scheme:

$$\gamma = \varphi = \phi, \quad s = 0, \quad V^j = Y^j, \quad X^j = \operatorname{span}\{\eta_k^j : k \in \mathbb{Z}^{n,j}\},$$

where $\eta \in L_2(\mathbb{R}^n)$ is a compactly supported refinable function which is biorthogonal to φ, i.e.,

$$(\varphi, \eta(\cdot - \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n, \tag{4.37}$$

(cf. [9, 16]) so that

$$[\hat{\varphi}\bar{\hat{\eta}}](\omega) = [\hat{\gamma}\bar{\hat{\eta}}](\omega) \equiv 1, \quad g_\xi = \delta_{0,\xi},\ \xi \in \mathbb{Z}^n.$$

(III) Collocation scheme:

$$\eta(g) = g(\omega_0) \quad \text{for some} \quad \omega_0 \in \mathcal{T}^n.$$

In this case

$$\eta \in H^{-s}([0,1]^n) \quad \text{for any} \quad s > n/2. \tag{4.38}$$

Here one has to find a compactly supported continuous function γ (which could differ from φ) such that the corresponding spaces Y^j have suitable approximation properties and such that

$$\sum_{\xi \in \mathbb{Z}^n} \gamma(\omega_0 + \xi) e^{2\pi i \langle \omega, \xi \rangle} \neq 0, \quad \omega \in \mathcal{T}^n, \tag{4.39}$$

to which (4.31) specializes in this case. In the context of cardinal spline interpolation (4.39) has been established for various choices of ω_0 and several types of multivariate B-splines and box splines (see e.g. [6]) which suggests classical spline spaces as typical candidates for Y^j.

(IV) Quasiinterpolation: When γ is a tensor product B-spline or box spline explicit dual functionals of the form

$$\eta(g) := \sum_{\nu \in \mathbb{N}_0^n, |\nu| \leq d'} c_\nu (D^\nu g)(0) \tag{4.40}$$

such that

$$\eta(\gamma(\cdot - \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n,$$

are known (see e.g. [15]). Hence one can take

$$\phi = \gamma, \quad s > d' + \frac{n}{2}.$$

5 Discrete Commutator Property and Super-Approximation

Throughout this section we will assume that the refinable function γ satisfies $\mathrm{C}_0^{\mathrm{d}}$ for some $d \in \mathbb{N}_0$ and that the projectors Q_j are defined by (4.28), (4.31) and (4.33), where $\eta \in H^{-s'}(\Gamma)$ for some $s' \geq 0$ is some fixed functional as described in (3.12). We will continue denoting by $\rho \in (0,1)$ the Hölder coefficient of the dth order derivatives of γ whose existence is asserted by Proposition 2.1 (ii). We will reserve the special notation P_j for the orthogonal projectors (relative to the inner product $(\cdot,\cdot)_0$) onto the corresponding spaces Y^j, defined by (4.29). If there is any reason to distinguish the range of P_j we will write P_{Y^j}.

The objective of this section is to develop various approximation properties of the operators Q_j or P_j which will form the corner stones for our subsequent analysis of the schemes (3.15) for the case of variable symbols.

It will be convenient to employ certain equivalent norms on $H^s(\mathcal{T}^n)$ which allow us to describe local properties of the operators Q_j. The analysis is complicated somewhat by the fact that in many cases of interest the functions ϕ, defined by (4.34), do not have compact support. We will always assume this to be the case in the following and will not comment on possible simpler arguments for the special case of compact support.

To this end, define for $h \in \mathbb{R}^n$ the ℓth order forward differences of u by

$$(\Delta_h^\ell u)(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j} u(x+jh).$$

The corresponding ℓth order L_2-modulus of continuity is then given as

$$\omega_\ell(u,t,\Omega)_2 := \sup_{|h|\le t} \|\Delta_h^\ell u\|_0(\Omega_{h,l}), \tag{5.1}$$

where $\|\cdot\|_0(\Omega)$ denotes the usual L_2-norm relative to some domain $\Omega \subseteq \mathbb{R}^n$ and $\Omega_{h,l} := \{x \in \mathbb{R}^n : x+jh \in \Omega,\ j=0,\dots,\ell\}$. In addition we will need the corresponding *modified* modulus

$$w_\ell(u,t,\Omega)_2 := \left((2t)^{-n} \int_{[-t,t]^n} \|\Delta_h^\ell u\|_0^2(\Omega_{h,\ell})dh \right)^{1/2}. \tag{5.2}$$

It is known [19] that

$$w_\ell(u,t,\Omega)_2 \sim \omega_\ell(u,t,\Omega)_2, \quad t \to 0, \tag{5.3}$$

We are now ready to introduce the Besov-norm

$$\|u\|^2_{B^t_{2,2}}(\Omega) := \|u\|_0^2(\Omega) + |u|_{(t)}(\Omega), \tag{5.4}$$

where for any fixed $\ell \in \mathbb{N},\ \ell > t$

$$|u|^2_{(t)}(\Omega) := \sum_{j=0}^{\infty} 2^{2jt} \omega_\ell(u, 2^{-j}, \Omega)_2^2. \tag{5.5}$$

Here, according to (5.3), we could replace ω_ℓ in (5.5) by w_ℓ whenever this is convenient.

It is known that for domains Ω with sufficiently regular boundary (for instance 'minimally smooth' in the sense of Stein [47] will do for our purposes) and for $0 < t < \ell$, the set of all functions in $L_2(\Omega)$ for which the above expression is finite, agrees with $H^t(\Omega)$ and that

$$\|\cdot\|_t(\Omega) \sim \|\cdot\|_{B^t_{2,2}}(\Omega), \quad 0 < t < \ell, \tag{5.6}$$

(see e.g. [19, 49]).

We will omit any reference to the underlying domain when working on the torus $\mathcal{T}^n$.

Finally, the norm

$$\|u\|^2_{Y,t} := \|u\|_0^2 + \sum_{j\in\mathbb{N}} 2^{2jt} \|(P_j - P_{j-1})u\|_0^2 \tag{5.7}$$

satisfies [14]

$$\|\cdot\|_{Y,t} \sim \|\cdot\|_{B^t_{2,2}} \quad \text{for} \quad 0 < t < d+\rho, \tag{5.8}$$

which, in particular, implies that $Y^j \subset H^t(\mathcal{T}^n)$ for $t < d + \rho$.

We begin presenting direct and inverse estimates for an essentially symmetric scale of fractional order Sobolev spaces. The basic form of these estimates is, of course, quite familiar and their validity has been established for various special types of approximation spaces and Sobolev norms. Classical versions involve Sobolev norms of integer order while the general case follows usually by interpolation techniques. We will include here a proof without making explicit use of interpolation theorems since, on one hand, we are not aware of any reference which would be suitable for the present general setting and since, on the other hand, the ingredients of the proof will be of immediate use for the subsequent developments.

A key step is the following basic estimate.

Lemma 5.1 *For any $0 < t \leq d + 1$ there exists a constant c such that for all $u \in H^t(\mathcal{T}^n)$ and $j \in \mathbb{N}$*

$$\|u - P_j u\|_0 \leq c\, 2^{-jt} \|u\|_t. \tag{5.9}$$

Proof: The same arguments as in the proof of Theorem 6.1 in [14] confirm that, under the given assumptions on φ, one has

$$\|u - P_j u\|_0 \leq c\, \omega_{d+1}(u, 2^{-j})_2, \quad u \in L_2(\mathcal{T}^n), \tag{5.10}$$

where c is independent of u and $j \in \mathbb{N}_0$. Thus, when $t = d + 1$ the assertion follows from the well-known estimate

$$\omega_{d+1}(u, \tau)_2 \leq c\, \tau^{d+1} \|u\|_{d+1} \tag{5.11}$$

which holds uniformly in $\tau > 0$ and $u \in H^{d+1}(\mathcal{T}^n)$ (see e.g. [24]). If $t < d + 1$ we have

$$\begin{aligned} \|P_j u - u\|_0^2 &\leq c\, 2^{-2jt} \left(2^{2jt} \omega_{d+1}(u, 2^{-j})_2^2\right) \\ &\leq c\, 2^{-2jt} \left(\sum_{m \in \mathbb{N}_0} 2^{2mt} \omega_{d+1}(u, 2^{-m})_2^2\right) \\ &\leq c\, 2^{-2jt} \|u\|_{B^t_{2,2}}^2, \end{aligned}$$

so that the assertion follows from (5.6). □

It is now an easy matter to prove the following direct and inverse estimates.

Theorem 5.1 *Let $-d - 1 \leq s < d + \rho$, $-d - \rho < t \leq d + 1$ and $s \leq t$. Then the* Jackson *estimate*

$$\|u - P_j u\|_s \leq c 2^{j(s-t)} \|u\|_t \tag{5.12}$$

holds for all $u \in H^t(\mathcal{T}^n)$, where c is independent of j and u.

Moreover, when $s \leq t < d + \rho$ there exists a constant c such that for all $u^j \in V^j$, $j \in \mathbb{N}_0$ the Bernstein *estimate*

$$\|u^j\|_t \leq c 2^{j(t-s)} \|u^j\|_s \tag{5.13}$$

is valid.

