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Propagation of sound and surface waves in porous materials

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In memory of Professor Ekkehart Kröner

Propagation of sound and surface waves in porous materials

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Summary

We review main properties of acoustic waves which are described by continuous models of saturated porous materials. Bulk waves are only mentioned in the introduction, and we concentrate on the propagation of surface waves. We demonstrate differences between a one-component approach to the problem typical for seismology, and a two-component approach. In the latter case we rely on a model proposed by K. Wilmanski in the recent ten years, and we indicate similarities, and differences of results obtained within this model and that of the classical Biot's model of porous materials. The main conclusion is that both models predict the same modes of propagation, and differences are only quantitative in spite of flaws of the Biot's model which contains contributions violating the second law of thermodynamics, and the principle of material objectivity.

1. Introduction, bulk waves

Weak discontinuity waves in multicomponent continua differ considerably from those in single component media. The main difference is related to the fact that systems with multiple velocity fields (relative motions of components), and described by hyperbolic field equations lead to multiple modes of propagation. For instance a mixture of N ideal gaseous components yields N longitudinal waves. In the most prominent example of the Biot's model of two-component saturated poroelastic materials there exist three modes of propagation: two longitudinal P1 and P2 waves, and a transversal shear S wave [1]. The latter appears while the skeleton of the porous material is considered as an elastic material, and consequently it admits shear stresses.

In general continuous models of porous materials are constructed as a particular case of an immiscible mixture of continua in which one of the components – the skeleton – is a solid. In the above mentioned simplest case such a model contains a description of two components.

In this work we rely on such a model which differs however from that of M. A. Biot in three essential points:

a) source of momentum in momentum balance equations does not depend on the relative acceleration of components. It means that the coefficient ρ^{12} of the Biot's model (e.g. [2]) is identically zero. The reason for this change is that such contributions violate the principle of material objectivity [3];
b) partial stresses do not contain interaction terms related to volume changes of the other component. In the notation of the present work it means that the partial pressure of the fluid component, p^F, does not depend on volume changes of the skeleton, $\mathrm{tr}\,\mathbf{e}^S$, where $\mathbf{e}^S$ is the Almansi-Hamel deformation tensor of small deformations (i.e. $\max(|\lambda^1|,|\lambda^2|,|\lambda^3|)\;\; 1$, λ^i – eigenvalues of $\mathbf{e}^S$, i=1,2,3), and the Cauchy stress tensor in the skeleton, $\mathbf{T}^S$, does not depend on the current mass density of the fluid component, ρ^F. Such dependencies are impossible without violating the second law of thermodynamics within a model which does not contain higher gradients among constitutive variables [4]. This is an analogue of the simple mixture of miscible (fluid) components. In the Biot's notation this conclusion means that his constant Q must be identically zero;

c) changes of porosity are described by a balance equation of porosity rather than to be entirely neglected as it is the case for the Biot's model. Let us mention that these changes can be divided into two contributions: n_E describing changes in the thermodynamical equilibrium (such changes have been mentioned in the early Biot's paper [5]), and Δ_n which describes nonequilibrium changes, i.e. the porosity is given by $n = n_E + \Delta_n$. It can be shown [4] that the former depends in general on the fraction of partial mass densities $\frac{\rho^F}{\rho^S}$, and consequently it must be nonlinear, and the latter satisfies a balance equation

$$\frac{\partial \Delta_n}{\partial t} + n_E \operatorname{div}(\mathbf{v}^F - \mathbf{v}^S) = -\frac{\Delta_n}{\tau}, \tag{1.1}$$

where τ is the relaxation time of porosity [7]. There exists a coupling of partial stresses due to Δ_n which for the linear model has the following form

$$\begin{aligned} &\mathbf{T}^S = \lambda^S \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S + \beta \Delta_n \mathbf{1}, \\ &\mathbf{T}^F = -\left[p_0^F + \kappa\left(\rho^F - \rho_0^F\right) + \beta \Delta_n\right]\mathbf{1}, \quad \Delta_n := n - n_E, \quad n_E = \text{const}, \end{aligned} \tag{1.2}$$

where the effective Lame parameters λ^S, μ^S, and the compressibility coefficient κ as well as the coupling coefficient β are constant. They depend parametrically on the equilibrium porosity n_E.

