Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Singular limit in parabolic differential inclusions and the stop operator

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submitted: September 11, 2001

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Preprint No. 678
Berlin 2001

2000 *Mathematics Subject Classification.* 35K85, 35B25, 47J40.

Key words and phrases. Parabolic differential inclusion, singular limit, hysteresis operators, penalty approximation, phase transitions.

Edited by
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E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
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Abstract

Parabolic differential inclusions with convex constraints in a finite-dimensional space are considered with a small "diffusion" coefficient ε in the elliptic term. This problem arises for instance in multicomponent phase-field systems. We prove the strong convergence of solutions as $\varepsilon \to 0$ to the solution of the singular limit equation and show the connection to elementary hysteresis operators.

1 Introduction

This paper is motivated by problems arising in phase transition models described by systems of equations involving parabolic inclusions of the form

$$w_t - \varepsilon\,\Delta w + \partial I_K(w) \;\ni\; \gamma(w, u^\varepsilon) \quad \text{for} \;\; (x,t) \in Q_T := \Omega \times \,]0,T[\tag{1.1}$$

with appropriate initial and boundary conditions, where $\Omega \subset \mathbb{R}^n$ is a Lipschitzian domain, Δ is the Laplace operator in Ω, ∂I_K is the subdifferential of the indicator function I_K of a convex closed set $K \subset \mathbb{R}^N$, $w : Q_T \to \mathbb{R}^N$ is the unknown function, $u^\varepsilon : Q_T \to \mathbb{R}^\ell$ is a control variable, $\gamma : K \times \mathbb{R}^\ell \to \mathbb{R}^N$ is a given Lipschitz continuous mapping, and $\varepsilon > 0$ is a small constant. This 'diffusion' parameter ε is often physically controversial, and its value cannot be identified in a straightforward way. A natural question therefore concerns the stability of the model with respect to the transition $\varepsilon \to 0+$. The case $N = 1$ and $K = [0,1]$ was solved in [2], where w played the role of order parameter (phase fraction) and u^ε was the inverse temperature in a phase-field system of Penrose-Fife type. The well-posedness of phase-field systems with a vector order parameter in the limit case $\varepsilon = 0$ in a hysteresis setting has been established in [6, 7]. The idea consists in reformulating the inclusion (1.1) as an equation involving the so-called *stop operator with characteristic* K with a possible extension to more general hysteresis operators.

This is also our strategy here. We propose a 'hysteresis' framework for the transition $\varepsilon \to 0+$, and show that solutions of Eq. (1.1) converge strongly in the L^2-norm to the solution of the formal limit equation provided $\{u^\varepsilon\}$ converges strongly to u^0.

The paper is divided into Sections 2 – 5. In Section 2 we state Theorem 2.2 as our main result. Section 3 is devoted to a short survey of basic concepts from convex analysis, in Section 4 we give an overview of results on the stop operator, and using a suitable penalty approximation of ∂I_K, we justify a formal integration-by-parts formula in Lemma 4.2 which constitutes a substantial step in our argument. This result is of independent interest for applications in the theory of partial differential equations with hysteresis. The proof of Theorem 2.2 is given in Section 5.

2 Statement of the problem

Throughout the paper, we make the following hypotheses with fixed integers $n, N, \ell \in \mathbb{N}$.

Hypothesis 2.1

(i) $\Omega \subset \mathbb{R}^n$ *is a bounded open domain with a Lipschitzian boundary,* $T > 0$ *is a given final time, and we set* $Q_T := \Omega \times]0,T[$ *;*

(ii) $0 \in K \subset \mathbb{R}^N$ *is a given convex closed (not necessarily bounded) set;*

(iii) $\varphi \in W^{1,2}(\Omega;\mathbb{R}^N)\,, \quad \varphi(x) \in K$ *for a. e.* $x \in \Omega$*;*

(iv) $u^\varepsilon \in L^2(Q_T;\mathbb{R}^\ell)$ *for all* $\varepsilon \ge 0$*,* $u^\varepsilon \to u^0$ *strongly in* $L^2(Q_T;\mathbb{R}^\ell)$ *as* $\varepsilon \to 0+$*;*

(v) *There exists a constant* $L > 0$ *such that the function* $\gamma : K \times \mathbb{R}^\ell \to \mathbb{R}^N$ *satisfies the inequality*

$$|\gamma(w,u) - \gamma(\tilde{w},\tilde{u})| \;\le\; L(|w-\tilde{w}| + |u-\tilde{u}|) \quad \forall w,\tilde{w} \in K\,,\ u,\tilde{u} \in \mathbb{R}^\ell\,. \tag{2.1}$$

Under the above hypotheses, we consider the system

$$w_t - \varepsilon\,\Delta w + \partial I_K(w) \;\ni\; \gamma(w,u^\varepsilon) \qquad \text{for} \quad \text{a. e.} \quad (x,t) \in Q_T\,, \tag{2.2}$$

$$\frac{\partial w}{\partial \nu} = 0 \qquad \text{for} \quad \text{a. e.} \quad (x,t) \in \partial\Omega \times]0,T[\ , \tag{2.3}$$

$$w(x,0) = \varphi(x) \qquad \text{for} \quad \text{a. e.} \quad x \in \Omega\,. \tag{2.4}$$

We rewrite Eq. (2.2) in the form

$$w(x,t) \in K \qquad \text{for} \quad \text{a. e.} \quad (x,t) \in Q_T\,, \tag{2.5}$$

$$\langle w_t - \varepsilon\,\Delta w - \gamma(w,u^\varepsilon), z - w\rangle \ge 0 \quad \text{a. e.} \qquad \forall z \in K\,, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^N$.

The main result of this paper can be stated as follows.

