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A new type of travelling wave solutions

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Abstract

We study the existence of combustion waves for an autocatalytic reaction in the non-adiabatic case. Basing on the fact that the reaction system has canard solutions separating the slow combustion regime from the explosive one, we prove by applying the geometric theory of singularly perturbed differential equations the existence of a new type of travelling waves solutions, the so-called canard travelling waves.

1 Introduction

Let us consider the singularly perturbed system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,\varepsilon), \\ \varepsilon\,\frac{dy}{dt} &= g(x,y,\alpha,\varepsilon) \end{aligned} \tag{1}$$

with $x \in R^n, y \in R^m$, ε and α are parameters, where ε is small and positive, f and g are sufficiently smooth.

Let $y = \phi(x,\alpha)$ be an isolated simple root of the *degenerate equation*

$$g(x,y,\alpha,0) = 0$$

to system (1). We denote the graph of ϕ by S_α and call it *slow manifold* of (1). S_α consists of equilibria of the *associated equation* to (1)

$$\frac{dy}{d\tau} = g(x,y,\alpha,0). \tag{2}$$

Let $B(x,\alpha)$ be the Jacobian matrix of $g(x,y,\alpha,0)$ with respect to y at the root $y = \phi(x,\alpha)$. We denote by $\sigma(B(x,\alpha))$ the spectrum of $B(x,\alpha)$. We define the subsets S^s_α and S^u_α of S_α by

$$S^s_\alpha := \{(x,y) \in S_\alpha : Re\,\sigma\Big(B(x,\alpha)\Big) < 0\},$$

$$S^u_\alpha := \{(x,y) \in S_\alpha : \sigma\Big(B(x,\alpha)\Big) \cap \Big(Re\,z > 0\Big) \neq \emptyset\}.$$

S^s_α (S^u_α) consists of stable (unstable) equilibria of (2), therefore we call S^s_α (S^u_α) the stable (unstable) slow manifold of (1). A point $(x,\phi(x,\alpha))$ of S_α satisfying $\det B(x,\alpha) = 0$ is called *impass point*.

According to the geometric theory of singularly perturbed systems (see e.g. [7, 16, 25]), there is to given α a sufficiently small positive number ε_0 such that for $0 < \varepsilon \le \varepsilon_0$ system (1) has in a small neighborhood of S^s_α (S^u_α) an attracting (repelling) locally invariant manifold $S^s_{\alpha,\varepsilon}$ ($S^u_{\alpha,\varepsilon}$). Usually, $S^s_{\alpha,\varepsilon}$ and $S^u_{\alpha,\varepsilon}$ do not belong to the same integral manifold of (1). Under some additional conditions, to any sufficiently small fixed ε there exists a $\alpha = \alpha^*(\varepsilon)$ such that there are a trajectory in $S^s_{\alpha^*(\varepsilon),\varepsilon}$ and a trajectory in $S^u_{\alpha^*(\varepsilon),\varepsilon}$ which can be glued together near an impass point [24]. As a result we get a trajectory containing an attracting and a repelling part. We call such a trajectory *canard solution* (or French duck solution) [1, 6, 18]. We note that the existence of a canard trajectory can imply the phenomenon of *delayed loss of stability* [19]. Concerning the existence of canards in chemical systems see e.g. [3, 10] and references therein.

In modeling the self–ignition regime in case of an autocatalytic combustion reaction the following interesting fact was discovered: the occurrence of a critical regime can be characterized by the existence of a canard solution (see [12, 13]).

In this paper we shall consider the problem of thermal explosion in case of an autocatalytic combustion reaction. Taking into account heat convection and diffusion of the reacting substances and restricting to one space dimension we get the following model ([9, 13])

$$\begin{aligned}\gamma\frac{\partial\theta}{\partial t} &= \eta(1-\eta)\exp(\frac{\theta}{1+\beta\theta}) - \alpha\theta + \delta\frac{\partial^2\theta}{\partial\xi^2}\ ,\\ \gamma\frac{\partial\eta}{\partial t} &= \gamma\eta(1-\eta)\exp(\frac{\theta}{1+\beta\theta}) + \mu\frac{\partial^2\eta}{\partial\xi^2}\ .\end{aligned}$$

