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# Longtime dynamics in adaptive gain control systems


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**Abstract**

We study the longtime dynamics of a nonlinear adaptive control system introduced by Mareels et al. [10] to control the behavior of a plant which can be described by a finite dimensional SISO linear time invariant system stabilizable by a high gain output feedback. We apply frequency domain methods to derive conditions for global stability, to approximate the region containing the global attractor and to estimate its Hausdorff dimension.

# 1 Introduction

Adaptive output gain control has been considered by I. Mareels [8], A.S. Morse [11], C.I. Byrnes and J.C. Willems [1], A. Ilchmann [4], H. Kaufman, I. Bar–Kana and K. Sobel [5] and I. Mareels et al. [10] to name but a few. The goal of this paper is to study the longtime dynamics of a class of adaptive gain control systems considered in [10].

We assume that the plant to be controlled can be described by a finite dimensional single input single output linear time invariant system that can be stabilized by a high gain output feedback. Such systems have a transfer function with stable zeroes and relative degree one. As has been proved in [9, 10], the class of systems under consideration can be transformed into the form

$$\begin{aligned} \frac{dx}{dt} &= Ax + by, \\ \frac{dy}{dt} &= -c^T x - dy + u, \end{aligned} \tag{1.1}$$

where $u$ is the input, $y$ the output, $(x, y) \in \mathrm{R}^n \times \mathrm{R}$ is the state of the system, $A$ is an $n \times n$-matrix, i.e. $A \in L(\mathrm{R}^n, \mathrm{R}^n)$, $b, c \in \mathrm{R}^n$, $d \in \mathrm{R}$. In [10] the adaptive feedback law

$$\begin{aligned} u &= -zy + e, \\ \frac{dz}{dt} &= -\sigma z + y^2, \quad z(0) > 0 \end{aligned} \tag{1.2}$$

has been applied to (1.1). Here, $\sigma$ is a positive constant representing the so-called sigma-modification, and $e$ characterizes the control offset error. Substituting (1.2) into (1.1) we obtain after some rescaling

$$\begin{aligned} \frac{dx}{dt} &= Ax + by, \\ \frac{dy}{dt} &= -c^T x - dy - zy + e, \\ \frac{dz}{dt} &= -\sigma z + y^2, \quad z(0) > 0. \end{aligned} \tag{1.3}$$

Under the assumptions that $(A, b)$ is controllable, $A$ is Hurwitz, and $\sigma > 0$, it has been proved in [10] that system (1.3) is dissipative in the sense of Levinson, that is, every trajectory enters finally a uniformly bounded region $G$ of the phase space, moreover an estimate of $G$ and conditions for global stability has been derived. An essential aim of [10] was to show by a bifurcation analysis and by numerical investigations that, for $n = 1$, the longtime dynamics of system (1.3) can be very rich, including chaotic behavior. Therefore, from the point of control theory it is desirable to find conditions for (1.3) to be globally stable or to minimize the region $G$ containing the global attractor.

The goal of this paper is to study the longtime dynamics of (1.3) by frequency methods. We derive estimates for the global attractor and give conditions for global asymptotic stability which improve corresponding results in [10] at least for the case $n = 1$, furthermore, we derive an upper bound for the Hausdorff dimension of the global attractor.

## 2 Assumptions, Preliminaries

Throughout this paper we assume

$(A_1)$. The matrix $A$ is Hurwitz, that is, all eigenvalues of $A$ are located in the left half plane.

$(A_2)$. The pair $(A, b)$ is controllable.