Theorem 2.2 *Let Hypothesis 2.1 hold. Then Problem (2.3) – (2.6) has for every* $\varepsilon > 0$ *a unique solution* $w = w^\varepsilon \in L^2(Q_T;\mathbb{R}^N)$ *such that* $w^\varepsilon_t, \Delta w^\varepsilon \in L^2(Q_T;\mathbb{R}^N)$*, Problem (2.4) – (2.6) has a unique solution* $w = w^0 \in L^2(Q_T;\mathbb{R}^N)$ *such that* $w^0_t \in L^2(Q_T;\mathbb{R}^N)$ *for* $\varepsilon = 0$*, and we have*

$$\lim_{\varepsilon\to 0+} \varepsilon \int_0^T \int_\Omega \|\nabla w^\varepsilon\|^2 \,dx\,dt \;=\; 0\,, \tag{2.7}$$

$$\lim_{\varepsilon\to 0+}\ \sup_{s\in[0,T]} \int_\Omega |w^\varepsilon - w^0|^2(x,s)\,dx \;=\; 0\,, \tag{2.8}$$

where $\|\cdot\|$ *denotes the norm in* $\mathbb{R}^{nN}$.

3 Convex sets

In this section, we recall some elements of convex analysis which are needed in the sequel. We use the notation from Part II of [3].

For any $r > 0$ we denote by $B_r(z_0)$ the ball in $\mathbb{R}^N$ centered in $z_0 \in \mathbb{R}^N$ with radius r. By $P, Q : \mathbb{R}^N \to \mathbb{R}^N$ we denote the *projection pair associated with* K according to the formula

$$z = Pz + Qz\,, \quad Qz \in K\,, \quad |Pz| = \operatorname{dist}(z, K) \qquad \forall z \in \mathbb{R}^N\,. \tag{3.1}$$

We then have

$$\langle Pz, Qz - \zeta\rangle \geq 0 \qquad \forall z \in \mathbb{R}^N\,, \ \ \forall \zeta \in K\,, \tag{3.2}$$

in particular

$$\langle Pz_1 - Pz_2, Qz_1 - Qz_2\rangle \geq 0 \qquad \forall z_1, z_2 \in \mathbb{R}^N\,. \tag{3.3}$$

We further introduce the *Minkowski functional* (or *gauge*) of the set K by the formula

$$M(z) := \inf\left\{s > 0; \frac{1}{s} z \in K\right\} \qquad \text{for} \ \ z \in \mathbb{R}^N\,. \tag{3.4}$$

The subdifferential $\partial M(z)$ of M at a point $z \in \operatorname{Dom}(M) := \{z \in \mathbb{R}^N\,;\, M(z) < \infty\}$ is defined in a usual way as the set of all $y \in \mathbb{R}^N$ such that

$$\langle y, z - \tilde{z}\rangle \geq M(z) - M(\tilde{z}) \qquad \forall \tilde{z} \in \mathbb{R}^N\,. \tag{3.5}$$

We list the following straightforward consequences of (3.4), (3.5).

Lemma 3.1 *The mapping $M : \mathbb{R}^N \to [0, \infty]$ is convex, and we have*

$$|M(z_1) - M(z_2)| \leq \bar{M}(z_1 - z_2) \qquad \forall z_1, z_2 \in \mathbb{R}^N\,, \tag{3.6}$$

$$M(\lambda z) = \lambda\, M(z) \qquad \forall z \in \mathbb{R}^N \ \ \forall \lambda > 0\,, \tag{3.7}$$

$$\partial M(\lambda z) = \partial M(z) \qquad \forall z \in \operatorname{Dom}(M) \ \ \forall \lambda > 0\,, \tag{3.8}$$

$$\langle y, z\rangle = M(z) \qquad \forall z \in \operatorname{Dom}(M) \ \ \forall y \in \partial M(z)\,, \tag{3.9}$$

where we set $\bar{M}(z) := \max\{M(z), M(-z)\}$ for $z \in \mathbb{R}^N$. If moreover $B_r(0) \subset K \subset B_R(0)$ for some $R > r > 0$, then

$$\frac{|z|}{R} \leq M(z) \leq \frac{|z|}{r} \qquad \forall z \in \mathbb{R}^N\,. \tag{3.10}$$

The following result is on approximation of the domain K by smooth bounded convex sets.

Lemma 3.2 *For $\delta > 0$ put $\tilde{K}_\delta := K \cap B_{1/\delta^2}(0)$, $K_\delta := \tilde{K}_\delta + B_\delta(0)$. Let M_δ be the Minkowski functional associated with K_δ. Then $\partial M_\delta(z)$ contains for every $z \neq 0$ a single point denoted again by $\partial M_\delta(z)$, and we have*

$$|\partial M_\delta(z)| \leq 1/\delta \qquad \forall z \neq 0\,, \tag{3.11}$$

$$|\partial M_\delta(z_1) - \partial M_\delta(z_2)| \leq \delta^{-8}(1+2\delta^3)^2|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{R}^N \setminus \mathrm{Int}\, K_\delta\,. \tag{3.12}$$

Proof. Let us first note that $B_\delta(0) \subset K_\delta \subset B_{\delta+(1/\delta^2)}(0)$, and (3.10) yields that

$$\frac{\delta^2}{1+\delta^3}|z| \leq M_\delta(z) \leq \frac{1}{\delta}|z| \qquad \forall z \in \mathbb{R}^N\,. \tag{3.13}$$

Let P_δ, Q_δ be the projections associated with $\tilde{K}_\delta$ according to (3.1), and let $z \in \partial K_\delta$, $\zeta \in K_\delta$ be arbitrary. We then have $|P_\delta z| = \delta$, $|P_\delta \zeta| \leq \delta$, and from (3.2) we obtain that

$$\langle P_\delta z, z - \zeta\rangle = \langle P_\delta z, Q_\delta z - Q_\delta \zeta\rangle + \langle P_\delta z, P_\delta z - P_\delta \zeta\rangle \geq 0\,. \tag{3.14}$$

Assume that a unit vector $\eta \in \mathbb{R}^N$ belongs to the outward normal cone to K_δ at the point z, that is, $\langle \eta, z - \xi\rangle \geq 0$ for every $\xi \in K_\delta$. Then putting $\xi := Q_\delta z + \delta\eta$ we obtain that $\delta \leq \langle P_\delta z, \eta\rangle$, hence $\eta = (1/\delta)P_\delta z$. We thus conclude that $\nu_\delta(z) = (1/\delta)P_\delta z$ is the uniquely determined unit outward normal to K_δ at the point z, and by (3.3) we have

$$|\nu_\delta(z_1) - \nu_\delta(z_2)| \leq \frac{1}{\delta}\,|z_1 - z_2| \qquad \forall z_1, z_2 \in \partial K_\delta\,. \tag{3.15}$$