Here, θ denotes the dimensionless temperature, η is the dimensionless depth of conversion of the gas mixture, $-\alpha\theta$ describes the volumetric heat loss, γ and β are parameters which are small in case of a highly exothermic reactions. In what follows we suppose for simplicity $\beta = 0$, i.e. we shall investigate the system

$$\begin{aligned}\gamma\frac{\partial\theta}{\partial t} &= \eta(1-\eta)e^\theta - \alpha\theta + \delta\frac{\partial^2\theta}{\partial\xi^2}\ ,\\ \gamma\frac{\partial\eta}{\partial t} &= \gamma\eta(1-\eta)e^\theta + \mu\frac{\partial^2\eta}{\partial\xi^2}\ .\end{aligned} \tag{3}$$

The goal of this paper is to study travelling wave solutions of (3) connecting the steady states O($\eta = 0, \theta = 0$) and P($\eta = 1, \theta = 0$). Analyzing the corresponding boundary value problem we will show that it is possible to choose the parameters in such a way that the projection of the associated heteroclinic trajectory into the θ, η-plane is located in a small neighborhood of the canard trajectory characterizing the occurrence of a critical regime. We call the corresponding travelling wave solution a canard travelling wave solution.

Note that combustion waves have been extensively studied during the last three decades (see [2, 4, 8, 9, 17, 20, 21, 22, 23, 26, 28] and references therein). Most of research has been focussed on the adiabatic case for first order combustion reactions (for n-th order reactions see [2, 4]). The non-adiabatic case for a first order reaction has been studied in [27]. In the present paper we investigate the non-adiabatic case ($\alpha > 0$) in case of an autocatalytic reaction.

2 Canard solutions in a self-ignition problem

First we study the reaction–diffusion system (3) in a homogeneous medium, that is, we investigate the reaction system

$$\begin{aligned}\gamma\frac{d\theta}{dt} &= \eta(1-\eta)e^{\theta} - \alpha\theta,\\ \frac{d\eta}{dt} &= \eta(1-\eta)e^{\theta},\end{aligned} \tag{4}$$

where we suppose that γ is a small positive parameter. The chemically relevant phase space of (4) is defined by $\theta \geq 0, 0 \leq \eta \leq 1$. A qualitative investigation of system (4) has been performed in [12, 13, 14, 15]. In what follows we recall the main results.

The degenerate equation to (4) reads

$$0 = \eta(1-\eta) - \alpha\theta e^{-\theta}.$$

Its solution set is called the slow manifold S_α of (4) which is depictured in the figures Fig. 1 – Fig. 3 for different values of α.

It can be easily verified that the stable slow manifold S_α^s (unstable slow manifold S_α^u) of (4) is located in the region $0 \leq \theta < 1$ ($\theta > 1$). We represent S_α^s (S_α^u) by a solid (dashed) line.