Since we are using frequency methods we have to introduce some transfer functions. First we introduce the function $\chi : \mathbb{C} \to \mathbb{C}$ by

$$\chi(s) := c^T (sI - A)^{-1} b \tag{2.1}$$

which is the transfer function of the input $y$ to the output $v$ of the system

$$\begin{aligned} \frac{dx}{dt} &= Ax + by, \\ v &= c^T x. \end{aligned}$$

By $\psi : \mathbb{C} \to \mathbb{C}$ we denote the transfer function of system (1.1) which can be represented in the form

$$\psi(s) := \frac{1}{s + d + \chi(s)}. \tag{2.2}$$

Using the notation

$$\eta(s) := det(sI - A), \quad p(s) := det \begin{pmatrix} sI - A & -b \\ c & s \end{pmatrix} \tag{2.3}$$

$\psi(s)$ can be represented also in the form

$$\psi(s) = \frac{\eta(s)}{p(s) + d\eta(s)}. \tag{2.4}$$

The investigation of the longtime behavior of system (1.3) will be based on the construction of appropriate Lyapunov functions. An essential part of these functions is some quadratic form defined by means of a symmetric positive definite matrix $H$. For the existence and also for the construction of $H$ we use frequency domain methods, in particular, we will apply results of V.A. Yakubovich, R.E. Kalman and V.M. Popov. For convenience of the reader we recall these results, also a theorem due to A. Douady and J. Oesterlé that will be used to estimate the Hausdorff dimension of the global attractor.

The following result represents a version of the Yakubovich - Kalman frequency domain theorem (see Theorem 1.10.1 in [7]).

**Theorem 2.1** *Let $P \in L(\mathrm{R}^n, \mathrm{R}^n)$ be Hurwitz, let $q, \delta \in \mathrm{R}^n$, let $g \in \mathrm{R}$. We assume $(P, q)$ to be controllable, and $(P, \delta)$ to be observable. Let $\mathcal{G}(\xi, \eta)$ be the Hermitian form defined by*

$$\mathcal{G}(\xi, \eta) := 2Re\, \xi^* \delta \eta + g\ |\eta|^2, \quad \xi \in \mathrm{C}^n,\ \eta \in \mathrm{C}. \tag{2.5}$$

*Then there is a positive definite symmetric matrix $H \in L(\mathrm{R}^n, \mathrm{R}^n)$ satisfying*

$$2Re\ \xi^* H(P\xi + q\eta) + \mathcal{G}(\xi, \eta) \le 0 \quad \forall\, \xi \in \mathrm{C}^n, \quad \forall \eta \in \mathrm{C}$$

*if and only if*

$$Re\ \mathcal{G}\Big((i\omega I - P)^{-1} q\eta, \eta\Big) \le 0 \quad \forall \eta \in \mathrm{C}, \forall\, \omega \in \mathrm{R}.$$

The following result is basically an application of Theorem 2.1 (see Theorem 1.12.1 in [7]).

**Theorem 2.2** *Let $P \in L(\mathrm{R}^n, \mathrm{R}^n)$ be Hurwitz, let $q, r \in \mathrm{R}^n$. We assume the pair $(P, q)$ to be controllable, and the pair $(P, r)$ to be observable. Let $\kappa : \mathrm{C} \to \mathrm{C}$ be the transfer function defined by*

$$\kappa(s) := r^T (P^T - sI) q. \tag{2.6}$$

*Then there exists a positive definite symmetric matrix $H \in L(\mathrm{R}^n, \mathrm{R}^n)$ satisfying the relations*

$$HP + P^T H \le 0 \quad and \quad Hq + r = 0$$

*if and only if*

$$Re\, [\kappa(i\omega)] > 0 \quad \forall \omega \in R.$$

The next result represents a special form of the criterion of Popov and coincides essentially with the circle criterion (see Theorem 1.14.1 in [7]).

**Theorem 2.3** *Let the matrix $P \in L(\mathrm{R}^n, \mathrm{R}^n)$ be Hurwitz, let $r, q \in \mathrm{R}^n$, let the pair $(P, q)$ be controllable. Suppose that for a certain number $\mu > 0$ the following inequality holds*

$$\mu^{-1} + Re[\zeta(i\omega)] > 0 \quad \forall \omega \in \mathrm{R},$$

*where $\zeta(s) := r^T(P - sI)^{-1}q$. Then the system*

$$\begin{aligned} \frac{dx}{dt} &= Px + qy, \quad \sigma = r^T x, \\ y &= \varphi(t, \sigma), \end{aligned} \tag{2.7}$$