Proof: In fact, note first that

$$(P_m - P_{m-1})(P_j u - u) = \begin{cases} 0, & m \le j; \\ (P_{m-1} - P_m)u, & m > j. \end{cases} \tag{5.14}$$

Thus, using again (5.6) and (5.7), we get for $s < d + \rho$

$$\begin{aligned} \|P_j u - u\|_s^2 &\sim \|P_j u - u\|_0^2 + \sum_{m=1}^{\infty} 2^{2ms} \|(P_m - P_{m-1})(P_j u - u)\|_0^2 \\ &= \|P_j u - u\|_0^2 + \sum_{m=j+1}^{\infty} 2^{2ms} \|(P_m - P_{m-1})u\|_0^2 \\ &\le c \sum_{m=j+1}^{\infty} 2^{2m(s-t)} \|u\|_t^2, \end{aligned}$$

where we have used Lemma 5.1 in the last step. This proves (5.12) for $0 \le s < t$. The case $s < t \le 0$ follows by a simple duality argument:

$$\begin{aligned} \sup_{v \in H^{-s}(\mathcal{T}^n)} \frac{(u - P_j u, v)_{\mathcal{T}^n}}{\|v\|_{-s}} &= \sup_{v \in H^{-s}(\mathcal{T}^n)} \frac{(u, v - P_j^* v)_{\mathcal{T}^n}}{\|v\|_{-s}} \\ &= \sup_{v \in H^{-s}(\mathcal{T}^n)} \frac{(u, v - P_j^* v)_{\mathcal{T}^n}}{\|v - P_j^* v\|_{-t}} \frac{\|v - P_j^* v\|_{-t}}{\|v\|_{-s}} \\ &\le c\, \|u\|_t 2^{j(s-t)}, \end{aligned}$$

which confirms (5.12) in this case as well. In the case $s < 0 < t$ one argues similarly

$$\begin{aligned} \sup_{v \in H^{-s}(\mathcal{T}^n)} \frac{(u - P_j u, v)_{\mathcal{T}^n}}{\|v\|_{-s}} &= \sup_{v \in H^{-s}(\mathcal{T}^n)} \frac{(u - P_j u, v - P_j^* v)_{\mathcal{T}^n}}{\|v - P_j^* v\|_0} \frac{\|v - P_j^* v\|_0}{\|v\|_{-s}} \\ &\le c\, \|u - P_j u\|_0 2^{js} \\ &\le c\, 2^{j(s-t)} \|u\|_t, \end{aligned}$$

where we have used Lemma 5.1 in the last step. This completes the proof of (5.12) for the asserted range of s, t.

Likewise (5.8) and Lemma 5.1 provide

$$\begin{aligned} \|u^j\|_t^2 &\le c\, \|u^j\|_0^2 + \sum_{m=1}^{j} 2^{2mt} \|(P_m - P_{m-1})u^j\|_0^2 \\ &\le c \sum_{m=0}^{j} 2^{2m(t-s)} \|u^j\|_s^2, \end{aligned}$$

which proves (5.13) thereby finishing the proof of Theorem 5.1 □

Remark 5.1 *The direct estimate (5.12) implies that the orthogonal projectors P_j are uniformly bounded in $H^s(\mathcal{T}^n)$ for $|s| < d + \rho$.*

We will determine next to what extent the Jackson estimate (5.12) and Remark 5.1 remain valid for the larger class of operators Q_j. The analysis will, however, be much more involved and will make explicit use of the representation (4.28). We begin with collecting some auxiliary facts.

Lemma 5.2 *Under the above assumptions on η there exists for any $s \geq s'$ a constant c such that for all $u \in H^s(\mathcal{T}^n), j \in \mathbb{N}$ and $k \in \mathbb{Z}^{n,j}$*

$$|\eta_k^j(u)|^2 \leq c \left(\|u\|_0^2(\Gamma_k^j) + 2^{-2sj}\|u\|_s^2(\Gamma_k^j)\right).$$

Proof: For any $s \geq s'$ we have

$$|\eta_k^j(u)| = |2^{-nj/2}\eta(u(2^{-j}(\cdot + k)))| \leq c\, \|2^{-nj/2}u(2^{-j}(\cdot + k))\|_s(\Gamma). \tag{5.15}$$

Noting that $\Delta_h^\ell u(2^{-j}(x+k)) = \Delta_{2^{-j}h}^\ell u(y)$ when $y = 2^{-j}(x+k)$, we obtain

$$\omega_{d+1}(2^{-nj/2}u(2^{-j}(\cdot + k)), 2^{-l}, \Gamma)_2 = \omega_{d+1}(u, 2^{-j-l}, \Gamma_k^j)_2. \tag{5.16}$$

Thus we conclude from (5.6) and (5.15) that

$$\begin{aligned} |\eta_k^j(u)|^2 &\leq c \left(\|u\|_0^2(\Gamma_k^j) + \sum_{l=0}^{\infty} 2^{2sl}\omega_{d+1}(u, 2^{-j-l}, \Gamma_k^j)_2^2\right) \\ &= c \left(\|u\|_0^2(\Gamma_k^j) + 2^{-2js}\sum_{l=0}^{\infty} 2^{2s(l+j)}\omega_{d+1}(u, 2^{-(l+j)}, \Gamma_k^j)_2^2\right) \\ &\leq c \left(\|u\|_0^2(\Gamma_k^j) + 2^{-2sj}\|u\|_s^2(\Gamma_k^j)\right), \end{aligned} \tag{5.17}$$

as claimed. $\square$

Next, note that, by definition of ϕ (4.33), a straightforward calculation yields

$$\phi_l^j = \sum_{k \in \mathbb{Z}^{n,j}} g_{l-k}^j \gamma_k^j, \tag{5.18}$$

where

$$g_k^j := \sum_{\xi \in \mathbb{Z}^n} g_{2^j\xi - k}, \quad k \in \mathbb{Z}^{n,j}, \tag{5.19}$$

and g_ξ are the coefficients appearing in (4.33). It will be convenient to use the following notation. For any domain $\Omega \subseteq \mathcal{T}^n$ let

$$\Omega_k^j := 2^{-j}(k + \Omega), \quad \Omega_{k,\gamma}^j := \{l \in \mathbb{Z}^{n,j} : \Omega_k^j \cap (\operatorname{supp} \gamma_l^j) \neq \emptyset\}.$$

Specifically, we will briefly write $\square := [0,1]^n$. One easily concludes now from the exponential decay of the coefficients g_ξ that there exists some constant c, independent of $j \in \mathbb{N}_0$, $k \in \mathbb{Z}^{n,j}$, for which

$$\sum_{m \in \mathbb{Z}^{n,j}} \sum_{l \in \square_{k,\gamma}^j} |g_{m-l}^j| \leq c, \quad \sum_{k \in \mathbb{Z}^{n,j}} \sum_{l \in \square_{k,\gamma}^j} |g_{m-l}^j| \leq c. \tag{5.20}$$

Lemma 5.3 *Suppose $s' < d + \rho$ (cf. (3.12)). Then for any $0 \le t < d + \rho, s' \le s$ there exists a constant c such that for $u \in H^s(\mathcal{T}^n)$*

$$\|Q_j u\|_t(\Box_k^j) \le c\, 2^{jt} \sum_{l \in \mathbb{Z}^{n,j}} \sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \left(\|u\|_0(\Gamma_l^j) + 2^{-js}\|u\|_s(\Gamma_l^j) \right),$$

where Γ is the domain from (3.12).

Proof: We infer from (5.18) that

$$\begin{aligned} \|\phi_l^j\|_t(\Box_k^j) &\le \sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j|\, \|\gamma_m^j\|_t \\ &\le c\, 2^{jt} \sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j|, \end{aligned} \tag{5.21}$$

where we have used the inverse estimate (5.13) and the fact that $\|\gamma_m^j\|_0 \le c$ in the last step. The assertion follows now from (5.20), (5.21), Lemma 5.2, and (4.28). □

Let F_k^j be the dual functionals appearing in Lemma 2.1 relative to γ satisfying $\mathbf{C}_0^{\mathbf{d}}$ and consider the associated projectors

$$G_j u := \sum_{k \in \mathbb{Z}^{n,j}} F_k^j(u)\gamma_k^j.$$

Lemma 5.4 *Let $\Omega \subseteq \mathbb{R}^n$ be some fixed bounded domain satisfying the uniform cone property (cf. [24]). Then there exists a constant c such that*

$$\|G_j u - u\|_0(\Omega_k^j) \le c\, w_{d+1}(u, 2^{-j}, \tilde{\Omega}_k^j)_2, \quad u \in L_2(\mathcal{T}^n), \tag{5.22}$$

where

$$\tilde{\Omega}_k^j := \bigcup \{\Box_l^j : (\operatorname{supp} \gamma_l^j) \cap \Omega_k^j \ne \emptyset\}.$$

Moreover, for $0 \le s \le t,\ s < d + \rho,\ t \le d + 1$, one has

$$\|G_j u - u\|_s(\Omega_k^j) \le c\, 2^{-j(t-s)} \|u\|_t(\tilde{\Omega}_k^j), \quad u \in H^t(\mathcal{T}^n). \tag{5.23}$$

Proof: Without loss of generality, we may assume that j is sufficiently large so that $\tilde{\Omega}_k^j \subset\subset \mathcal{T}^n$. Since γ satisfies $\mathbf{C}_0^{\mathbf{d}}$ and since G_j is a projector, we infer from Proposition 2.1 that G_j reproduces all polynomials of degree d on $\tilde{\Omega}_k^j$, so that

$$\|G_j u - u\|_s(\Omega_k^j) \le \|G_j(u - p)\|_s(\Omega_k^j) + \|u - p\|_s(\Omega_k^j), \tag{5.24}$$

and therefore, on account of (2.23), when $s = 0$,

$$\|G_j u - u\|_0(\Omega_k^j) \le c \inf_{p \in \Pi_d(\mathbb{R}^n)} \|u - p\|_0(\tilde{\Omega}_k^j). \tag{5.25}$$

A Whitney type estimate (cf. [19]) ensures the existence of a polynomial $p_0 \in \Pi_d(\mathbb{R}^n)$ such that

$$\|u - p_0\|_0(\tilde{\Omega}_k^j) \le c\, w_{d+1}(u, 2^{-j}, \tilde{\Omega}_k^j)_2, \quad u \in L_2(\tilde{\Omega}_k^j), \tag{5.26}$$

which, in view of (5.25), proves (5.22).

Now we can use again (5.6) for the standard interpolation argument, i.e., (5.26) yields

$$
\begin{aligned}
\|u-p_0\|_0(\tilde{\Omega}_k^j) &\leq c\, 2^{-jt}\left(\sum_{l=0}^{\infty} 2^{2lt} w_{d+1}(u,2^{-l},\tilde{\Omega}_k^j)_2^2\right)^{1/2} \\
&\leq c\, 2^{-jt}\|u\|_t(\tilde{\Omega}_k^j), \qquad (5.27)
\end{aligned}
$$

which, by (5.26), confirms (5.23) for $s=0$.

