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The asymptotic behavior of semi-invariants for linear stochastic systems

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ABSTRACT. The asymptotic behavior of semi-invariants of the random variable $\ln|X(t,x)|$, where $X(t,x)$ is a solution of a linear system of stochastic differential equations, is connected with the moment Lyapunov exponent $g(p)$. Namely, it is obtained that the n-th semi-invariant is asymptotically proportional to the time t with the coefficient of proportionallity $g^{(n)}(0)$. The proof is based on the concept of analytic characteristic functions. It is also shown that the asymptotic behavior of the analytic characteristic function of $\ln|X(t,x)|$ in a neighbourhood of the origin on the complex plane is controlled by the extension $g(iz)$ of $g(p)$.

1. INTRODUCTION

Consider an autonomous linear d-dimensional system of stochastic differential equations in the sense of Ito

$$dX = A_0 X dt + \sum_{r=1}^{q} A_r X dw_r(t),\ X(0) = x. \tag{1.1}$$

Let $X(t,x)$ be the solution of (1.1). It is known [1], [2] that under some nondegeneracy conditions the moment Lyapunov exponent

$$g(p) := \lim_{t\to\infty} \frac{1}{t} \ln E|X(t,x)|^p,\ x \neq 0,$$

exists and is independent of $x \neq 0$. The function $g(p)$ is a convex analytic function of p :

$$g(p) = \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} p^n.$$

We have

$$\ln E|X(t,x)|^p = \ln E e^{p\xi(t,x)} = \sum_{n=1}^{\infty} \frac{\gamma_n(t,x)}{n!} p^n,$$

where $\gamma_n := \gamma_n(t,x)$ is the n-th semi-invariant of the random variable

$$\xi := \xi(t,x) := \ln|X(t,x)|,\ x \neq 0.$$

Therefore

$$\lim_{t\to\infty} \frac{1}{t} \ln E|X(t,x)|^p = \lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{\gamma_n(t,x)/t}{n!} p^n = \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} p^n,$$

whence the following conjecture arises

$$\lim_{t\to\infty} \frac{\gamma_n(t,x)}{t} = g^{(n)}(0),\ n = 1, 2, \dots\ . \tag{1.2}$$

The main aim of the present paper is to prove (1.2). For $n = 1$ the first semi-invariant γ_1 is equal to $E\ln|X(t,x)|$ and the well-known connection between the Lyapunov exponent λ and $g'(0)$,

$$\lambda = \lim_{t\to\infty} \frac{E\ln|X(t,x)|}{t} = g'(0), \tag{1.3}$$

confirms this assertion. The second semi-invariant is equal to $\gamma_2(t,x) = E\ln^2|X(t,x)| - (E\ln|X(t,x)|)^2 = D\ln|X(t,x)|$. The equality (1.2) for $n=2$ can be proved without any serious difficulties (see Section 2). To prove (1.2) in a general case, we study some properties of the analytic characteristic function $\psi(z;t,x) = E\exp\{iz\ln|X(t,x)|\}$ (see Section 3). This function is an extension of the function $\ln E|X(t,x)|^p$. Since $\psi(z;t,x)$ takes, as a rule, zero values if $d>1$, the cumulant generating function $\ln\psi(z;t,x)$ is not defined everywhere. At the same time for every $t\ge 0$ there exists $\delta_t>0$ such that for any $x\in\mathbf{R}^d$ the function $\ln\psi(z;t,x)$ is analytic in $C_{\delta_t} := \{z : |z| < \delta_t\}$. Much more complicated assertion consists in the fact that there exists such $\delta>0$ independently of t. Moreover we prove (see Lemma 4.2) that under usual nondegeneracy conditions for (1.1) there exists C_δ such that the function $(\ln\psi(z;t,x))/t$ is analytic in C_δ and uniformly bounded with respect to $t>0$ and x with $|x|=1$. Due to this fact, we are able to use the classical Vitali convergence theorem and prove the basic result (1.2) (see Section 4). It is also shown that the asymptotic behavior of $\ln\psi(z;t,x)$ in a neighborhood of the origin on the complex plane is controlled by the extension $g(iz)$ of $g(p)$.

