Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Asymptotic Analysis of Surface Waves at Vacuum/Porous Medium and Liquid/Porous Medium Interfaces

I.Edelman[1 2], K.Wilmanski[1]

submitted: 1st November 2001

[1] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
E-Mail: edelman@wias-berlin.de
E-Mail: wilmansk@wias-berlin.de

[2] United Institute of Physics of the Earth
Russian Academy of Sciences
B.Gruzinskaya 10
Moscow, 123810
Russia

Preprint No. 695
Berlin 2001

2000 *Mathematics Subject Classification.* 35C20, 35L50, 74J15.

Key words and phrases. asymptotic expansions, waves in porous media.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

dedicated to Professor Ingo Müller on the occasion oh his 65th birthday

Abstract

Surface waves at a free interface of a saturated porous medium and at an interface between a porous medium and a liquid are investigated. Existence and peculiarities of such surface waves are revealed. At a free interface two types of surface waves are proved to be possible: the true Stoneley wave, propagating almost without attenuation, and the leaky generalized Rayleigh wave, which reradiates a part of its energy into interior of a medium. At a porous medium/liquid interface three types of surface waves are expected. These are the true Stoneley wave, the pseudo Stoneley wave, and the generalized Rayleigh wave.

Introduction

The theory of propagation of mechanical surface waves in porous and granular materials has developed primarily for single component models (see: [1, 2] for the extensive review of results). This is related to the fact that the most important practical applications appear in the field of seismology. Observations of surface waves after earthquackes, nuclear explosions or explosions of volcanos are limited to Rayleigh waves (e.g. [3]). Solely on interfaces between solids and fluids the second mode of surface waves – Stoneley waves are also registered, and described theoretically.

Most of the contemporary effort in extension of the classical theory is directed to describe properties of surface waves for heterogeneous materials. This is related to applications in nondestructive testing of soils. Rather little has been done in the analysis of additional modes of propagation of surface waves for multicomponent systems.

Let us recall briefly important features of the surface modes on the boundaries of an elastic solid. It is well known that at an interface between an isotropic solid and vacuum, there is only one surface wave—the Rayleigh wave [1,4,5]. This wave is a nondispersive plane inhomogeneous wave, nonattenuated in its direction of propagation along the surface, and damped normal to the boundary. Its phase velocity c_R is a single-valued function of parameters of an elastic half-space. It does not depend on its frequency and is close to but somewhat less than the velocity of shear wave in unbounded media. The Rayleigh wave is a coupled compressional-shear system, propagating with unique velocity c_R.

The Rayleigh waves are modified if the vacuum-bounding plane elastic half-space is replaced by a liquid or by another solid. Early studies on this subject are due to Love [6] and Stoneley [7]. The essential results are that a wave corresponding to Rayleigh wave on a free surface, due to seepage of energy into another medium becomes exponentially attenuated along its direction of propagation, while simultaneously other new modes of surface waves appear. Whereas for certain values of elastic parameters these generalized Rayleigh waves cannot exist on a plane surface between two solids, they are always possible on a liquid-solid interface [1,8]. At a solid-liquid interface the phase velocity of the generalized Rayleigh wave, which is a system of three waves (one in the liquid and two in the solid), is higher than the wave velocity in the fluid. This surface wave radiates energy continuously into the liquid, forming therein an inhomogeneous wave departing from the boundary. Since the energy flows across the interface (leaky wave), the wave attenuates along the propagation direction. At the same time on a solid-liquid interface there exists a true Stoneley surface wave (sometimes called Scholte wave) [8,9], consisting of an inhomogeneous wave in the liquid and two inhomogeneous waves in the solid, and propagating parallel to the boundary without attenuation and being exponentially damped in both directions perpendicular to the interface. Its velocity is lower than all the bulk velocities in the solid and in the liquid.

Due to the presence of a second compressional wave in a fluid-saturated porous medium, properties of surface waves at interfaces of fluid-filled porous solid in contrast to either free interface of elastic half-space or liquid-solid interface should be different. Theoretical works on surface waves in multicomponent systems are based solely on the Biot's model of porous materials. There are only a few papers published on this subject.

Standard boundary conditions of poroelasticity have been introduced by Deresiewicz and Skalak [10] and later by Rosenbaum [11]. Recently (see [12]) interface conditions at a boundary between two porous media were derived directly from Biot's equations by replacing the discontinuity surface with a thin transition layer. These conditions are identical to those introduced in [10] for the open-pore interface. Interface conditions for closed or partially open interface violate Biot's equations at the interface.

The pioneering work devoted to the research of surface waves on a free boundary of a porous medium has been done by Deresiewicz [13]. A numerical analysis of the dispersion equation revealed that there is always a complex root which corresponds to the velocity of the generalized Rayleigh wave. Hence, contrary to the elastic medium, the Rayleigh wave in a saturated porous medium is attenuated. For low frequencies, the velocity of this wave tends to the velocity of the Rayleigh wave in an elastic medium. Feng and Johnson [14,15] have extended the Biot theory to numerically predict the velocities of various surface modes at an interface between a fluid half-space and a half-space of a fluid-saturated porous medium. Using standard boundary conditions [10,11] it has been shown that a slowest surface wave is expected only if a closed-pore boundary condition applies. It was found that three different types of surface modes can exist on a fluid-porous solid interface depending on the shear velocity of the frame and whether the pores are closed or open on the interface. It was discovered experimentally [16,17] that in the ideal case of completely closed surface pores and viscosity-free fluid, at a free interface of a porous medium two types of surface waves can propagate: there is pseudo-Rayleigh mode, which leaks its energy into the slow compressional wave, and a true surface mode with a velocity slightly below that of the slow wave. The second mode is a simple form of a slowest surface wave predicted in [14,15]. However, as it was noticed in [18], yet, available experimental data [19] seem to show that such slow surface mode does in fact propagate when the open-pore boundary condition applies. Experimental results [19] for excitation of surface waves of different modes at fluid-porous solid interface have revealed that for an open pore interface at least two, and in some cases three, types of surface waves are expected. Namely these are pseudo Rayleigh wave, pseudo Stoneley wave, slower than the shear and fluid bulk modes, but higher than the slow bulk mode, and true Stoneley wave, with speed slightly below the slow wave velocity (existing only for certain parameters when the shear wave velocity is not much higher than the slow wave velocity).

Some attention has been devoted to surface waves on curved interfaces. The problem becomes mathematically very involved, and even the existence of surface waves has not been fully investigated. This type of the surface wave problem has an important practical bearing. For instance, it appears in the investigation of boreholes. An account of results obtained within the frame of Biot's model can be found, for example, in the paper by A.N.Norris [20].

In the same paper A.N.Norris shows that the existence of surface waves is not related to values of the shear modulus in contrast to the above mentioned claims in the earlier paper on this subject (e.g. [14], [15]).

The focus of this paper is on the research of the surface waves at a free interface of a porous medium and at an interface between a porous medium and a liquid. The aim of the present work is to reveal an existence of possible surface modes and their characteristic features. In contrast to the earlier theoretical works on this subject we rely on the model which is simpler than the Biot model. Simultaneously this model satisfies all invariance and thermodynamical conditions, which the Biot model does not do, and, in spite of its simplicity leads to similar results as the classical Biot's model. We use a systematic mathematical approach based on the asymptotic analysis. This method proves to be effective and very convenient not only in the case considered in the present work but it enables the analysis of much more complex problems of interfacial waves. The latter shall be the subject of separate papers.

1. A new linear model of a saturated poroelastic material vs. Biot's model

We rely in this paper on a model of saturated poroelastic materials proposed in a series of papers by K. Wilmanski (e.g. [21-26]). These papers contain a full thermodynamical construction of a two-component model which is nonlinear with respect to deformations of components, and linear with respect to deviations from the thermodynamical equilibrium. We do not present any details of such a general model in this work, and limit our attention to the particular case of the fully linear model. Such a model reminds the classical model proposed by Biot (see: the collection of papers of M. A. Biot published by Tolstoi [27]). However there exist also substantial differences between these models, and therefore we present below the governing equations in a juxtaposition.

We use solely macroscopic fields in the formulation of both models, and they are denoted as follows:

ρ^S, ρ^F – partial mass densities of both components; they are connected with the so called realistic (true) mass densities ρ^{SR}, ρ^{FR} by the relations $\rho^S = (1-n)\rho^{SR}$, $\rho^F = n\rho^{FR}$, where n denotes the porosity (volume fraction of the fluid component),

$\mathbf{v}^S, \mathbf{v}^F$ – velocities of components; multiplied by the corresponding partial mass densities they give (macroscopic) partial momentum densities,

$\mathbf{u}^S, \mathbf{u}^F$ – displacements of both components with $\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}, \mathbf{v}^F = \frac{\partial \mathbf{u}^F}{\partial t}$,

$\mathbf{e}^S = \frac{1}{2}\left(\operatorname{grad}\mathbf{u}^S + \left(\operatorname{grad}\mathbf{u}^S\right)^T\right)$ – Almansi-Hamel deformation tensor of small deformations (i.e. $\max(|\lambda^1|, |\lambda^2|, |\lambda^3|) \ll 1$, where $\lambda^i, i = 1,2,3$ denote eigenvalues of $\mathbf{e}^S$),

$\Delta_n \equiv n - n_E$ – dynamic changes of porosity; n_E is a constant equilibrium value of porosity.