As for $0 \leq s \leq t$, note first that (2.23) yields

$$
\begin{aligned}
\|G_j(u-p_0)\|_s(\Omega_k^j) &\leq c \sum_{k' \in \Box_{k,\gamma}^j} \|u-p_0\|_0(\Box_{k'}^j)\|\gamma_k^j\|_s \\
&\leq c\, 2^{js}\|u-p_0\|_0)(\tilde{\Omega}_k^j) \leq c\, 2^{-j(t-s)}\|u\|_t(\tilde{\Omega}_k^j), \qquad (5.28)
\end{aligned}
$$

where we have used (5.13) and (5.27) in the last two steps, respectively. Furthermore, by (5.6),

$$
\begin{aligned}
\|u-p_0\|_s^2(\Omega_k^j) &\leq c\left(\|u-p_0\|_0^2(\Omega_k^j) + \sum_{l=0}^{j} 2^{2sl} w_{d+2}(u-p_0,2^{-l},\Omega_k^j)_2^2\right. \\
&\quad + \left.\sum_{l=j+1}^{\infty} 2^{2sl} w_{d+2}(u-p_0,2^{-l},\Omega_k^j)_2^2\right) \qquad (5.29) \\
&=: c\,(T_1+T_2+T_3).
\end{aligned}
$$

Taking $w_\ell(u,t,\Omega)_2 \leq c\,\|u\|_0(\Omega)$, (5.27) provides for $t < d+2$,

$$
T_2 \leq c\sum_{l=0}^{j} 2^{2sl}\|u-p_0\|_0^2(\Omega_k^j) \leq c\, 2^{-2j(t-s)}\|u\|_t^2(\tilde{\Omega}_k^j). \qquad (5.30)
$$

Observing that $\Delta_h^{d+2}p_0 = 0$, we note next that for $s \leq t$

$$
\begin{aligned}
T_3 &\leq c\, 2^{-2j(t-s)} \sum_{l=j+1}^{\infty} 2^{2l(t-s)} 2^{2ls} w_{d+2}(u,2^{-l},\Omega_k^j)_2^2 \\
&\leq c\, 2^{-2j(t-s)}\|u\|_t^2(\Omega_k^j). \qquad (5.31)
\end{aligned}
$$

The assertion follows now from (5.25), (5.28), (5.29), (5.30), and (5.31). □

Lemma 5.5 *For $0 \leq s \leq t$, $s < d+\rho$ and $t \leq d+1$ there exists a constant c independent of $j \in \mathbb{N}_0, k \in \mathbb{Z}^{n,j}$ such that for any $u \in H^t(\mathcal{T}^n)$*

$$
\|Q_j u - u\|_s(\Box_k^j) \leq c\, 2^{j(s-t)} \sum_{l \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j|\right) \|u\|_t(\tilde{\Gamma}_l^j).
$$

Proof: Since $Q_j u - u = Q_j(u - G_j u) + (G_j u - u)$ we obtain

$$\|Q_j u - u\|_s(\Box_k^j) \leq \|Q_j(u - G_j u)\|_s(\Box_k^j) + \|u - G_j u\|_s(\Box_k^j). \tag{5.32}$$

By Lemma 5.3 we get

$$\begin{aligned} \|Q_j(u - G_j u)\|_s(\Box_k^j) \ \leq \ & c\, 2^{js} \sum_{l \in \mathbb{Z}^{n,j}} \Big(\|G_j u - u\|_0(\Gamma_l^j) \\ & + \ 2^{-js'} \|G_j u - u\|_{s'}(\Gamma_l^j) \Big) \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right). \end{aligned} \tag{5.33}$$

Invoking Lemma 5.4, the right hand side of (5.33) can be estimated by

$$c\, 2^{js} \sum_{l \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right) \left(2^{-jt} \|u\|_t(\tilde{\Gamma}_l^j) \right),$$

which proves the assertion of Lemma 5.5. $\Box$

We are now ready to prove the following extension of Theorem 5.1 and Remark 5.1.

Theorem 5.2 *Let $s, s' < d + \rho$, $0 \leq s \leq t$ and $s' \leq t \leq d + 1$. Then there exists a constant $c < \infty$, independent of j, such that*

$$\|Q_j u - u\|_s \leq c\, 2^{-j(t-s)} \|u\|_t, \quad u \in H^t(\mathcal{T}^n), \quad j \in \mathbb{N}_0, \tag{5.34}$$

as well as

$$\|Q_j u\|_t \leq c\, \|u\|_t, \quad u \in H^t(\mathcal{T}^n), \quad j \in \mathbb{N}_0. \tag{5.35}$$

Proof: By (5.3), (5.6) and the fact that the modified moduli of smoothness permit summing over the respective domains we may invoke Lemma 5.5 and obtain

$$\begin{aligned} & \|Q_j u - u\|_s^2 \leq c \sum_{k \in \mathbb{Z}^{n,j}} \|Q_j u - u\|_s^2(\Box_k^j) \\ \leq \ & c\, 2^{-2j(t-s)} \sum_{k \in \mathbb{Z}^{n,j}} \left(\sum_{l \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right) \|u\|_t(\tilde{\Gamma}_l^j) \right)^2 \\ \leq \ & c\, 2^{-2j(t-s)} \sum_{k \in \mathbb{Z}^{n,j}} \left\{ \sum_{l \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right) \right\} \left\{ \sum_{l \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right) \|u\|_t^2(\tilde{\Gamma}_l^j) \right\}. \end{aligned} \tag{5.36}$$

By (5.20), the right hand side of (5.36) can be bounded by

$$c\, 2^{-2j(t-s)} \sum_{l \in \mathbb{Z}^{n,j}} \left\{ \sum_{k \in \mathbb{Z}^{n,j}} \left(\sum_{m \in \Box_{k,\gamma}^j} |g_{l-m}^j| \right) \right\} \|u\|_t^2(\tilde{\Gamma}_l^j) \leq c\, 2^{-2j(t-s)} \|u\|_t^2,$$

where we have used the fact that $\operatorname{diam}\tilde{\Gamma}_l^j \sim 2^{-j}$. This proves (5.34). The bound (5.35) follows now from (5.34) with $s = t$. $\square$

The subsequent considerations are motivated by the following observation. If $\eta(g) = g(\omega_0)$, for some $\omega_0 \in \mathcal{T}^n$, i.e., if Q_j is a Lagrange interpolation projector, one trivially has

$$Q_j f u = Q_j f Q_j u \tag{5.37}$$

for every fixed cut-off function f and any $u \in C(\mathcal{T}^n)$.

In general, the projectors Q_j will not satisfy (5.37) but we will show that under certain circumstances (5.37) will hold at least *asymptotically* for the whole class of projectors considered above. As a first step in this direction we will establish a super-approximation result which, due to its close connection with (5.37), is sometimes referred to as *discrete commutator property.*

Theorem 5.3 *Let $-d-1 \leq s < d+\rho$, $s \leq t < d+\rho$. There exists a constant $0 < \delta < 1$ and a constant $c = c(s,t,\delta)$ such that one has for any $f \in C^\infty(\mathcal{T}^n)$ and any $u^j \in V^j$*

$$\|(I - P_j)(fu^j)\|_s \leq c\, 2^{-j\delta} 2^{-j(t-s)} \|f\|_{d+1,\infty} \|u^j\|_t, \tag{5.38}$$

where $\|f\|_{l,\infty}(\Omega) := \max_{|\nu|\leq l} \sup_{x\in\Omega} |\partial^\nu f(x)|$ and $\|\cdot\|_{l,\infty} := \|\cdot\|_{l,\infty}(\mathcal{T}^n)$. Specifically, one can take

$$\delta := 1 - \rho.$$

Proof: As before let

$$G_j(u) := \sum_{k\in\mathbb{Z}^{n,j}} F_k^j(u)\gamma_k^j \tag{5.39}$$

define the projectors onto the spaces Y^j which appear in Lemma 5.4. For a given $f \in C^\infty(\mathcal{T}^n)$, $u^j \in Y^j$, and any constant b let $g^j := fu^j$ and $g_b^j := (f-b)u^j$. One readily verifies that

$$G_j g^j - g^j = G_j g_b^j - g_b^j,$$

so that we conclude from (5.22) in Lemma 5.4 that

$$\|G_j g^j - g^j\|_0(\square_k^j) \leq c\, w_{d+1}(g_b^j, 2^{-j}, \square_k^j)_2. \tag{5.40}$$

Next note that

$$\begin{aligned}\|\Delta_h^{d+1} g_b^j\|_0^2(\Omega) &\leq c\,(\|f-b\|_{0,\infty}^2(\Omega)\|\Delta_h^{d+1}u^j\|_0^2(\Omega) \\ &\quad + \sum_{q=1}^{d+1} \|\Delta_h^q f\|_{0,\infty}^2 \|\Delta_h^{d+1-q} u^j\|_0^2(\Omega_{q,h}))\end{aligned}$$

where $\Omega_{q,h} := \cup_{l\leq q}(lh + \Omega)$. Thus, since the constant b is arbitrary, we get

$$\begin{aligned}\|G_j g^j - g^j\|_0^2(\square_k^j) &\leq c\,(2^{-2j}\|f\|_{1,\infty}^2 w_{d+1}(u^j, 2^{-j}, \square_k^j)_2^2 \\ &\quad + \sum_{l=1}^{d+1} 2^{-2lj}\|f\|_{l,\infty}^2 w_{d+1-l}(u^j, 2^{-j}, \square_k^j)_2^2).\end{aligned} \tag{5.41}$$

Since $u^j \in C^d(\mathcal{T}^n)$ we may use (5.11) again to conclude that

$$w_{d+1-l}(u^j, 2^{-j}, \square_k^j)_2^2 \leq \omega_{d+1-l}(u^j, 2^{-j}, \square_k^j)_2^2 \leq c\, 2^{-2j(d+1-l)} \|u^j\|_{d+1-l}, \quad 1 \leq l \leq d+1.$$

Thus, upon summing over $k \in \mathbb{Z}^{n,j}$, we infer from (5.41) that

$$\|G_j g^j - g^j\|_0^2 \leq c\, \|f\|_{d+1,\infty}^2 (2^{-2j} w_{d+1}(u^j, 2^{-j}, \mathcal{T}^n)_2^2 + 2^{-2j(d+1)} \|u^j\|_d^2). \tag{5.42}$$

Now fix some s'' with $d < s'' < d + \rho$. Since obviously

$$w_{d+1}(u, t, \mathcal{T}^n)_2 \leq \omega_{d+1}(u, t)_2$$

we derive from (5.42) the estimate

$$\begin{aligned} \|G_j g^j - g^j\|_0^2 &\leq c\, \|f\|_{d+1,\infty}^2 \left(2^{-2j(s''+1)}(2^{2js''} \omega_{d+1}(u^j, 2^{-j})_2^2) + 2^{-2j(d+1)} \|u^j\|_d^2\right) \\ &\leq c\, \|f\|_{d+1,\infty}^2 2^{-2j(d+1)} \left(\sum_{l \in \mathbb{N}} 2^{2ls''} \omega_{d+1}(u^j, 2^{-l})_2^2 + \|u^j\|_d^2 \right). \end{aligned}$$

Hence, by (5.6), we obtain

$$\|G_j g^j - g^j\|_0 \leq c\, \|f\|_{d+1,\infty} 2^{-j(d+1)} \|u^j\|_{s''}. \tag{5.43}$$

Note that $\delta := 1 - \rho < d + 1 - s''$ so that, in view of the inverse estimate (5.13), the estimate (5.43) implies, in particular,

$$\|G_j g^j - g^j\|_0 \leq c\, \|f\|_{d+1,\infty} 2^{-j\delta} 2^{-jt} \|u^j\|_t, \tag{5.44}$$

for $t \leq s''$.