2. The asymptotic behavior of the second semi-invariant

The diffusion process

$$\Lambda(t,\lambda) := \frac{X(t,x)}{|X(t,x)|}\ ,\ \lambda = \frac{x}{|x|}\ ,\ x\neq 0,$$

defined on the unit sphere $\mathbf{S}^{d-1}$ with center at the origin satisfies the Khasminskii system

$$d\Lambda = h_0(\Lambda)dt + \sum_{r=1}^{q} h_r(\Lambda)dw_r(t), \tag{2.1}$$

where the vector fields $h_r(\lambda)$, $r=0,1,...,q$, on $\mathbf{S}^{d-1}$ are equal to

$$h_0(\lambda) = A_0\lambda - (A_0\lambda,\lambda)\lambda$$

$$-\frac{1}{2}\sum_{r=1}^{q}(A_r\lambda, A_r\lambda)\lambda - \sum_{r=1}^{q}(A_r\lambda,\lambda)A_r\lambda + \frac{3}{2}\sum_{r=1}^{q}(A_r\lambda,\lambda)^2\lambda,$$

$$h_r(\lambda) = A_r\lambda - (A_r\lambda,\lambda)\lambda,\ r=1,...,q.$$

It is assumed that the following condition of nondegeneracy is fulfilled:

$$\dim LA\{\tilde h_0, h_1, ..., h_q\} = d-1 \text{ for all } \lambda = x/|x| \in \mathbf{S}^{d-1}, \tag{2.2}$$

where

$$\tilde h_0(\lambda) = \tilde A_0\lambda - (\tilde A_0\lambda,\lambda)\lambda,\ \tilde A_0 = A_0 - \frac{1}{2}\sum_{r=1}^{q} A_r^2,$$

$LA\{\}$ denotes the Lie algebra generated by the vector fields which occur in the brackets (see [2]).

The following semigroup of positive operators $T_t(p)$ (which depends on the parameter p) is defined on $\mathbf{C}(\mathbf{S}^{d-1})$:

$$T_t(p)f(\lambda) = Ef(\Lambda(t,\lambda))|X(t,\lambda)|^p,\ \lambda \in \mathbf{S}^{d-1},\ f \in \mathbf{C}(\mathbf{S}^{d-1}). \tag{2.3}$$

It is well known [6], [5], [2] that under the nondegeneracy condition (2.2) the process Λ is ergodic and for any $t > 0$, $-\infty < p < \infty$, the operator $T_t(p)$ is compact and irreducible, even strongly positive. We recall that a positive operator Q on $\mathbf{C}(\mathbf{K})$ ($\mathbf{K}$ is a compact set) is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{K})$ are the only Q-invariant closed ideals, and Q is called strongly positive if $Qf(x) > 0$, $x \in \mathbf{K}$, for any nontrivial $f \geq 0$. The generalized Perron-Frobenius theorem ensures that for each $p \in \mathbf{R}$ the operator $T_t(p)$ and consequently its generator $L(p)$ have a strictly positive eigenfunction corresponding to the principal eigenvalue $g(p)$, which is real, simple, and strictly dominates the real part of any other point of the spectrum of $L(p)$. This principal eigenvalue coincides with the moment Lyapunov exponent $g(p)$. So, we have

$$L(p)e(p;\lambda) = g(p)e(p;\lambda),\ T_t(p)e(p;\lambda) = \exp(g(p)t)e(p;\lambda), \tag{2.4}$$

where $e \in \mathbf{C}(\mathbf{S}^{d-1})$, $e(p;\lambda) > 0$, and the eigenfunction $e(p;\lambda)$ is chosen so that $||e(p;\cdot)|| = \max_\lambda |e(p;\lambda)| = 1$ for any $-\infty < p < \infty$. Clearly $e(0;\lambda) = 1$, $\lambda \in \mathbf{S}^{d-1}$.