By (3.5), (3.9) we have

$$\partial M_\delta(z) = \frac{\nu_\delta(z)}{\langle \nu_\delta(z), z\rangle} \qquad \forall z \in \partial K_\delta\,, \tag{3.16}$$

where (3.14) with $\zeta = P_\delta z$ implies that $\langle \nu_\delta(z), z\rangle \geq \delta$. From (3.8) we thus obtain that for $z \neq 0$ we have $|\partial M_\delta(z)| = |\partial M_\delta(z/M_\delta(z))| \leq 1/\delta$, and (3.11) follows.

To prove (3.12), consider $z_1, z_2 \in \mathbb{R}^N \setminus \mathrm{Int}\, K_\delta$, and put $\hat{z}_1 := z_1/M_\delta(z_1) \in \partial K_\delta$, $\hat{z}_2 := z_2/M_\delta(z_2) \in \partial K_\delta$. Then $|\hat{z}_i| \leq (1+\delta^3)/\delta^2$, $M_\delta(z_i) \geq 1$ for $i = 1, 2$. By (3.8) we have that

$$\begin{aligned}
|\partial M_\delta(z_1) - \partial M_\delta(z_2)| &= |\partial M_\delta(\hat{z}_1) - \partial M_\delta(\hat{z}_2)| \\
&\leq \frac{1}{\delta^2}\,|\nu_\delta(\hat{z}_1)\,\langle \nu_\delta(\hat{z}_2), \hat{z}_2\rangle - \nu_\delta(\hat{z}_2)\,\langle \nu_\delta(\hat{z}_1), \hat{z}_1\rangle| \\
&\leq \frac{1}{\delta^2}\,(|\langle \nu_\delta(\hat{z}_2), \hat{z}_1 - \hat{z}_2\rangle| + |\nu_\delta(\hat{z}_1)\,\langle \nu_\delta(\hat{z}_2), \hat{z}_1\rangle - \nu_\delta(\hat{z}_2)\,\langle \nu_\delta(\hat{z}_1), \hat{z}_1\rangle|) \\
&\leq \frac{1}{\delta^2}\,(|\hat{z}_1 - \hat{z}_2| + |\hat{z}_1||\nu_\delta(\hat{z}_1) - \nu_\delta(\hat{z}_2)|) \leq \frac{1}{\delta^5}(1+2\delta^3)|\hat{z}_1 - \hat{z}_2|\,,
\end{aligned} \tag{3.17}$$

where we used (3.15) and the elementary inequality

$$|\langle b, c\rangle a - \langle a, c\rangle b| \le \frac{1}{2}|a-b|\,|a+b|$$

for every $a, b, c \in \mathbb{R}^N$, $|a| = |b| = |c| = 1$. Furthermore, (3.6) and (3.10) yield that

$$|\hat{z}_1 - \hat{z}_2| \le \frac{1}{M_\delta(z_2)}\Big(|z_1 - z_2| + |\hat{z}_1|\bar{M}_\delta(z_1 - z_2)\Big) \le \Big(2 + \frac{1}{\delta^3}\Big)|z_1 - z_2|, \tag{3.18}$$

and the proof follows easily. ∎

In the next section we apply the penalty argument based on the following Lemma.

Lemma 3.3 *For any $\delta > 0$ we define the functional $\Psi_\delta : \mathbb{R}^N \to [0, \infty[$ by the formula*

$$\Psi_\delta(z) := \begin{cases} \dfrac{(M_\delta(z) - 1)^2}{M_\delta(z)} & \text{for} \quad z \in \mathbb{R}^N \setminus K_\delta\,, \\ 0 & \text{for} \quad z \in K_\delta\,. \end{cases} \tag{3.19}$$

Then Ψ_δ is a convex functional of class C^1, and its derivative

$$\psi_\delta(z) = \partial\Psi_\delta(z) = \begin{cases} \partial M_\delta(z)\left(1 - \dfrac{1}{M_\delta^2(z)}\right) & \text{for} \quad z \in \mathbb{R}^N \setminus K_\delta\,, \\ 0 & \text{for} \quad z \in K_\delta \end{cases} \tag{3.20}$$

is a bounded monotone Lipschitz continuous mapping from $\mathbb{R}^N$ into $\mathbb{R}^N$.

Proof. We only have to check that ψ_δ is Lipschitz continuous, that is, find a constant $L_\delta > 0$ such that

$$|\psi_\delta(z_1) - \psi_\delta(z_2)| \le L_\delta |z_1 - z_2| \qquad \forall z_1, z_2 \in \mathbb{R}^N\,. \tag{3.21}$$

Let $z_1, z_2 \in \mathbb{R}^N$ be arbitrary. Inequality (3.21) is trivial if both z_1, z_2 belong to K_δ. If both $z_1, z_2 \in \mathbb{R}^N \setminus K_\delta$, then $M_\delta(z_i) > 1$ for $i = 1, 2$, and using Lemma 3.2 we obtain that

$$\begin{aligned} |\psi_\delta(z_1) - \psi_\delta(z_2)| &\le \left(1 - \frac{1}{M_\delta(z_2)^2}\right)|\partial M_\delta(z_1) - \partial M_\delta(z_2)| \\ &\quad + |\partial M_\delta(z_1)| \frac{M_\delta(z_1)^2 - M_\delta(z_2)^2}{M_\delta(z_1)^2 M_\delta(z_2)^2} \\ &\le |\partial M_\delta(z_1) - \partial M_\delta(z_2)| + \frac{2}{\delta}\bar{M}_\delta(z_1 - z_2) \\ &\le \Big(2\delta^{-2} + \delta^{-8}(1 + 2\delta^3)^2\Big)|z_1 - z_2|\,, \end{aligned} \tag{3.22}$$