Both models are based on the following field equations:

1. Partial mass balance equations

   - K. Wilmanski

$$\frac{\partial \rho^S}{\partial t} + \operatorname{div}\left(\rho^S \mathbf{v}^S\right) = 0,$$
$$\frac{\partial \rho^F}{\partial t} + \operatorname{div}\left(\rho^F \mathbf{v}^F\right) = 0, \tag{1.1}$$

   - M. A. Biot – the same;

2. Momentum balance equations

   - K. Wilmanski

$$\rho^S \frac{\partial \mathbf{v}^S}{\partial t} = \lambda^S \mathrm{grad}\left(\mathrm{tr}\mathbf{e}^S\right) + 2\mu^S \mathrm{div}\mathbf{e}^S + \beta \mathrm{grad}\Delta_n + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

$$\mathbf{v}^S := \frac{\partial \mathbf{u}^S}{\partial t}, \qquad \mathbf{e}^S := \frac{1}{2}\left(\mathrm{grad}\mathbf{u}^S + \left(\mathrm{grad}\mathbf{u}^S\right)^T\right),$$

$$\rho^F \frac{\partial \mathbf{v}^F}{\partial t} = -\kappa \mathrm{grad}\rho^F - \beta \mathrm{grad}\Delta_n - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \tag{1.2}$$

   - M. A. Biot

$$\rho^{11} \frac{\partial^2 \mathbf{u}^S}{\partial t^2} + \rho^{12} \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = (P - 2N)\,\mathrm{graddiv}\mathbf{u}^S + 2N \mathrm{divgrad}\mathbf{u}^S +$$

$$+Q \mathrm{graddiv}\mathbf{u}^F + bF\left(\frac{\partial \mathbf{u}^F}{\partial t} - \frac{\partial \mathbf{u}^S}{\partial t}\right),$$

$$\rho^{12} \frac{\partial^2 \mathbf{u}^S}{\partial t^2} + \rho^{22} \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = Q \mathrm{graddiv}\mathbf{u}^S + R \mathrm{graddiv}\mathbf{u}^F -$$

$$-bF\left(\frac{\partial \mathbf{u}^F}{\partial t} - \frac{\partial \mathbf{u}^S}{\partial t}\right); \tag{1.3}$$

3. Balance equation of porosity

   - K. Wilmanski

$$\frac{\partial \Delta_n}{\partial t} + n_E \mathrm{div}\left(\mathbf{v}^F - \mathbf{v}^S\right) = -\frac{\Delta_n}{\tau}, \qquad n_E = const. \tag{1.4}$$

   - M. A. Biot – none (porosity is constant). Let us mention in passing that in his first work from the year 1941 (eqn. (4.3)) Biot considered the balance equation of porosity similar to (1.4) without the right hand side, i.e without relaxation properties.

The notation of the Biot's model is fully explained, for instance, in the work [28]. However it is easy to see that the following identification of material parameters

$$\rho^S = \rho^{11} + \rho^{12}, \qquad \rho^F = \rho^{22} + \rho^{12},$$

$$N = \mu^S, \qquad P - 2N = \lambda^S, \qquad R = -\kappa\rho_0^F, \qquad bF = \pi, \tag{1.5}$$

where ρ_0^F denotes the reference value of the partial mass density of the fluid component, makes the two models identical provided the following conditions are satisfied

1. the coupling through acceleration in the Biot's model is ignored:
   $\rho^{12} \equiv 0$,

2. the coupling through partial stresses in the Biot's model is ignored: $Q \equiv 0$,

3. dynamical changes of porosity Δ_n in Wilmanski's model are ignored.

The structure of balance and field equations of our model will be explained again in Section 2.

We proceed to discuss the differences of the models in some details.

In order to discuss the first condition we write Biot's equations (1.3) in the following form

$$\rho^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} = (P - 2N)\,\mathrm{graddiv}\mathbf{u}^S + 2N\mathrm{divgrad}\mathbf{u}^S +$$
$$+Q\mathrm{graddiv}\mathbf{u}^F + bF\left(\frac{\partial \mathbf{u}^F}{\partial t} - \frac{\partial \mathbf{u}^S}{\partial t}\right) - \rho^{12}\left(\frac{\partial^2 \mathbf{u}^F}{\partial t^2} - \frac{\partial^2 \mathbf{u}^S}{\partial t^2}\right),$$

$$\rho^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} = Q\mathrm{graddiv}\mathbf{u}^S + R\mathrm{graddiv}\mathbf{u}^F -$$
$$-bF\left(\frac{\partial \mathbf{u}^F}{\partial t} - \frac{\partial \mathbf{u}^S}{\partial t}\right) + \rho^{12}\left(\frac{\partial^2 \mathbf{u}^F}{\partial t^2} - \frac{\partial^2 \mathbf{u}^S}{\partial t^2}\right). \tag{1.6}$$

Consequently the momentum sources in the Biot's model consists of the two following parts: proportional to the relative velocity of components with the permeability coefficient bF, and proportional to the difference of accelerations with the coefficient ρ^{12}. The latter is nonobjective, and, consequently the constitutive assumption on the momentum source in this model violates the principle of material objectivity (e.g. see the work [29]). Commonly used arguments advocating for such terms refer either to the so called added mass effect or to the influence of tortuosity. We do not share the opinion that such a nonobjective contribution would indeed account for those microscopic effects. In the first case, even if added mass effects may appear in fast drainage processes they would be certainly dominated by a viscosity of the fluid component on the microlevel of description, and by microvorticities. Added mass effects yield considerable contributions to solutions of field equations solely under very special conditions on the type of an obstacle, and on flow conditions. Even less plausible seems to us a relation to tortuosity. As a geometrical property of porous and granular materials relating length scales the tortuosity should be reflected on the macroscoic level by the existence of a characteristic length which, of course, the relative acceleration does not introduce. An appropriate model describing tortuosity should contain a microstructural variable additional to the porosity with a corresponding field equation.

In our opinion these arguments justify the assumption $\rho^{12} \equiv 0$.

More sophisticated is the argument leading to the second condition. Such couplings between partial stresses as these descibed by the constant Q are, of course, plausible in general. However within the frame of poroelastic models in which constitutive relations do not contain higher gradients a coupling by volume changes of components violates the second law of thermodynamics. This conclusion holds true even in the much larger class of models with large deformations (for the proof see Appendix in [26]). Such an observation has been made by Ingo Müller [30] within the frame of his theory of miscible mixtures. Some details concerning contributions of higher gradients – in his case – gradients of mass density – can be found in the book [31]. Without a dependence on higher gradients the mixture is called simple, and its partial free energies depend solely on corresponding partial mass densities ruling out the coupling between partial pressures. Such higher gradients are absent in both models of poroelastic materials.

It can be easily checked that the above assumptions do not influence the number of modes of bulk waves which are described by both models. The existence of an additional longitudinal mode of propagation of sound waves in porous materials which is called a P2-wave or Biot's wave is solely related to two kinematics (two velocity fields $\mathbf{v}^S$, and $\mathbf{v}^F$) of the model. Such modes are known also in theories of mixtures of miscible components (for instance fluids). Certainly additional couplings in the model influence speeds of propagation, and the attenuation of waves but they do not lead to any qualitative changes in spite of rather common claims in some papers on the Biot's model.

Let us mention that the third condition may be accepted on the ground of the analysis of orders of magnitude. This analysis leads to the conclusion that in the zeroth approximation the coefficient β may be put equal to zero. In effect the field Δ_n is not identically zero but its equation becomes decoupled. Dynamic changes of porosity are important for propagation of nonlinear waves but they yield small quantitative changes for weak discontinuity waves. Some results concerning this point may be found in the works [25,32].

Let us mention that apart from the above described differences in field equations there are also some differences in the form of boundary conditions in both models. It seems more natural to use the field of velocity $\mathbf{v}^F$ rather than the field of displacement $\mathbf{u}^F$ for the fluid component. Therefore the volume changes of the fluid are described in the model of K.Wilmanski by the fraction $\frac{\rho^F - \rho_0^F}{\rho_0^F}$, and not by $\operatorname{div}\mathbf{u}^F$. For the same reason the boundary condition on the permeable boundary differs in this model from the condition proposed by Deresiewicz and Skalak [10], and used rather commonly in works based on the Biot's model.

In Figure 1 we explain notations related to the geometry of the problem which are relevant for the formulation of boundary conditions. The boundary Γ which is assumed to be orientable divides the domain Ω into two parts: Ω^-, Ω^+. The domain Ω^- is assumed to be occupied by the porous material, and the domain Ω^+ contains either vacuum or a liquid. These two cases are considered in this work. The unit vector $\mathbf{n}$ normal to Γ is oriented in the external direction with respect to Ω^-. This orientation specifies the signature of various quantities of the model.

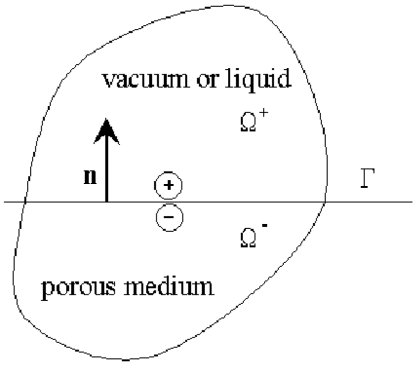

Figure 1: *Geometry of the problem*

The above mentioned boundary condition on the permeable boundary has the following form

- K. Wilmanski

$$\rho^F \left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \mathbf{n} = -\alpha n^- \left[\left[\frac{p^F}{n}\right]\right], \tag{1.7}$$

with $n^+ = 1$ for interfaces between porous medium and a fluid, n^- is a porosity of the porous material in Ω^-, $[[\ldots]] = (\ldots)^+\big|_\Gamma - (\ldots)^-\big|_\Gamma$ denotes the jump of the limits on the interface, and α denotes the surface permeability coefficient;

- M. A. Biot (after [14], in the notation of the present work)

$$T n_E \left(\mathbf{u}^F - \mathbf{u}^S\right) \cdot \mathbf{n} = -\left[\left[\frac{p^F}{n_E}\right]\right]. \tag{1.8}$$

For small changes of fluid mass density ρ^F, and porosity n these two conditions differ solely in the description of the fluid flow through the interface. In the first model it is the rate of flow which is determined by jump of the pore pressure, and in the second model it is the difference of fluid displacements. The latter is not well defined in the fluid dynamics, and this is the reason for using the condition (1.7) in this work.

Let us mention in passing that the model which we investigate in this paper can be, in contrast to the Biot's model, easily extended to more complicated models in which such nonlinear effects as changes of equilibrium porosity $n_E = n_E\left(\frac{\rho^F}{\rho^S}\right)$ or large deformations of the skeleton, and of the fluid as well as nonisothermal effects are incorporated. The research of waves in such models is in progress.