Using the perturbation theory of linear operators [4], it is possible to prove that the function $e(p;\lambda)$ has derivatives with respect to p.

For $x \neq 0$ we have

$$\xi = \xi(t,x) = \ln|X(t,x)| = \ln|x| + \int_0^t Q(\Lambda)ds$$
$$+ \int_0^t \sum_{r=1}^q (A_r\Lambda,\Lambda)dw_r(s),\ \Lambda = \Lambda_\lambda(s),\ \lambda = \frac{x}{|x|},$$

where

$$Q(\lambda) = (A_0\lambda,\lambda) + \frac{1}{2}\sum_{r=1}^q (A_r\lambda, A_r\lambda) - \sum_{r=1}^q (A_r\lambda,\lambda)^2.$$

Proposition 2.1. *Let the nondegeneracy condition* (2.2) *be fulfilled. Then the equality*

$$\lim_{t\to\infty} \frac{\gamma_2(t,x)}{t} = \lim_{t\to\infty} \frac{E\ln^2|X(t,x)| - (E\ln|X(t,x)|)^2}{t} = g''(0) \tag{2.5}$$

holds. The limit in (2.5) *does not depend on* x.

Proof. The second equality in (2.4) can be rewritten in the form:

$$E[e(p;\Lambda(t,\lambda))\exp(p\xi(t,\lambda))] = e(p;\lambda)\exp(g(p)t). \tag{2.6}$$

Differentiating with respect to p (it is not difficult to justify the differentiation under the sign of mathematical expectation), we get

$$E[e'_p(p;\Lambda)\exp(p\xi)] + E[e(p;\Lambda)\xi\exp(p\xi)] \tag{2.7}$$
$$= e'_p(p;\lambda)\exp(g(p)t) + e(p;\lambda)g'(p)t\exp(g(p)t).$$

Let $p = 0$. Then (we recall that $g(0) = 0,\ e(0;\lambda) = 1$)

$$Ee'_p(0;\Lambda) + E\xi = e'_p(0;\lambda) + g'(0)t. \tag{2.8}$$

Since λ belongs to the compact set $\mathbf{S}^{d-1}$, $Ee'_p(0;\Lambda)$ and $e'_p(0;\lambda)$ are bounded. Now we see from (2.8)

$$E\xi(t;\lambda) = g'(0)t + O(1). \tag{2.9}$$

We remark that the assertion (2.9) is stronger than (1.3).

Differentiating (2.7), we obtain for $p = 0$

$$\begin{aligned} &Ee''_{p^2}(0;\Lambda) + 2Ee'_p(0;\Lambda)\xi + E\xi^2 \\ &= e''_{p^2}(0;\lambda) + 2e'_p(0;\lambda)g'(0)t + g''(0)t + [g'(0)t]^2. \end{aligned} \tag{2.10}$$

We note that (2.9) implies $\lim_{t\to\infty}(E\xi)^2/t^2 = [g'(0)]^2$, and (2.10) implies

$$\lim_{t\to\infty} E\xi^2/t^2 = [g'(0)]^2. \tag{2.11}$$

Now we find $g'(0)t$ from (2.8) and substitute it in (2.10). As a result we get

$$\lim_{t\to\infty}\frac{E\xi^2 - (E\xi)^2}{t} - g''(0) = \lim_{t\to\infty}\frac{2}{t}(Ee'_p(0;\Lambda)\xi - Ee'_p(0;\Lambda)E\xi), \tag{2.12}$$

provided the limit in the right-hand side of (2.12) exists. We have

$$\begin{aligned} &\frac{Ee'_p(0;\Lambda)\xi - Ee'_p(0;\Lambda)E\xi}{t} \\ &= E[e'_p(0;\Lambda)(\frac{\xi}{t} - g'(0))] + Ee'_p(0;\Lambda)(g'(0) - \frac{E\xi}{t}). \end{aligned} \tag{2.13}$$