Since one trivially has

$$\|P_j g^j - g^j\|_0 \leq \|G_j g^j - g^j\|_0,$$

and since $s'' < d + \rho$ may be chosen arbitrarily close to $d + \rho$, the assertion of Theorem 5.3 for $s = 0$ follows now from (5.44).

Since $u^j \in V^m$ for all $m \geq j$, (5.43) implies

$$\|P_m g^j - g^j\|_0 \leq c\, \|f\|_{d+1,\infty} 2^{-m(d+1)} \|u^j\|_{s''}, \tag{5.45}$$

where c is independent of $m \geq j, j$ and u^j. Recalling (5.14) and using (5.6) and (5.7) again, provides for $s \geq 0$

$$\begin{aligned} \|P_j g^j - g^j\|_s^2 &\leq c \sum_{m=0}^{\infty} 2^{2ms} \|(P_m - P_{m-1})(P_j g^j - g^j)\|_0^2 \\ &\leq c \sum_{m=j}^{\infty} 2^{2ms} \|P_m g^j - g^j\|_0^2 \\ &\leq c\, \|f\|_{d+1,\infty}^2 \left(\sum_{m=j}^{\infty} 2^{-2m(d+1-s)} \right) \|u^j\|_{s''}^2, \end{aligned} \tag{5.46}$$

where we have used (5.45) in the last step. Hence

$$\|P_j g^j - g^j\|_s \leq c\, 2^{-j(d+1-s)} \|f\|_{d+1,\infty} \|u^j\|_{s''} \tag{5.47}$$

Thus (5.13) provides for any $s \leq t \leq s''$

$$\begin{aligned} \|P_j g^j - g^j\|_s &\leq c\, 2^{-j(d+1-s)} 2^{j(s''-t)} \|f\|_{d+1,\infty} \|u^j\|_t \\ &\leq c\, 2^{-j\delta} 2^{-j(t-s)} \|f\|_{d+1,\infty} \|u^j\|_t, \end{aligned} \tag{5.48}$$

where we have set again $\delta = 1 - \rho > 0$. This proves the assertion for $s \geq 0$.

Now suppose $-d-1 \leq s < 0$. Since $I - P_j = (I - P_j)^2$ the Jackson estimate (5.12) yields

$$\|P_j g^j - g^j\|_s \leq c\, 2^{js} \|(P_j - I) g^j\|_0.$$

Thus applying (5.45) to the right hand side of the above estimate and employing again the inverse estimate (5.13), confirms (5.48) also for the negative range of $s \leq t$. This completes the proof. □

Observing that for every projector Q_j onto V^j

$$Q_j u - u = Q_j(u - P_j u) + P_j u - u, \tag{5.49}$$

and recalling Theorem 5.2 yields

Corollary 5.1 *Let $s' \leq s \leq t < d + \rho$. Then there exists a constant $0 < \delta < 1$ and a constant $c = c(s, t, \delta)$ such that for any $f \in C^\infty(\mathcal{T}^n)$ and any $u^j \in V^j$*

$$\|(I - Q_j)(f u^j)\|_s \leq c\, 2^{-j\delta} 2^{-j(t-s)} \|f\|_{d+1,\infty} \|u^j\|_t.$$

The following duality argument shows that (5.38) is closely related to another super-approximation result for the orthogonal projectors P_j. Using selfadjointness gives

$$\begin{aligned} \|P_j f(I - P_j) u\|_s &= \sup_{\|w\|_{-s}=1} (u, (I - P_j) \overline{f} P_j w)_0 &(5.50) \\ &\leq \sup_{\|w\|_{-s}=1} \|u - P_j u\|_s \; \|(I - P_j) \overline{f} P_j w\|_{-s}. &(5.51) \end{aligned}$$

Thus applying the direct estimate (5.12) to the first factor and (5.38) to the second factor on the right hand side of (5.50) provides

Corollary 5.2 *Let $|s| < d + \rho$ and $s \leq t < d + \rho$. Then there exists $\delta \in (0, 1)$ and some constant $c = c(s, t, \delta)$ such that*

$$\|P_j f(I - P_j) u\|_s \leq c\, 2^{-j\delta} 2^{-j(t-s)} \|f\|_{d+1,\infty} \|u\|_t, \quad u \in H^t(\mathcal{T}^n). \tag{5.52}$$

Corollary 5.2 shows the full Sobolev scale for which (5.37) holds in an asymptotic sense for orthogonal projectors. The remainder of this section is devoted to establishing a similar result also for the projectors Q_j, which will be the second important ingredient for the subsequent stability analysis.

Theorem 5.4 *Let $s' \leq t \leq d+1$, $0 \leq s < d+\rho$ and $s \leq t$. Then there exists a sequence $\{\delta_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}\delta_j = 0$ and a constant c such that for any $f \in C^\infty(\mathcal{T}^n)$, $u \in H^t(\mathcal{T}^n)$*

$$\|Q_j f(I-Q_j)u\|_s \leq c\,\delta_j 2^{-j(t-s)}\|f\|_{d+1,\infty}\|u\|_t. \tag{5.53}$$

Proof: Let

$$r_{l,k} := \sum_{m\in\square^j_{k,\gamma}} |g^j_{l-m}|$$

and let $\{N_j\}_{j\in\mathbb{N}}$ be an increasing sequence of integers such that

$$\lim_{j\to\infty} N_j = \infty, \quad \lim_{j\to\infty} N_j 2^{-j} = 0.$$

It follows from the exponential decay of the coefficients g_ξ, $|\xi| \to \infty$ that

$$\epsilon_j := \sum_{l\in\mathbb{Z}^{n,j}\setminus K_{k,j}} r_{l,k} \to 0, \quad j\to\infty, \tag{5.54}$$

where

$$K_{k,j} := \{l \in \mathbb{Z}^{n,j} : N_j 2^j|\theta(2^{-j}(l-k))| < 2^j\}.$$

Now note that for any $b \in \mathbb{R}$

$$Q_j f(I-Q_j) = Q_j(f-b)(I-Q_j).$$

Thus Lemma 5.3 yields

$$\begin{aligned}
&\|Q_j f(I-Q_j)u\|_s(\square^j_k) = \|Q_j(f-b_k)(I-Q_j)u\|_s(\square^j_k) \qquad (5.55)\\
\leq\ & c\,2^{js}\sum_{l\in\mathbb{Z}^{n,j}} r_{l,k}\left(\|(f-b_k)(I-Q_j)u\|_0(\Gamma^j_l) + 2^{-js'}\|(f-b_k)(I-Q_j)u\|_{s'}(\Gamma^j_l)\right)\\
=\ & c\,2^{js}\left\{\sum_{l\in K_{k,j}} + \sum_{l\in\mathbb{Z}^{n,j}\setminus K_{k,j}}\right\} r_{l,k}\left(\|(f-b_k)(I-Q_j)u\|_0(\Gamma^j_l)\right.\\
+\ & \left.2^{-js'}\|(f-b_k)(I-Q_j)u\|_{s'}(\Gamma^j_l)\right)\\
=:\ & c\,2^{js}\left(\Sigma_1+\Sigma_2\right).
\end{aligned}$$

Next note that

$$\begin{aligned}
&\left\{\sum_{l\in\mathbb{Z}^{n,j}\setminus K_{k,j}} r_{l,k}\left(\|(f-b_k)(I-Q_j)u\|_0(\Gamma^j_l) + 2^{-js'}\|(f-b_k)(I-Q_j)u\|_{s'}(\Gamma^j_l)\right)\right\}^2\\
&\qquad \leq 2\,\epsilon_j \sum_{l\in\mathbb{Z}^{n,j}\setminus K_{k,j}} r_{l,k}\left(\|(f-b_k)(I-Q_j)u\|_0^2(\Gamma^j_l)\right. \qquad (5.56)\\
&\qquad\qquad \left.+\,2^{-2js'}\|(f-b_k)(I-Q_j)u\|_{s'}^2(\Gamma^j_l)\right).
\end{aligned}$$

Choosing some fixed point $x_k \in \square^j_k$ and setting $b_k := f(x_k)$, we obtain

$$\begin{aligned}
\|(f-b_k)(I-Q_j)u\|_0^2(\Gamma^j_l) &\leq \|f-b_k\|_{0,\infty}(\Gamma^j_l)\|u-Q_ju\|_0^2(\Gamma^j_l)\\
&\leq 2\|f\|^2_{0,\infty}\|u-Q_ju\|_0^2(\Gamma^j_l),
\end{aligned} \tag{5.57}$$

while by (5.6),

$$\|(f-b_k)(I-Q_j)u\|^2_{s'}(\Gamma^j_l) \leq c\,(\|(f-b_k)(I-Q_j)u\|^2_0(\Gamma^j_l) + \sum_{q=1}^{\infty} 2^{2qs'} w_{d+1}((f-b_k)(I-Q_j)u, 2^{-q}, \Gamma^j_l)^2_2). \tag{5.58}$$