*where $\varphi : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ is continuous and such that*

$$0 \leq \frac{\varphi(t, \sigma)}{\sigma} \leq \mu \quad \forall t, \sigma \in \mathrm{R},$$

*is globally asymptotically stable.*

It is well-known [3] that a dissipative autonomous system

$$\frac{dx}{dt} = f(x) \tag{2.8}$$

with $f \in C^1(\mathrm{R}^n, \mathrm{R}^n)$ has a global attractor $K$. Let $J(x)$ be the Jacobian of $f(x)$. The following theorem due to A. Douady and J. Oesterlé [2] aims to estimate the Hausdorff dimension of $K$ by means of the eigenvalues $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ of the symmetric matrix

$$M(x) := \frac{1}{2}\Big[J(x) + J(x)^T\Big]. \tag{2.9}$$

It follows from a more general result (see Theorem 5.5.1 in [7]).

**Theorem 2.4** *Assume $f \in C^1(\mathrm{R}^n, \mathrm{R}^n)$ and (2.8) to be dissipative. Let $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ be the eigenvalues of the symmetric matrix $M(x)$ defined in (2.9). Furthermore, we suppose that for $x \in G$, where $G$ is an open bounded region in $\mathrm{R}^n$, and for some $s \in [0, 1]$ and some $j$, $1 \leq j < n$, the following inequality holds*

$$\lambda_1(x) + \ldots + \lambda_j(x) + s\lambda_{j+1}(x) < 0. \tag{2.10}$$

*Then the Hausdorff dimension $\dim_H K$ of the global attractor $K$ of system (2.8) can be estimated by*

$$\dim_H K \leq j + s.$$

Under some additional conditions, Theorem 2.4 yields a criterion for global stability (see Theorem 3.1.1 in [6]).

**Theorem 2.5** *Suppose $f \in C^1(\mathrm{R}^n, \mathrm{R}^n)$ and that there exists a bounded region $G$ with smooth boundary $\partial G$ such that the trajectories of (2.8) transversally enter $G$ for increasing $t$. Furthermore, we assume that $G$ contains only a finite set of equilibria of (2.8) and that for all $x \in G$ the following inequality holds*

$$\lambda_1(x) + \lambda_2(x) < 0.$$

*Then any solution $x(t, x_0)$ of system (2.8) with initial data $x_0 \in G$ tends to some equilibrium as $t$ tends to $+\infty$.*

In the next section we derive some estimates for the region where the global attractor K of (1.3) is located.

# 3 Localization of the global attractor

First we note that from the last equation in (1.3) we get

$$z(t) = e^{-\sigma t} z(0) + \int_0^t e^{-\sigma(t-\tau)} y^2(\tau) d\tau \geq e^{-\sigma t} z(0).$$

Thus, if the $z$-component of a solution of system (1.3) satisfies $z(0) > 0$ then $z(t) > 0$ holds for all $t \geq 0$. This implies

$$\liminf_{t \to \infty} z(t) \geq 0 \quad \text{for} \quad z(0) \geq 0. \tag{3.1}$$

**Theorem 3.1** *Suppose the hypotheses* $(\mathrm{A}_1)$, $(\mathrm{A}_2)$ *and $\sigma > 0$ to be valid. Moreover, we assume the pair $(A, c)$ to be observable and that for some numbers $\gamma, \nu, \lambda$ satisfying $\nu \geq 0,\ \lambda \in (0, \sigma], \gamma \in R$ the following relations hold*

(i) *all eigenvalues of the matrix $A + \lambda I$ have negative real parts.*

(ii) $\nu + Re\ [\chi(i\omega - \lambda)] > 0 \quad \forall\ \omega \in R.$

(iii)

$$\begin{aligned} (2\lambda - \sigma)\gamma &\leq 0, && (3.2) \\ 2(d - \nu - \lambda) - \gamma &> 0. && (3.3) \end{aligned}$$