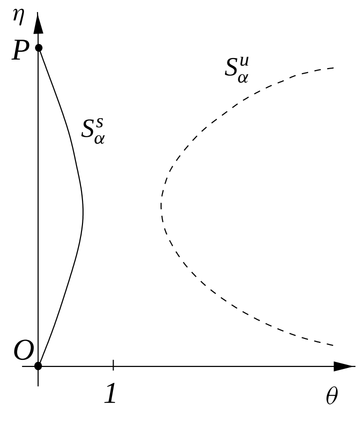

Fig. 1. $\alpha > e/4$

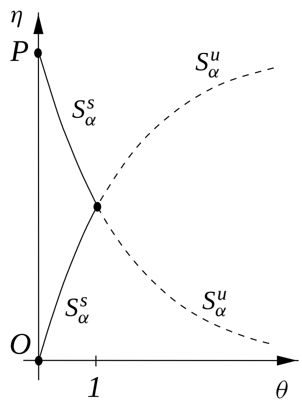

Fig. 2. $\alpha = e/4$

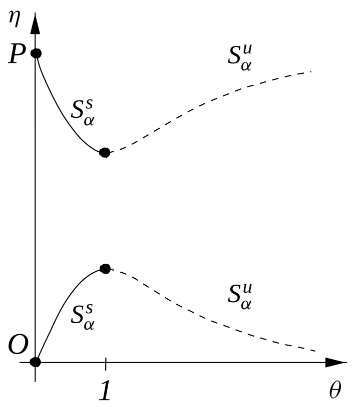

Fig. 3. $\alpha < e/4$

Fig. 1 shows S_α for some $\alpha > e/4$. Here, each set S^s_α and S^u_α consists of a single connected curve. For $\alpha > e/4$ and γ sufficiently small, system (4) has an attracting invariant manifold $S^s_{\alpha,\gamma}$ and a repelling invariant manifold $S^u_{\alpha,\gamma}$ near the slow manifold S^s_α and S^u_α, respectively. To study the behavior of (4) we consider the initial value problem

$$\begin{aligned} \gamma\frac{d\theta}{dt} &= \eta(1-\eta)e^\theta - \alpha\theta, \\ \frac{d\eta}{dt} &= \eta(1-\eta)e^\theta,\ \ t > 0, \\ \\ \theta(0) &= 0,\ \eta(0) = \eta_0,\ \ 0 < \eta_0 \ll 1. \end{aligned} \tag{5}$$

Since the initial point $(0, \eta_0)$ belongs to the basin of attraction of the set S^s_α, after some short transition period the solution of (5) follows the attracting slow invariant manifold $S^s_{\alpha,\gamma}$ and tends to the equilibrium P as t tends to ∞. We call this behavior the slow combustion regime.

Fig. 3 shows S_α for some $\alpha < e/4$. Each set S^s_α and S^u_α consists of two different components. For $\alpha < e/4$ and γ sufficiently small, system (4) has an attracting invariant manifold $S^s_{\alpha,\gamma}$ (repelling invariant manifold $S^u_{\alpha,\gamma}$) near each component of the slow invariant manifold S^s_α (S^u_α). For γ sufficiently small and after some short transition period, the solution of (5) will follow the component of $S^s_{\alpha,\gamma}$ which is related to the origin until it reaches the value $\theta = 1$. After this moment, $\theta(t)$ will increase very fast. This behavior characterizes the explosive regime.

Fig. 2 depictures the special case $\alpha = e/4$ where the slow manifold possesses a singularity which corresponds to a transcritical bifurcation of the set of equilibria of the associated system (2). For $\alpha = e/4$ and γ sufficiently small, system (4) has an attracting invariant manifold $S^s_{\alpha,\gamma}$ (repelling invariant manifold $S^u_{\alpha,\gamma}$) near each component of the slow manifold S^s_α (S^u_α). If we study the initial value problem (5), then to given small γ and for α near $e/4$ but less than $e/4$ we can observe the existence of canard solutions which describe the transition between the slow combustion regime and the explosive regime. That means, to given small γ there is an α-interval $(\alpha_e(\gamma), \alpha_c(\gamma))$ such that for $\alpha > \alpha_c(\gamma)$ ($\alpha < \alpha_e(\gamma)$) the solution of (5) belongs to the slow regime (explosive regime). The interval $(\alpha_e(\gamma), \alpha_c(\gamma))$ characterizes the critical regime, that is, after some short transition time, for $\alpha \in (\alpha_e(\gamma), \alpha_c(\gamma))$ the solution of (5) follows the component of S^s_α which is related to the origin until it reaches the value $\theta = 1$. After this moment, it follows the component S^u_α which is located in the region $\eta > \frac{1}{2}$ up to some point J from which the solution "jumps" towards the attracting manifold S^s_α related to the equilibrium P. Then it follows this manifold to approach P as t tends to ∞ (see Fig. 4).