2. The mathematical model

2.1. Governing equations

Consider two semi-infinite spaces Ω^- and Ω^+ having a common interface Γ (Fig. 1). Let the region Ω^- be occupied by a saturated porous medium. The set of balance equations describing the porous two-component medium has the following general form ($x \in \Omega^-,\ t \in [0,T]$) [21-26]:

Mass conservation equations

$$\frac{\partial \rho^F}{\partial t} + \operatorname{div}(\rho^F \mathbf{v}^F) = 0,$$

$$\frac{\partial \rho^S}{\partial t} + \operatorname{div}(\rho^S \mathbf{v}^S) = 0. \tag{2.1}$$

Here ρ is the mass density, $\mathbf{v}$ is the velocity vector and indices F and S indicate fluid or solid phases, respectively.

Momentum conservation equations

$$\rho^F \left[\frac{\partial}{\partial t} + (v_j^F, \frac{\partial}{\partial x_j})\right] v_i^F - \frac{\partial}{\partial x_j} T_{ij}^F + \pi(v_i^F - v_i^S) = 0,$$

$$\rho^S \left[\frac{\partial}{\partial t} + (v_j^S, \frac{\partial}{\partial x_j})\right] v_i^S - \frac{\partial}{\partial x_j} T_{ij}^S - \pi(v_i^F - v_i^S) = 0, \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the inner product.

Balance equation for the change of porosity

$$\frac{\partial \Delta_n}{\partial t} + (v_i^S, \frac{\partial}{\partial x_i})\Delta_n + n_E \operatorname{div}(\mathbf{v}^F - \mathbf{v}^S) = -\frac{\Delta_n}{\tau}, \tag{2.3}$$

where τ is the relaxation time of porosity, assumed to be constant.

Here $\mathbf{T}^F$ and $\mathbf{T}^S$ are the partial stress tensors, π is a positive parameter which is constant in the model used in this paper.

Constitutive relations for linear poroelastic materials

$$\mathbf{T}^F = -p^F \mathbf{1} - \beta \Delta_n \mathbf{1}, \quad p^F = p_0^F + \kappa(\rho^F - \rho_0^F), \tag{2.4}$$

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S \operatorname{div}\mathbf{u}^S \mathbf{1} + 2\mu^S \operatorname{symgrad}\mathbf{u}^S + \beta \Delta_n \mathbf{1}, \tag{2.5}$$

where p^F is the pore pressure, p_0^F and ρ_0^F are the initial values of pore pressure and fluid mass density, respectively, κ is the constant compressibility coefficient of the fluid depending only on the equilibrium value of porosity n_E. $\Delta_n = n - n_E$ is the change of the porosity, β denotes the coupling coefficient of the components. $\mathbf{T}_0^S$

denotes constant reference value of the partial stress tensor in the skeleton, λ^S and μ^S are the Lame constants of the skeleton, which depend only on n_E, and $\mathbf{u}^S$ is the displacement vector for the solid phase with

$$\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}. \tag{2.6}$$

2.2. Boundary conditions

For the general case, when the region Ω^+ is occupied by, for instance, a saturated porous medium as well, the boundary conditions at the interface Γ are:

1) the continuity of total stresses

$$\left[\left[(\mathbf{T}^S + \mathbf{T}^F)\mathbf{n}\right]\right] = 0, \tag{2.7}$$

where $[[\ldots]] = (\ldots)^+ |_\Gamma - (\ldots)^- |_\Gamma$ denotes the jump of the limits on the interface,

2) the continuity of displacements of the solid phases (i.e. the boundary Γ is material with respect to the skeleton)

$$\left[\left[\mathbf{u}^S\right]\right] = 0, \tag{2.8}$$

3) the continuity of the mass flux across the interface

$$\left[\left[\rho^F(\mathbf{v}^F - \mathbf{v}^S)\cdot\mathbf{n}\right]\right] = 0, \tag{2.9}$$

4) proportionality between a discontinuity in pore pressures and the mass flux of the fluid through the interface [33]

$$\rho^F(\mathbf{v}^F - \mathbf{v}^S)\cdot\mathbf{n} \,|_\Gamma = -\alpha n^- \left[\left[\frac{p^F}{n}\right]\right] \tag{2.10}$$

Condition (2.10) reflects in a phenomenological way the existence of a boundary layer and relates the rate at which saturating fluid flows relative to the solid at the interface due to the pressure drop across the surface. Thus experimental constant α is a kind of surface porosity and the case $\alpha = 0$ corresponds to completely closed surface pores (impermeable boundaries), while the case $\alpha = \infty$ corresponds to the dynamical compatibility condition for partial tractions used in composits. For porous and granular materials it yields the pore pressure continuity condition. Such a condition is commonly used in models of porous materials based on Darcy's law.

2.3. Dimensionless variables and parameters

Let us rewrite the system of equation (2.1)-(2.6) in a dimensionless form. For this purpose we introduce the following dimensionless variables and parameters:

$$\hat{\rho}^F = \frac{\rho^F}{\rho_0^S}, \qquad \hat{\rho}^S = \frac{\rho^S}{\rho_0^S},$$

$$\hat{\mathbf{v}}^F = \frac{\mathbf{v}^F}{U_{||}^S}, \qquad \hat{\mathbf{v}}^S = \frac{\mathbf{v}^S}{U_{||}^S},$$

where ρ_0^S is the initial value of skeleton mass density and $U_{||}^S = \sqrt{(\lambda^S + 2\mu^S)/\rho_0^S}$ is a velocity of a longitudinal wave in an unbounded elastic medium. Also one has:

$$\hat{x} = \frac{x}{U_{||}^S \tau}, \qquad \hat{t} = \frac{t}{\tau}, \qquad \hat{\mathbf{u}} = \frac{\mathbf{u}}{U_{||}^S \tau},$$

$$\hat{p}^F = \frac{p^F}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\kappa} = \frac{\kappa}{(U_{||}^S)^2},$$

$$\hat{\pi} = \frac{\pi\tau}{\rho_0^S}, \qquad \hat{\beta} = \frac{\beta}{\rho_0^S (U_{||}^S)^2},$$

$$\hat{\lambda}^S = \frac{\lambda^S}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\mu}^S = \frac{\mu^S}{\rho_0^S (U_{||}^S)^2},$$

$$\hat{\alpha} = \alpha U_{||}^S.$$

After the change of variables and parameters the original system (2.1)-(2.6) keeps its form except of the right-hand side in the equation for the change of porosity. One gets there $-\Delta_n$. For typographical reasons we omit below symbol $\hat{}$ above the expressions.

2.4. Formulation of the problem

In what follows we consider two typical cases, namely we investigate surface waves at an interface of a porous medium and vacuum, and at an interface, separating a porous medium and a liquid. Basic conservation equations describing the liquid in the region Ω^+ have the form (below upper index "+" indicates the region Ω^+):

Mass conservation equation

$$\frac{\partial \rho^{F+}}{\partial t} + \operatorname{div}\left(\rho^{F+} \mathbf{v}^{F+}\right) = 0. \tag{2.11}$$

Momentum conservation equation

$$\rho^{F+} \left[\frac{\partial}{\partial t} + \left(v_j^{F+}, \frac{\partial}{\partial x_j} \right) \right] v_i^{F+} - \frac{\partial}{\partial x_j} \mathbf{T}_{ij}^{F+} = 0. \tag{2.12}$$

Here $\mathbf{T}^{F+}$ is the stress tensor:

$$\mathbf{T}^{F+} = -p^{F+} \mathbf{1}, \qquad p^{F+} = p_0^{F+} + \kappa^+ \left(\rho^{F+} - \rho_0^{F+} \right). \tag{2.13}$$

The liquid is assumed to be compressible with constant compressibility coefficients κ^+. p_0^{F+} and ρ_0^{F+} are the initial values of pressure and liquid density. After introduction of the following dimensionless variables

$$\hat{\rho}^{F+} = \frac{\rho^{F+}}{\rho_0^S}, \qquad \hat{\mathbf{v}}^{F+} = \frac{\mathbf{v}^{F+}}{U_{||}^S},$$

$$\hat{p}^{F+} = \frac{p^{F+}}{\rho_0^S (U_{||}^S)^2}, \qquad \hat{\kappa}^+ = \frac{\kappa^+}{(U_{||}^S)^2}$$

equations (2.11)-(2.13) preserve their form. Again below the symbol $\hat{}$ is omitted.

2.5. Linearization

Let us linearize the system of equation (2.1)-(2.6) about some equilibrium state. The simplest case arises when in the equilibrium state the fields have the following constant values: $\rho^F = \rho_0^F$, $\rho^S = \rho_0^S$, $\mathbf{v}^F = 0$, $\mathbf{v}^S = 0$ and $\Delta_n = 0$. After the introduction of displacement vector for the fluid phase $\mathbf{u}^F$ and linearization, the system (2.1)-(2.6) takes the following form:

$$\frac{\partial \rho^F}{\partial t} + r \mathrm{div} \frac{\partial \mathbf{u}^F}{\partial t} = 0, \tag{2.14}$$

$$\frac{\partial \rho^S}{\partial t} + \mathrm{div} \frac{\partial \mathbf{u}^S}{\partial t} = 0, \tag{2.15}$$

$$r \frac{\partial^2 \mathbf{u}^F}{\partial t^2} + \mathrm{grad}(p^F + \beta \Delta_n) + \pi \frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{2.16}$$

$$\frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \mu^S \Delta \mathbf{u}^S - (\lambda^S + \mu^S) \mathrm{graddiv} \mathbf{u}^S - \beta \mathrm{grad} \Delta_n - \pi \frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = 0, \tag{2.17}$$

$$\frac{\partial \Delta_n}{\partial t} + n_E \mathrm{div} \frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S) = -\Delta_n, \tag{2.18}$$

where $r = \rho_0^F / \rho_0^S$.

After linearization about equilibrium state with constant values $\rho^{F+} = \rho_0^{F+}$ and $\mathbf{v}^{F+} = 0$, equations (2.11), (2.12) take the following form:

$$\frac{\partial \rho^{F+}}{\partial t} + r^+ \mathrm{div} \frac{\partial \mathbf{u}^{F+}}{\partial t} = 0, \tag{2.19}$$

$$r^+ \frac{\partial^2 \mathbf{u}^{F+}}{\partial t^2} + \mathrm{grad} p^{F+} = 0, \tag{2.20}$$

where $\mathbf{u}^{F+}$ is the displacement vector for the liquid and $r^+ = \rho_0^{F+} / \rho_0^S$.