In such a model mechanical processes are described by two scalar fields of partial mass densities, ρ^F, ρ^S, two vector fields of velocities, $\mathbf{v}^F$, $\mathbf{v}^S$, and the field of porosity, n. Field equations follow from balance equations for those quantities. Namely

$$\begin{aligned} &\frac{\partial \rho^S}{\partial t} + \rho_0^S \operatorname{div} \mathbf{v}^S = 0, \quad \frac{\partial \rho^F}{\partial t} + \rho_0^F \operatorname{div} \mathbf{v}^F = 0, \\ &\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div} \mathbf{T}^S + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \quad \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = -\operatorname{grad} p^F - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \end{aligned} \tag{1.3}$$

where ρ_0^S, ρ_0^F are constant reference values of partial mass densities, and $\mathbf{T}^S, p^F$ are given by the constitutive relations (1.2).

It can be shown that such a set of field equations is hyperbolic, and that there exist three modes of propagation of sound waves: P1 – a fast longitudinal wave, S – a shear wave, P2 – a slow (Biot's) wave. The latter is very strongly attenuated. Their speeds of propagation are given by the relations

$$U_{\parallel}^S = \sqrt{\frac{\lambda^S + 2\mu^S}{\rho_0^S}}, \quad U_{\perp}^S = \sqrt{\frac{\mu^S}{\rho_0^S}}, \quad U^F = \sqrt{\kappa}. \tag{1.4}$$

Properties of these waves as well as of monochromatic waves have been extensively investigated, and can be found elsewhere [6,8].

From the practical point of view even more important are surface waves. In classical seismological applications solely the so-called Rayleigh and Love waves were investigated. Their theoretical description is based on the classical elasticity. In modern applications of those waves one should mention the nondestructive testing of soils where the modification of the classical model to cover heterogeneous materials is of the primary importance. We present a few aspects of this topic in the next section of the paper.

For multicomponent systems the situation changes dramatically. The number of modes of surface waves increases, and their properties differ considerably from those of the classical surface waves. The number of modes depends on the type of boundary conditions. We show in the paper two most important cases:

1. a boundary between a saturated porous material, and the vacuum. In this case there exist two surface modes: a classical Rayleigh wave which is attenuated, and a Stoneley wave which propagates with the speed almost equal to this of the P2 wave,
2. a boundary between a saturated porous material, and the liquid (permeable boundary). In such a case there exist three surface modes whose speeds of propagation are strongly dependent on the permeability of the boundary.

2. Surface waves in single component elastic materials

We present here solely a few most important features of this problem. An excellent presentation of the state of art can be found in the PhD-thesis of Carlo Lai [9].

Let us begin with the problem of classical homogeneous elastic materials described by Lame constants, λ, μ. The field equations of linear elasticity for the displacement, $\mathbf{u}$ can be written in the form

$$
\begin{aligned}
&\frac{\partial^2 \mathbf{u}^L}{\partial t^2} = U_\parallel^2 \Delta \mathbf{u}^L, \quad U_\parallel^2 := \frac{\lambda + 2\mu}{\rho}, \quad \operatorname{rot} \mathbf{u}^L = 0, \\
&\frac{\partial^2 \mathbf{u}^T}{\partial t^2} = U_\perp^2 \Delta \mathbf{u}^T, \quad U_\perp^2 := \frac{\mu}{\rho}, \quad \operatorname{div} \mathbf{u}^T = 0, \\
&\mathbf{u} = \mathbf{u}^L + \mathbf{u}^T,
\end{aligned}
\tag{2.1}
$$

where ρ denotes the constant mass density. We construct the solution of 2D boundary value problem for the semispace with the boundary z=0 free of stresses, i.e.

$$
U_\parallel^2 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - 2U_\perp^2 \frac{\partial u}{\partial x} \bigg|_{z=0} = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \bigg|_{z=0} = 0,
\tag{2.2}
$$

where u,w denote the components of the displacement $\mathbf{u}$ in the direction of x-axis (along the boundary), and in the direction of z-axis (perpendicular to the boundary), respectively. We construct the solution in the form

$$
\begin{aligned}
&u^L = A^L e^{-\beta z} \sin(\omega t - kz), \quad w^L = -A^L \frac{\beta}{k} e^{-\beta z} \cos(\omega t - kz), \quad u = u^L + u^T, \\
&u^T = A^T e^{-\gamma z} \sin(\omega t - kz), \quad w^T = -A^T \frac{k}{\gamma} e^{-\gamma z} \cos(\omega t - kz), \quad w = w^L + w^T,
\end{aligned}
\tag{2.3}
$$

where the conditions $(2.1)_3$, and $(2.1)_6$ have been already used. Substitution in equations $(2.1)_1$, and $(2.1)_4$ yields the compatibility conditions

$$
\left(\frac{\beta}{k} \right)^2 = 1 - \frac{U_R^2}{U_\parallel^2}, \quad \left(\frac{\gamma}{k} \right)^2 = 1 - \frac{U_R^2}{U_\perp^2}, \quad U_R^2 := \left(\frac{\omega}{k} \right)^2.
\tag{2.4}
$$

