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THE SPIN-GLASS PHASE-TRANSITION IN THE HOPFIELD MODEL WITH p-SPIN INTERACTIONS

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Abstract: We study the Hopfield model with pure p-spin interactions with even $p \geq 4$, and a number of patterns, $M(N)$ growing with the system size, N, as $M(N) = \alpha N^{p-1}$. We prove the existence of a critical temperature β_p characterized as the first time quenched and annealed free energy differ. We prove that as $p \uparrow \infty$, $\beta_p \to \sqrt{\alpha 2 \ln 2}$. Moreover, we show that for any $\alpha > 0$ and for all inverse temperatures β, the free energy converges to that of the REM at inverse temperature $\beta/\sqrt{\alpha}$. Moreover, above the critical temperature the distribution of the *replica overlap* is concentrated at zero. We show that for large enough α, there exists a non-empty interval of in the low temperature regime where the distribution has mass both near zero and near ± 1. As was first shown by M. Talagrand in the case of the p-spin SK model, this implies the the Gibbs measure at low temperatures is concentrated, asymptotically for large N, on a countable union of disjoint sets, no finite subset of which has full mass. Finally, we show that there is $\alpha_p \sim 1/p!$ such that for $\alpha > \alpha_p$ the set carrying almost all mass does not contain the original patterns. In this sense we describe a genuine spin glass transition.

Our approach follows that of Talagrand's analysis of the p-spin SK-model. The more complex structure of the random interactions necessitates, however, considerable technical modifications. In particular, various results that follow easily in the Gaussian case from integration by parts formulas have to be derived by expansion techniques.

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1. Introduction and Results

In a recent paper [T4] (see also [T6] for a more pedagogical exposition) Talagrand has presented for the first time a rigorous analysis of a phase transition from a high temperature phase to what could be called a "spin glass phase". This was done in the context of the so called p-spin Sherrington-Kirkpatrick (SK) model [SK] for $p \geq 3$. From the heuristic analysis on the basis of the replica method (see [MPV]), it is known that this model should have a spin glass phase that is much simpler than in the case $p = 2$, the standard SK model, and this fact is to be expected to be related to the success of Talagrand's approach. In any event, this important new result has highlighted the p-spin interaction model as an important playground to develop new techniques and to gain more insight into the fascinating world of spin glasses.

The Hamiltonian of the p-spin SK model can most simply be described as a Gaussian process X_σ on the hypercube $\mathcal{S}_N \equiv \{-1,1\}^N$ with mean zero and covariance function

$$\mathbb{E} X_\sigma X_{\sigma'} = N R_N(\sigma,\sigma')^p \tag{1.1}$$

where $R_N(\sigma,\sigma') \equiv \frac{1}{N}\sum_{i=1}^N = 1 - dist_{Ham}(\sigma,\sigma')$ where d_{Ham} denotes the Hamming distance. Seen from this point of view, the distinction between different values of p is in the speed of decrease of the correlation of the process X_σ with distance.

Talagrand's methods use heavily the Gaussian nature of the SK model, and in particular the fact the X_σ can be represented in the form

$$X_\sigma = \sum_{1\leq i_1<i_2<\cdots<i_p\leq N} J_{1_i,\dots,i_p}\sigma_{i_1},\dots\sigma_{i_p} \tag{1.2}$$

where $J_{1_i,\dots,i_p}$ is a family of i.i.d. standard Gaussian random variables. It is therefore natural to ask whether and to what extent his approach can be generalized to other models that have similar correlation decay properties as processes on $\mathcal{S}_N$, but that are not Gaussian and do not have the simple structure as (1.2). A natural candidate to test this question on and whose investigation has considerable interest in its own right, is the so-called p-spin Hopfield model which we shall describe below. These models have been introduced in the context of neural networks by Peretto and Niez [PN] and Lee et al. [Lee] as generalizations of the standard Hopfield model [Ho] which corresponds to the case $p = 2$. This latter case has been studied heavily and since its first introduction by Figotin and Pastur [FP1,FP2] has become, on the rigorous level, one of the best understood mean field spin glass models [N1,ST,Ko,BGP1,BGP2,BG1,BG2,BG3,BG4,T3,T7]. It should be noted, however, that all the results obtained for this model so far concern the high-temperatures phase and the so-called retrieval phase, while next to nothing is known about the supposedly existing *spin glass phase*. The investigation of this phase in the $p \geq 4$ version of the model is the main concern of the present paper.