The general problem of propagation of elastic waves through a bounded space is complicated. We confine ourselves to the consideration of a 2D problem (xy plane). This assumption does not limit the generality for the plane boundary Γ. We investigate surface waves on the interface of a porous medium which occupies the semi-infinite space $y > 0$ (region Ω^-) and bounded either by the vacuum or by the liquid. Vacuum or liquid fills the semi-infinite space $y < 0$ (region Ω^+).

On the interface $y = 0$, separating porous medium and vacuum, the following linearized boundary conditions, which are the consequence of the general conditions (2.7)-(2.10), have to be satisfied:

1) the total stress vector vanishes

$$\Big(\frac{\partial u_1^S}{\partial y} + \frac{\partial u_2^S}{\partial x}\Big)|_{y=0} = 0, \tag{2.21}$$

$$\Big(\lambda^S \operatorname{div} \mathbf{u}^S + 2\mu^S \frac{\partial u_2^S}{\partial y} - \kappa(\rho^F - \rho_0^F)\Big)|_{y=0} = 0, \tag{2.22}$$

2) the relative normal velocity is equal to zero, i.e. $\alpha = 0$. The latter means that the pores at the interface are completely closed

$$\frac{\partial(u_2^F - u_2^S)}{\partial t}|_{y=0} = 0. \tag{2.23}$$

If the interface $y = 0$ separates a porous medium and a liquid, then following boundary conditions are valid:

1) the continuity of the total stress vector

$$\Big(\frac{\partial u_1^S}{\partial y} + \frac{\partial u_2^S}{\partial x}\Big)\Big|_{y=0} = 0 \tag{2.24}$$

and

$$\Big(\lambda^S \text{div } \mathbf{u}^S + 2\mu^S \frac{\partial u_2^S}{\partial y} - \kappa(\rho^F - \rho_0^F)\Big)\Big|_{y=0} = -\kappa^+(\rho^{F+} - \rho_0^{F+})\Big|_{y=0}, \tag{2.25}$$

2) the continuity of the mass flux across the interface

$$\rho_0^F \frac{\partial}{\partial t}(u_2^F - u_2^S)\Big|_{y=0} = \rho_0^{F+} \frac{\partial}{\partial t}(u_2^{F+} - u_2^S)\Big|_{y=0}, \tag{2.26}$$

3) proportionality between discontinuity in pressures and mass flux of the fluid through the interface [33]

$$-\rho_0^F(v_2^F - v_2^S)\Big|_{y=0} = \alpha(p^F - n_E p^{F+})\Big|_{y=0}. \tag{2.27}$$

Our goal is to prove that the boundary value problem (2.14)-(2.18), (2.21)-(2.23) as well as the boundary value problem (2.14)-(2.20), (2.24)-(2.27) has solutions in the form of surface waves, i.e. solutions which decrease sufficiently fast as $|y| \to \infty$. For this purpose we will investigate the propagation of a harmonic wave whose frequency is ω, wave number is k, and its amplitude depends on y. It should be noted here that as in [34-36] we consider the solutions of (2.1)-(2.10) in the absence of external forces. Then they are defined uniquely by Cauchy data. In this case it is natural to derive ω as a function of the real wave number $k \in R^1$. Thus, $\mathrm{Re}\omega/k$ defines the phase velocity of waves, while $\mathrm{Im}\omega$ defines the attenuation. Below we use following dimensionless parameters: $\hat{\omega} = \omega\tau$ and $\hat{k} = kU_{\|}^S\tau$ (the upper symbol $\hat{}$ is again omitted in further consideration).

3. Construction of solution

Solution in the region Ω^- (porous medium half-space) is sought in the following form [34]-[36]:

$$\mathbf{u}^F = \mathrm{grad}\varphi^F + \mathrm{rot}\Psi^F, \quad \mathbf{u}^S = \mathrm{grad}\varphi^S + \mathrm{rot}\Psi^S, \tag{3.1}$$

where $\Psi^F = (0, 0, \psi^F)$ and $\Psi^S = (0, 0, \psi^S)$. Concequently, in the explicit form one has:

$$u_1^F = \frac{\partial\varphi^F}{\partial x} + \frac{\partial\psi^F}{\partial y}, \quad u_2^F = \frac{\partial\varphi^F}{\partial y} - \frac{\partial\psi^F}{\partial x},$$

$$u_1^S = \frac{\partial\varphi^S}{\partial x} + \frac{\partial\psi^S}{\partial y}, \quad u_2^S = \frac{\partial\varphi^S}{\partial y} - \frac{\partial\psi^S}{\partial x}$$

Here unknown potentials are sought as:

$$\varphi^F = A^F(y)\exp(i(kx - \omega t)), \quad \varphi^S = A^S(y)\exp(i(kx - \omega t)),$$

$$\psi^F = B^F(y)\exp(i(kx - \omega t)), \quad \psi^S = B^S(y)\exp(i(kx - \omega t)). \tag{3.2}$$

Simultaneously

$$\rho^F - \rho_0^F = A_\rho^F(y)\exp(i(kx - \omega t)), \quad \rho^S - \rho_0^S = A_\rho^S(y)\exp(i(kx - \omega t)),$$

$$\Delta_n = A_\Delta \exp(i(kx - \omega t)). \tag{3.3}$$

Substitution of (3.1) into (2.14)-(2.18) and the following insertion of expressions (3.2), (3.3) result in three equations for unknown amplitudes $A^F(y)$, $A^S(y)$, and $B^S(y)$

$$\Big(c_f^2(\frac{d^2}{d\,y^2} - k^2) + \omega^2\Big)A_F$$

$$+\left(\frac{\beta\omega n_E}{r\,(i+\omega)}\left(\frac{d^2}{dy^2} - k^2\right) + \frac{i\pi\omega}{r}\right)\left(A^F - A^S\right) = 0, \tag{3.4}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \omega^2\right)A^S - \left(\frac{\beta\omega n_E}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\pi\omega\right)\left(A^F - A^S\right) = 0, \tag{3.5}$$

$$\left(\frac{d^2}{dy^2} - k^2 + \frac{\omega^2}{c_s^2} - \frac{i\pi\omega^2 r}{c_s^2\,(\omega r + i\pi)}\right) B^S = 0 \tag{3.6}$$

and in four algebraic relations for $B^F(y)$, $A_\Delta(y)$, $A_\rho^S(y)$, and $A_\rho^F(y)$:

$$B^F = \frac{i\pi}{\omega r + i\pi} B^S, \tag{3.7}$$

$$A_\Delta = -\frac{m_E\omega}{i+\omega}(\frac{d^2}{d\,y^2} - k^2)(A^F - A^S), \tag{3.8}$$

$$A_\rho^S = -(\frac{d^2}{d\,y^2} - k^2)A^S, \tag{3.9}$$

$$A_\rho^F = \frac{r\omega^2}{c_f^2}A^F + \frac{1}{c_f^2}\left(\frac{\beta\omega n_E}{i+\omega}\left(\frac{d^2}{dy^2} - k^2\right) + i\pi\omega\right)\left(A^F - A^S\right) = 0. \tag{3.10}$$

Here $c_f = U^F/U_{||}^S < 1$, $c_s = U_\perp^S/U_{||}^S < 1$ and $U^F = \sqrt{\kappa}$, $U_\perp^S = \sqrt{\mu^S/\rho_0^S}$. Let us remind that $U_\perp^S$ is a velocity of shear wave in an unbounded elastic medium, while U^F is the sound speed in a liquid.

Next let us prove the existence of solutions for the system (3.4)-(3.5) and for the equation (3.6) decaying with y. First consider (3.6). The solution has the following form:

$$B^S = C_s \exp(\pm\gamma_s\, y) \tag{3.11}$$

with

$$\gamma_s = \sqrt{k^2 - \frac{\omega^2}{c_s^2} + \frac{i\pi\omega^2 r}{c_s^2\,(\omega r + i\pi)}}. \tag{3.12}$$

Let us define the following condition:

Condition 1.

$$\mathrm{Re}\Big[k^2 - \frac{\omega^2}{c_s^2} + \frac{i\pi\omega^2 r}{c_s^2\,(\omega r + i\pi)}\Big] > 0. \tag{3.13}$$

As we will show below, this condition is indeed fulfilled by all surface waves which are proven to be possible on the free interface of a porous medium and on the interface separating a porous medium and a liquid. It is also quite natural. Namely, a similar condition in the classical elasticity theory yields the conclusion that the phase velocity of a surface wave should be less than the velocity of a shear wave.

Then, the square root in (3.12) is defined as $\sqrt{1} = 1$ and in order to get a bounded solution we choose

$$B^S = C_s \exp(-\gamma_s\, y). \tag{3.14}$$

We proceed to prove the existence of solution for the system (3.4)-(3.5). The solution is sought in the following form:

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C_j \begin{pmatrix} R_j^F \\ R_j^S \end{pmatrix} \exp(\pm\gamma_j\, y). \tag{3.15}$$

Substituting (3.15) into (3.4),(3.5), one obtains the following eigenvalues problem:

$$\begin{aligned} d_1^F(j) R_j^F + d_1^S(j) R_j^S &= 0, \\ d_2^F(j) R_j^F + d_2^S(j) R_j^S &= 0, \end{aligned} \tag{3.16}$$

where

$$d_1^F(j) = \Big(rc_f^2 + \frac{\beta\, n_E \omega}{\omega + i}\Big)\Big(\frac{\gamma_j^2}{k^2} - 1\Big) + \frac{\omega}{k}\big(r\frac{\omega}{k} + i\frac{\pi}{k}\big),$$

$$d_1^S(j) = -\frac{\beta\, n_E \omega}{\omega + i}\Big(\frac{\gamma_j^2}{k^2} - 1\Big) - i\frac{\pi\omega}{k^2},$$

$$d_2^F(j) = -\frac{\beta\, n_E \omega}{\omega + i}\Big(\frac{\gamma_j^2}{k^2} - 1\Big) - i\frac{\pi\omega}{k^2},$$

$$d_2^S(j) = \Big(1 + \frac{\beta\, n_E \omega}{\omega + i}\Big)\Big(\frac{\gamma_j^2}{k^2} - 1\Big) + \frac{\omega}{k}\big(\frac{\omega}{k} + i\frac{\pi}{k}\big), \tag{3.17}$$

and eigenvalues γ_j and eigenvectors $(R_j^F, R_j^S)^T$ have to be found. Obviously, γ_j are defined from the condition that determinant of (3.16) must vanish. Consequently, one can derive eigenvectors $(R_j^F, R_j^S)^T$.