*Then there exists a positive definite symmetric matrix $H$ such that the global attractor of system (1.3) is contained in*

$$\Omega := \left\{ (x, y, z) \in \mathrm{R}^{n+2} : x^T H x + y^2 + z^2 + \gamma z \leq \frac{e^2}{2\lambda(2(d - \nu - \lambda) - \gamma)} \right\}.$$

**Theorem 3.2** *Suppose the hypotheses of Theorem 3.1 are valid except condition* (3.2). *Then there exists a positive definite matrix $H$ such that the global attractor of system* (1.3) *is contained in the set*

$$\Phi := \Big\{ \quad (x,y,z) \in \mathrm{R}^{n+2} : x^T Hx + y^2 + z^2 + \gamma z \\ \leq \frac{1}{2\lambda}\left[\frac{e^2}{2(d-\nu-\lambda)-\gamma} + \frac{(2\lambda-\sigma)^2\gamma^2}{8(\sigma-\lambda)}\right]\Big\}.$$

**Theorem 3.3** *Suppose the hypotheses* ($\mathrm{A}_1$),($\mathrm{A}_2$) *and $\sigma > 0$ to be valid. Additionally, we assume $e = 0$ and that for $\lambda = 0$ and for some numbers $\nu \geq 0$, $\gamma \in R$ the relations* (i) - (iii) *of Theorem* 3.1 *are valid. Then, any solution of* (1.3) *tends to the origin as $t$ tends to $+\infty$.*

**Proofs of Theorem 3.1 - Theorem 3.3.**

First we prove Theorem 3.1. To this end we construct a Lyapunov function in the form

$$V(x,y,z) := x^T Hx + y^2 + z^2 + \gamma z \tag{3.4}$$

where $H$ is a real positive definite symmetric matrix with some special property. We will apply Theorem 2.1 to establish its existence. To this end we set $P = A + \lambda I, q = b, \delta = -c,\ g = -2\nu$. From (2.5) we get

$$\begin{aligned} \mathcal{G}((i\omega I - P)^{-1}q\eta, \eta) &= \mathcal{G}(((i\omega-\lambda)I - A)^{-1}b\eta, \eta) \\ &= -2\Big(c^T((i\omega-\lambda)I-A)^{-1}b + \nu\Big)|\eta|^2. \end{aligned}$$

Taking into account the definition of the transfer function $\chi$ in (2.1) we obtain

$$-Re\,[\mathcal{G}((i\omega I - P)^{-1}q\eta, \eta)] = 2(Re\,[\chi(i\omega - \lambda)] + \nu)|\eta|^2.$$

Applying Theorem 2.1 we get that under the conditions (i) and (ii) of Theorem 3.1 there exists a positive symmetric matrix $H$ satisfying

$$2x^T H[(A+\lambda I)x + by] - 2c^T xy - 2\nu y^2 \leq 0 \quad \forall\ x \in \mathrm{R}^n, \quad \forall\ y \in \mathrm{R}. \tag{3.5}$$

An algorithm to construct the matrix $H$ satisfying (3.5) can be found in [7].

Using the inequality (3.5) we get from (3.4) and (1.3)

$$\begin{aligned} \frac{dV}{dt} + 2\lambda V &= 2x^T H[(A+\lambda I)x + by] - \\ &\quad 2c^T xy - 2\nu y^2 + 2(\nu - d)y^2 + 2ey - 2\sigma z^2 \\ &\quad -\sigma\gamma z + \gamma y^2 + 2\lambda(y^2 + z^2 + \gamma z) \\ &\leq -[2(d-\nu-\lambda) - \gamma]y^2 + 2ey \\ &\quad -2(\sigma-\lambda)z^2 + (2\lambda-\sigma)\gamma z. \end{aligned}$$

From the validity of the relations (3.2) and (3.3) and taking into account (3.1) we obtain

$$\frac{dV}{dt} + 2\lambda V \leq \frac{e^2}{2(d-\nu-\lambda)-\gamma}. \tag{3.6}$$

Therefore, $dV/dt$ is negative outside $\Omega$, and the global attractor $K$ is located in $\Omega$. This proves Theorem 3.1.
In case that only the inequality (3.3) holds we have

$$\frac{dV}{dt} + 2\lambda V \leq \frac{e^2}{2(d-\nu-\lambda)-\gamma} + \frac{(2\lambda-\sigma)^2\gamma^2}{8(\sigma-\lambda)}. \tag{3.7}$$

This inequality implies the validity of Theorem 3.2.