The second term here evidently tends to zero as $t \to \infty$. Further

$$|E[e'_p(0;\Lambda)(\frac{\xi}{t} - g'(0))]| \le [E(e'_p(0;\Lambda))^2]^{1/2}[E(\frac{\xi}{t} - g'(0))^2]^{1/2}.$$

Due to (2.9) and (2.11), the first term in the right-hand side of (2.13) also tends to zero as $t \to \infty$. Consequently, the limit in the right-hand side of (2.12) is equal to zero. □

3. The characteristic function, moments, and semi-invariants for $\xi = \ln|X(t,x)|$

The characteristic function of $\xi = \ln|X(t,x)|$ can be considered as a function of complex variable z :

$$\psi(z) = \psi(z;t,x) := E\exp\{iz\xi\} = E\exp\{iz\ln|X(t,x)|\}.$$

If $z = p$ is real, we get the classical characteristic function of the random variable $\xi = \ln|X(t,x)|$:

$$\psi(p) = \psi(p;t,x) = E\exp\{ip\ln|X(t,x)|\} = E|X(t,x)|^{ip},\ |x| \ne 0. \tag{3.1}$$

If $z = -ip$ is pure imaginary, we get

$$\psi(-ip) = \psi(-ip; t, x) = E \exp\{p \ln |X(t, x)|\} = E|X(t, x)|^p, \ |x| \neq 0,$$

i.e., the p-th moment of the random variable $\xi = \ln |X(t, x)|$.

Clearly, $\psi(-ip), \ -\infty < p < \infty$, takes positive values and

$$|\psi(q + ip)| \leq \psi(ip). \tag{3.2}$$

Further, there exists $\psi'(z)$, i.e., for every $t, x \neq 0$ the function $\psi(z)$ is entire. Existence of the derivative $\psi'(z) = \psi'_z(z; t, \lambda)$ with respect to z and the equality $\psi'_z(z; t, \lambda) = iE(\ln |X(t, \lambda)| \exp\{iz \ln |X(t, \lambda)|\})$ can be proved in the standard way by differentiation under the sign of mathematical expectation. The knowledge about analytic characteristic functions can be found in [7], [8], [9].

Moments m_n of ξ. They can be expressed in terms of the coefficients of the Taylor-series expansion for $\psi(z) = \psi(z; t, x)$:

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} z^n, \ m_n(t, x) = m_n := E\xi^n = \frac{1}{i^n} \psi^{(n)}(0).$$

Semi-invariants (cumulants) γ_n of ξ. They are equal to (in taking the logarithm $\ln \psi(z)$ of the characteristic function we choose a branch such that $\ln \psi(z) = 0$ at $z = 0$)

$$\gamma_n(t, x) := \gamma_n = \frac{1}{i^n} \frac{d^n}{dz^n} \ln \psi(z) \ |_{z=0} = \frac{d^n}{dp^n} \ln \psi(-ip) \ |_{p=0} = \frac{d^n}{dp^n} \ln E|X(t, x)|^p \ |_{p=0} \ .$$

This notion is well-defined since for every $t, x \neq 0$ there exists a sufficiently small neigbourhood $|z| < \delta$ (where δ in general depends on t, x) in which the function $\psi(z) = \psi(z; t, x)$ does not vanish. We have (as $\gamma_0 = 0$)

$$\ln \psi(z) = \sum_{n=1}^{\infty} \frac{i^n \gamma_n}{n!} z^n, \ |z| < \delta, \ \ln E|X(t, x)|^p = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!} p^n, \ |p| < \delta.$$

The connection between the moments and the semi-invariants can be obtained in the following well known way. Put $iz = w$ and $\varphi(w) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} w^n$. Then $\ln \varphi(w) = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!} w^n$. We have $\varphi \cdot (\ln \varphi)' = \varphi'$. This is equivalent to

$$(1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} w^n) \cdot \sum_{n=1}^{\infty} \frac{\gamma_n}{(n-1)!} w^{n-1} = \sum_{n=1}^{\infty} \frac{m_n}{(n-1)!} w^{n-1}. \tag{3.3}$$

Putting $w = 0$ in (3.3), we find $\gamma_1 = m_1$. Subsequently differentiating (3.3) with respect to w and putting $w = 0$, we obtain $\gamma_2 = m_2 - m_1^2, \ \gamma_3 = m_3 - 3m_1 m_2 + 2m_1^3, \ \gamma_4 = m_4 - 4m_1 m_3 - 3m_2^2 + 12m_1^2 m_2 - 6m_1^4$, and so on.