We omit here lengthy formulae for these eigenvalues and eigenvectors in the general case. Let us consider the simplified case when $\beta = 0$. The assumption on the vanishing coefficient β means that we neglect a static coupling between components. This happens to be justified for sufficiently small changes of porosity Δ_n even though these changes influence, however to much lesser extent than relative motion of phases, both speeds of propagation and attenuation [33]. The vanishing of the determinant of the system (3.16) yields biquadratic equation for unknown functions γ_j:

$$\begin{aligned} \left(\frac{\gamma_j^2}{k^2} - 1\right)^2 + \frac{\omega}{k}\left(\frac{\omega}{k}\Big(1 + \frac{1}{c_f^2}\Big) + i\frac{\pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right)\left(\frac{\gamma_j^2}{k^2} - 1\right) \\ + \frac{1}{c_f^2}\frac{\omega^3}{k^3}\left(\frac{\omega}{k} + i\frac{\pi}{k}\Big(1 + \frac{1}{r}\Big)\right) = 0. \end{aligned} \tag{3.18}$$

Hence we have the solution:

$$\frac{\gamma_j^2}{k^2} = 1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big) \pm \frac{1}{2}\mathrm{Re}\,\delta + \frac{i}{2}\left(\pm \mathrm{Im}\,\delta - \frac{\omega}{k}\frac{\pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right), \tag{3.19}$$

where

$$\delta = \sqrt{\frac{\omega^4}{k^4}\Big(1 - \frac{1}{c_f^2}\Big)^2 - \frac{\omega^2}{k^2}\frac{\pi^2}{k^2}\Big(1 + \frac{1}{rc_f^2}\Big)^2 + 2i\frac{\omega^3}{k^3}\frac{\pi}{k}\Big(1 - \frac{1}{rc_f^2}\Big)\Big(1 - \frac{1}{c_f^2}\Big)}.$$

Similarly to the analysis of the equation (3.6) we assume that the following condition holds:

Condition 2.

$$\mathrm{Re}\left[1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big)\right] \pm \frac{1}{2}\mathrm{Re}\,\delta > 0. \tag{3.20}$$

Then there exist two roots γ_1 and γ_2, such that

$$\frac{\gamma_{1,2}}{k} = \sqrt{1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big) \mp \frac{1}{2}\mathrm{Re}\,\delta + \frac{i}{2}\left(\mp \mathrm{Im}\,\delta - \frac{\omega}{k}\frac{\pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right)}, \tag{3.21}$$

with

$$\mathrm{Re}\,\gamma_{1,2} > 0.$$

We will demonstrate below that physically the condition (3.21) means that the phase velocity of a surface wave should be less then velocities of longitudinal waves both in a solid and in a liquid. Otherwise the following condition holds true:

Condition 3.

$$\mathrm{Re}\left[1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big)\right] + \frac{1}{2}\mathrm{Re}\,\delta > 0, \quad \mathrm{Re}\left[1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big)\right] - \frac{1}{2}\mathrm{Re}\,\delta < 0. \tag{3.22}$$

Then $\mathrm{Re}\gamma_2 > 0$ and γ_2 is defined as above. But the square root in the expression for γ_1 is defined in such a way that the so-called radiation condition [1], i.e. the condition of boundedness of solutions, is satisfied:

$$\gamma_1 = i\sqrt{1 - \frac{1}{2}\frac{\omega^2}{k^2}\Big(1 + \frac{1}{c_f^2}\Big) + \frac{1}{2}\mathrm{Re}\,\delta + \frac{i}{2}\left(\mathrm{Im}\,\delta - \frac{\omega}{k}\frac{\pi}{k}\Big(1 + \frac{1}{rc_f^2}\Big)\right)}. \tag{3.23}$$

Corresponding eigenvectors are given by:

$$(R_1^F,\ R_1^S) = \Big(R_1^F,\ \frac{i\frac{\pi\omega}{k^2}}{\frac{\gamma_1^2}{k^2} - 1 + \frac{\omega}{k}\Big(\frac{\omega}{k} + i\frac{\pi}{k}\Big)}R_1^F\Big)$$

and

$$(R_2^F,\ R_2^S) = \Big(\frac{i\frac{\pi\omega}{k^2}}{rc_f^2\Big(\frac{\gamma_1^2}{k^2} - 1\Big) + \frac{\omega}{k}\Big(r\frac{\omega}{k} + i\frac{\pi}{k}\Big)}R_2^S,\ R_2^S\Big) \tag{3.24}$$

Thus, a bounded solution to (3.4)-(3.6) exists and it has the form:

$$\left(\begin{array}{c} A^F \\ A^S \end{array} \right) = C_1 \left(\begin{array}{c} R_1^F \\ R_1^S \end{array} \right) \exp(-\gamma_1\, y) + C_2 \left(\begin{array}{c} R_2^F \\ R_2^S \end{array} \right) \exp(-\gamma_2\, y),$$

$$B^S = C_s \exp(-\gamma_s\, y). \tag{3.25}$$

Here the constants C_1, C_2, and C_s are still unknown and have to be defined from the boundary conditions.

Let us clarify the above-obtained results by considering a particular case of short waves, i.e. $|k| \gg 1$. In this case radicals $\gamma_{1,2}$ and γ_s take rather simple forms, namely:

$$\tilde{\gamma}_1 = \sqrt{1 - \frac{\tilde{\omega}^2}{c_f^2}}, \qquad \tilde{\gamma}_2 = \sqrt{1 - \tilde{\omega}^2}, \qquad \tilde{\gamma}_s = \sqrt{1 - \frac{\tilde{\omega}^2}{c_s^2}}, \tag{3.26}$$

where $\tilde{\omega} = \omega/k$, $\tilde{\gamma}_{1,2} = \gamma_{1,2}/|k|$, and $\tilde{\gamma}_s = \gamma_s/|k|$. Concequently, eigenvectors for (3.17) are also simplified:

$$(R_1^F, R_1^S) = (1, 0), \quad (R_2^F, R_2^S) = (0, 1).$$

It is evident that radicals $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_s$ are multi-valued functions. In order to make these function single-valued, consider the Riemann surface of $\tilde{\omega}$ with the cuts outgoing from the points $\pm c_f$, $\pm c_s$, ± 1. From now on we consider this Riemann surface, where the signs at radicals are defined uniquely (depending on the strip of the Riemann surface [35]) in such a way that on the real axis the radiation condition [1] is satisfied.

Let one of the following conditions hold true:

Condition 4.

$$1 > \max \mathrm{Re}\left(\frac{\tilde{\omega}^2}{c_f^2}, \frac{\tilde{\omega}^2}{c_s^2}, \tilde{\omega}^2 \right). \tag{3.27}$$

Then $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_s$ are defined as in (3.26).

Condition 5.

$$\mathrm{Re}\frac{\tilde{\omega}^2}{c_f^2} > 1 > \max \mathrm{Re}\left(\frac{\tilde{\omega}^2}{c_s^2}, \tilde{\omega}^2 \right). \tag{3.28}$$

Then $\tilde{\gamma}_2$ and $\tilde{\gamma}_s$ are defined as in (3.26). However

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1} \tag{3.29}$$

on the first strip of the Riemann surface [35].

It should be noted also that in case $|k| \gg 1$ the conditions (3.13), (3.20) are reduced to condition (3.27). The latter means that the phase velocity of a surface wave should be less than velocities of all bulk waves. Condition (3.22) is transformed to (3.28). It says that if there exists a surface mode whose phase velocity is bigger than the sound velocity in a liquid then signs at radicals $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_s$ are chosen depending on the Riemann surface. We prove below that phase velocities of surface waves, existing on the free interface of a porous medium and on the interface separating a porous medium and a liquid, satisfy either (3.27) or (3.28).

In order to derive a dispersion relation for the definition of velocities of the surface waves, one should substitute solution (3.25) into the boundary conditions. First let us consider surface waves at the interface separating a porous medium and vacuum.

4. Surface waves at a free interface of a porous medium

4.1. Dispersion relation

Substituting solution (3.25) into the boundary conditions (2.21)-(2.23) for the case $\beta = 0$ one gets the following system of equations with respect to unknown constants C_1, C_2, C_s:

$$\tilde{\gamma}_1 C_1 R_1^S + \tilde{\gamma}_2 C_2 R_2^S + \frac{i}{2}\Big(\tilde{\gamma}_s^2 + 1\Big) C_s = 0, \tag{4.1}$$

$$(\lambda^S + 2\mu^S)\Big((\tilde{\gamma}_1^2 - 1) C_1 R_1^S + (\tilde{\gamma}_2^2 - 1) C_2 R_2^S\Big)$$

$$+2\mu^S (C_1 R_1^S + C_2 R_2^S) + 2i\mu^S \tilde{\gamma}_s C_s - \Big(\tilde{\omega}^2 r + i\frac{\pi\tilde{\omega}}{k}\Big)(C_1 R_1^F + C_2 R_2^F)$$

$$+i\frac{\pi\tilde{\omega}}{k}(C_1 R_{S1} + C_2 R_2^S) = 0, \tag{4.2}$$

$$\tilde{\gamma}_1 C_1 (R_1^F - R_1^S) + \tilde{\gamma}_2 C_2 (R_2^F - R_2^S) - iC_s\Big(1 - \frac{i\pi/k}{\tilde{\omega} r + i\pi/k}\Big) = 0. \tag{4.3}$$

The condition that the determinant of the system (4.1)-(4.3) must vanish yields the dispersion equation for the definition of frequencies of surface waves. For the case $|k| \gg 1$ one gets:

$$\frac{i\mu^S}{2\tilde{\gamma}_1\tilde{\gamma}_2}\mathcal{P}_v(\tilde{\omega}) = 0, \tag{4.4}$$

where

$$\mathcal{P}_v(\tilde{\omega}) = \tilde{\gamma}_1 \mathcal{P}_R(\tilde{\omega}) + \tilde{\gamma}_2 r \frac{\tilde{\omega}^4}{c_s^4} \tag{4.5}$$

and $\mathcal{P}_R$ corresponds to the classical Rayleigh equation [1,4]:

$$\mathcal{P}_R(\tilde{\omega}) = \left(2 - \frac{\tilde{\omega}^2}{c_s^2}\right)^2 - 4\tilde{\gamma}_2\tilde{\gamma}_s. \tag{4.6}$$

Equality

$$\mathcal{P}_v(\tilde{\omega}) = 0 \tag{4.7}$$

is the equation for the phase velocities of surface waves and their attenuation. It is interesting to note that (4.7) corresponds exactly to the dispersion equation for the case of surface waves at the interface between liquid and solid half-spaces [1]. Next let us construct asymptotic expansions for the roots of equation (4.7).