In case $\lambda = 0,\ e = 0$ we have the inequality

$$\frac{dV}{dt} \leq -\Big[2(d-\nu)-\gamma\Big]y^2 - 2\sigma z^2.$$

From this inequality and from the relation

$$V(x,y,x) \longrightarrow +\infty \quad \text{as} \quad |x|+|y|+|z| \longrightarrow \infty$$

we get that any solution $(x(t),y(t),z(t))$ of system (1.3) is uniformly bounded for $t \geq 0$. Obviously all conditions of the theorem of LaSalle (see [7]) are satisfied. Hence, the $\omega$-limit set of any trajectory of system (1.3) is contained in the subspace $\{y=0,\ z=0\}$. From the invariance of the $\omega$-limit set and from the first differential equation in (1.3) we get that for the $\omega$-limit set the relation $x=0$ is valid. Therefore, the $\omega$-limit set of any trajectory of system (1.3) consists of the equilibrium point $x=y=z=0$. This completes the proof of Theorem 3.3.

**Remark 1.** *It is easy to see that the conditions (i) and (ii) of Theorem 3.1 can be satisfied if we choose $\lambda$ sufficiently small and $\nu$ sufficiently large. Then, for negative $\gamma$ and for sufficiently large $|\gamma|$ condition (3.3) can be fulfilled. By this way, we can always find parameters $\lambda,\nu,\gamma$ such that the hypotheses of Theorem 3.2 are satisfied. Thus, system (1.3) is dissipative. From this point of view, Theorem 3.1 yields an improvement of the region of dissipativity compared with Theorem 3.2.* Theorem 3.1 is of special interest in case $e=0$. Here, we can draw the following conclusion.

**Corollary 3.4** *Let the hypotheses of Theorem 3.1 be valid. Additionally we assume $e=0$. Then, on the global attractor of system (1.3) we have*

$$0 \leq z \leq |\gamma|. \tag{3.8}$$

# 4 Longtime behavior and estimates of the Hausdorff dimension of the global attractor

In this section we estimate the Hausdorff dimension of the global attractor $K$ of system (1.3) by means of Theorem 2.4. At the same time we derive conditions for global stability.

To be able to apply Theorem 2.4 to system (1.3) we first derive conditions for the existence of a coordinate transformation such that the Jacobian $J(x)$ of the transformed system has the property that $J(x) + J(x)^T$ possesses a block-diagonal structure.

Let $S$ be an invertible $n \times n$-matrix. By means of the coordinate transformation

$$x \to Sx, \; z \to \sqrt{2}\, z, \; y \to y \tag{4.1}$$

we obtain from (1.3)

$$\begin{aligned} \frac{dx}{dt} &= S^{-1}ASx + S^{-1}by, \\ \frac{dy}{dt} &= -c^T Sx - dy - \sqrt{2}zy + e, \\ \frac{dz}{dt} &= -\sigma z + \frac{\sqrt{2}}{2}y^2. \end{aligned} \tag{4.2}$$

The Jacobian of (4.2) reads

$$J(x) \;:=\; \begin{pmatrix} S^{-1}AS & S^{-1}b & 0 \\ -c^T S & -d-\sqrt{2}z & -\sqrt{2}y \\ 0 & \sqrt{2}y & -\sigma \end{pmatrix}.$$