Frequency ω, and the wave number, k, as well as the constants, A^L, A^T have to satisfy the homogeneous boundary conditions (2.2), and this yields the dispersion relation

$$\mathsf{D}_R := \left(2-\frac{U_R^2}{U_\perp^2}\right)^4 - 4\left(1-\frac{U_R^2}{U_\parallel^2}\right)\left(1-\frac{U_R^2}{U_\perp^2}\right) = 0. \tag{2.5}$$

It can be shown that this equation has a single solution $U_R < U_\perp$. This is the speed of propagation of the Rayleigh wave described by the solution (2.3). Obviously the above Lane cnaonsc t, λaonsLo nsnoλsμonλ, ans, ns . etaμaλsL e nL nμ sonsλT shc f i nμl dsq su TaμTsr , nm that its solutions, U_R are the same for all frequencies, i.e. Rayleigh waves in homogeneous elastic materials are nondispersive. This makes them so important in practical applications.

The amplitude of this wave decays exponentially with the depth, z, and hence the disturbance is essentially different from zero solely in a small vicinity of the boundary. This is the reason for calling such a solution the surface wave.

In addition the solution (2.3) yields the following relation between components u, and w of the displacement

$$\frac{u^2}{\left(A^L e^{-\beta z} + A^T e^{-\gamma z}\right)^2} + \frac{w^2}{\left(A^L \frac{\beta}{k} e^{-\beta z} + A^T \frac{k}{\gamma} e^{-\gamma z}\right)^2} = 1. \tag{2.6}$$

It means that trajectories of material points are elliptic with radii decaying exponentially with the depth z.

This classical solution changes dramatically for heterogeneous materials. Let us μonnaL cs, snar etaha Lsμ, msansu TaμTsλT sr , λ ca, tse, c, r λ cndsydspdsws, c sontl sL e nL nλsonsλT depth variable, z. Then the field equation for the displacement, $\mathbf{u}$ has the form

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\,\Delta\mathbf{u} + (\lambda+\mu)\operatorname{grad}(\operatorname{div}\mathbf{u}) + \mathbf{e}_z\frac{d\lambda}{dz}\operatorname{div}\mathbf{u} + \frac{d\mu}{dz}\left(\mathbf{e}_z \times \operatorname{rot}\mathbf{u} + 2\frac{\partial\mathbf{u}}{\partial z}\right), \tag{2.7}$$

where $\mathbf{e}_z$ denotes the unit vector in the direction of z-axis (perpendicular to the boundary). A solution for harmonic Rayleigh waves is sought in the form

$$u = U(z,k,\omega)e^{i(\omega t - kx)}, \quad w = W(z,k,\omega)e^{i(\omega t - kx)}. \tag{2.8}$$

Substitution of this ansatz in the equation (2.7) yields the following set of ordinary differential equations

$$\frac{d\mathbf{f}}{dz} = \mathbf{A}(z)\mathbf{f}, \quad \mathbf{f} \in \Re^4, \quad \mathbf{A} \in \Re^4 \times \Re^4, \tag{2.9}$$

where the vector $\mathbf{f}$, and the matrix $\mathbf{A}$ are defined as follows

$$\mathbf{f} := (U, W, f_3, f_4)^T, \quad f_3 := \mu\left(\frac{dU}{dz} - kW\right), \quad f_4 := (\lambda+2\mu)\frac{dW}{dz} + k\lambda U, \tag{2.10}$$

$$\mathbf{A} := \begin{pmatrix} 0 & k & \mu^{-1} & 0 \\ -k\lambda(\lambda+2\mu)^{-1} & 0 & 0 & (\lambda+2\mu)^{-1} \\ k^2\zeta - \omega^2\rho & 0 & 0 & k\lambda(\lambda+2\mu)^{-1} \\ 0 & -\omega^2\rho & -k & 0 \end{pmatrix}, \quad \zeta := 4\mu\frac{\lambda+\mu}{\lambda+2\mu}. \tag{2.11}$$