4.2. Asymptotics of the roots

First we prove that dispersion equation (4.7) has a root $\tilde{\omega}_1$ satisfying condition (3.27), i.e $\mathrm{Re}\tilde{\omega}_1 \in (0, c_f)$. Because of the fact that velocities U^F and $U^S_\perp$ differ significantly, dimensionless speed $c_f = U^F/U^S_\perp$ can be considered as a small parameter: $\varepsilon \equiv c_f$. The asymptotic expansion of $\tilde{\omega}_1$ is sought in the following form [34-36]:

$$\tilde{\omega} = \varepsilon(1 - c_1\varepsilon^4 + \cdots). \tag{4.8}$$

Substituting (4.8) into (4.7) one specifies from the lowest appoximation mod$O(\varepsilon^4)$:

$$\sqrt{2c_1} = \frac{r}{2c_s^2(1 - c_s^2)} \tag{4.9}$$

and, consequently,

$$c_1 = \frac{r^2}{8c_s^4\left(1 - c_s^2\right)^2}. \tag{4.10}$$

It is obvious that in order to satisfy condition (3.27) both $\mathrm{Re}c_1$ and $\mathrm{Re}\sqrt{c_1}$ should be positive. This is indeed fulfilled since $c_s < 1$. Therefore

$$\mathrm{Re}\tilde{\omega}_1 = \varepsilon\Big(1 - c_1\varepsilon^4 + O(\varepsilon^5)\Big) \in [0, c_f). \tag{4.11}$$

This phase velocity corresponds to very slow surface wave (true Stoneley wave), propagating almost without attenuation. Its speed is less than the velocities of all bulk waves in the porous media and has order $O(c_f)$.

Next, let us show that dispersion equation (4.7) has also a complex root, satisfying condition (3.28). It belongs to the interval (c_f, c_s) and corresponds to the generalized Rayleigh wave whose phase velocity $c_{R'} \to c_R$ as $\rho_0^F \to 0$, where c_R is a velocity of the classical Rayleigh wave in elastic half-space. It should be reminded here that for $\tilde{\gamma}_1$ the following branch is taken (see condition (3.28)): $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1}$. Also, since

$c_{R'}$ should be close to c_s, than $\sqrt{\frac{\tilde{\omega}^2}{c_f^2}-1} \approx \frac{\tilde{\omega}}{c_f}$. Thus (4.7) can be rewritten in the form:

$$c_f r \frac{\tilde{\omega}^3}{c_s^4}\sqrt{1-\tilde{\omega}^2} - i\mathcal{P}_R = 0. \tag{4.12}$$

Let

$$\tilde{\omega} = \Omega_0 + \varepsilon\Omega_1 + \dots \tag{4.13}$$

It is easy to show that the leading part Ω_0 of (4.13) satisfies the Rayleigh equation $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$. For the definition of the next term Ω_1 of expansion (4.13) one gets the following equation:

$$\Big[\frac{4}{c_s^4}\Omega_0^3 - 8\frac{1}{c_s^2}\Omega_0 - 4\frac{d}{d\tilde{\omega}}\Big(\sqrt{1-\frac{\tilde{\omega}^2}{c_s^2}}\sqrt{1-\tilde{\omega}^2}\Big)\Big|_{\tilde{\omega}=\Omega_0}\Big]\Omega_1$$

$$= ir\frac{\Omega_0^3}{c_s^4}\sqrt{1-\Omega_0^2}. \tag{4.14}$$

Finally one has:

$$\tilde{\omega}_{R'} = c_R + \varepsilon\Omega_1 + O(\varepsilon^2), \tag{4.15}$$

where Ω_1 is imaginary and is determined by (4.14). The real part of (4.15), i.e. c_R, defines the phase velocity of generalized Rayleigh wave and $k\mathrm{Im}\Omega_1$ corresponds to the attenuation of this wave. Thus the reradiation of the energy into the medium occurs. It is the so-called leaky wave. Since $\mathrm{Im}\Omega_1 > 0$ one can prove, estimating the amplitudes of the bulk waves, that part of the energy of this surface wave is absorbed by the slow compressional wave. But in contrast to the generalized Rayleigh wave at the interface between liquid and solid half-spaces, where energy is radiated from solid into liquid, leaky wave at the free interface of a porous medium radiates energy into the half-space, where the wave is localized, i.e. into Ω^-. The first example of such type of surface waves was described at the concave cylindrical interface of elastic solid [1]. As it is clear from the research presented, such leaky waves, radiating energy into the medium where they are located, exist at the plane interface of a porous saturated medium.

5. Surface waves at an interface separating a saturated porous medium and a liquid

5.1. Construction of solution

Next consider surface waves at an interface separating a saturated porous medium and a liquid. As in the case of a free interface of a porous medium, solution in the region Ω^- $(y > 0)$, occupied by a porous medium, is sought in the form (3.1)-(3.3) and the form (3.25) of a bounded solution remains to be valid. For the liquid, occupying region Ω^+ $(y < 0)$, the solution for (2.11)-(2.13) has the form:

$$\mathbf{u}^{F+} = \mathrm{grad}\varphi^{F+},$$

$$\varphi^{F+} = A^{F+}(y)\exp\Big(i(kx - \omega t)\Big),$$

$$\rho^{F+} - \rho_0^{F+} = A_\rho^{F+}(y)\exp\Big(i(kx - \omega t)\Big). \tag{5.1}$$

Substituting (5.1) into (2.11)-(2.13) one gets:

$$A_\rho^{F+} = \frac{r^+\omega^2}{(c_f^+)^2}A^{F+} \tag{5.2}$$

and

$$A_\rho^{F+} + r^+\left(\frac{d^2}{dy^2} - k^2\right)A^{F+} = 0, \tag{5.3}$$

where $c_f^+ = U^{F+}/U_{||}^S < 1$ and $U^{F+} = \sqrt{\kappa^+}$. Obviously, equation (5.3) has the following bounded solution:

$$A^{F+} = C_1^+\exp(\gamma_1^+ y), \quad \gamma_1^+ = |k|\sqrt{1 - \frac{\tilde{\omega}^2}{(c_f^+)^2}}. \tag{5.4}$$

In order to derive dispersion relation and to define the phase velocities of the surface waves, one should substitute solutions (3.25) and (5.4) into boundary conditions (2.24)-(2.27). We proceed to do so.

5.2. Dispersion relation

Substituting the solution into boundary conditions (2.24)-(2.27) for the case $\beta = 0$ and $|k| \gg 1$ one gets the following system of equations with respect to unknown constants C_1, C_2, C_s, and C_1^- :

$$(\lambda^S + 2\mu^S)(\tilde{\gamma}_2^2 - 1)C_2 + 2\mu^S C_2 + 2\mu^S i\tilde{\gamma}_s C_s - \tilde{\omega}^2 r C_1 = -\tilde{\omega}^2 r^- C_1^+, \tag{5.5}$$

$$\tilde{\gamma}_2 C_2 + \frac{i}{2}\Big(\tilde{\gamma}_s^2 + 1\Big)C_s = 0, \tag{5.6}$$

$$-\tilde{\gamma}_1 C_1 + \tilde{\gamma}_2 C_2 + iC_s = \frac{r^+}{r}\tilde{\gamma}_1^+ C_1^+, \tag{5.7}$$

$$i\Big(\tilde{\gamma}_1^+ C_1^+ + \tilde{\gamma}_2 C_2 + iC_s\Big) = \alpha\tilde{\omega}\frac{r}{r^+}\Big(C_1 - C_1^+\Big), \tag{5.8}$$

where

$$\tilde{\gamma}_1 = \sqrt{1 - \frac{\tilde{\omega}^2}{c_f^2}}, \qquad \tilde{\gamma}_2 = \sqrt{1 - \tilde{\omega}^2},$$

$$\tilde{\gamma}_s = \sqrt{1 - \frac{\tilde{\omega}^2}{c_s^2}}, \qquad \tilde{\gamma}_1^+ = \sqrt{1 - \frac{\tilde{\omega}^2}{(c_f^-)^2}}. \tag{5.9}$$

The condition that the determinant of the system (5.5)-(5.8) must vanish yields the dispersion equation for the complex speeds of the surface waves:

$$\Big(- \frac{\mu^S}{\tilde{\gamma}_1\tilde{\gamma}_2}\mathcal{P}_v + r^+\frac{\tilde{\omega}^4}{\tilde{\gamma}_1 c_s^2}\Big)\Big(\tilde{\gamma}_1^+ - i\alpha\tilde{\omega}\Big(\frac{\tilde{\gamma}_1^+}{\tilde{\gamma}_1} + n_E\Big)\Big)$$

$$-r^+\Big(1 + \frac{\tilde{\gamma}_1^+}{\tilde{\gamma}_1}\Big)\frac{\tilde{\omega}^4}{c_s^2}\Big(1 - \frac{i}{\tilde{\gamma}_1}\alpha\tilde{\omega}(1 - n_E)\Big) = 0. \tag{5.10}$$

Here $\mathcal{P}_v$ is the dispersion relation, corresponding to the case of surface waves at a free interface of a porous medium, whereas $\mathcal{P}_R$ corresponds to the classical Rayleigh equation (see (4.5), (4.6)).

Obviously, (5.10) includes radicals $\tilde{\gamma}_1, \tilde{\gamma}_1^+, \tilde{\gamma}_2, \tilde{\gamma}_s$, which are multi-valued functions. In order to make these function single-valued, consider Riemann surface of $\tilde{\omega}$ with the cuts outgoing from the points $\pm c_f, \pm c_f^+, \pm c_s, \pm 1$. Later on we consider this Riemann surface, where the signs at radicals are defined uniquely (depending on the strip of the Riemann surface [35]) in such a way that on the real axis the radiation condition [1] is satisfied. The latter means that solutions (3.25) and (5.4) are bounded in the regions $y > 0$ and $y < 0$, respectively.