hence (3.21) holds. Finally, if $z_1 \notin K_\delta$, $z_2 \in K_\delta$, then

$$
\begin{aligned}
|\psi_\delta(z_1) - \psi_\delta(z_2)| \ &\leq \ |\partial M_\delta(z_1)| \frac{M_\delta(z_1)^2 - 1}{M_\delta(z_1)^2} \ \leq \ \frac{2}{\delta}(M_\delta(z_1) - 1) \\
&\leq \ \frac{2}{\delta}(M_\delta(z_1) - M_\delta(z_2)) \ \leq \ \frac{2}{\delta}\bar{M}_\delta(z_1 - z_2) \ \leq \ \frac{2}{\delta^2}|z_1 - z_2|, \qquad (3.23)
\end{aligned}
$$

and Lemma 3.3 is proved. ■

4 The stop operator

Let us first consider the variational inequality

$$w(t) \in K \qquad \forall t \in [0,T], \tag{4.1}$$

$$w(0) = \varphi, \tag{4.2}$$

$$\langle \dot{w}(t) - \dot{v}(t), z - w(t) \rangle \geq 0 \quad \text{a.e.} \qquad \forall z \in K, \tag{4.3}$$

independently of the space variable x, assuming that $v \in W^{1,1}(0,T;\mathbb{R}^N)$ and $\varphi \in K$ are given, and denoting by a dot the derivative with respect to t.

The solution operator

$$\mathcal{S}_K : K \times W^{1,1}(0,T;\mathbb{R}^N) \to W^{1,1}(0,T;\mathbb{R}^N)$$

defined by the formula $\mathcal{S}_K[\varphi, v](t) := w(t)$ for $t \in [0,T]$ constitutes one of the main building blocks in the theory of hysteresis operators, and its analytical properties have been studied in detail in [4, 8, 1, 5] in connection with complex hysteresis models.

We list here only a few results which are needed in the sequel. In particular, if $v, v_1, v_2 \in W^{1,1}(0,T;\mathbb{R}^N)$ are input functions, $\varphi, \varphi_1, \varphi_2 \in K$ are initial conditions, and $w, w_1, w_2 \in W^{1,1}(0,T;\mathbb{R}^N)$ are the corresponding solutions to (4.1) – (4.3), $w(t) = \mathcal{S}_K[\varphi, v](t)$, $w_i(t) = \mathcal{S}_K[\varphi_i, v_i](t)$, $i = 1, 2$, then we have

$$|\dot{w}(t)| \leq |\dot{v}(t)| \quad \text{a.e.}\ , \tag{4.4}$$

$$\langle \dot{v}_1(t) - \dot{v}_2(t), w_1(t) - w_2(t) \rangle \ \geq \ \frac{1}{2}\frac{d}{dt}|w_1(t) - w_2(t)|^2 \quad \text{a.e.} \tag{4.5}$$

From (4.5) it follows in particular that $\mathcal{S}_K$ maps the set $K \times W^{1,1}(0,T;\mathbb{R}^N)$ Lipschitz continuously into $C([0,T];\mathbb{R}^N)$. This rough property will be sufficient here due to the regularizing effect of the parabolic equation. In other applications, finer continuity results are required, and we refer the reader e.g. to [5].

We now define the output of the stop for input functions $\varphi(x)$, $v(x,t)$ depending also on x, using the same symbol $\mathcal{S}_K$ for the mapping

$$\mathcal{S}_K[\varphi, v](x,t) \ := \ \mathcal{S}_K[\varphi(x), v(x,\cdot)](t)$$

whenever $\varphi(x) \in K$ and $v(x,\cdot) \in W^{1,1}(0,T;\mathbb{R}^N)$.

Especially, if $\varphi \in C(\bar\Omega;K)$, and $v \in C(\bar\Omega;W^{1,1}(0,T;\mathbb{R}^N))$, then (4.5) yields that $w = \mathcal{S}_K[\varphi,v] \in C(\bar Q_T)$. If $v \in L^q(\Omega;W^{1,1}(0,T;\mathbb{R}^N))$ for some $1 \le q < \infty$ and $\varphi \in L^q(\Omega;K)$, then by density of $C(\bar Q_T)$ in $L^q(0,1;C([0,T];\mathbb{R}^N)$ we conclude that w as a mapping $\Omega \to C([0,T];\mathbb{R}^N)$ is strongly measurable, and (4.5) entails that the operator $\mathcal{S}_K : L^q(\Omega;K) \times L^q(\Omega;W^{1,1}(0,T;\mathbb{R}^N)) \to L^q(\Omega;C([0,T];K))$ is Lipschitz continuous.

We are now ready to solve Problem (2.4) – (2.6) for $\varepsilon = 0$.

Lemma 4.1 *Let Hypothesis 2.1 hold. Then there exists a unique* $w^0 \in L^2(\Omega;C([0,T];\mathbb{R}^N))$ *such that* $w^0_t \in L^2(Q_T;\mathbb{R}^N)$, *and*

$$w^0(x,0) = \varphi(x) \qquad \textit{for a. e. } x \in \Omega, \tag{4.6}$$

$$w^0(x,t) \in K \qquad \textit{for a. e. } (x,t) \in Q_T\,, \tag{4.7}$$

$$\langle w^0_t - \gamma(w^0,u^0), z - w^0\rangle \ge 0 \qquad \textit{a. e.} \quad \forall z \in K\,. \tag{4.8}$$

Proof. We define the set $U := \{x \in \Omega\,;\ u^0(x,\cdot) \in L^2(0,T;\mathbb{R}^N)\} \subset \Omega$, meas $(\Omega\backslash U) = 0$. For fixed $x \in U$ we consider the equation

$$v^0_t(x,t) = \gamma(\mathcal{S}_K[\varphi(x),v^0(x,\cdot)](t),u^0(x,t))\,, \qquad v^0(x,0) = 0\,. \tag{4.9}$$