For the sake of brevity let $f_k := f - b_k$, $v_j := (I-Q_j)u$ and note that as in the proof of Theorem 5.3

$$\|\Delta_h^{d+1}(f_k v_j)\|^2_0(\Omega) \leq c\,(\|f_k\|^2_{0,\infty}(\Omega)\|\Delta_h^{d+1}v_j\|^2_0(\Omega) + \sum_{p=1}^{d+1}\|\Delta_h^p f\|^2_{0,\infty}(\Omega)\|\Delta_h^{d+1-p}v_j\|^2_0(\Omega)), \tag{5.59}$$

so that by (5.58) and (5.57)

$$\begin{aligned} \|(f-b_k)(I-Q_j)u\|^2_{s'}(\Gamma^j_l) &\leq c\,(\|f_k\|^2_{0,\infty}(\Gamma^j_l)\|u-Q_ju\|^2_0(\Gamma^j_l) \\ &+ \|f_k\|^2_{0,\infty}(\Gamma^j_l)\sum_{q=1}^{\infty}2^{2qs'}w_{d+1}(v_j,2^{-q},\Gamma^j_l)^2_2 \\ &+ \|f\|^2_{d+1,\infty}\sum_{q=1}^{\infty}\sum_{p=1}^{d+1}2^{2q(s'-p)}w_{d+1-p}(v_j,2^{-q},\Gamma^j_l)^2_2 \\ &\leq c\,\|f_k\|^2_{0,\infty}(\Gamma^j_l)\left(\|(I-Q_j)u\|^2_{s'}(\Gamma^j_l)+\|u-Q_ju\|^2_0(\Gamma^j_l)\right) \\ &+ c\,\|f\|^2_{d+1,\infty}\left(\|u-Q_ju\|^2_{(s'-1)_+}(\Gamma^j_l)\right). \end{aligned} \tag{5.60}$$

Of course, when $s'=0$, one simply gets

$$\|(f-b_k)(I-Q_j)u\|^2_0(\Gamma^j_l) \leq c\,\|f_k\|^2_{0,\infty}(\Gamma^j_l)\|(I-Q_j)u\|^2_0(\Gamma^j_l). \tag{5.61}$$

At any rate, substituting (5.57) and (5.60) into (5.56) provides

$$\Sigma_2 \leq c\,\|f\|_{d+1,\infty}\left(\epsilon_j \sum_{l\in\mathbb{Z}^{n,j}\setminus K_{k,j}} r_{l,k}\left(\|u-Q_ju\|^2_0(\Gamma^j_l)+2^{-2js'}\|u-Q_ju\|^2_{s'}(\Gamma^j_l)\right)\right)^{1/2}. \tag{5.62}$$

As for Σ_1 we employ a similar argument but distinguish two cases. If $s'=0$ we get

$$\begin{aligned} \Sigma_1 &\leq c \sum_{l\in K_{k,j}} r_{l,k}\|(f-f(x_k))(I-Q_j)u\|_0(\Gamma^j_k) \\ &\leq c\,N_j 2^{-j}\|f\|_{1,\infty}\left(\sum_{l\in K_{k,j}} r_{l,k}\|Q_ju-u\|^2_0(\Gamma^j_k)\right)^{1/2}, \end{aligned} \tag{5.63}$$

where we have used in the last step Schwartz's inequality and the fact that the sum over $r_{l,k}$ remains bounded. If $s'>0$, the term $\|f_k\|_{0,\infty}(\Gamma^j_l)$ can be estimated by $N_j2^{-j}\|f\|_{1,\infty}$ since $\operatorname{diam}(\cup\{\Gamma^j_l : l\in K_{k,j}\}) \sim N_j 2^{-j}$. Hence, as in (5.60), we obtain for $l\in K_{k,j}$.

$$\begin{aligned} \|(f-b_k)(I-Q_j)u\|^2_{s'}(\Gamma^j_l) &\leq c\,\|f\|^2_{1,\infty}(N_j2^{-j})^2\left(\|u-Q_ju\|^2_0(\Gamma^j_l)\right. \\ &+ \left.\|u-Q_ju\|^2_{s'}(\Gamma^j_l)\right) + c\,\|f\|^2_{d+1,\infty}\|u-Q_ju\|^2_{(s'-1)_+}(\Gamma^j_l). \end{aligned} \tag{5.64}$$

Hence, by Schwartz's inequality,

$$\Sigma_1^2 \leq c \left(\sum_{l\in K_{k,j}} r_{l,k} \|(f - f(x_k))(I - Q_j)u\|_0^2(\Gamma_l^j) \right. \tag{5.65}$$
$$\left. + \quad 2^{-2js'} \sum_{l\in K_{k,j}} r_{l,k} \|(f - f(x_k))(I - Q_j)u\|_{s'}^2(\Gamma_l^j) \right).$$

Thus, using (5.63) and (5.64) yields

$$\Sigma_1^2 \leq c\,(N_j 2^{-j})^2 \|f\|_{1,\infty}^2 \sum_{l\in K_{k,j}} r_{l,k} \left(\|u - Q_j u\|_0^2(\Gamma_l^j) + 2^{-2js'} \|u - Q_j u\|_{s'}^2(\Gamma_l^j) \right)$$
$$+ \quad c\, 2^{-2js'} \|f\|_{d+1,\infty}^2 \sum_{l\in K_{k,j}} r_{l,k} \|u - Q_j u\|_{(s'-1)_+}^2(\Gamma_l^j). \tag{5.66}$$

Thus, setting

$$\delta_j := \begin{cases} \max\{\sqrt{\epsilon_j}, 2^{-(s'-(s'-1)_+)j}, N_j 2^{-j}\} & \text{if } s' > 0, \\ \max\{\sqrt{\epsilon_j}, N_j 2^{-j}\} & \text{if } s' = 0, \end{cases}$$

we obtain from (5.20), (5.55), (5.62), and (5.66)

$$\|Q_j f(I - Q_j)u\|_s^2(\square_k^j) \leq c\, 2^{2sj} \delta_j^2 \|f\|_{d+1,\infty}^2 \sum_{l\in \mathbb{Z}^{n,j}} r_{l,k} \left(\|u - Q_j u\|_0^2(\Gamma_l^j) \right.$$
$$\left. + \quad 2^{-2j(s'-1)_+} \|u - Q_j u\|_{(s'-1)_+}^2(\Gamma_l^j) + 2^{-2js'} \|u - Q_j u\|_{s'}^2(\Gamma_l^j) \right). \tag{5.67}$$

Now the same argument as in the proof of Theorem 5.2 yields, upon summing over $k \in \mathbb{Z}^{n,j}$

$$\|Q_j f(I - Q_j)u\|_s^2 \leq c\, \delta_j^2 2^{2sj} \|f\|_{d+1,\infty}^2 \left(\|u - Q_j u\|_0^2 + 2^{-2j(s'-1)_+} \|u - Q_j u\|_{(s'-1)_+}^2 \right.$$
$$\left. + \quad 2^{-2js'} \|u - Q_j u\|_{s'}^2(\Gamma_l^j) \right),$$

so that the assertion follows from Theorem 5.2 (5.34). This completes the proof. □

6 Stability Analysis for Variable Symbols

This section is concerned with a stability and convergence analysis for the class (3.15) of generalized Galerkin-Petrov schemes applied to the pseudodifferential equation

$$Au = f \tag{6.1}$$

on $\mathcal{T}^n$, where $A \in \Psi^\mu(\mathcal{T}^n)$, $\mathrm{Re}\,\mu = r$. Our strategy is to reduce the problem to the case of a homogeneous constant coefficient operator studied in the previous section.

Throughout this section we will assume that the function φ, which generates the trial spaces V^j, satisfies $\mathbf{C}_0^d$ for some $d \in \mathbb{N}_0$ (see Section 2). We will continue denoting by Q_j the projectors from Sections 4 and 5, defined relative to some function γ satisfying now $\mathrm{C}_0^{d'}$ for some $d' \in I\!N_0$ which may differ from d. Accordingly, we denote by $\rho, \rho' \in (0,1)$ the Hölder coefficients of the d, d'th order derivatives of φ, γ, respectively. Since we will be concerned with the spaces V^j generated by φ as well as with the spaces Y^j generated by γ we will distinguish the corresponding orthogonal projectors by writing P_{V^j}, P_{Y^j}, respectively.

In Section 5 we have collected some prerequisites to prove a series of approximation properties involving the pseudodifferential operator $A \in \Psi^r(\mathcal{T}^n)$. These facts, in turn, will allow us to characterize stability and convergence properties for the above class of Petrov-Galerkin schemes. As before we will always assume in the following that (3.12) holds.

For a linear operator A from a normed linear space X into a normed linear space Y we denote its norm by $\|A\|_{(X:Y)} := \sup_{\|u\|_X \leq 1} \|Au\|_Y$.

Proposition 6.1 *Suppose $s' < d + \rho - r, s' \leq d' + 1$ and $-d - 1 \leq s < d + \rho$. The sequence $\{A_j\}_{j \in I\!N_0}$ of finite dimensional operators*

$$A_j := Q_j A P_{V^j}$$

converges strongly to the operator

$$A : H^s(\mathcal{T}^n) \to H^{s-r}(\mathcal{T}^n) ,$$

whenever $0 \leq s - r < d' + \rho'$. Moreover, when Q_j agrees with the orthogonal projector P_{V^j}, the assertion remains valid for $-d - 1 \leq s - r < d' + \rho'$.