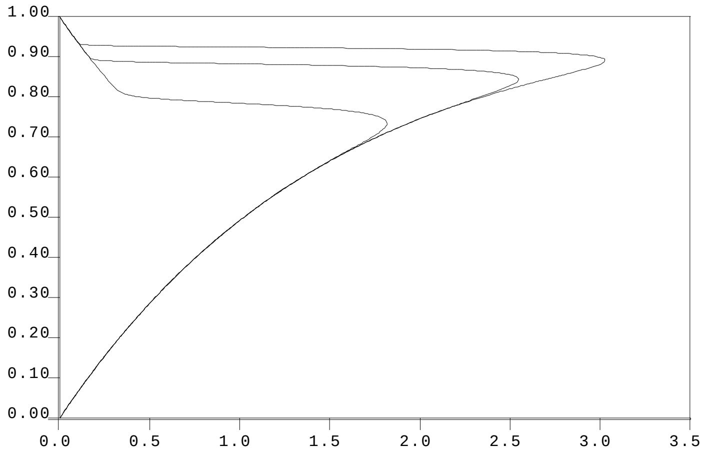

Fig. 4. Canard trajectories of system (4) for $\gamma = 0.05$, $\alpha' = 0.659941603$, $\alpha'' = 0.659941646$, $\alpha''' = 0.659952218$.

It is known that the interval $(\alpha_e(\gamma), \alpha_c(\gamma))$ satisfies $\alpha_e(\gamma) - \alpha_c(\gamma) = O(e^{-k/\gamma})$ as $\gamma \to 0$ for some $k > 0$ such that for $\alpha \in (\alpha_e(\gamma), \alpha_c(\gamma))$ it holds the representation

$$\alpha = \alpha^*(\gamma) + O(e^{-k/\gamma}) \quad \text{as } \gamma \to 0,$$

where the critical value $\alpha^*(\gamma)$ has the asymptotic expansion

$$\alpha^*(\gamma) = \alpha_0 + \gamma\alpha_1 + \gamma^2\alpha_2 + \dots. \tag{6}$$

In what follows we look for an asymptotic representation of the canard solution $\eta = H(\theta, \gamma)$ of (4) as long as it follows the slow manifold S_α and is continuous at $\theta = 1$

$$\eta = H(\theta, \gamma) \equiv H_0(\theta) + \gamma H_1(\theta) + \dots. \tag{7}$$

In order to find the coefficients α_i and $H_i(\theta)$ we substitute the expansions (6), (7) into the equation

$$\frac{d\eta}{dt} = H'(\theta)\frac{d\theta}{dt}$$

expressing the invariance of the canard trajectory. Equating the coefficients corresponding to the same powers of γ we get

$$\alpha_0 = \left.\frac{H_0(1 - H_0)e^\theta}{\theta}\right|_{\theta=1} = \frac{e}{4},$$

$$H_0(\theta) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha_0\theta e^{-\theta}},$$

$$\alpha_1 = \left.\frac{-\alpha_0}{H_0'}\right|_{\theta=1} = -\frac{e}{\sqrt{2}},$$

$$H_1(\theta) = \frac{\theta(\alpha_1 H_0' + \alpha_0)}{H_0'(1-2H_0)e^{\theta}},$$

$$\alpha_2 = -\left.\frac{\alpha_1 H_1' + H_0' H_1^2 e}{H_0'}\right|_{\theta=1} = -\frac{49}{36}e,$$

$$H_2(\theta) = \frac{\theta\left(\alpha_1 H_1' + \alpha_2 H_0'\right) + H_0' H_1^2 e^{\theta} + H_1\left(1 - H_1'\right)\left(1-2H_0\right)e^{\theta}}{H_0'(1-2H_0)e^{\theta}}.$$

Here, the values α_i are chosen in such a way that H_i are continuous at $\theta = 1$. All details can be found in [12, 13]. Thus, the critical values $\alpha^*(\gamma)$ has the representation

$$\alpha^*(\gamma) = \frac{e}{4}\left(1 - 2\sqrt{2}\gamma - \frac{49}{9}\gamma^2\right) + O(\gamma^3).$$

3 Canard travelling waves

We are interested in travelling wave solutions of (3) with speed c and which connect the steady states O and P. That means we are looking for solutions to (3) of the type

$$\theta(t,\xi) = \tilde{\theta}(\xi + ct) \equiv \theta(x), \quad \eta(t,\xi) = \tilde{\eta}(\xi + ct) \equiv \eta(x) \tag{8}$$

satisfying

$$\lim_{x\to-\infty} \eta(x) = \lim_{x\to-\infty} \theta(x) = 0,$$
$$\lim_{x\to+\infty} \eta(x) = 1, \ \lim_{x\to+\infty} \theta(x) = 0,$$

and where $x = \xi + ct$ is the phase of the wave. Such solution corresponds to a one-dimensional propagating flame.
Substituting (8) into (3) we get

$$\begin{aligned} \gamma c\,\frac{d\theta}{dx} &= \eta(1-\eta)e^{\theta} - \alpha\theta + \delta\,\frac{d^2\theta}{dx^2}\,, \\ \gamma c\,\frac{d\eta}{dx} &= \gamma\eta(1-\eta)e^{\theta} + \mu\frac{d^2\eta}{dx^2}. \end{aligned} \tag{9}$$