We define a mapping $G_x : L^1(0,T;\mathbb{R}^N) \to L^1(0,T;\mathbb{R}^N)$ in the following way. For an arbitrary $\zeta \in L^1(0,T;\mathbb{R}^N)$ and $t \in [0,T]$ put

$$v(t) \ := \ \int_0^t \zeta(\tau)\,d\tau\,, \tag{4.10}$$

$$G_x[\zeta](t) \ := \ \gamma(\mathcal{S}_K[\varphi(x),v](t),u^0(x,t))\,. \tag{4.11}$$

Then $v^0(x,t) := v(t)$ given by (4.10) is a solution of (4.9) if and only if ζ is a fixed point of the mapping G_x. For each $\zeta_1,\zeta_2 \in L^1(0,T;\mathbb{R}^N)$ we have by Hypothesis 2.1 (v) and inequality (4.5) that

$$\begin{aligned} 2ee7|G_x[\zeta_1](t) - G_x[\zeta_2](t)| \ &\le \ L\,|\mathcal{S}_K[\varphi(x),v_1](t) - \mathcal{S}_K[\varphi(x),v_2](t)| \\ &\le \ 2L\int_0^t |\zeta_1(\tau) - \zeta_2(\tau)|\,d\tau\,. \end{aligned} \tag{4.12}$$

Denoting by G^k_x the k-th iteration of G_x, that is, $G^1_x = G_x$, $G^{k+1}_x = G_x[G^k_x]$ for $k = 1,2,\ldots$, we easily obtain by induction that

$$|G^k_x[\zeta_1](t) - G^k_x[\zeta_2](t)| \ \le \ \frac{(2L)^k t^{k-1}}{(k-1)!}\int_0^T |\zeta_1(\tau) - \zeta_2(\tau)|\,d\tau\,, \tag{4.13}$$

hence G^k_x is a contraction for sufficiently large k. By the Banach Contraction Principle, G_x admits a unique fixed point $\zeta \in L^1(0,T;\mathbb{R}^N)$, hence Eq. (4.9) has a unique solution, and the function

$$w^0 := \mathcal{S}_K[\varphi,v^0] \tag{4.14}$$

has the properties (4.6) – (4.8). The uniqueness is obtained in a standard way: let $w^0, \hat{w}^0$ be two solutions. Putting $z := (1/2)(w^0 + \hat{w}^0)$ in the inequality (4.8) successively for w^0 and $\hat{w}^0$ and summing the resulting inequalities up, we obtain the assertion from the Gronwall argument. Using (4.9), (4.4), and again Gronwall's inequality, we easily check that $v^0, v_t^0, w^0, w_t^0 \in L^2(Q_T; \mathbb{R}^N)$, and Lemma 4.1 is proved. ■

The main result of this section which will play a crucial role in the proof of Theorem 2.2 reads as follows.

Lemma 4.2 *Let Hypothesis 2.1 (i) – (iii) hold, and let $v, w \in L^2(Q_T; \mathbb{R}^N)$ be such that*

(i) $v_t, \Delta w \in L^2(Q_T; \mathbb{R}^N)$,

(ii) $w = \mathcal{S}_K[\varphi, v]$,

(iii) $\partial w/\partial\nu(x,t) = 0$ *for a. e.* $(x,t) \in \partial\Omega\times\,]0,T[$.

Then for every $s \in [0,T]$ we have that

$$-\int_0^s \int_\Omega \langle v_t, \Delta w\rangle\,(x,t)\,dx\,dt \;\geq\; \frac{1}{2}\left(\int_\Omega \|\nabla w\|^2(x,s)\,dx - \int_\Omega \|\nabla\varphi\|^2(x)\,dx\right). \quad (4.15)$$

Proof. We introduce the function $f := v_t - \Delta w \in L^2(Q_T; \mathbb{R}^N)$. Inequality (4.15) can be written equivalently in the form

$$\int_0^s \int_\Omega |v_t|^2\,dx\,dt + \frac{1}{2}\int_\Omega \|\nabla w\|^2(x,s)\,dx \;\leq\; \frac{1}{2}\int_\Omega \|\nabla\varphi\|^2(x)\,dx + \int_0^s \int_\Omega \langle v_t, f\rangle\;dx\,dt \quad (4.16)$$

for every $s \in [0,T]$.

Using Lemma 3.3, we consider the penalized problem

$$\begin{cases} w_t^{(\delta)} - \Delta w^{(\delta)} + \frac{1}{\delta}\psi_\delta(w^{(\delta)}) \;=\; f & \text{in } Q_T\,, \\ \dfrac{\partial w^{(\delta)}}{\partial\nu} = 0 & \text{on } \partial\Omega\times\,]0,T[\;, \\ w^{(\delta)}(x,0) = \varphi(x) & \text{in } \Omega \end{cases} \quad (4.17)$$

with the intention to let δ tend to $0+$. The mapping ψ_δ is for every fixed $\delta > 0$ bounded, monotone, and Lipschitz continuous, hence Problem (4.17) admits a unique solution $w^{(\delta)} \in L^2(Q_T; \mathbb{R}^N)$ such that $w_t^{(\delta)}, \Delta w^{(\delta)} \in L^2(Q_T; \mathbb{R}^N)$. In order to derive suitable a priori estimates, we denote by $C_1, C_2, \ldots$ any positive constant independent of δ.