Proof: By (3.10), Theorem 5.2 (5.34) we obtain for $d' + 1 \geq t - r \geq s'$ and $s \leq t < d + \rho$

$$\begin{aligned} \|A_j u\|_{s-r} &\leq \|AP_{V^j}u\|_{s-r} + \|(I - Q_j)AP_{V^j}u\|_{s-r} \\ &\leq c(\|u\|_s + 2^{-j(t-s)}\|AP_{V^j}u\|_{t-r}) \\ &\leq (\|u\|_s + 2^{-j(t-s)}\|P_{V^j}u\|_t) \leq c\|u\|_s \end{aligned}$$

where we have used the inverse estimate (5.13) and Remark 5.1 in the last step. Therefore it suffices to establish the convergence on a dense subset of $H^s(\mathcal{T}^n)$. Thus for $u \in C^\infty(\mathcal{T}^n)$ we obtain as before for $s \leq t < d + \rho$

$$\begin{aligned} \|(A_j - A)u\|_{s-r} &\leq \|(I - Q_j)AP_{V^j}u\|_{s-r} + \|A(P_{V^j} - I)u\|_{s-r} \\ &\leq c\,(2^{-j(t-s)}\|u\|_t + \|P_{V^j}u - u\|_s) \\ &\leq c\,2^{-j(t-s)}\|u\|_t \end{aligned} \tag{6.2}$$

where we have used again Theorem 5.2, or (5.12) when $Q_j = P_{V^j}$, (3.10), and in the last step again the Jackson estimate (5.12). Since the right hand side tends to zero for fixed u the assertion follows. □

Corollary 6.1 *Let $0 \leq s - r < d' + \rho'$, or $-d - 1 \leq s - r < d' + \rho'$ when $Q_j = P_{V^j}$. For any cut-off function $f \in C^\infty(\mathcal{T}^n)$ with compact support one has*

$$\sup_{j \in \mathbb{N}_0} \|Q_j f P_{Y^j}\|_{(H^{s-r}(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} < \infty \tag{6.3}$$

and

$$\|P_{Y^j} f v^j\|_s \leq c \|v^j\|_s \tag{6.4}$$

for all $v^j \in Y^j$, $j \in \mathbb{N}$ and $-d - 1 \leq s < d' + \rho'$.

Proof: Since $f \in \Psi^0(\mathcal{T}^n)$ (6.3) is covered by Proposition 6.1, while (6.4) follows from (5.12) and (3.10) since $f \in \Psi^0(\mathcal{T}^n)$. □

Proposition 6.2 *Let $f \in C^\infty(\mathcal{T}^n)$ be a fixed cut-off function. Then for $-d-1 \leq s < d + \rho$, $s' < d + \rho - r$, $0 \leq s - r < d' + \rho'$ and $-d - 1 \leq s - r < d' + \rho'$, respectively, when $Q_j = P_{V^j}$, one has*

$$\|Q_j A(I - P_{V^j}) f P_{V^j}\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} \to 0 \quad as \quad j \to \infty \,. \tag{6.5}$$

Proof: The triangle inequality and (3.10) yield

$$\begin{aligned} \|Q_j A(I - P_{V^j}) f P_{V^j} u\|_{s-r} &\leq \|A(I - P_{V^j}) f P_{V^j} u\|_{s-r} \\ + \|(I - Q_j) A(I - P_{V^j}) f P_{V^j} u\|_{s-r} &\leq c\, (\|(I - P_{V^j}) f P_{V^j} u\|_s \\ &+ \|(I - Q_j) A(I - P_{V^j}) f P_{V^j} u\|_{s-r}). \end{aligned} \tag{6.6}$$

Applying Theorem 5.3 to the first summand and Theorem 5.2 for $d+\rho > t \geq s$, $t - r \geq s'$, to the second summand above, (6.6) can be estimated by

$$\begin{aligned} & c\, (2^{-j\delta} \|u\|_s + 2^{-j(t-s)} \|A(I - P_{V^j}) f P_{V^j} u\|_{t-r}) \\ \leq \; & c\, (2^{-j\delta} \|u\|_s + 2^{-j(t-s)} \|(I - P_{V^j}) f P_{V^j} u\|_t) \leq c\, (2^{-j\delta} \|u\|_s + 2^{-j\delta} 2^{-j(t-s)} \|P_{V^j} u\|_t) \,, \end{aligned} \tag{6.7}$$

where we have used Theorem 5.3 in the last step again. Applying the inverse estimate (5.13) to the second summand on the right hand side of (6.7) yields, in view of Remark 5.1, finally,

$$\|Q_j A(I - P_{V^j}) f P_{V^j} u\|_{s-r} \leq c 2^{-j\delta} \|u\|_s$$

where c is independent of j. This completes the proof. □

Let us denote by $L_{2,0}(\mathcal{T}^n)$ the set of those functions g in $L_2(\mathcal{T}^n)$ for which there exists some $y \in \mathcal{T}^n$ such that g restricted to the unit n-cube with center y has support strictly contained in the interior of that cube. Following [38] we denote for $y \in \mathcal{T}^n$ by $\mathcal{M}_y$ the *localizing* classes of functions in $C^\infty(\mathcal{T}^n) \cap L_{2,0}(\mathcal{T}^n)$ which are equal to one in some neighborhood of y. It is clear that for every $f_y \in \mathcal{M}_y$ there exists a $g_y \in \mathcal{M}_y$ such that $f_y(x) g_y(x) = g_y(x)$, $x \in \mathcal{T}^n$. Moreover, for every set $\{f_y\}_{y \in \mathcal{T}^n}$ there exists a finite subcollection $\{f_{y_i}\}_{i \in I}$ and a periodic function $g \in C^\infty(\mathcal{T}^n)$ such that

$$(\sum_{i \in I} f_{y_i}) g = 1 \quad \text{on} \quad \mathcal{T}^n \,. \tag{6.8}$$

Finally, we will need the following version of Seeley's lemma [21, 44].

Theorem 6.1 *Let $\mu \in \mathbb{C}$, $s \in \mathbb{R}$, $Re\mu = r$ and $\sigma = \sigma_0 \in \sum^\mu(\mathcal{T}^n)$. Then*

$$\inf_{K \in \Psi^{r'}(\mathcal{T}^n)} \|\sigma(x,D) + K\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} \le \sup_{x \in \mathcal{T}^n, |\xi|=1} |\sigma_0(x,\xi)|$$

holds for every $r' < r$.

We are now in a position to prove the following local equivalence of approximation operators.

Proposition 6.3 *Suppose s, s', r satisfy the hypotheses of Proposition 6.2. Let $A \in \Psi^\mu(\mathcal{T}^n)$, $Re\mu = r$ and $y \in \mathcal{T}^n$. Then there exists a homogeneous function $\sigma_y(\xi)$ of degree μ such that for each $\varepsilon > 0$, there are $T_y \in \Psi^{r'}(\mathcal{T}^n)$, $r' < r$, and $f_y \in \mathcal{M}_y$ satisfying*

$$\|Q_j((A - \sigma_y(D))f_y + T_y)P_{Vj}\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} < \varepsilon \ , \tag{6.9}$$

for all $j > j_0(\varepsilon)$.

Proof: For $\sigma_A = \sigma_0 + \sigma_1 \in \sum^\mu(\mathcal{T}^n)$ and fixed $y \in \mathcal{T}^n$, let

$$\sigma_y(\xi) = \sigma_0(y,\xi) \tag{6.10}$$

which, by assumption is a positively homogeneous function of degree μ. More precisely, σ_y is determined by

$$\lim_{R\to 0} \sup_{|\xi|>R} |\sigma_A(y,\xi) - \sigma_y(\xi)| \ |\xi|^{-r} = 0 \ . \tag{6.11}$$

Thus, Seeley's lemma Theorem 6.1 ensures that for each $\varepsilon > 0$ there exists a $T_y \in \Psi^{r'}(\mathcal{T}^n)$, $r' \le r-1$, and $f_y \in L_{2,0}(\mathcal{T}^n) \cap C^\infty(\mathcal{T}^n)$ such that

$$\|(A - \sigma_y(D))f_y + T_y\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} < \varepsilon \ . \tag{6.12}$$

Now we proceed as in previous proofs estimating for any $u \in H^s(\mathcal{T}^n)$

$$\begin{aligned} \|Q_j((A-\sigma_y(D))f_y + T_y)P_{Vj}u\|_{s-r} &\le \|(\sigma_A(\cdot,D) - \sigma_y(D))f_y + T_y)P_{Vj}u\|_{s-r} \\ + & \ \ \|(I-Q_j)((\sigma_A(\cdot,D) - \sigma_y(D))f_y + T_y)P_{Vj}u\|_{s-r} \\ \le & \ \ \varepsilon\|P_{Vj}u\|_s + c2^{-j(t-s)}\|((\sigma_A(\cdot,D) - \sigma_y(D))f_y + T_y)P_{Vj}\|_{t-r}, \end{aligned} \tag{6.13}$$

where we have used (6.12) and Theorem 5.2 for $s' \le t - r < d + \rho - r$. The second term on the right hand side of (6.13) can be bounded by

$$\begin{aligned} & c\,2^{-j(t-s)}\Big(\|(\langle D\rangle^{t-s}(A-\sigma_y(D))f_y - (A-\sigma_y(D))f_y\langle D\rangle^{t-s} \\ +(\langle D\rangle^{t-s}T_y & - T_y\langle D\rangle^{t-s}))P_{Vj}u\|_{s-r} + \|((A-\sigma_y(D))f_y + T_y)\langle D\rangle^{t-s}P_{Vj}u\|_{s-r}\Big) \ . \end{aligned} \tag{6.14}$$

Note that $\langle D\rangle^{t-s}T_y - T_y\langle D\rangle^{t-s} = \langle D\rangle^{t-s}T'$ where $T' = T_y - \langle D\rangle^{s-t}T_y\langle D\rangle^{t-s} \in \Psi^{r'}(\mathcal{T}^n)$ for $r' \le r-1$. The operator

$$\tilde{A} := \langle D\rangle^{t-s}(A-\sigma_y(D))f_y - (A-\sigma_y(D))f_y\langle D\rangle^{t-s} \ ,$$

in turn, is of order $r+t-s-1$ (see e.g. [21, 48, 26]). Thus, on account of the inverse estimate (5.13), the first summand on the right hand side of (6.14) can be bounded in terms of

$$\begin{aligned} c\, 2^{-j(t-s)} \|\tilde{A} P_{V^j} u\|_{s-r} &+ \|\langle D\rangle^{t-s} T' P_{V^j} u\|_{s-r} \leq c\, 2^{-j(t-s)} \|P_{V^j} u\|_{t-1} \\ &\leq c\, 2^{-j(t-s)} 2^{j(t-1-s)} \|P_{V^j} u\|_s = c\, 2^{-j} \|u\|_s\,. \end{aligned} \tag{6.15}$$

The second summand on the right hand side of (6.14) can be estimated again with the aid of (6.12) and the inverse estimate (5.13) giving

$$\varepsilon 2^{-j(t-s)} \|\langle D\rangle^{t-s} P_{V^j} u\|_s \leq c\, \varepsilon 2^{-j(t-s)} \|P_{V^j} u\|_t \leq c\varepsilon \|P_{V^j} u\|_s\,. \tag{6.16}$$

Combining (6.15) and (6.16) and choosing $j_0 = \lceil \log_2 c/\varepsilon \rceil$ where c is the constant in (6.15) completes the proof. □

According to (4.7), let

$$\alpha_\eta(\omega, y) := [\sigma_y \hat{\varphi} \overline{\hat{\eta}}](\omega) \tag{6.17}$$

denote the numerical symbol relative to the constant coefficient operator $\sigma_y(D)$ defined above in (6.10).