At first we consider the case of a travelling wave solution with speed $c = \nu/\gamma$ where ν does not depend on γ. Since γ is a small parameter we are looking for travelling waves with high speed.

In that case, (9) takes the form

$$
\begin{aligned}
\nu \frac{d\theta}{dx} &= \eta(1-\eta)e^{\theta} - \alpha\theta + \delta \, \frac{d^2\theta}{dx^2} \ , \\
\nu \frac{d\eta}{dx} &= \gamma\eta(1-\eta)e^{\theta} + \mu \frac{d^2\eta}{dx^2} .
\end{aligned}
$$

This system is equivalent to the system

$$
\begin{aligned}
\frac{d\eta}{dx} &= \gamma p \ , \\
\frac{d\theta}{dx} &= q \ , \\
\delta \frac{dq}{dx} &= \nu q - \eta(1-\eta)e^{\theta} + \alpha\theta \ , \\
\mu \frac{dp}{dx} &= \nu p - \eta(1-\eta)e^{\theta} \ .
\end{aligned}
\tag{10}
$$

Introducing the new independent variable s by $s = \gamma x$ $(\gamma \neq 0)$ we obtain

$$
\begin{aligned}
\frac{d\eta}{ds} &= p \ , \\
\gamma \frac{d\theta}{ds} &= q \ , \\
\gamma\delta \frac{dq}{ds} &= \nu q - \eta(1-\eta)e^{\theta} + \alpha\theta \ , \\
\gamma\mu \frac{dp}{ds} &= \nu p - \eta(1-\eta)e^{\theta} \ .
\end{aligned}
\tag{11}
$$

Since γ is assumed to be small, (11) is a singularly perturbed system with the slow variable η and the fast variables θ, p, q. We are interested in a solution of (11) satisfying the boundary conditions

$$
\lim_{s\to-\infty} \eta(s) = \lim_{s\to-\infty} p(s) = \lim_{s\to-\infty} \theta(s) = 0, \ \lim_{s\to-\infty} q(s) = 0,
$$
$$
\lim_{s\to+\infty} \eta(s) = 1, \ \lim_{s\to+\infty} p(s) = \lim_{s\to+\infty} \theta(s) = 0, \ \lim_{s\to+\infty} q(s) = 0,
$$

that is, we are looking for a heteroclinic trajectory of the singularly perturbed system (11) connecting the equilibria O and P.

The degenerate equations of (11) read

$$
\begin{aligned}
0 &= q \ , \\
0 &= \nu q - \eta(1-\eta)e^{\theta} + \alpha\theta \ , \\
0 &= \nu p - \eta(1-\eta)e^{\theta} \ ,
\end{aligned}
$$

or, in more convenient form,

$$
\begin{aligned}
0 &= q \, , \\
0 &= -\eta(1-\eta)e^{\theta} + \alpha\theta \, , \\
\nu p &= \alpha\theta \, .
\end{aligned} \tag{12}
$$

System (12) defines the slow manifold $\tilde{S}_\alpha$ of system (11) in R^4. It is easy to see that $\tilde{S}_\alpha$ is a differentiable curve located in the plane $q = 0, p = \alpha\theta/c$ and that its projection into the θ, η-plane coincides with the manifold S_α introduced in the previous section. $\tilde{S}_\alpha$ represents the set of equilibria of the associated system to (3)

$$
\begin{aligned}
\frac{d\theta}{d\tau} &= q \, , \\
\delta \frac{dq}{d\tau} &= \nu q - \eta(1-\eta)e^{\theta} + \alpha\theta \, , \\
\mu \frac{dp}{d\tau} &= \nu p - \eta(1-\eta)e^{\theta} \, .
\end{aligned} \tag{13}
$$

It can be checked that for $\nu < 0$ the slow manifold $\tilde{S}_\alpha$ consists of stable (unstable) equilibria of (13) in the region $\theta > 1$ ($\theta < 1$).