Let us note a remarkable feature of semi-invariants: any semi-invariant of a sum of independent random variables is equal to the sum of the semi-invariants of these variables. As against, the second and higher moments do not possess this property.

The order of the entire function $\psi(z;t,x)$. Without loss of generality the function $\psi(z;t,\lambda)$ with $|\lambda|=1$ can be considered.

Proposition 3.1. *The order* ρ *of the entire function* $\psi(z;t,\lambda) = E\exp\{iz\ln|X(t,\lambda)|\}$ *under any* $t\geq 0,\ \lambda\in S^{d-1}$ *is not more than 2:*

$$\rho := \limsup_{r\to\infty}\{\ln\ln M(r,\psi)/\ln r\} \leq 2,$$

where

$$M(r,\psi) := \max_{|z|=r}|\psi(z;t,\lambda)|.$$

If

$$\sum_{r=1}^{q}(A_r\lambda,\lambda)^2 \geq A_* > 0,\ \lambda\in\mathbf{S}^{d-1}, \tag{3.4}$$

then $\rho=2$.

Proof. For real q and p we have (3.2). From here and from the maximum principle we get

$$M(r,\psi) = \max(\psi(-ir),\psi(ir)) = \max(E|X(t,\lambda)|^r, E|X(t,\lambda)|^{-r}).$$

We have

$$|X(t,\lambda)|^{\pm r} = \exp\{\pm r(\int_0^t Q(\Lambda)ds + \int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)dw_r(s))\}$$

$$= \exp\{\pm r\int_0^t Q(\Lambda)ds + \frac{r^2}{2}\int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2 ds\}$$

$$\times\exp\{\pm r\int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)dw_r(s) - \frac{r^2}{2}\int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2 ds\}.$$

Let $|Q(\lambda)|\leq Q^*,\ \sum_{r=1}^{q}(A_r\lambda,\lambda)^2\leq A^*$ on $\mathbf{S}^{d-1}$, where $Q^*>0$ and $A^*>0$ are constants. Then

$$|X(t,\lambda)|^{\pm r} \leq \exp\{(rQ^* + \frac{r^2}{2}A^*)t\}$$

$$\times\exp\{\pm r\int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)dw_r(s) - \frac{r^2}{2}\int_0^t\sum_{r=1}^{q}(A_r\Lambda,\Lambda)^2 ds\}$$

and consequently (as the second exponent here is a martingale)

$$E|X(t,\lambda)|^{\pm r} \leq \exp\{(rQ^* + \frac{r^2}{2}A^*)t\}.$$

Then $M(r,\psi) \le \exp\{(rQ^* + \frac{r^2}{2}A^*)t\}$ and the first conclusion of the lemma follows from the definition of ρ. If (3.4) is fulfilled, then

$$|X(t,\lambda)|^{\pm r} \ge \exp\{(-rQ^* + \frac{r^2}{2}A_*)t\}$$

$$\times \exp\{\pm r \int_0^t \sum_{r=1}^q (A_r\Lambda,\Lambda)dw_r(s) - \frac{r^2}{2}\int_0^t \sum_{r=1}^q (A_r\Lambda,\Lambda)^2 ds\}$$

and therefore we obtain

$$\exp\{(-rQ^* + \frac{r^2}{2}A_*)t\} \le E|X(t,\lambda)|^{\pm r} \le \exp\{(rQ^* + \frac{r^2}{2}A^*)t\}.$$