ρ aλTsλTansnoλ, λaonsλT smT , csmλc mndsW$_{xz}$ds, nLsλT smλc mnsμor eon nλsnocr , tsλosλT s2oi nL, cl dsD$_z$ can be written in the form

$$\tau_{xz} = f_3 e^{i(\omega t - kx)}, \quad \sigma_z = f_4 e^{i(\omega t - kx)}. \tag{2.12}$$

The set of equations (2.9) defines a linear differential eigenvalue problem with eigenfunctions **f**. Boundary conditions associated to this problem follow from the requirement that the stress components (2.12) vanish for z=0, and the eigenvector **f** vanishes as $z \rightarrow \infty$.

Lame const, asλeam,taμte c,tTchTnonsλTtf ai sTl tμa tnthoTntμ TdλTnqutr tTyc,etamsutμa some values of the wave number k, say k_j, j=1,...M which are called eigenvalues of the problem. The relation between the frequency, and eigenvalues is known only in the implicit form D_R**pwpoW2DppW2bppW2v**$_j$**2r WotqnsTktAnusTch. tkc,f T ,camt TsneamtSasλeam,taμte c,t Tsneam** are complex which means that Rayleigh waves in heterogeneous materials are attenuated in contrast to the Rayleigh waves in homogeneous materials. In addition they depend on the frequency which means that Rayleigh waves in heterogeneous materials are dispersive.

Several numerical techniques are used to solve the above eigenvalue problem. We shall not present here any details of those techniques refering to the work of C. Lai [9] for their presentation with corresponding references.

3. Surface waves in two-component saturated poroelastic materials – field equations

We proceed to the problem of two-component materials described by field equations presented in section 1 of this work. In the 2D case of the semispace we seek the solution by means of the following potentials for the displacements $\mathbf{u}^S$, $\mathbf{u}^F$

$$\begin{aligned} \mathbf{u}^S &= \operatorname{grad} \varphi^S + \operatorname{rot} \boldsymbol{\psi}^S, \quad \mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}, \quad \mathbf{e}^S := \operatorname{sym} \operatorname{grad} \mathbf{u}^S, \\ \mathbf{u}^F &= \operatorname{grad} \varphi^F + \operatorname{rot} \boldsymbol{\psi}^F, \quad \mathbf{v}^F = \frac{\partial \mathbf{u}^F}{\partial t}. \end{aligned} \tag{3.1}$$

We make the following ansatz for solutions harmonic in the x-direction along the boundary z=0

$$\begin{aligned} &\varphi^S = A^S(z) \exp(i(kx - \omega t)), \quad \varphi^F = A^F(z) \exp(i(kx - \omega t)), \\ &\psi_z^S = B^S(z) \exp(i(kx - \omega t)), \quad \psi_z^F = B^F(z) \exp(i(kx - \omega t)), \\ &\psi_x^S = \psi_y^S = \psi_x^F = \psi_y^F = 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\rho^S - \rho_0^S = A_\rho^S(z) \exp(i(kx - \omega t)), \quad \rho^F - \rho_0^F = A_\rho^F(z) \exp(i(kx - \omega t)), \\ &\Delta_n = A^\Delta \exp(i(kx - \omega t)). \end{aligned} \tag{3.3}$$

Substitution in field equations leads after straightforward calculations to the following compatibility conditions

$$\begin{aligned}
&B^F = \frac{i\pi}{\rho_0^F\omega + i\pi} B^S, \\
&A_\rho^S = -\rho_0^S\left(\frac{d^2}{dz^2} - k^2\right)A^S, \quad A_\rho^F = -\rho_0^F\left(\frac{d^2}{dz^2} - k^2\right)A^F, \\
&A^\Delta = -\frac{n_E\omega\tau}{i+\omega\tau}\left(\frac{d^2}{dz^2} - k^2\right)\left(A^F - A^S\right),
\end{aligned} \tag{3.4}$$

as well as

$$\left[\kappa\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]A^F + \left[\frac{n_E\beta\omega\tau}{\rho_0^F(i+\omega\tau)}\left(\frac{d^2}{dz^2} - k^2\right) + \frac{i\pi}{\rho_0^F}\omega\right]\left(A^F - A^S\right) = 0, \qquad (3.5)_1$$

$$\left[\frac{\lambda^S + 2\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]A^S - \left[\frac{n_E\beta\omega\tau}{\rho_0^S(i+\omega\tau)}\left(\frac{d^2}{dz^2} - k^2\right) + \frac{i\pi}{\rho_0^S}\omega\right]\left(A^F - A^S\right) = 0, \qquad (3.6)_2$$

$$\left[\frac{\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]B^S + \frac{i\pi\rho_0^F}{\rho_0^S\left(\rho_0^F\omega + i\pi\right)}\omega^2 B^S = 0. \qquad (3.6)_3$$