Lemma 1. *To given ν, δ, μ there is a sufficiently small positive number γ_0 such that for $\gamma \in (0, \gamma_0)$ there is an $\tilde{\alpha}^* = \tilde{\alpha}^*(\gamma)$ such that (10) has a canard trajectory. $\tilde{\alpha}^*(\gamma)$ has the asymptotic representation*

$$
\tilde{\alpha}^* = \tilde{\alpha}_0 + \gamma\tilde{\alpha}_1 + \gamma^2\tilde{\alpha}_2 + \dots . \tag{14}
$$

Near the slow manifold the canard trajectory can be approximated by the asymptotic series

$$
\begin{aligned}
\eta &= \tilde{H}(\theta,\gamma) = \tilde{H}_0(\theta) + \gamma\tilde{H}_1(\theta) + \gamma^2\tilde{H}_2(\theta) + \dots \, , \\
q &= \gamma\tilde{Q}(\theta,\gamma) = \gamma\tilde{Q}_1(\theta) + \gamma^2\tilde{Q}_2(\theta) + \dots \, , \\
p &= \tilde{P}(\theta,\gamma) = \tilde{P}_0(\theta) + \gamma\tilde{P}_1(\theta) + \gamma^2 P_2(\theta) + \dots \, .
\end{aligned} \tag{15}
$$

Proof. We give only a sketch of the proof. First we linearize system (11) for $\alpha = e/4$ at the impass point $\eta = 0.5, \theta = 1, q = 0, p = \alpha/\nu$. The corresonding Jacobian matrix has the eigenvalues $\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = \nu \neq 0$. For α near $e/4$ we may reduce system (11) near the impass point by means of the center manifold theorem to a two-dimensional system which is singularly perturbed with the small parameter γ. Then, we apply the same technique as we have used in [14] to prove the existence of an α-interval near $\alpha = e/4$ such that (11) has a canard trajectory if α belongs to that interval.

To calculate the coefficients of the asymptotic expansions we substitute these series into (11) and equate the coefficients with the same power of γ. We get

$$
\tilde{\alpha}_0 = \alpha_0, \;\; \tilde{\alpha}_1 = \alpha_1, \;\; \tilde{\alpha}_2 = \alpha_2 + \frac{e^2}{2c^2}\left(\frac{5}{3}\delta - \mu\right),
$$

$$\tilde{H}_0 = H_0,\ \tilde{H}_1 = H_1,$$

where α_i and H_i are defined in section 2.

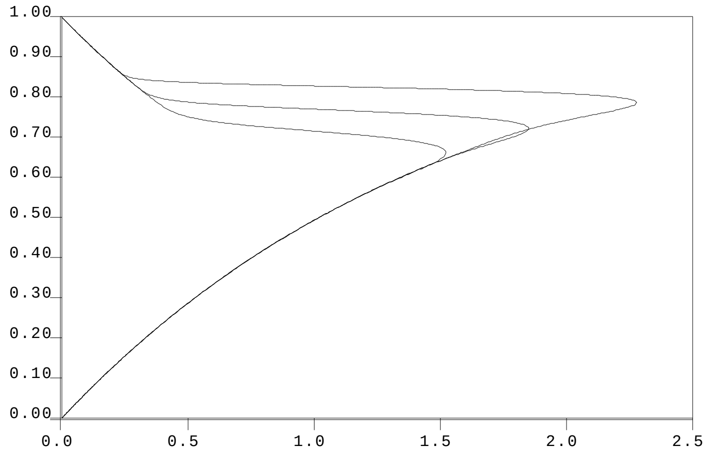

Fig. 5. Projection of canard trajectories of system (10) for $\gamma = 0.05$, $\alpha' = 0.66022803,\ \alpha'' = 0.66025281,\ \alpha''' = 0.66024835$.