If we assume

$$S^{-1}b = (c^T S)^T = S^T c$$

which is equivalent to

$$b = SS^T c \tag{4.3}$$

then $J(x) + J(x)^T$ has the block diagonal structure

$$J(x) + J(x)^T \;=\; \begin{pmatrix} S^{-1}AS + (S^{-1}AS)^T & 0 & 0 \\ 0 & -2(d+\sqrt{2}z) & 0 \\ 0 & 0 & -2\sigma \end{pmatrix}. \tag{4.4}$$

Our goal is to guarantee the existence of a positive definite symmetric matrix $H$ such hat

$$b = Hc, \quad (A+\mu I)H + H(A+\mu I)^T \le 0. \tag{4.5}$$

It is clear that the existence of a symmetric positive definite matrix $H$ satisfying (4.5) implies the existence of a regular matrix $S$ ($H = SS^T$) satisfying (4.3).

The proof of the existence of the matrix $H$ is based on the application of Theorem 2.2. To this end we set in Theorem 2.2 $P = (A + \mu I)^T$, $q = c$, $r = -b$ and assume

$(H_1)$. There is a positive number $\mu$ such that

(i). $A + \mu I$ is Hurwitz.

(ii). $(A + \mu I, c)$ is controllable

(iii). $(A + \mu I, b)$ is observable.

$(H_2)$.

$$Re\,[\kappa(i\omega - \mu)] < 0 \quad \forall\, \omega \in \mathrm{R}, \tag{4.6}$$

where $\kappa(s)$ is defined according to (2.6) by

$$\kappa(s) := b^T(A^T - sI)^{-1}c. \tag{4.7}$$

Under the assumptions $(H_1)$ and $(H_2)$, it follows from Theorem 2.2 that there exists a positive definite matrix $H$ satisfying (4.5). Thus, the following lemma is valid.

**Lemma 4.1** *Assume the hypotheses* $(H_1)$ *and* $(H_2)$ *hold. Then there exists a regular matrix* $S$ *such that by means of the coordinate transformation* (4.1) *system* (1.3) *can be mapped into system* (4.2) *whose Jacobian* $J(x)$ *satisfies the relation* (4.4), *moreover the inequality*

$$S^{-1}AS + (S^{-1}AS)^T \leq -2\mu I \tag{4.8}$$

*is valid.*

We note that (4.8) is equivalent to

$$ASS^T + SS^TA^T \leq -2\mu SS^T \tag{4.9}$$

which follows from (4.5) by setting $H = SS^T$.

Now we are able to apply Theorem 2.4 to system (4.2) in order to estimate the Hausdorff dimension of the global attractor $K$.

**Theorem 4.2** *Suppose the hypotheses of Lemma* 4.1 *hold. Then, under the additional condition*

$$\min\,(\mu, \sigma) + d > 0 \tag{4.10}$$

*any solution of system* (1.3) *tends to a stationary solution for* $t \to +\infty$. *Under the condition*

$$\min\,(\mu, \sigma) + d \leq 0,\ \ d + \sigma + \mu \geq 0$$

*the Hausdorff dimension* $\dim_H K$ *of the global attractor* $K$ *satisfies*

$$\dim_H K \leq 2 - \frac{\min(\mu, \sigma) + d}{\max(\mu, \sigma)}. \tag{4.11}$$

*The same estimate holds for $\sigma < \mu, \sigma + d \leq 0, \sigma + n\mu + d > 0$.*
*In case $\sigma \geq \mu, d + n\mu > 0$ we have*

$$\dim_H K \leq 1 - \frac{d}{\mu}.$$

*For $\sigma \geq \mu, d + n\mu \leq 0, d + n\mu + \sigma > 0$ it holds*

$$\dim_H K \leq n + 1 - \frac{d + n\mu}{\sigma}.$$

**Proof.** Under our assumptions, we get from (4.4) and (4.9)

$$\begin{aligned} J(x) + J(x)^T \;\; &:= \begin{pmatrix} S^{-1}AS + (S^{-1}AS)^T & 0 & 0 \\ 0 & -2(d + \sqrt{2}\ z) & 0 \\ 0 & 0 & -2\sigma \end{pmatrix} \\ &\leq \begin{pmatrix} -2\mu I & 0 & 0 \\ 0 & -2d & 0 \\ 0 & 0 & -2\sigma \end{pmatrix}. \end{aligned} \tag{4.12}$$