From here $\rho = 2$. □

Proposition 3.2. *If the characteristic function $\psi(z;t,\lambda)$ of the random variable $\xi = \ln|X(t,\lambda)|$ has no zeros, then ξ has either a normal or a degenerate distribution.*

Proof. The following result of H'Adamard is well-known: if an entire function is of order not more than ρ and has no zeros, then it has the form $\exp\{Q(z)\}$, where Q is a polynomial of degree not more than ρ. Therefore, due to Proposition 3.1, the following representation takes place

$$\psi(z;t,\lambda) = \exp\{Q(z;t,\lambda)\},$$

where $Q(z;t,\lambda)$ is a polynomial of degree not more than 2. Since $\psi(p;t,\lambda)$, $p \in \mathbf{R}$, is the characteristic function of $\xi = \ln|X(t,\lambda)|$, this random variable has either a normal or a degenerate distribution. □

In the one-dimensional case the variable $\ln|X(t,\lambda)|$ is gaussian if $\sigma \ne 0$. However, if $d > 1$ and, for example, the strong nondegeneracy condition (3.4) is fulfilled, the random variable is neither normal nor degenerate. Therefore, as a rule, the function $\psi(z;t,\lambda) = E\exp\{iz\ln|X(t,\lambda)|\}$ has zeros which, of course, depend on t,λ. Consequently, there does not exist the function $\ln\psi(z;t,\lambda)$ for all z. Nevertheless, because $\psi_{t,\lambda}(0) = 1$ and λ belongs to the compact set $\mathbf{S}^{d-1}$, it is clear that for every $t \ge 0$ there exists $\delta_t > 0$ such that for any $\lambda \in \mathbf{S}^{d-1}$ the function $\ln\psi(z;t,\lambda)$ is analytic in $C_{\delta_t} := \{z : |z| < \delta_t\}$. In the next section we prove that there exists an analogous δ, but which is independent of t, whereupon the proof of (1.2) is carried out.

4. The main theorem

Lemma 4.1. *Let there exist $\delta > 0$ such that for any $t > 0$, $\lambda \in \mathbf{S}^{d-1}$ the function $\frac{1}{t}\ln\psi(z;t,\lambda)$ is analytic in C_δ and bounded uniformly with respect to the t,λ. Then for $z \in C_\delta$*

$$\lim_{t\to\infty}\frac{1}{t}\ln\psi(z;t,\lambda) = \lim_{t\to\infty}\frac{1}{t}\ln E\exp\{iz\ln|X(t,\lambda)|\} = g(iz), \tag{4.1}$$

$$\lim_{t\to\infty} \frac{1}{t}\frac{d^n}{dz^n} \ln \psi(z;t,\lambda) = i^n g^{(n)}(iz), \tag{4.2}$$

and, consequently, (1.2) *is fulfilled.*

Proof. Recall the Vitali convergence theorem (see [10], p.168): "Let $f_n(z)$ be a sequence of functions, each regular in a region D; let $|f_n(z)| \leq M$ for every n and z in D; and let $f_n(z)$ tend to a limit, as $n \to \infty$, at a set of points having a limit point inside D. Then $f_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D, the limit being, therefore, an analytic function of z." The lemma follows from this theorem, well-known properties of analytic functions, and from the fact that the convergence takes place for $z = -ip \in C_\delta$, p is real. $\Box$

Lemma 4.2. *Let the nondegeneracy condition* (2.2) *be fulfilled. Then there exists* $\delta > 0$ *such that for any* $t > 0$, $\lambda \in \mathbf{S}^{d-1}$ *the function* $\frac{1}{t}\ln\psi(z;t,\lambda)$ *is analytic in* C_δ, *and bounded uniformly with respect to the* t, λ.