Let us return now to system (9). In what follows we assume

$$\delta = \kappa\mu, \tag{16}$$

where κ is some positive constant.

Introducing the new variable z by $x = -cz$ we get from (9) and taking into account (16)

$$\begin{aligned} -\gamma \frac{d\theta}{dz} &= \eta(1-\eta)e^{\theta} - \alpha\theta + \frac{\kappa\mu}{c^2}\frac{d^2\theta}{dz^2}\ , \\ -\gamma \frac{d\eta}{dz} &= \gamma\eta(1-\eta)e^{\theta} + \frac{\mu}{c^2}\frac{d^2\eta}{dz^2}. \end{aligned} \tag{17}$$

For the sequel we set

$$\varepsilon := \mu/c^2$$

and assume ε to be small, that is, we assume that the quotient of the diffusivity and the square of the velocity is small. System (17) is equivalent to the singularly perturbed system

$$\begin{aligned}
\frac{d\eta}{dz} &= -p, \\
\frac{d\theta}{dz} &= -q\ , \\
\kappa\varepsilon\,\frac{dq}{dz} &= -\gamma + q\eta(1-\eta)e^{\theta} - \alpha\theta\ , \\
\varepsilon\,\frac{dp}{dz} &= -\gamma p + \gamma\eta(1-\eta)e^{\theta}.
\end{aligned} \tag{18}$$

Assuming $\alpha > 0$, system (18) has the equilibria $O_1 := (p = q = \eta = \theta = 0)$ and $P_1 := (p = q = \theta = 0, \eta = 1)$ which do not depend on any parameter.

The corresponding degenerate equations are

$$\begin{aligned}
0 &= \gamma q - \eta(1-\eta)e^{\theta} + \alpha\theta\ , \\
0 &= p - \eta(1-\eta)e^{\theta}\ ,
\end{aligned}$$

their solution set S_α can be represented in the form

$$\begin{aligned}
q &= \gamma^{-1}(\eta(1-\eta)e^{\theta} - \alpha\theta)\ , \\
p &= \eta(1-\eta)e^{\theta}\ .
\end{aligned}$$

It is easy to check that S_α contains no impass point. According to a fundamental result of the geometric theory of singularly perturbed differential equations [7, 25], there exists for sufficiently small ε a smooth invariant manifold $S_{\alpha,\varepsilon}$ of (18) which is close to S_α, contains the equilibria O_1 and P_1 and can be represented in the form

$$\begin{aligned}
q &= \varphi(\eta,\theta,\varepsilon) = \gamma^{-1}\Big[\eta(1-\eta)e^{\theta} - \alpha\theta + \varepsilon\varphi_1(\eta,\theta) + O(\varepsilon^2)\Big], \\
p &= \psi(\eta,\theta,\varepsilon) = \eta(1-\eta)e^{\theta} + \varepsilon\psi_1(\eta,\theta) + O(\varepsilon^2)\ .
\end{aligned}$$

On $S_{\alpha,\varepsilon}$ system (18) can be written as

$$\begin{aligned}
\frac{d\eta}{dz} &= \eta(1-\eta)e^{\theta} + \varepsilon\psi_1(\eta,\theta) + O(\varepsilon^2), \\
\gamma\,\frac{d\theta}{dz} &= \eta(1-\eta)e^{\theta} - \alpha\theta + \varepsilon\varphi_1(\eta,\theta) + O(\varepsilon^2).
\end{aligned} \tag{19}$$

Lemma 2. *To given sufficiently small γ there is a an exponentially small α-interval $I_\alpha(\gamma)$ and a sufficiently small ε_0 such that for $0 \le \varepsilon < \varepsilon_0$ and for $\tilde{\alpha}^* \in I_\alpha(\gamma)$ system (19) has a canard trajectory connecting the equilibria O and P.*