As the analysis of the problem of surface waves for these systems requires some asymptotic methods we transform the above relations to a dimensionles form. Let us define

$$\begin{aligned}
&c_S := \frac{U_\perp^S}{U_\parallel^S} < 1, \quad c_f := \frac{U^F}{U_\parallel^S} < 1, \quad \pi' := \frac{\pi\tau}{\rho_0^S} > 0, \quad \beta' := \frac{n_E\beta}{\rho_0^S U_\parallel^{S2}} > 0, \quad r := \frac{\rho_0^F}{\rho_0^S} < 1, \\
&z' := \frac{z}{U_\parallel^S\tau}, \quad k' := kU_\parallel^S\tau, \quad \omega' := \omega\tau,
\end{aligned} \tag{3.7}$$

where the speeds $U_\parallel^S, U_\perp^S, U^F$ are defined by relations (1.4). For typographical reasons we leave out the primes in the remaining part of the paper. Substitution of (3.7) in equations (3.6) yields the following set of ordinary differential equations for the function A^S, A^F, B^S

$$\begin{aligned}
&\left(c_f^2\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right)A^F + \left[\frac{\beta\omega}{r(i+\omega)}\left(\frac{d^2}{dz^2} - k^2\right) + i\frac{\pi}{r}\omega\right]\left(A^F - A^S\right) = 0, \\
&\left(\frac{d^2}{dz^2} - k^2 + \omega^2\right)A^S - \left[\frac{\beta\omega}{i+\omega}\left(\frac{d^2}{dz^2} - k^2\right) + i\pi\omega\right]\left(A^F - A^S\right) = 0, \\
&\left(c_s^2\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2 + \frac{i\pi\omega^2}{\omega + \frac{1}{r}i\pi}\right)B^S = 0.
\end{aligned} \tag{3.8}$$

This differential egenvalue problem can be solved easily because the matrix of coefficients is independent of z. We look for the solution in the form

$$A^F = A_f^1 e^{\gamma_1 z} + A_f^2 e^{\gamma_2 z}, \quad A^S = A_s^1 e^{\gamma_1 z} + A_s^2 e^{\gamma_2 z}, \quad B^S = B_s e^{\zeta z}, \tag{3.9}$$

with the exponents chosen in such a way that their real parts are negative. This follows from the boundedness of solutions in the limit $z \to \infty$.

Substitution in (3.8) yields the following relations for the exponents

$$\left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(1 + \frac{i\pi}{\omega + \frac{1}{r} i\pi}\right)\left(\frac{\omega}{k}\right)^2, \tag{3.10}$$

and

$$\begin{aligned} &\left[c_f^2 + \left(c_f^2 + \frac{1}{r}\right)\frac{\beta\omega}{i+\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]^2 + \\ &+\left\{\left[1 + c_f^2 + \left(1 + \frac{1}{r}\right)\frac{\beta\omega}{i+\omega}\right]\left(\frac{\omega}{k}\right)^2 + \left(c_f^2 + \frac{1}{r}\right)\frac{i\pi\omega}{k^2}\right\}\left[\left(\frac{\gamma}{k}\right)^2 - 1\right] + \\ &+\left[\left(\frac{\omega}{k}\right)^4 + \left(1 + \frac{1}{r}\right)\frac{i\pi}{k}\left(\frac{\omega}{k}\right)^3\right] = 0. \end{aligned} \tag{3.11}$$

Simultaneously we obtain the following eigenvectors

$$\mathbf{R}^1 = \left(B_s, A_s^1, A_f^1\right)^T, \quad \mathbf{R}^2 = \left(B_s, A_s^2, A_f^2\right)^T, \tag{3.12}$$

where

$$\begin{aligned} &A_f^1 = \delta_f A_s^1, \\ &\delta_f := \frac{1}{r}\frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \frac{i\pi\omega}{k^2}}{\left(c_f^2 + \frac{1}{r}\frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi\omega}{rk^2}}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &A_s^2 = \delta_s A_f^2, \\ &\delta_s := \frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \frac{i\pi\omega}{k^2}}{\left(1 + \frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi\omega}{k^2}}, \end{aligned} \tag{3.14}$$

and, certainly, γ_1, γ_2 are the solutions of the biquadratic equation (3.11) whose real parts are negative.