Testing Eq. (4.17) by $w_t^{(\delta)}$ we see that the identity

$$\int_0^s \int_\Omega |w_t^{(\delta)}|^2(x,t)\,dx\,dt \quad + \quad \int_\Omega \left(\frac{1}{2}\|\nabla w^{(\delta)}\|^2 + \frac{1}{\delta}\Psi_\delta(w^{(\delta)})\right)(x,s)\,dx \tag{4.18}$$
$$= \int_\Omega \frac{1}{\delta}\Psi_\delta(\varphi(x))\,dx \quad + \quad \frac{1}{2}\int_\Omega \|\nabla\varphi(x)\|^2\,dx \; + \; \int_0^s \int_\Omega \left\langle w_t^{(\delta)}, f\right\rangle(x,t)\,dx\,dt\,,$$

holds for every $\delta > 0$ and $s \in [0,T]$. Let us check that

$$\lim_{\delta\to 0+} \int_\Omega \frac{1}{\delta}\Psi_\delta(\varphi(x))\,dx \;=\; 0\,. \tag{4.19}$$

Indeed, for $\delta > 0$ we define the sets

$$F_\delta := \{x \in \Omega\,;\, |\varphi(x)| > \delta^{-2}\}\,. \tag{4.20}$$

By the Sobolev Embedding Theorem, $\varphi \in L^p(\Omega)$ for some $p > 2$ (more precisely, $p = 2n/(n-2)$ if $n \geq 3$, $p > 2$ arbitrary if $n \leq 2$). This yields that

$$C_1 \geq \int_{F_\delta} |\varphi(x)|^p\,dx \geq \delta^{-2p}\text{meas}\,(F_\delta)\,,$$

hence $\text{meas}\,(F_\delta) \leq C_1\,\delta^{2p}$. By definition of Ψ_δ, we have $\Psi_\delta(\varphi(x)) = 0$ whenever $\varphi(x) \leq \delta^{-2}$. This yields that

$$\int_\Omega \frac{1}{\delta}\Psi_\delta(\varphi(x))\,dx \;=\; \int_{F_\delta} \frac{1}{\delta}\Psi_\delta(\varphi(x))\,dx \;\leq\; \frac{1}{\delta}\int_{F_\delta} M_\delta(\varphi(x))\,dx \;\leq\; \frac{1}{\delta^2}\int_{F_\delta} |\varphi(x)|\,dx$$
$$\leq\; \delta^{-2}\left(\int_{F_\delta} |\varphi(x)|^p\,dx\right)^{1/p} (\text{meas}\,(F_\delta))^{(p-1)/p} \;\leq\; C_2\,\delta^{2(p-2)} \tag{4.21}$$

and (4.19) follows. Using the Cauchy-Schwarz inequality we thus obtain for every $s \in [0,T]$ the estimate

$$\int_0^s \int_\Omega |w_t^{(\delta)}|^2(x,t)\,dx\,dt + \int_\Omega \left(\|\nabla w^{(\delta)}\|^2 + \frac{1}{\delta}\Psi_\delta(w^{(\delta)})\right)(x,s)\,dx \;\leq\; C_3\,. \tag{4.22}$$

We further test Eq. (4.17) by $-\Delta w^{(\delta)}$. Then we have

$$\frac{1}{2}\int_\Omega \|\nabla w^{(\delta)}\|^2(x,s)\,dx \quad + \quad \int_0^s \int_\Omega \left(|\Delta w^{(\delta)}|^2 + \frac{1}{\delta}\left\langle\!\left\langle \nabla\psi_\delta(w^{(\delta)}), \nabla w^{(\delta)}\right\rangle\!\right\rangle\right)(x,t)\,dx\,dt$$
$$\leq\; \frac{1}{2}\int_\Omega \|\nabla\varphi(x)\|^2\,dx \quad + \quad \int_0^s \int_\Omega \left\langle \Delta w^{(\delta)}, f\right\rangle(x,t)\,dx\,dt\,, \tag{4.23}$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the scalar product in $\mathbb{R}^{nN}$. The monotonicity and Lipschitz continuity of ψ_δ entails that

$$\int_0^s \int_\Omega \left\langle\!\left\langle \nabla\psi_\delta(w^{(\delta)}), \nabla w^{(\delta)}\right\rangle\!\right\rangle(x,t)\,dx\,dt \;\geq\; 0\,,$$

and we obtain the estimate

$$\int_0^s \int_\Omega \left(|w_t^{(\delta)}|^2 + |\Delta w^{(\delta)}|^2\right)(x,t)\,dx\,dt + \int_\Omega \|\nabla w^{(\delta)}\|^2(x,s)\,dx \ \le\ C_4\,. \tag{4.24}$$

We finally test Eq. (4.17) by $w_t^{(\delta)} + \frac{1}{\delta}\psi_\delta(w^{(\delta)})$ and obtain analogously as above that

$$\begin{aligned}\int_0^s \int_\Omega \left|w_t^{(\delta)} + \frac{1}{\delta}\psi_\delta(w^{(\delta)})\right|^2 (x,t)\,dx\,dt \ &+\ \frac{1}{2}\int_\Omega \|\nabla w^{(\delta)}\|^2(x,s)\,dx \\ \le\ \frac{1}{2}\int_\Omega \|\nabla\varphi(x)\|^2\,dx \ &+\ \int_0^s \int_\Omega \left\langle w_t^{(\delta)} + \frac{1}{\delta}\psi_\delta(w^{(\delta)}), f\right\rangle(x,t)\,dx\,dt\,.\end{aligned} \tag{4.25}$$

Combining the above estimates we conclude that for every $s \in [0,T]$ we have

$$\int_\Omega \left(\|\nabla w^{(\delta)}\|^2 + \frac{1}{\delta}\Psi_\delta(w^{(\delta)})\right)(x,s)\,dx \ \le\ C_5\,, \tag{4.26}$$

$$\int_0^s \int_\Omega \left(\left|w_t^{(\delta)}\right|^2 + \left|\Delta w^{(\delta)}\right|^2 + \left|\frac{1}{\delta}\psi_\delta(w^{(\delta)})\right|^2\right)(x,t)\,dx\,dt \ \le\ C_6\,. \tag{4.27}$$

We now let δ tend to $0+$. Passing to a subsequence, if necessary, we find functions $\bar\psi, \bar w \in L^2(Q_T;\mathbb{R}^N)$ such that $\bar w_t, \Delta\bar w \in L^2(Q_T;\mathbb{R}^N)$, and

$$w_t^{(\delta)} \ \to\ \bar w_t, \quad \Delta w^{(\delta)} \to \Delta\bar w, \quad \frac{1}{\delta}\psi_\delta(w^{(\delta)}) \to \bar\psi \quad \text{weakly in } L^2(Q_T;\mathbb{R}^N), \tag{4.28}$$

$$w^{(\delta)} \ \to\ \bar w \quad \text{strongly in } L^2(Q_T;\mathbb{R}^N)\,. \tag{4.29}$$