Proof. We note that for $\varepsilon = 0$ system (19) coincides with system (4) having O as a saddle equilibrium and P as a stable node. Moreover, it has been proven that for sufficiently small γ and if α belongs to some exponentially small interval $(a_e(\gamma), a_c(\gamma))$, there is canard trajectory $\mathcal{T}_h(\gamma)$ of (4) connecting the equilibria O and P. As the stable manifold of P and the unstable manifold of O intersect transversally, $\mathcal{T}_h(\gamma)$ is a transversal heteroclinic trajectory. Thus, sufficiently small perturbations of (4) do not destroy the existence of $\mathcal{T}_h(\gamma)$. Since the right hand side of (19) depends smoothly on ε we can conclude that to given small γ there is a sufficiently small $\varepsilon_0(\gamma)$ such that for $\varepsilon \in (0, \varepsilon_0(\gamma))$ system (19) has a heteroclinic trajectory $\mathcal{T}_h(\gamma, \varepsilon)$ connecting O and P. Moreover, since under our conditions a solution of (19) depends continuously on the right hand side, there is an exponentially small α-interval $I_\alpha(\gamma)$ such that for $\alpha \in (I_\alpha(\gamma))$ $\mathcal{T}_h(\gamma, \varepsilon)$ is a canard trajectory. This completes the proof of the lemma.

From Lemma 2 we get immediately

Theorem 1. *To given sufficiently small γ there is an exponentially small α-interval $I_\alpha(\gamma)$ and a sufficiently small $\varepsilon_0(\gamma)$ such that for $0 \leq \mu(c^2) < \varepsilon_0(\gamma)$ and for $\tilde{\alpha}^* \in I_\alpha(\gamma)$ system (3) has a canard travelling wave solution connecting the equilibria O and P.*

The canard value $\tilde{\alpha}^*(\gamma)$ separates two types of waves corresponding to the slow combustion regime and to the thermal explosion (self-ignition) one, respectively. Like in section 2, the case $\alpha > \tilde{\alpha}^*(\gamma)$ corresponds to slow combustion profiles, while the case $\alpha < \tilde{\alpha}^*(\gamma)$ characterizes to self-ignition profiles.

The following figures show numerical investigations of the travelling wave solution of system (3) in the case of critical regime.

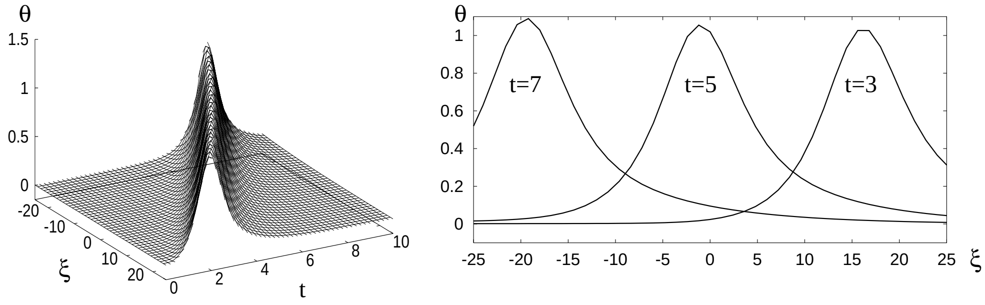

Fig. 6. θ–profiles of the canard travelling wave solution of system (3) for $\gamma = 0.05, \alpha = 0.58443, \delta = \mu = 1$.

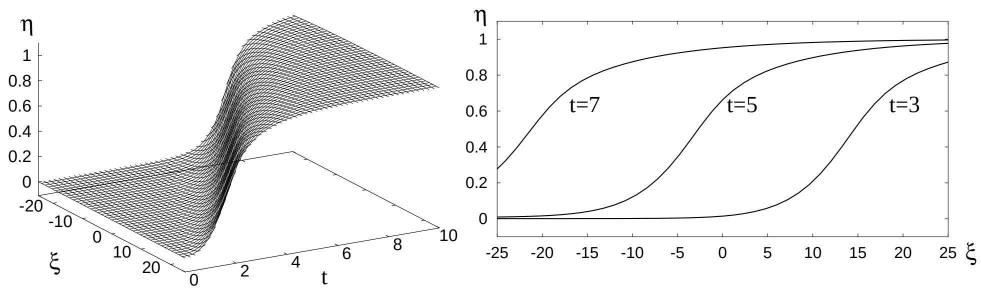

Fig. 7. η–profiles of the canard travelling wave solution of system (3) for $\gamma = 0.05, \alpha = 0.58443, \delta = \mu = 1$.

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