Further (section 5) we investigate the limit case $\omega \to \infty, |k| \to \infty$ with $\frac{\omega}{k}$ being finite. It is seen that in this limit the contributions of the bulk porosity vanish ($\frac{\pi}{k} \to 0$) whereas these

Lam mc onst ,aλμ onc..aoacλmT hcf iaλ aλ mc cl t hcddanλ .nh cl t nλcλmdqu ead ad iλ af t nhmiλm conclusion for the speeds of propagation of the fronts of surface waves which are not influenced by the diffusion reflected through the bulk permeability.

The above solution for the exponents still leaves three unknown constants B_s, A_f^2, A_s^1 in the solution (3.9) which must be determined from boundary conditions on the boundary z=0. We proceed to investigate this problem for two cases of boundary conditions.

4. Surface waves in two-component saturated poroelastic materials – boundary conditions

We consider two cases of boundary conditions: the contact of the saturated poroelastic material with the vacuum, and the contact of the saturated poroelastic material with the liquid. In the first case we deal with a medium in the exterior whose impedance is so low in comparison with the porous material that we may consider all waves confine to the porous semispace. The boundary is not permeable for the fluid. In the second case the boundary is permeable, and there is a transmission of waves between both semispaces.

In both cases the components of the total stress are continuous on the boundary. This is a safe approximation for flows in porous, and granular materials in which we can neglect quadratic contributions of relative velocities to the total momentum flux. Hence we have

1. boundary conditions on an impermeable boundary between a porous body, and vacuum

$$\left.\frac{\partial u^S}{\partial z}+\frac{\partial w^S}{\partial x}\right|_{z=0}=0,$$
$$U_{\parallel}^{S2}\rho_0^S\left(\frac{\partial u^S}{\partial x}+\frac{\partial w^S}{\partial z}\right)-2U_{\perp}^{S2}\rho_0^S-U^{F2}\left(\rho^F-\rho_0^F\right)\Big|_{z=0}=0, \tag{4.1}$$
$$\left.\frac{\partial}{\partial t}\left(w^F-w^S\right)\right|_{z=0}=0,$$

2. boundary conditions on a permeable boundary between a porous body and a liquid

$$\left.\frac{\partial u^S}{\partial z}+\frac{\partial w^S}{\partial x}\right|_{z=0}=0,$$
$$U_{\parallel}^{S2}\rho_0^S\left(\frac{\partial u^S}{\partial x}+\frac{\partial w^S}{\partial z}\right)-2U_{\perp}^{S2}\rho_0^S-U^{F2}\left(\rho^F-\rho_0^F\right)\Big|_{z=0}=-U^2\left.\left(\rho-\frac{\rho_0^F}{n_E}\right)\right|_{z=0}, \tag{4.2}$$
$$\rho_0^F\left.\frac{\partial}{\partial t}\left(w^F-w^S\right)\right|_{z=o}=\alpha\left.\left(p^F-pn_E\right)\right|_{z=0},$$

and we require the continuity of mass flux through the boundary.

In the above relations u^S, u^F, w^S, w^F denote the components of the displacement vectors of the skeleton, and of the fluid in the directions of x-axis, and z-axis, respectively.

The first condition (4.1) as well as the first condition (4.2) mean that the shear stress on the boundary is zero. This condition holds true also in the second case because we assume the fluid in the exterior to be ideal, i.e. the pressure in this region is given by the constitutive relation

$$p=\frac{p_0^F}{n_E}+U^2\left(\rho-\frac{\rho_0^F}{n_E}\right), \tag{4.3}$$

(compare the relation $(1.2)_2$). The second conditions in (4.1), and (4.2), respectively, describe the continuity of the total stress component normal to the boundary.

Finally the condition $(4.1)_3$ means that the boundary is impermeable for the fluid while the condition $(4.2)_3$ describes the flow through the permeable boundary. The "force" responsible for this flow is equal to the difference of real pressures in the fluid, i.e. in the linear approximation it is equal to $\frac{p^F}{n_E}$ in the porous material, and to p – in the exterior fluid. Lane cnoon ms tecoe, λc, cλtrcs μ.rtTehef μλmied mtd mns etanef μ.qneuecλetanerr , mλr mpl .ned cqs yμλTp and ∞ for the ideally permeable boundary (continuity of real pressures).