Consequently, the function $\bar w$ satisfies the same initial and boundary conditions as w. We now use (4.26) to check that $\bar w(x,t) \in K$ a.e. To this end, assume that there exists a set $A \subset Q_T$, $\operatorname{meas}(A) > 0$, such that $\bar w(x,t) \notin K$ for $(x,t) \in A$. Putting for $k \in \mathbb{N}$

$$A_k := \{(x,t) \in A\,;\ |\bar w(x,t)| \le k\,,\ \operatorname{dist}(\bar w(x,t), K) \ge 1/k\}\,,$$

we have $A = \cup_{k=1}^\infty A_k$, hence there exist $\mu > 0$ and $k_0 \in \mathbb{N}$ such that $\operatorname{meas}(A_{k_0}) = \mu > 0$. Put

$$\kappa(\delta) := \int_0^T \int_\Omega |w^{(\delta)} - \bar w|^2(x,t)\,dx\,dt\,.$$

Then $\lim_{\delta\to 0+}\kappa(\delta) = 0$, and we may find $\delta_0 > 0$ such that

$$\kappa(\delta) \ <\ \frac{\mu}{8k_0^2} \qquad \text{for } \delta \le \delta_0\,. \tag{4.30}$$

Put $B_\delta := \{(x,t) \in Q_T\,;\ |w^{(\delta)}(x,t) - \bar w(x,t)| > 1/(2k_0)\}$. Then

$$\operatorname{meas}(B_\delta) \ \le\ 4k_0^2\kappa(\delta) \ <\ \frac{\mu}{2} \qquad \text{for } \delta \le \delta_0\,, \tag{4.31}$$

and there exists a set $\bar{A} \subset A_{k_0}$, $\operatorname{meas}(\bar{A}) \geq \mu/2$, such that

$$|w^{(\delta)}(x,t) - \bar{w}(x,t)| \ \leq \ \frac{1}{2k_0} \qquad \forall (x,t) \in \bar{A} \ \ \forall \delta \leq \delta_0\,, \tag{4.32}$$

hence

$$\operatorname{dist}(w^{(\delta)}(x,t), K) \ \geq \ \frac{1}{2k_0} \qquad \forall (x,t) \in \bar{A} \ \ \forall \delta \leq \delta_0\,, \tag{4.33}$$

and

$$\operatorname{dist}(w^{(\delta)}(x,t), K_\delta) \ \geq \ \frac{1}{4k_0} \qquad \forall (x,t) \in \bar{A} \ \ \forall \delta \leq \delta_1 := \min\{\delta_0, 1/(4k_0)\}\,. \tag{4.34}$$

We have $|w^{(\delta)}(x,t)| \leq k_0 + 1/(2k_0)$ for $(x,t) \in \bar{A}$ and $\delta \leq \delta_1$, hence

$$\left(1 - \frac{1}{4k_0^2 + 2}\right) w^{(\delta)}(x,t) \notin \operatorname{Int} K_\delta \qquad \forall (x,t) \in \bar{A} \ \ \forall \delta \leq \delta_1\,. \tag{4.35}$$

This yields for every $(x,t) \in \bar{A}$ and $\delta \leq \delta_1$ that

$$M_\delta(w^{(\delta)}(x,t)) \ \geq \ 1 + \frac{1}{4k_0^2 + 1}\,, \qquad \Psi_\delta(w^{(\delta)}(x,t)) \ \geq \ \frac{1}{(4k_0^2+1)(4k_0^2+2)}\,, \tag{4.36}$$

which contradicts (4.26), and we thus checked that $\bar{w}(x,t) \in K$ a. e.

We continue by putting

$$\bar{v}(x,t) \ := \ v(x,0) + \bar{w}(x,t) + \int_0^t \bar{\psi}(x,\tau)\, d\tau. \tag{4.37}$$

We see that

$$w_t^{(\delta)} + \frac{1}{\delta}\psi_\delta(w^{(\delta)}) \to \bar{v}_t \quad \text{weakly in} \ \ L^2(Q_T; \mathbb{R}^N)\,, \tag{4.38}$$

and passing to the limit in (4.17), (4.25) we obtain that

$$\bar{v}_t - \Delta \bar{w} \ = \ f\,, \tag{4.39}$$

$$\begin{aligned} &\int_0^s \int_\Omega |\bar{v}_t|^2\, dx\, dt + \frac{1}{2}\int_\Omega \|\nabla \bar{w}\|^2(x,s)\, dx \\ \leq \ & \frac{1}{2}\int_\Omega \|\nabla \varphi\|^2(x)\, dx + \int_0^s \int_\Omega \langle \bar{v}_t, f\rangle \ dx\, dt\,. \end{aligned} \tag{4.40}$$

We now claim that

$$\left\langle \frac{1}{\delta}\psi_\delta(w^{(\delta)}), w^{(\delta)} - \tilde{w}\right\rangle \ \geq \ 0 \qquad \text{a. e.} \ \ \forall \tilde{w} \in \tilde{K}_\delta\,. \tag{4.41}$$

Indeed, if $w^{(\delta)}(x,t) \in K_\delta$, then $\psi_\delta(w^{(\delta)}(x,t)) = 0$, and if $w^{(\delta)}(x,t) \notin K_\delta$, then we have that $M_\delta(w^{(\delta)}(x,t)) > 1$. The definition of the subdifferential yields that

$$\left\langle \partial M_\delta(w^{(\delta)}(x,t)), w^{(\delta)}(x,t) - \tilde{w}\right\rangle \geq M_\delta(w^{(\delta)}(x,t)) - M_\delta(\tilde{w}) > 0\,, \tag{4.42}$$

and (4.41) follows. Passing to the limit in (4.41) we obtain that

$$\langle \bar{v}_t - \bar{w}_t, \bar{w} - \tilde{w} \rangle \ \geq \ 0 \qquad \text{a.e.} \quad \forall \tilde{w} \in K\,, \tag{4.43}$$

that is, $\bar{w} = \mathcal{S}_K[\varphi, \bar{v}]$. Testing the identity

$$v_t - \bar{v}_t = \Delta(w - \bar{w}) \tag{4.44}$$

by $w - \bar{w}$ and using the inequality (4.5) we conclude that $w = \bar{w}$, $v = \bar{v}$, and the assertion follows from (4.40) and (4.16). ∎

5 Proof of Theorem 2.2

The existence and uniqueness result for $\varepsilon = 0$ has been established in Lemma 4.1. For each fixed $\varepsilon > 0$, the unique solution can be constructed by the penalty method with the same penalty function ψ_δ as in the proof of Lemma 4.2 and we do not repeat the standard argument here. Instead, we derive a priori estimates which will enable us to pass to the limit as $\varepsilon \to 0+$. We continue to denote by C_i any positive constant independent of ε.