We consider condition (4.10) and assume $\min(\mu, \sigma) = \sigma$. Than we obtain from (4.10) and (4.12)

$$\lambda_1(x) + \lambda_2(x) \leq -2(d + \sigma) < 0.$$

Thus, according to Theorem 2.5, any solution of (1.3) tends to an equilibrium point as $t$ tends to $+\infty$. The case $min(\mu, \sigma) = \mu$ is treated analogously.
Let $\min(\mu, \sigma) + d \leq 0, d + \sigma + \mu \geq 0$ and $\min(\mu, \sigma) = \mu$. In that case we have for $s > -(d + \mu)/\sigma$

$$\lambda_1(x) + \lambda_2(x) + s\lambda_3(x) \leq -2(d + \mu + s\sigma) < 0.$$

This proves the estimate (4.9). The other cases can be treated similarly. This completes the proof of the theorem.

For $n = 1,\ A = -a < 0,\ b = 1$ we obtain from (4.7)

$$\kappa(s) = \frac{-c}{s + a}.$$

In that case it is easy to see that the relations (4.6) holds for $c > 0$ and $\mu \in (0, a)$. Thus, we have

**Corollary 4.3** *Assume $n = 1,\ a > 0,\ c > 0,\ \sigma > 0$ and*

$$\min\ (a, \sigma) + d > 0.$$

*Then any solution of system (1.3) tends to an equilibrium for $t \to +\infty$.*

*In case*

$$\min\,(a,\sigma)+d<0$$

*the Hausdorff dimension of the global attractor $K$ can be estimated by*

$$\dim_H K \le 2-\frac{\min(a,\sigma)+d}{\max(a,\sigma)}.$$

We note that Mareels et al. [10] in case $a=1,\ \sigma=0.1,\ d=-\alpha,\ c=3/4-\alpha/4,\ \alpha\in[0,1]$ have got numerically for $e=0$ that the origin is globally stable for $\alpha\in(0,0.6)$. In this case we obtain from Corollary 4.3 that for any $e$ the origin is globally stable for $\alpha\in(0,0.1)$. For $\alpha>0.1$ we obtain the following estimate of the Hausdorff dimension of the global attractor $K$

$$\dim_H K \le \alpha+1.9.$$

We wish to underline that this result holds true for any $e$.

In what follows we consider system (1.3) in case $e=0$, and under the condition $\gamma<0$. Our goal is to derive a frequency criterion for the global asymptotical stability of the origin which extends a corresponding result in [10].

For this purpose we study the system

$$\begin{aligned}\dot x &= Ax+by,\\ \dot y &= -c^Tx-dy-z(t)y,\end{aligned}\tag{4.13}$$

where we assume

$$0\le z(t)\le -\gamma \quad \text{for} \quad t\in R.\tag{4.14}$$

We will apply Theorem 2.3 to system (4.13) in order to get a criterion guaranteeing the global asymptotic stability of the origin.

First we note that the transfer function of system (4.13) with the input $z(t)y$ and the output $-y$ coincides with the function $\psi(s)$ defined in (2.2).

To satisfy the assumptions of Theorem 2.3 we have to assume

**($\tilde A_3$)**. The matrix

$$\tilde A=\begin{pmatrix} A & b\\ -c^T & -d\end{pmatrix}$$

is Hurwitz.