In addition we have the following continuity condition for the mass flux in the second case

$$\rho_0^F \frac{\partial}{\partial t}\left(w^F - w^S\right)\Big|_{z=0^-} = \frac{\rho_0^F}{n_E}\left(v - \frac{\partial w^S}{\partial t}\right)\Big|_{z=0^+}, \tag{4.4}$$

where v is the z-component of the velocity in the exterior fluid. This condition has been already used in the formulation of the condition $(4.2)_3$.

In order to construct solutions of the boundary value problem we have to substitute relations for the fields discussed in section 3 into the above boundary conditions. The additional boundary condition (4.4) in the second case is needed to find one arbitrary constant appearing in the solution for the exterior. We shall not present here general relations following from this substitution. It is sufficient to note that in each case they form a homogeneous set of algebraic relations. Consequently we obtain dispersion relations which should determine the speeds of propagation of surface waves. This problem has not been yet investigated in the full generality. Some properties of these dispersion relations are known in the limit of high frequencies which we present in the next section.

5. High frequency approximations

The eigenvalue problem presented in the last two sections cannot be solved analytically in the full generality. Therefore we proceed to some simplifications.

The most natural assumption is based on the fact that the phase speeds of monochromatic waves are either constant or grow with the growing frequency. For this reason one can expect that the speed of the signal or, in another words, the speed of propagation of tand μf neoλcs tecc..cd i equi etane.rr ntewe→∞. As the system of field equations is hyperbolic the limit of the phase velocity $\lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} < \infty$ must be finite which means that $|k| \to \infty$ as the frequency goes to infinity. We investigate this case.

Let us begin with the first boundary value problem. After easy calculations we obtain the following dispersion relation for large values of $|k|$

$$
\begin{aligned}
&\left\{\left[2c_s^2+\left(\frac{\gamma_2}{k}\right)-1\right]\left(2c_s^2-\left(\frac{\omega}{k}\right)^2\right)\delta_s+rc_f^2\left[\left(\frac{\gamma_2}{k}\right)^2-1\right]-4c_s^2\frac{\gamma_2}{k}\frac{\zeta}{k}\delta_s\right\}\times\\
&\times\left\{\delta_f\left(2c_s^2-\left(\frac{\omega}{k}\right)^2\right)+\left(\frac{\omega}{k}\right)^2\right\}\left(\frac{\gamma_1}{k}\right)-\\
&-\left\{\left[2c_s^2+\left(\frac{\gamma_1}{k}\right)-1\right]\left(2c_s^2-\left(\frac{\omega}{k}\right)^2\right)\delta_s+rc_f^2\left[\left(\frac{\gamma_1}{k}\right)^2-1\right]\left(2c_s^2-\left(\frac{\omega}{k}\right)^2\right)\delta_f-\right.\\
&\left.-4c_s^2\frac{\gamma_1}{k}\frac{\zeta}{k}\delta_s\right\}\times\left\{2c_s^2-\left(\frac{\omega}{k}\right)^2+\delta_s\left(\frac{\omega}{k}\right)^2\right\}\left(\frac{\gamma_2}{k}\right)=0,
\end{aligned}
\tag{5.1}
$$

where

$$
\delta_s=\frac{\beta\left[\left(\frac{\gamma_2}{k}\right)^2-1\right]}{(1+\beta)\left[\left(\frac{\gamma_2}{k}\right)^2-1\right]+\left(\frac{\omega}{k}\right)^2},\quad \delta_f=\frac{1}{r}\frac{\beta\left[\left(\frac{\gamma_1}{k}\right)^2-1\right]}{\left(c_f^2+\frac{1}{r}\beta\right)\left[\left(\frac{\gamma_1}{k}\right)^2-1\right]+\left(\frac{\omega}{k}\right)^2},
\tag{5.2}
$$

and the eigenvalues reduce to the following form

$$
\begin{aligned}
&\left(\frac{\zeta}{k}\right)^2=1-\frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2,\\
&\left(\frac{\gamma_{1,2}}{k}\right)^2=1-\frac{\left[1+c_f^2+\left(1+\frac{1}{r}\right)\beta\right]\pm\sqrt{D}}{2\left[c_f^2+\left(c_f^2+\frac{1}{r}\right)\beta\right]}\left(\frac{\omega}{k}\right)^2,\\
&D:=\left[1+c_f^2+\left(1+\frac{1}{r}\right)\beta\right]^2-4\left[c_f^2+\left(c_f^2+\frac{1}{r}\right)\beta\right].
\end{aligned}
\tag{5.3}
$$

Lame const,λμt acn tTμhf μi aene,st l t d μot q ,t equnc μqr μt ,λμt deoTμhoe qt hμri ,e qt eqt ,λeo limit.