Proof. Introduce the strongly continuous semigroup $T_t(z)$ (which depends on the complex parameter z) on the Banach space $\mathbf{C}(\mathbf{S}^{d-1})$ of complex-valued functions $f(\lambda)$, $\lambda \in \mathbf{S}^{d-1}$:

$$T_t(z)f(\lambda) = E[f(\Lambda_\lambda(t)) \exp\{z \ln |X(t,\lambda)|\}].$$

Let $L(z)$ be the infinitesimal operator of the semigroup $T_t(z)$. For $z = p$ real, a number of properties were mentioned in Section 2. Not all of them are fulfilled for arbitrary complex z. For example, the property of positivity is broken. At the same time many of them remain true. In particular, the operator $T_t(z)$ for any z and $t > 0$ is compact as well. This fact can be proved analogously to [2].

Clearly,

$$\psi(z;t,\lambda) = T_t(iz)\mathbf{1}(\lambda) = E\exp\{iz \ln |X(t,\lambda)|\},$$

where the function $\mathbf{1}(\lambda)$ is identically equal to 1.

Making use of infinitesimal generators in spectral theory of semigroups is a rather usual matter. However the perturbation theory is more advanced for bounded operators. That is why we use both $L(z)$ and $T_t(z)$ in our proof below.

Let us fix $t = 1$ and consider the family $T_1(z)$ for z belonging to a sufficiently small neighborhood of the origin $z = 0$. This family analytically depends on z [4]. The operator $T_1(0)$ has $\exp(g(0)) = 1$ as an eigenvalue with the eigenfunction $e(0;\lambda) = \mathbf{1}(\lambda)$: $T_1(0)\mathbf{1}(\lambda) = \mathbf{1}(\lambda)$. It was noted in Section 2 that the eigenvalue $g(p)$ of $L(p)$ is simple and $g(p)$ strictly dominates the real part of any other point of the spectrum of $L(p)$. Therefore the spectrum of $T_1(0)$ is equal to $\sigma[T_1(0)] = \exp(\sigma[L(0)]) = \{1\} \cup \exp(\sigma[L(0)]\setminus\{0\})$, where the set $\exp(\sigma[L(0)]\setminus\{0\})$ lies in a circle of the radius $\exp(-r) < 1$, $r > 0$. Since the family $T_1(z)$ analytically depends on z, the spectrum $\sigma[T_1(z)]$ for sufficiently small z consists of an eigenvalue, which is close to 1 and lies outside a circle containing the rest of the spectrum. And both the eigenvalue and a corresponding eigenvector $e(z;\lambda)$ depend on z analytically [4]. Therefore the eigenvalue is equal to $\exp(g(z))$. Choose the eigenvector $e(z;\lambda)$ so that $|e| = 1$, $e(0;\lambda) = \mathbf{1}(\lambda)$. Clearly, $L(z)e(z;\lambda) = g(z)e(z;\lambda)$, and the spectrum $\sigma[L(z)] = \sigma_1(z) \cup \sigma_2(z)$, where $\sigma_1(z) = \{g(z)\}$ and there exists $\delta > 0$ such that if $\zeta \in \sigma_2(z)$ and $|z| < \delta$, then $\mathrm{Re}\,\zeta < -r/2$. Below we consider z with $|z| < \delta$. For any such z there exists (see [3]) a spectral decomposition of the space: $\mathbf{C}(\mathbf{S}^{d-1}) = \mathbf{C}_1(z) \oplus \mathbf{C}_2(z)$, of the semigroup: $T_t(z) = T_t^{(1)}(z) \oplus T_t^{(2)}(z)$, and of the generator: $L(z) = L^{(1)}(z) \oplus L^{(2)}(z)$, where $\mathbf{C}_1(z)$ is the one-dimensional space generated by the eigenvector $e(z;\lambda)$, $T_t^{(1)}(z)e(z;\lambda) = \exp(g(z)t)e(z;\lambda)$, $L^{(1)}(z)e(z;\lambda) = g(z)e(z;\lambda)$, $\sigma[L^{(1)}(z)] = \sigma_1(z)$, $\sigma[L^{(2)}(z)] = \sigma_2(z)$. This follows from compactness of $\sigma_1(z)$ (we recall that the set $\sigma_1(z)$ is one-point). A projection $P(z)$ such that $P(z)\mathbf{C}(\mathbf{S}^{d-1}) = \mathbf{C}_1(z)$ and $[P(z)]^{-1}(0) = \mathbf{C}_2(z)$ corresponds to the decomposition. The projection $P(z)$ analytically depends on z [4]. Further, $T_t^{(1)}(z) = P(z)T_t(z) = T_t(z)P(z) = T_t^{(1)}(z)P(z)$ and $T_t^{(2)}(z) = (I - P(z))T_t(z) = T_t(z)(I - P(z)) = T_t^{(2)}(z)(I - P(z))$. Since $\mathrm{Re}\,\zeta < -r/2$ for all $\zeta \in \sigma_2(z)$ if only $|z| < \delta$, we get