The numerical symbol α_η is called *elliptic* if there exists a constant $c > 0$ such that

$$|\alpha_\eta(\omega, y)| \geq c|\omega|^r, \;\; \omega \in [-\frac{1}{2}, \frac{1}{2}]^n, \;\; y \in \mathcal{T}^n\,. \tag{6.18}$$

We will now begin to formulate stability properties of the scheme (3.15). Adhering to the terminology of [32, 33]), we call the scheme (3.15) or the sequence $\{Q_j A P_{V^j}\}_{j\in\mathbb{N}}$ *locally* (s,r)*-stable* if for each $y \in \mathcal{T}^n$ there exist $g_y \in \mathcal{M}_y$ and operators $T_y, T'_y \in \Psi^{r'}(\mathcal{T}^n)$, $r' < r$, and bounded linear operators $C_{y,j}, D_{y,j}$ mapping V^j into itself such that

$$Q_j g_y(\sigma_y(D) + T_y) C_{y,j} \simeq_{s-r} Q_j g_y P_{V^j}, \tag{6.19}$$
$$D_{y,j} Q_j (\sigma_y(D) + T'_y) g_y P_{V^j} \simeq_s P_{V^j} g_y P_{V^j}, \tag{6.20}$$

and

$$\sup_{j\in\mathbb{N}} \|C_{y,j}\|_{(H^{s-r}(\mathcal{T}^n):H^s(\mathcal{T}^n))} < \infty, \quad \sup_{j\in\mathbb{N}} \|D_{y,j}\|_{(H^{s-r}(\mathcal{T}^n):H^s(\mathcal{T}^n))} < \infty, \tag{6.21}$$

where for any two sequences of operators B_j, C_j the notation $B_j \simeq_s C_j$ stands for

$$\lim_{j\to\infty} \|B_j - C_j\|_{(H^s(\mathcal{T}^n):H^s(\mathcal{T}^n))} = 0.$$

Proposition 6.4 *Let s', s, r satisfy the assumptions of Proposition 6.2. Suppose $A \in \Psi^\mu(\mathcal{T}^n)$, $Re\mu = r$ is invertible as an operator from $H^s(\mathcal{T}^n)$ to $H^{s-r}(\mathcal{T}^n)$. Then the scheme (3.15) is (s,r)-stable if and only if it is locally (s,r)- stable.*

Proof: First suppose (3.15) is (s,r)-stable and write briefly $B_j := Q_j B P_{V^j}$ whenever B is some operator with appropriate domain and range. Thus, by Remark 4.3, the operators A_j satisfy

$$\|A_j u^j\|_{s-r} \geq c_0 \, \|u^j\|_s, \quad u^j \in V^j, \; j \geq j_0 \in \mathbb{N}_0,$$

for some constant c_0 independent of $j \geq j_0$. Let us also abbreviate for any $f \in C^\infty(\mathcal{T}^n)$

$$f_j^P := P_{V^j} f P_{V^j}, \qquad f_j^Q := Q_j f P_{V^j}.$$

In view of Proposition 6.2 and Proposition 6.3, for each $q \in (0,1)$ and any $y \in \mathcal{T}^n$ there exist $f_y \in \mathcal{M}_y$ and $T_y \in \Psi^{r'}(\mathcal{T}^n), r' < r$ such that

$$\|\{A_j - (\sigma_y(D))_j\}(f_y)_j^P + (T_y)_j\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} \leq q/c_0 \tag{6.22}$$

for $j \geq j_0$. Choose $g_y \in \mathcal{M}_y$ so that $g_y = g_y f_y$. We infer from Theorem 5.3 that

$$\begin{aligned} A_j^{-1}(\sigma_y(D))_j(g_y)_j^P &= (g_y)_j^P + A_j^{-1}\{(\sigma_y(D))_j - A_j\}(g_y)_j^P \\ &\simeq_s (I + B_{y,j})(g_y)_j^P + A_j^{-1}(T_y)_j(g_y)_j^P, \end{aligned} \tag{6.23}$$

where

$$B_{y,j} := A_j^{-1}\left\{((\sigma_y(D))_j - A_j)(f_y)_j^P - (T_y)_j\right\}.$$

Since by (6.22)

$$\|B_{y,j}\|_{(H^s(\mathcal{T}^n):H^s(\mathcal{T}^n))} \leq q < 1,$$

the linear operators $I + B_{y,j}$ are invertible and

$$\|(I + B_{y,j})^{-1}\|_{(H^s(\mathcal{T}^n):H^s(\mathcal{T}^n))} \leq (1-q)^{-1}$$

whenever $j \geq j_0$. Hence (6.23) implies (6.20) with $D_{y,j} := (I + B_{y,j})^{-1} A_j^{-1}$. In an analogous way one derives relation (6.19) In order to prove the converse, suppose now that (6.19) and (6.20) hold for all $y \in \mathcal{T}^n$ and $j \geq j_0$. Then we conclude from Theorem 5.4 that

$$D_{y,j}\left(\sigma_y(D) + T_y'\right)_j (g_y)_j^P \simeq_s (g_y)_j^P, \tag{6.24}$$

and

$$(g_y)_j^Q \left(\sigma_y(D) + T_y\right)_j C_{y,j} \simeq_{s-r} (g_y)_j^Q. \tag{6.25}$$

Recall that for $a_y \in \mathcal{M}_y$ one has

$$A a_y - a_y A \in \Psi^{r'}(\mathcal{T}^n) \quad \text{for} \quad r' < r.$$

Thus we infer from Proposition 6.3 and Theorem 5.4 that for each $y \in \mathcal{T}^n$ there exist $a_y \in \mathcal{M}_y$, $T_y \in \Psi^{r'}(\mathcal{T}^n)$, $r' < r$ and a continuous operator $B_{y,j}$ on V^j satisfying

$$\lim_{j\to\infty} \|(a_y)_j^Q\{A_j - (\sigma_y(D))_j\} - (B_{y,j} - (T_y)_j)\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} = 0 \tag{6.26}$$

and

$$\|B_{y,j} C_{y,j}\|_{(H^{s-r}(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} \leq q < 1.$$

Now choose $f_y \in \mathcal{M}_y$ such that $f_y a_y = f_y g_y = f_y$, we see from (6.26) that

$$\lim_{j\to\infty} \|(f_y)_j^Q \{A_j - (\sigma_y(D))_j\} - (f_y)_j^Q \{B_{y,j} - (T_y)_j\}\|_{(H^s(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} = 0. \qquad (6.27)$$

Combining (6.25) and (6.27), we find that

$$(f_y)_j^Q A_j C_{y,j} \simeq_{s-r} (f_y)_j^Q \left((I + B_{y,j}C_{y,j}) - (T_y)_j C_{y,j}\right).$$

Thus

$$(f_y)_j^Q A_j G_{y,j} \simeq_{s-r} (f_y)_j^Q \left(I - (T_y)_j G_{y,j}\right), \qquad (6.28)$$

where

$$G_{y,j} := C_{y,j}(I + B_{y,j}C_{y,j})^{-1}.$$

Obviously, one has

$$\sup_{j\in\mathbb{N}} \|G_{y,j}\|_{(H^{s-r}(\mathcal{T}^n):H^s(\mathcal{T}^n))} < \infty.$$

We select now a finite number of functions $f_{y_i} \in \mathcal{M}_{y_i}$, $i = 1\ldots, N$ such that the function $f := \sum_{i=1}^N f_{y_i} \in C^\infty(\mathcal{T}^n)$ is invertible. Setting

$$C_j := \sum_{i=1}^N P_{Vj} f_{y_i} G_{y_i,j},$$

and employing Proposition 6.2, we obtain that

$$A_j C_j \simeq_{s-r} \sum_{i=1}^N (f_{y_i})_j^Q A_j G_{y_i,j} + \sum_{i=1}^N Q_j T_i G_{y_i,j},$$

where

$$T_i := A f_{y_i} - f_{y_i} A \in \Psi^{r'}(\mathcal{T}^n).$$

Setting $T_i' := T_i - f_{y_i} T_{y_i} \in \Psi^{r'}(\mathcal{T}^n)$, we apply (6.28) and get

$$A_j C_j \simeq_{s-r} f_j^Q + \sum_{i=1}^N Q_j T_i' G_{y_i,j}. \qquad (6.29)$$

Finally, let

$$\tilde{C}_j := C_j - \sum_{i=1}^N P_{Vj} A^{-1} T_i' G_{y_i,j}.$$

It follows from (6.29) that

$$A_j \tilde{C}_j \simeq_{s-r} f_j^Q + W_j,$$

where

$$W_j := \sum_{i=1}^N (Q_j - A_j A^{-1}) T_i' G_{y_i,j}.$$

On account of Theorem 5.2, Corollary 5.1 and Proposition 6.1, the operators $Q_j - A_j A^{-1}$ converge strongly to zero in $H^{s-r}(\mathcal{T}^n)$. Thus, because the operators $T_i' : H^s(\mathcal{T}^n) \to H^{s-r}(\mathcal{T}^n)$ are compact,

$$\lim_{j\to\infty} \|W_j\|_{(H^{s-r}(\mathcal{T}^n):H^{s-r}(\mathcal{T}^n))} = 0.$$

Hence $A_j\tilde{C}_j \simeq_{s-r} f_j^Q$. Since, in view of Theorem 5.4 and Corollary 6.1, $(f_j^Q)^{-1} \simeq_{s-r} (f^{-1})_j^Q$ and $\sup_j \|(f^{-1})_j^Q\| < \infty$, there exist operators D_j such that

$$\lim_{j\to\infty} \|D_j - \tilde{C}_j(f^{-1})_j^Q\|_{(H^{s-r}(\mathcal{T}^n):H^s(\mathcal{T}^n))} = 0$$

and

$$A_jD_ju^j = u^j, \quad u^j \in V^j, \quad \sup_{j\in\mathbb{N}} \|D_j\|_{(H^{s-r}(\mathcal{T}^n):H^s(\mathcal{T}^n))} < \infty.$$

Hence, because V^j has finite dimension, A_j is invertible and $(A_j)^{-1} = D_j$. This proves the (s,r)-stability of the sequence $\{A_j\}_{j\in\mathbb{N}_0}$. □

Proposition 6.5 *Suppose the hypotheses of Proposition 6.4 are satisfied. Then the Galerkin-Petrov scheme (3.15) for the operator A is (s,r)-stable if and only if the scheme is (s,r)-stable for the operator $\sigma_y(D)$ for all $y \in \mathcal{T}^n$.*

Proof: If the scheme (3.15) is (s,r)-stable for the operator $\sigma_y(D)$ for all $y \in \mathcal{T}^n$, the relations (6.19), (6.20) are fulfilled with $g_y \equiv 1$, $T_y = T'_y = 0$ and $C_{y,j} = D_{y,j} = ((\sigma_y(D))_j)^{-1}$. Thus, by Proposition 6.4, the scheme (3.15) is (s,r)-stable for the operator A.