Under the assumption ($\mathrm{A}_1$) we have due to Schur's lemma and taking into account the notation introduced in (2.3) and the relation (2.4)

$$\det(s\tilde I-\tilde A)=\det(sI-A)\det(s+d+c^T(A-pI)^{-1}b)=p(s)+d\eta(s)=\frac{\eta(s)}{\psi(s)}.$$

Thus, if assumption ($\mathrm{A}_1$) holds, then hypothesis ($\tilde{A}_3$) is equivalent to the following hypothesis

**($\mathbf{A}_3$).** $\psi(s)$ has only poles with negative real parts.

If we additionally assume

**($\mathbf{A}_4$).**

$$-\gamma^{-1} + \ \mathrm{Re}\,\psi(i\omega) > 0 \qquad \forall\, \omega \in R, \tag{4.15}$$

then Theorem 2.3 can be applied to system (4.13) and we get that the origin of system (4.13) is asymptotically stable, that is, any solution of system (4.13) satisfies

$$\lim_{t\to+\infty} x(t) = 0, \ \lim_{t\to+\infty} y(t) = 0. \tag{4.16}$$

Under the assumptions of Theorem 3.1, the $z$-component of system (1.3) satisfies by Corollary 1 the condition (4.14). Thus, from (4.16) and from the last equation in (1.3) we get that in case $\sigma > 0$ the relation

$$\lim_{t\to+\infty} z(t) = 0$$

holds and the origin of (1.3) is also asymptotically stable.

**Theorem 4.4** *Let all hypotheses of Theorem* 3.1 *be satisfied. Additionally we suppose* $e = 0$ *and that the assumptions* ($\mathrm{A}_3$) *and* ($\mathrm{A}_4$) *are valid. Then the origin of system* (1.3) *is globally asymptotically stable.*

**Remark 2.** We note that Theorem 4.4 improves Theorem 3.3 in [10] at least in the case $n = 1$ where instead of (4.15) the condition $Re\,\psi(i\omega) > 0$ is used.

Now we apply Theorem 4.4 to system (1.3) in the case $n = 1,\ b = 1,\ e = 0, c > 0, A = -a$.
By (2.1) and (2.2) the corresponding transfer function reads

$$\psi(s) = \frac{s+a}{(s+a)(s+d)+c}. \tag{4.17}$$

With $\nu = 0$, condition (ii) of Theorem 3.1 reads

$$\mathrm{Re}\,\frac{c}{i\omega + a - \lambda} > 0 \quad \forall\, \omega \in R.$$

This relation is valid for any $\lambda \in (0, a)$. For the same $\lambda$ also condition (i) of Theorem 3.1 holds.

If we assume

$$a > \sigma/2 > d,$$

then for

$$\lambda = \sigma/2,\ -\gamma > \sigma - 2d,\ \nu = 0$$

all conditions of Theorem 3.1 are satisfied. Taking into account the explicit form of the transfer function $\psi(s)$ defined in (4.17) we get the result:

**Corollary 4.5** *Consider the case* $n = 1,\ b = 1,\ e = 0,\ a > 0,\ \sigma > 0,\ c > 0,\ a > \sigma/2 > d$. *We assume that the polynomial*

$$(s+d)(s+a)+c \tag{4.18}$$

*has only zeros with negative real parts and that the frequency inequality*

$$\frac{1}{\sigma - 2d} + \ Re\ \left[\frac{i\omega + a}{(i\omega + a)(i\omega + d) + c}\right] > 0 \quad \forall\, \omega \in R \tag{4.19}$$

*holds true. Then system* (1.3) *is globally stable.*

It can be easily verified that all zeros of the polynomial (4.18) are located in the left half plane if we have

$$a + d > 0,\ \ ad + c > 0.$$

Condition (4.19) can be written in the form

$$\frac{1}{\sigma - 2d} + \frac{d\omega^2 + a(ad+c)}{(ad + c - \omega^2)^2 + (a+d)^2\omega^2} > 0.$$

Note that in [10] it has been shown numerically that in the case

$$e = 0,\ a = 1,\ \sigma = 0.1,\ c = \frac{-\alpha}{4} + \frac{3}{4},\ d = -\alpha,\ \alpha \in (0,1)$$

the origin is globally stable for $\alpha < 0.6$. From Corollary 4.5 we get that the origin is globally stable for $\alpha < 0.5463$.

We note that Theorem 3.3 in [10] is not applicable since from $d < 0$ it follows that the inequality $\psi(i\omega) > 0$ cannot be satisfied for sufficiently large $\omega$.

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