The above explicit relations allow to estimate the influence of the coupling of stresses through dynamical changes of porosity. For typical values of other material parameters one can show that this influence, at least for rocks, is negligible. For instance, for the data quoted below the influence of this coupling is of the order of magnitude of 1%. Consequently we can f i . μti tuch,λμhtoef Treus eqyti oocf T,e qptwp WtDλμqta ,λtb$_s$ti qdtb$_f$ are equal to zero, and the problem becomes identical with that investigated by I. Edelman, K. Wilmanski, and E. Radkevich in [10]. The dispersion relation reduces to the following equation

$$
\begin{aligned}
D_V:=&\left\{\left(2-\frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2}\right\}\sqrt{1-\frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2}+\\
&+\frac{r}{c_s^4}\left(\frac{\omega}{k}\right)^4\sqrt{1-\left(\frac{\omega}{k}\right)^2}=0.
\end{aligned}
\tag{5.4}
$$

Comparison with the classical Rayleigh dispersion relation (2.5) shows that it becomes only a coefficient in the new dispersion relation.

The analysis of the equation (5.4) yields the following conclusions:

- a/ there exists a Rayleigh wave whose speed satisfies the inequalities

$$U^F < U_R < U_\perp^S; \tag{5.5}$$

it is a so-called leaky wave which means that it is attenuated due to the exchange of energy between the surface wave, and the P2 longitudinal wave which is slower than the surface wave;

- b/ there exists a second surface mode of propagation – the so-called Stoneley wave which is almost not attenuated, and it propagates with the speed only slightly smaller than U^F.

For instance for the typical data reproduced in the table below the speeds of these two waves are 1.5754 km/s, and 0.8997 km/s, respectively.

Typical values of parameters for rocks

reference porosity n_E	0.3
L anme const e, λesμceλ.Tn eh^F	210 kg/m^3
L anme const e, λesμcenf c.cs, oeh^S	2100 kg/m^3
cλλci snl ceda L cei , cλλri rcosneq^{S}ur ^{S}	6.62 GPa, 6.14 GPa
i , cλλri rcose, λei , L ypcmmwn nst ep	0.81 10^6 m^2/s^2
i , cλλri rcose, λeycpL cawn nst eW	10^7 kg/m^3/s
pc.a2asn, oesnL ceD	10^{-3} s
i , Ty.robei , cλλi rcosev	100 MPa
speeds of bulk waves $U^S_\parallel$, $U^S_\perp$, U^F	3 km/s, 1.71 km/s, 0.9 km/s

For the second boundary conditions the problem becomes technically even more involved. In the work [11] it has been shown that the dispersion relation for this case yields the existence of the following modes of surface waves:

- a/ a generalized leaky Rayleigh wave with the speed satisfying the inequality (5.5);
- b/ a leaky pseudo-Stoneley wave whose speed of propagation satisfies the same inequality;
- c/ a true Stoneley wave whose speed is smaller than this of the P2 wave (i.e. U^F).

These waves exist for all values of material parameters, and this conclusion is different from that obtained in the work [2] by S. Feng and L. Johnson.

6. Final remarks

Results reviewed in this work indicate that the analysis of linear weak discontinuity waves in porous materials can be successfuly based on the model proposed by K. Wilmanski. This conclusion is important as the model is simpler than the model of Biot which is traditionally used in this analysis.

It should be stressed that the results are still rather preliminary because we do not know much about the behaviour of surface waves in the range of low frequencies, and the model in its present form does not describe a material heterogeneity. It follows from the analysis of classical Rayleigh waves that these two factors play an essential role in propagation properties, and they even influence the frequency range of existence of some monochromatic

Lame cons t oL,nλμn. mTcaf mnT onhμefaoμconl enT ondafqn. omu o dhfhT nynhLnhu . l mT μT,nλT doi t oLnfhqonT one cT mnyrpqns i l Lorhμefaoμconwnhs Lnel mfl μwns t oLpn μWcl μLo2aoμTfr nfl s frequencies.

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