Conversely, assume the scheme (3.15) is (s,r)-stable relative to A. Then by Proposition 6.4, the relations (6.19), (6.20) hold for each $y \in \mathcal{T}^n$. Since $\mathcal{T}^n$ is a compact manifold, there exists $N_0 \in \mathbb{N}$ such that for all $y, y' \in \mathcal{T}^n$ and $j \geq j_0$, there is a vector $k \in \mathbb{Z}^{n,j}$ such that $g_{y'} := g_y(\cdot - 2^{-j}k) \in \mathcal{M}_{y'}$. Note that the spaces V^j, Y^j as well as the projections P_{V^j}, Q_j are invariant under translation by $2^{-j}k$. Let $\tau u := u(\cdot + 2^{-j}k)$. Taking also the translation invariance of the operators $\sigma_y(D)$ into account, we deduce from (6.19) and (6.20) that for fixed $y \in \mathcal{T}^n$ and all $y' \in \mathcal{T}^n$, $j > j_0$

$$Q_jg_{y'}(\sigma_y(D) + T_{y'})C_{y',j} \simeq_{s-r} Q_jg_{y'}P_{V^j},$$

and

$$D_{y',j}Q_j(\sigma_y(D) + T'_{y'})g_{y'}P_{V^j} \simeq_s P_{V^j}g_{y'}P_{V^j},$$

where $D_{y',j} := \tau^{-1}D_{y,j}\tau, C_{y',j} := \tau^{-1}C_{y,j}\tau$ and $T_{y'} := \tau^{-1}T_y\tau, T'_{y'} := \tau^{-1}T'_y\tau$. Therefore the sequence $\{(\sigma_y(D))_j\}_{j\in\mathbb{N}}$ is locally (s,r)-stable for all $y \in \mathcal{T}^n$. Thus, by Proposition 6.4, $\{(\sigma_y(D))_j\}_{j\in\mathbb{N}}$ is (s,r)-stable for all $y \in \mathcal{T}^n$. □

Combining now Remark 4.1, Remark 4.2, Theorem 4.2 and Proposition 6.5 establishes the main result of this section.

Theorem 6.2 *Let s', s, r satisfy the hypotheses of Proposition 6.2. Suppose $A \in \Psi^\mu(\mathcal{T}^n)$, $Re\mu = r$ is invertible as an operator from $H^s(\mathcal{T}^n)$ to $H^{s-r}(\mathcal{T}^n)$. Then the scheme (3.15) is (s,r)-stable if and only if the numerical symbol α_η is elliptic.*

We are now in a position to estimate the convergence of the schemes (3.15).

Theorem 6.3 *Let* $A : H^s(\mathcal{T}^n) \to H^{s-r}(\mathcal{T}^n)$ *be an invertible pseudodifferential operator in* $\Psi^\mu(\mathcal{T}^n)$, $Re\mu = r$ *where* $-d-1 \leq s < \min\{d+\rho, d'+\rho'\}$, $0 \leq s-r < d'+\rho'$, $-d-1 \leq s-r < d'+\rho'$ *when* $Q_j = P_{V^j}$, *respectively, and* $s' < d+\rho-r$. *Suppose that for some* $t \geq s$ *such that* $d'+1 \geq t-r \geq s'$ *the right hand side* f *in (6.1) belongs to* $H^{t-r}(\mathcal{T}^n)$. *Finally, assume that the scheme (3.15) is* (s,r)*-stable. Let* u^* *denote the exact solution of (6.1) and let* u^j *denote the unique solution of (3.15) whose existence is asserted by Theorem 6.2. Then*

$$\|u^* - u^j\|_s \leq c\, 2^{-j(t-s)} \|u^*\|_t . \tag{6.30}$$

If, in addition $s' \leq s-r$, *one even has*

$$\|u^* - u^j\|_{t'} \leq c 2^{-j(t-t')} \|u^*\|_t, \quad \max\{-d-1, r\} \leq t' \leq s . \tag{6.31}$$

Finally, in case of the classical Galerkin scheme, i.e., $Q_j = P_{V^j}$, *(6.31) holds for* $\max\{-d-1, -d-1-r\} \leq t' \leq t$.

Proof: For $d'+1 \geq t-r \geq s'$ Theorem 5.2 yields

$$\begin{aligned} \|Q_j f - f\|_{s-r} &\leq c 2^{-j(t-s)} \|f\|_{t-r} = c 2^{-j(t-s)} \|Au^*\|_{t-r} \\ &\leq c 2^{-j(t-s)} \|u^*\|_t , \end{aligned} \tag{6.32}$$

where we have used, (3.10) in the last step.

Next, recall from (6.2) that

$$\|(Q_j A P_{V^j} - A)u^*\|_{s-r} \leq c\, 2^{-j(t-s)} \|u^*\|_t \tag{6.33}$$

whenever $0 \leq s-r < d'+\rho'$ or whenever $-d-1 \leq s-r < d'+\rho'$ for $Q_j = P_{V^j}$. Now note that

$$\|u^* - u^j\|_s \leq \|(P_{V^j} - I)u^*\|_s + \|P_{V^j} u^* - u^j\|_s . \tag{6.34}$$

On account of the stability (4.35) of our scheme (3.15), we have

$$\begin{aligned} \|P_{V^j} u^* - u^j\|_s &\leq c\, \|Q_j A(P_{V^j} u^* - u^j)\|_{s-r} \\ &\leq c\, \|Q_j A P_{V^j} u^* - Au^*\|_{s-r} + \|Au^* - Q_j A u^j\|_{s-r} . \end{aligned}$$

But $Q_j A u^j = Q_j f$ since $Au^* = f$. Thus

$$\|P_{V^j} u^* - u^j\|_s \leq c\, (\|(Q_j A P_{V^j} - A)u^*\|_{s-r} + \|Q_j f - f\|_{s-r})$$

so that (6.34) yields

$$\begin{aligned} \|u^* - u^j\|_s &\leq \|P_{V^j} u^* - u^*\|_s + c\, \|(Q_j A P_{V^j} - A)u^*\|_{s-r} \\ &+ c\, \|Q_j f - f\|_{s-r} . \end{aligned} \tag{6.35}$$

Substituting the estimates (5.12), (6.33), and (6.32) in the right hand side of (6.35) yields (6.30).

Now suppose that $s' \leq s-r$ and recall that the invertibility of A as an operator from $H^s(\mathcal{T}^n)$ to $H^{s-r}(\mathcal{T}^n)$ implies its invertibility as an operator from $H^{t'}(\mathcal{T}^n)$ to $H^{t'-r}(\mathcal{T}^n)$ for any $t' \in \mathbb{R}$, i.e., there exists a constant $c_{t'} > 0$ such that

$$\|Au\|_{t'-r} \geq c_{t'}\|u\|_{t'}, \quad u \in H^{t'}(\mathcal{T}^n) . \tag{6.36}$$

This regularity result yields for $t' \leq t$

$$\begin{aligned} \|u^* - u^j\|_{t'} &\leq \|u^* - P_{V^j}u^*\|_{t'} + \|P_{V^j}u^* - u^j\|_{t'} \\ &\leq c\, 2^{-j(t-t')}\|u^*\|_t + \|AP_{V^j}u^* - Au^j\|_{t'-r} , \end{aligned} \tag{6.37}$$

where we have used also (5.12). Moreover, using (3.10), we obtain

$$\begin{aligned} &\|AP_{V^j}u^* - Au^j\|_{t'-r} \leq \|A(P_{V^j}u^* - u^*)\|_{t'-r} + \|f - Au^j\|_{t'-r} \\ &\quad \leq c\, \|P_{V^j}u^* - u^*\|_{t'} + \|f - Q_jf\|_{t'-r} + \|Q_jAu^j - Au^j\|_{t'-r}. \end{aligned}$$

(5.12) and Theorem 5.2 provide now for $s - r \geq s',\ t' - r \geq 0$

$$\begin{aligned} \|AP_{V^j}u^* \;-\; & Au^j\|_{t'-r} \leq c2^{-j(s-t')}(\|u^*\|_s + \|f\|_{s-r} + \|Au^j\|_{s-r}) \\ \leq\; & c2^{-j(s-t')}(\|u^*\|_s + \|u^j\|_s) , \end{aligned} \tag{6.38}$$

since $\|f\|_{s-r} = \|Au^*\|_{s-r} \leq c\|u^*\|_s$ by (3.10). Again, when $Q_j = P_{V^j}$, the restriction $t' - r \geq 0$ may be relaxed to $-d - 1 \leq t' - r$.

Since by assumption $s - r \geq s'$ we conclude from (6.30) for $t = s$ that

$$\|u^j\|_s \leq c\|u^*\|_s ,$$

which, in view of (6.38) proves (6.31). The remaining part of the assertion follows from using Theorem 5.1 (5.12) instead of Theorem 5.2 in (6.38). □

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