Next consider for simplicity the case when the liquid, saturating a porous medium, and the liquid, occupying half-space $y < 0$, are the same. Thus $\kappa^- = \kappa$ and $\tilde{\gamma}_1^- = \tilde{\gamma}_1$.

Let either

Condition 1

$$1 > \max Re\Big(\frac{\tilde{\omega}^2}{c_f^2}, \frac{\tilde{\omega}^2}{c_s^2}, \tilde{\omega}^2\Big) \tag{5.11}$$

and, consequently, $\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\gamma}_s$ are defined as in (5.9),

or

Condition 2

$$Re\frac{\tilde{\omega}^2}{c_f^2} > 1 > \max Re\Big(\frac{\tilde{\omega}^2}{c_s^2}, \tilde{\omega}^2\Big) \tag{5.12}$$

holds. Then $\tilde{\gamma}_2$ and $\tilde{\gamma}_2$ are defined as in (5.9). However

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1} \tag{5.13}$$

for the first strip of the Riemann surface (see Appendix 1 [35] for technical details).

Next we will show that dispersion equation (5.10) has three roots satisfying either (5.11) or (5.12). In what follows we investigate the dependence of the roots of (5.10) on parameters α and ρ_0^{F+} and consider the correspondence of these roots to those obtained for the case: porous medium-vacuum.

5.3. Asymptotics of the roots

First let us prove that there exists a root $\tilde{\omega}_1$ of (5.10) satisfying (5.11), i.e. $\mathrm{Re}\tilde{\omega}_1 \in [0, c_f)$. Its asymptotic expansion is sought in the following form [4,5,7,8]:

$$\tilde{\omega} = \varepsilon(1 - c_1\varepsilon^4 + \cdots). \tag{5.14}$$

It should be noted here that we consider outer expansion $\alpha \sim 1$ of the roots with respect to ε. Substituting (5.14) into (5.10) one gets from the lowest approximation:

$$\sqrt{2c_1} = \frac{(1+n_E)r + (1-3n_E)r^+}{2\mu^S(1-c_s^2)(1+n_E)}. \tag{5.15}$$

It is obvious that in order to satisfy condition (5.11) both $\mathrm{Re}c_1$ and $\mathrm{Re}\sqrt{c_1}$ have to be positive. By virtue of physical reasons

$$(1+n_E)r + (1-3n_E)r^+ > 0 \tag{5.16}$$

and, consequently, $\sqrt{2c_1} > 0$ and $c_1 > 0$. Therefore

$$\mathrm{Re}\tilde{\omega}_1 = \varepsilon\Big(1 - c_1\varepsilon^4 + O(\varepsilon^5)\Big) \in [0, c_f). \tag{5.17}$$

As in the case of a free interface of a porous medium, this phase velocity corresponds to very slow surface wave (true Stoneley wave), propagating almost without attenuation. Its speed is less than the velocities of all bulk waves in the porous medium and in the liquid, and it has the order $O(c_f)$.

Next we will show that dispersion equation (5.10) has also two complex roots, satisfying condition (5.12). These roots correspond to the localized with respect to y surface waves whose phase velocities are close to c_f and c_s, respectively. First of them is sought in the following form [35]:

$$\tilde{\omega} = \varepsilon(1 + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots). \tag{5.18}$$

Substitution of (5.18) into (5.10) yields at the lowest $O(\varepsilon^4)$ approximation:

$$-\sqrt{2c_2} + \alpha(1+n_E) = 0, \tag{5.19}$$

i.e.

$$c_2 = \frac{\alpha^2}{2}(1+n_E)^2 > 0. \tag{5.20}$$

Let us recall, that in accordance with condition (5.12) for $\tilde{\gamma}_1$ the following branch is taken: $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1}$. From the next $O(\varepsilon^5)$ approximation one has:

$$c_3 = -i\frac{2}{\mu^S}\frac{\alpha n_E r^+}{(1-c_s^2)}. \tag{5.21}$$

Finally, one gets the following expansion for the second root of dispersion relation (5.10):

$$\tilde{\omega}_2 = \varepsilon\Big(1 + c_2\varepsilon^2 + c_3\varepsilon^3 + O(\varepsilon^4)\Big), \tag{5.22}$$

where coefficients c_2 and c_3 are defined above. This root defines slightly attenuated leaky surface wave (pseudo-Stoneley wave), whose phase velocity is close but somewhat bigger than c_f.

As it was mentioned already, dispersion equation (5.10) has one more complex root, satisfying also (5.12). It corresponds to the generalized Rayleigh wave with phase velocity $c_{R'} \to c_R$ as $\rho_0^{F+} \to 0$, where c_R is a velocity of the classical Rayleigh wave in an elastic half-space. Taking into account that here we have to choose $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1}$ and, due to the fact that $c_{R'}$ is close to c_s, $\sqrt{\frac{\tilde{\omega}^2}{c_f^2} - 1} \approx \frac{\tilde{\omega}}{c_f}$, equation (5.10) can be rewritten as:

$$\begin{aligned}\left(\mathcal{P}_R - ic_f\tilde{\gamma}_2 r + ic_f\tilde{\gamma}_2 r^+\frac{\tilde{\omega}^3}{c_s^4}\right)\Big(1 - c_f\alpha(1+n_E)\Big)&\\ -2i\tilde{\gamma}_2 c_f r^+\frac{\tilde{\omega}^3}{c_s^4}\Big(1 - c_f\alpha(1-n_E)\Big) &= 0.\end{aligned} \tag{5.23}$$

Let

$$\tilde{\omega} = \Omega_0 + \varepsilon\Omega_1 + \cdots \tag{5.24}$$

It is easy to see that the leading part Ω_0 of expansion (5.24) satisfies the Rayleigh equation: $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$. For the next term Ω_1 one gets the following equation:

$$\begin{aligned}&i\sqrt{1-\Omega_0^2}(r+r^+)\frac{\Omega_0^3}{c_s^4}\\ &-\left[\frac{4}{c_s^4}\Omega_0^3 - \frac{8}{c_s^2}\Omega_0 - 4\frac{d}{d\tilde{\omega}}\left(\sqrt{1-\tilde{\omega}^2}\sqrt{1-\frac{\tilde{\omega}^2}{c_s^2}}\right)\bigg|_{\tilde{\omega}=\Omega_0}\right]\Omega_1 = 0.\end{aligned} \tag{5.25}$$

Finally, one has:

$$\tilde{\omega}_{R'} = c_R + \varepsilon\Omega_1 + O(\varepsilon^2), \tag{5.26}$$

where Ω_1 is imaginary and it is determined by (5.25). Real part of (5.25), defines the phase velocity of the generalized Rayleigh wave and $k\mathrm{Im}\Omega_1$ corresponds to the attenuation of this wave. Thus the reradiation of the energy into the medium occurs, i.e. it is a leaky wave. Since $\mathrm{Im}\Omega_1 > 0$ then one can prove, estimating the amplitudes of the bulk waves, that a part of the energy of this surface wave is absorbed by the slow compressional wave.

It should be noted that complex roots $\tilde{\omega}_2$ and $\tilde{\omega}_{R'}$ lie at the first strip of the Riemann surface, while $\tilde{\omega}_1$ lies at the upper (second) strip of the Riemann surface. Moreover, if $\alpha \to 0$ and $\rho_0^{F+} \to 0$, i.e. $r^+ \to 0$, (limit passage to the vacuum), the roots $\tilde{\omega}_1$ and $\tilde{\omega}_{R'}$ continuously pass to the corresponding roots derived for the case of a free interface of a saturated porous medium. As for the generalized Rayleigh wave, this statement simply follows from (5.25) if $r^+ \to 0$. Then (5.26) coincides with (4.14).

In order to prove this statement for the true Stoneley wave, we have to construct inner asymptotic expansion of the root with respect to ε, i.e. $\alpha = \alpha_0\varepsilon, \quad \alpha_0 \geq 0$ (see Appendix 2 [35] for more details). Let us consider the first quadrant $R^{++} = \{\alpha_0 > 0,\ \rho_0^{F+} > 0\}$ of parametric plane $(\alpha_0,\ \rho_0^{F+})$ excluding the cut

$$\gamma = \{\alpha_0,\ \rho_0^{F+};\ \rho_0^{F+} \geq (\rho_0^{F+})^*,\ \alpha_0 = \alpha_0^*(\rho_0^{F+})\}, \tag{5.27}$$

where

$$\alpha_0^*(\rho_0^{F+}) = \frac{\rho_0^F + \rho_0^{F+}}{4\mu^S(1-c_s^2)(1+n_E)} \tag{5.28}$$

and

$$(\rho_0^{F+})^* = \frac{(1+n_E)\rho_0^F}{7n_E - 1}. \tag{5.29}$$

Then in the domain $R^{++}\backslash\gamma$ the root $\tilde{\omega}_1$ tends continuously to the analogous root $\tilde{\omega}_1$ of dispersion relation (4.7) which corresponds to a free interface of a porous medium. As it was proven in [35], an appearence of such a cut is stipulated by different limit values of $\tilde{\omega}_1$ from the left and right of $\alpha_0 = \alpha_0^*(\rho_0^{F+}), \quad \rho_0^{F+} > (\rho_0^{F+})^*$. Different behaviour of the root $\tilde{\omega}_1$ at the upper and lower sides of the cut γ can be interpreted as a hysteresis phenomenon.

Regarding pseudo-Stoneley surface wave, it is degenerated into the bulk compressional wave of the second kind as $\alpha \to 0$, i.e. $\tilde{\omega}_2$ tends to c_f (see Appendix 2 [35]).