Put $v^\varepsilon(x,t) := \int_0^t (\varepsilon\, \Delta w^\varepsilon + \gamma(w^\varepsilon, u^\varepsilon))(x,\tau)\, d\tau$ for $(x,t) \in Q_T$. By (2.6) we then have $w^\varepsilon = \mathcal{S}_K[\varphi, v^\varepsilon]$ according to the notation introduced in Section 2, and we obtain that

$$v^\varepsilon_t - \varepsilon\, \Delta w^\varepsilon \ = \ \gamma(w^\varepsilon, u^\varepsilon) \quad \text{a.e.} \quad \text{for all } \ \varepsilon \geq 0 \tag{5.1}$$

analogously as in the proof of Lemma 4.1 for $\varepsilon = 0$.

Lemma 4.2 enables us to test Eq. (5.1) for $\varepsilon > 0$ by v^ε_t, and obtain for every $s \in [0,T]$ that

$$\begin{aligned} \int_0^s \int_\Omega |v^\varepsilon_t|^2(x,t)\, dx\, dt \ &+ \ \varepsilon \int_\Omega \|\nabla w^\varepsilon\|^2(x,s)\, dx \\ &\leq \ \varepsilon \int_\Omega \|\nabla \varphi\|^2(x)\, dx + \int_0^s \int_\Omega |\gamma(w^\varepsilon, u^\varepsilon))|^2(x,t)\, dx\, dt \\ &\leq \ C_7 \left(1 + \int_0^s \int_\Omega |w^\varepsilon|^2(x,t)\, dx\, dt\right). \end{aligned} \tag{5.2}$$

From (4.4) and the Gronwall argument we thus obtain for every $s \in [0,T]$ the estimate

$$\int_0^s \int_\Omega \left(|w^\varepsilon|^2 + |w^\varepsilon_t|^2 + |v^\varepsilon_t|^2\right)(x,t)\, dx\, dt + \varepsilon \int_\Omega \|\nabla w^\varepsilon\|^2(x,s)\, dx \ \leq \ C_8\,. \tag{5.3}$$

From (5.1) and (5.3) it further follows that

$$\varepsilon^2 \int_0^T \int_\Omega |\Delta w^\varepsilon|^2(x,t)\, dx\, dt \ \leq \ C_9\,. \tag{5.4}$$

Let $\eta : \bar{Q}_T \to \mathbb{R}^N$ be any smooth test function. We have by (5.2) that

$$\left|\int_0^T \int_\Omega \varepsilon \left\langle \Delta w^\varepsilon, \eta \right\rangle \, dx\, dt\right| \le \sqrt{\varepsilon} \int_0^T \int_\Omega \sqrt{\varepsilon} \|\nabla w^\varepsilon\| \, \|\nabla \eta\| \, dx\, dt \tag{5.5}$$

$$\le \sqrt{\varepsilon}\, C_{10} \int_0^T \left(\int_\Omega \|\nabla \eta\|^2 \, dx\right)^{1/2} dt\,.$$

Together with (5.4), (5.2) this yields that

$$\varepsilon \Delta w^\varepsilon \to 0 \quad \text{weakly in} \quad L^2(Q_T; \mathbb{R}^N) \quad \text{as} \quad \varepsilon \to 0+\,. \tag{5.6}$$

The last step of the proof consists in testing Eq. (5.1) by $w^\varepsilon - w^0$ with w^0 as in Lemma 4.1. We then obtain for $s \in [0,T]$ that

$$\int_0^s \int_\Omega \left\langle v_t^\varepsilon - v_t^0, w^\varepsilon - w^0 \right\rangle (x,t)\, dx\, dt + \varepsilon \int_0^s \int_\Omega \|\nabla w^\varepsilon\|^2 (x,t)\, dx\, dt$$

$$= -\int_0^s \int_\Omega \varepsilon \left\langle \Delta w^\varepsilon, w^0 \right\rangle (x,t)\, dx\, dt$$

$$+ \int_0^s \int_\Omega \left\langle \gamma(w^\varepsilon, u^\varepsilon) - \gamma(w^0, u^0), w^\varepsilon - w^0 \right\rangle (x,t)\, dx\, dt\,. \tag{5.7}$$

Using (4.5) and Hypothesis 2.1 (v) we conclude that

$$\frac{1}{2} \int_\Omega |w^\varepsilon - w^0|^2 (x,s)\, dx \;+\; \varepsilon \int_0^s \int_\Omega \|\nabla w^\varepsilon\|^2 (x,t)\, dx\, dt$$

$$\le -\int_0^s \int_\Omega \varepsilon \left\langle \Delta w^\varepsilon, w^0 \right\rangle (x,t)\, dx\, dt$$

$$+ C_{11} \left(\int_0^s \int_\Omega |w^\varepsilon - w^0|^2 (x,t)\, dx\, dt \;+\; \int_0^s \int_\Omega |u^\varepsilon - u^0|^2 (x,t)\, dx\, dt \right) \tag{5.8}$$

for every $s \in [0,T]$. In order to pass to the limit in (5.8) as $\varepsilon \to 0+$, it suffices to use (5.6), Hypothesis 2.1 (iv), and Gronwall's argument. The proof of Theorem 2.2 is thus complete. ■

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