$$||T_t^{(2)}(z)|| \le M^{(2)} \exp(-\frac{r}{2}t), \tag{4.3}$$

where $M^{(2)}$ and r do not depend on z belonging to the δ-neighborhood C_δ of the origin. Besides, there exist constants M and ω such that

$$||T_t(z)|| \le M \exp(\omega t). \tag{4.4}$$

We have

$$\begin{aligned} \psi(z;t,\lambda) = E\exp\{iz\ln|X(t,\lambda)|\} = T_t(iz)\mathbf{1}(\lambda) &= T_t^{(1)}(iz)\mathbf{1}(\lambda) + T_t^{(2)}(iz)\mathbf{1}(\lambda) \\ &= T_t^{(1)}(iz)P(iz)\mathbf{1}(\lambda) + T_t^{(2)}(iz)(I - P(z))\mathbf{1}(\lambda). \end{aligned} \tag{4.5}$$

Further, $P(iz)\mathbf{1}(\lambda) = K(iz)e(\lambda;iz)$, where $K(iz)$ is a complex-valued scalar depending on z. Therefore $T_t^{(1)}(iz)P(iz)\mathbf{1}(\lambda) = K(iz)\exp(g(iz)t)e(iz;\lambda) = \exp(g(iz)t)P(iz)\mathbf{1}(\lambda)$. Since $g(iz)$ and $P(iz)\mathbf{1}(\lambda)$ analytically depend on z, $g(0) = 0$, and $P(0)\mathbf{1}(\lambda) = \mathbf{1}(\lambda)$, we obtain that for any $0 < \varepsilon < r/2$, $\varepsilon < 1$, there exists ρ, $0 < \rho \le \delta$, such that for $z \in C_\rho$

$$|T_t^{(1)}(iz)P(iz)\mathbf{1}(\lambda)| = |\exp(g(iz)t)| \cdot |P(iz)\mathbf{1}(\lambda)| \ge (1-\varepsilon)\exp(-\varepsilon t),$$

$$|(I - P(z))\mathbf{1}(\lambda)| \le \varepsilon.$$

Now from (4.3)-(4.5) we get

$$(1-\varepsilon)\exp(-\varepsilon t) - M^{(2)}\varepsilon\exp(-\frac{r}{2}t) \le |\psi(z;t,\lambda)| \le M\exp(\omega t).$$

Taking $\varepsilon > 0$ sufficiently small, we obtain the assertion of Lemma 4.2. □

Clearly, Lemmas 4.1 and 4.2 imply the following main theorem.

Theorem 4.1. *Let the nondegeneracy condition* (2.2) *be fulfilled. Then for big t the n-th semi-invariant $\gamma_n(t,x)$ of random variable $\xi = \ln|X(t,x)|$ is proportional to t for big t with the coefficient of proportionality $g^{(n)}(0)$. More precisely: there exists the limit*

$$\lim_{t\to\infty} \frac{1}{t}\gamma_n(t,x) = g^{(n)}(0). \tag{4.6}$$

The moment Lyapunov function $g(p)$ can be extended for complex z belonging to a circle $C_\delta = \{z : |z| < \delta\}$ in the sense that for such z (4.1) *is fulfilled. The limits in* (4.1) *and* (4.6) *do not depend on $x \in \mathbf{R}^d$.*

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