6. Conclusions

The results presented in the paper concern surface waves which appear at a free interfaces of saturated porous media and at an interface separating a saturated porous medium and a liquid. Such waves were not yet systematically investigated. The present research reveals new features of surface waves in porous media which do not appear for example in the case of classical liquid/elastic medium interface. They are connected with the presence of a slow compressional wave in a porous solid. In contrast to the free interface of elastic half-space, where only the Rayleigh wave exists, in porous materials two types of surface waves are proven to be possible. They are due to the combination of three waves in the porous medium: two longitudinal waves and one shear wave. Consequently, as in the case of an interface between liquid and elastic half-spaces, there are two surface waves. However, qualitatively the properties of the surface waves in porous media differ significantly.

1. The first mode, which exists at the free interface of a saturated porous medium, is a true surface wave (true Stoneley wave), which, as demonstrated by (4.11), propagates almost without attenuation. Asymptotic analysis showed that its velocity is less than the velocities of all bulk waves in unbounded porous medium and that it is influenced primarily by the compressibility coefficient of the liquid phase. This surface wave is much slower than an analogous one at the interface of liquid and elastic half-spaces.

2. The second type of surface waves, which appear on interfaces of porous media are the so-called leaky waves. These leaky waves are generalized Rayleigh waves since their phase velocity is close to the velocity of the classical Rayleigh wave. In the typical case of an interface between liquid and elastic half-spaces the generalized Rayleigh wave is carried mostly by the elastic half-space and radiates some of its energy into the liquid. This is not the case for porous materials in which the generalized Rayleigh wave is carried by the porous medium and radiates a part of its energy into the porous medium itself. Namely, this energy is absorbed by a slow compressional wave. Such leaky modes are transitional modes between surface and bulk waves. It is obvious that due to energy radiation into interior of the medium, they can exist only in the limited domain (localized waves). They are transformed into the bulk waves as soon as their surface component is transformed into the bulk one.

New results were obtained also concerning surface waves appearing at an interface separating a porous medium and a liquid. In contrast to the classical liquid/elastic medium case, where two surface waves exist, namely the true Stoneley wave and the generalized Rayleigh wave, in porous materials three types of surface waves are proven to be possible. They are due to the combination of four waves: three waves in a porous medium and one wave in a liquid.

1. The first mode is the true Stoneley wave. Similar to the analogous wave at a free interface of a porous medium, it propagates almost without attenuation and its velocity is less than the velocities of all bulk waves in unbounded porous medium and in a liquid. If $\alpha \to 0$ and $\rho_0^{F+} \to 0$ (limit passage to the vacuum), this wave passes to the analogous wave at a free interface of a porous medium.

2. The second type of surface waves is the leaky pseudo-Stoneley wave. Its velocity is close but somewhat bigger than velocity of the lowest logitudinal wave of the second kind and it is defined, as for the true Stoneley wave, by the compressibility coefficient of the liquid phase. If $\alpha \to 0$, than this wave is degenerated to the bulk logitudinal wave of the second kind.

3. The third mode is the leaky generalized Rayleigh wave since its phase velocity is close to the velocity of the classical Rayleigh wave. In the typical case of an interface between liquid and elastic half-spaces the generalized Rayleigh wave is carried mostly by the elastic half-space and radiates some of its energy into the liquid. In porous materials some part of its energy is absorbed by a slow compressional wave. If $\varrho_0^{F+} \to 0$, this wave is transformed into the generalized Rayleigh wave at a free interface of a porous medium.

Conclusions concerning the second case considered above have a particular practical bearing. As we have seen the properties of surface waves appearing in this case are strongly dependent on the surface permeability parameter α. Consequently measurements of such waves may determine this important parameter for boundary value problems for saturated poroelastic media with permeable boundaries.

Finally let us mention that some results of this work coincide to the great extent with those obtained numerically by Feng and Johnson [14,15] in spite of the fact that the model used in this paper is much simpler than the Biot's model applied in [14,15]. This seems to support our arguments made at the beginning of this work concerning flaws of the Biot model.

Acknowledgement

One of the authors (I.Edelman) acknowledges the support of the Alexander von Humboldt Foundation during the work on this paper.

References

[1] I.A. Viktorov (1967) *Rayleigh and Lamb waves*, Plenum, New York.

[2] L. M. Brekhovskih, O. A. Godin (1990) *Acoustics of Layered Media I,* Springer, Berlin.

[3] A. Udias (1999) *Principles of Seismology*, Cambridge University Press.

[4] Lord Rayleigh (1885) On waves propagated along the plane surface of an elastic solid, *Proc. London Math. Soc.* **17** 4-11.

[5] J.D.Achenbach (1973) *Wave Propagation in Elastic Solids*, North - Holland, Amsterdam.

[6] A.E.H. Love (1926) *Some Problems in Geodynamics*, Cambridge University Press, London.

[7] R. Stoneley (1924) Elastic waves at the surface of separation of two solids, *Proc. Roy. Soc. London*, **Ser.A 106** 416-428.

[8] W.M. Ewing, W.S. Jardetky, F. Press (1957) *Elastic Waves in Layered Media,* McGraw–Hill, New York.

[9] J.G. Scholte (1947) The range of existence of Rayleigh and Stoneley waves, *Mon. Not. R. Astr. Soc., Geophys. Suppl.* **5** 120-126.

[10] H. Deresiewicz, R.Skalak (1963) On uniqueness in dynamic poroelasticity, *Bull.Seismol. Soc. Am.* **53**, 783-788.

[11] J.H. Rosenbaum (1974) Synthetic microseismograms: logging in porous formations, *Geophysics* **39(1)**, 14-32.

[12] B.Gurevich, M.Schoenberg (1999) Interface conditions for Biot's equations of poroelasticity, *J. Acoust. Soc. Amer.* **105(5)** 2585-2589.

[13] H. Deresiewicz (1962) The effect of boundaries on wave propagation in a liquid-filled porous solid. IV. Surface waves in a half-space. *Bull. Seism. Soc. Am.* **52(3)** 627-638.

[14] S. Feng, D. L. Johnson (1983) High-frequency Acoustic Properties of a Fluid/Porous Solid Interface. I. New Surface Mode, *J. Acoust. Soc. Am.* **74(3)** 906-914.

[15] Feng, D. L. Johnson (1983) High-frequency Acoustic Properties of a Fluid/Porous Solid Interface. II. The 2D Reflection Green's Function, *J. Acoust. Soc. Am.* **74(3)** 915-924.

[16] P.B. Nagy (1992) Observation of a new surface mode on a fluid-saturated permeable solid. *Appl. Phys. Lett.* **60(22)** 2734-2737.

[17] P.B. Nagy (1999) Acoustics and ultrasonics, in: *Experimental methods in the physical sciences*, Academic Press, 161–221.

[18] J.G.Berryman (1992) Waves in partially saturated porous media, in: W.E. Fitzgibbon, M.F. Wheeler (eds.) *Wave Propagation and Inversion*, SIAM, Philadelphia, 1–25.

[19] M.J. Mayes, P.B. Nagy, L. Adler, B.P. Bonner, R.Streit (1986) Excitation of surface waves of different modes at fluid-porous solid interface, *J. Acoust. Soc. Am.* **79(2)** 249-252.

[20] A. Norris (1999) Stonely-wave attenuation and dispersion in permeable formations, *Geophysics* **54(3)** 330-341.

[21] K. Wilmanski (1995) Lagrangean Model of Two-Phase Porous Material, *J. Non-Equilib. Thermodyn.*, **20** 50–77.

[22] K. Wilmanski (1996) Porous Media at Finite Strains. The New Model with the Balance Equation for Porosity, *Arch. Mech.*, **48(4)** 591–628.

[23] K. Wilmanski (1998) *Thermomechanics of Continua*, Springer, Berlin, Chpt. 10.

[24] K. Wilmanski (1998) A Thermodynamic Model of Compressible Porous Materials with the Balance Equations of Porosity, *Transport in Porous Media*, **32** 21-47.

[25] K. Wilmanski (2001) Thermodynamics of Multicomponent Continua, in: R. Teisseyre, E. Majewski (eds.) *Earthquake Thermodynamics and Phase Transformations in the Earth's Interior*, Academic Press, 567–655.

[26] K. Wilmanski (2001, to appear) Mass Exchange, Diffusion, and Large Deformations of Poroelastic Materials, in: G. Capriz, V. Ghionna, P. Giovine (eds.), *Mechanics of Porous Media. Flow and Transport Phenomena in Particulate Materials*, Birkhäuser.

[27] I. Tolstoy (1991) Acoustics, Elasticity, and Thermodynamics of Porous Media: Twenty-one Papers by M. A. Biot, *Acoustical Society of America.*

[28] D. L. Johnson, T. J. Plona (1982) Acoustic slow waves and the consolidation transition, *J.Acoustic.Soc.Am.*, **72** 556-565.

[29] K. Wilmanski (2001, to appear) Some questions arising in models of porous materials, in: M.Brocato (ed.), *Rational Continua, Classical and New.* A collection of papers dedicated to G.Capriz, Springer-Verlag, Italy; see as well: K. Wilmanski, B. Albers (2000) Two notes on continuous modelling of porous media, *WIAS, Preprint No.579.*[3]

[30] I. Müller (1968) A Thermodynamic Theory of Mixtures of Fluid, *Arch. Rat. Mech. Anal.* **28** 1-39.

[31] I. Müller (1985) *Thermodynamics*, Pitman, Boston.

[32] K. Wilmanski (1999) Waves in Porous and Granular Materials, in: K. Hutter, K. Wilmanski (eds.), *Kinetic and Continuum Theories of Granular and Porous Media*, SpringerWienNewYork, CISM 400, 131-186.

[33] K. Wilmanski (2001) Sound and shock waves in porous and granular materials, *Proc. of "WASCOM 99", 10th Conference on Waves and Stability in Continuous Media*, V.Ciancio, A.Donato, F.Oliveri, S.Rionero (eds), World Scientific, 489-503.

[3]WIAS - Preprints are easily available from the webpage http://www.wias-berlin.de/publications/preprints as pdf or ps-files.

[34] I. Edelman, K. Wilmanski, E. Radkevich (1999) Surface waves at a free interface of a saturated porous medium, *WIAS, Preprint No.513.*[3]

[35] I. Edelman, K. Wilmanski (1999) Surface waves at an interface separating a saturated porous medium and a liquid, *WIAS, Preprint No.531.*[3]

[36] I. Edelman, K. Wilmanski (2000) Surface waves at an interface separating two saturated porous media, *WIAS, Preprint No.568.*[3]