We now give a precise definition of the models we will study. Let $(\Omega,\mathcal{F},\mathbb{P})$ be an abstract probability space and $\{\xi_i^\mu\}_{i,\mu\in\mathbb{N}}$ a family of i.i.d. Bernoulli variables, taking values 1 and -1 with equal probability.

Define for each $N \in \mathbb{N}$ a (finite) random *Hamiltonian*, that is, a function $H_N : \Omega \times \mathcal{S}_N \to \mathbb{R}$ by

$$H_N[\omega](\sigma) \equiv -\left(\frac{p!}{N^{2p-2}}\right)^{\frac{1}{2}} \sum_{\mu=1}^{M(N)} \sum_{i_1<\ldots<i_p} \prod_{l=1}^{p} \xi_{i_l}^{\mu} \sigma_{i_l}. \tag{1.3}$$

The value of p is considered a fixed parameter of the model, and will in the following be even and at least be 4. While this model can be analyzed rather easily along the lines of the standard Hopfield model if $M \sim N$ (see [BG1]), the results of Newman [N1] on the storage capacity suggest that the model should have a good behavior even if $M(N)$ scales as N^{p-1}, i.e.[5]

$$\lim_{N\uparrow\infty} \frac{M(N)}{N^{p-1}} = \alpha < \infty. \tag{1.4}$$

In this paper we will always be concerned with this case. The limit α will also turn out to be a crucial parameter for the behavior of the system. In the standard Hopfield model, it has been proven that for small values of α, the model at low temperatures is in a retrieval phase, where there are Gibbs measures that are concentrated on small neighborhoods of the stored patterns. It is believed that for large values of α (or smaller values of β) this property fails and that in fact the model should then be very similar to the Sherrington-Kirkpatrick model; however, there exist no rigorous results to that effect. While in the present paper we do not present results concerning the retrieval phase in the $p \geq 4$ case, the results we shall present show that for reasonably large values of α a phase transition occurs from the high-temperature phase to a "spin glass phase" that is strikingly similar to those of the corresponding SK models.

We will use the following multi-index notation. For finite subsets $\mathcal{I}$ of the natural numbers, and real numbers $(x_n)_{n\in\mathbb{N}}$, let by $x_{\mathcal{I}} = \prod_{l\in\mathcal{I}} x_l$. Let furthermore $\mathcal{P}_N$ be the set of subsets of $\mathcal{N} = \{1, \ldots, N\}$ of cardinality p. The Hamiltonian (1.3) can then be written as

$$H_N[\omega](\sigma) = -\left(\frac{p!}{N^{2p-2}}\right)^{\frac{1}{2}} \sum_{\mu=1}^{M(N)} \sum_{\mathcal{I}\in\mathcal{P}} \xi_{\mathcal{I}}^{\mu} \sigma_{\mathcal{I}}. \tag{1.5}$$

These Hamiltonians define random, finite volume Gibbs measures $\mathcal{G}_{N,\beta}[\omega]$ by assigning each configuration $\sigma \in \mathcal{S}_N$ a weight proportional to its Boltzmann factor, that is

$$\mathcal{G}_{N,\beta}[\omega](\sigma) = 2^{-N} \frac{e^{-\beta H_N[\omega](\sigma)}}{Z_{N,\beta}[\omega]}. \tag{1.6}$$

Consider now the Hamiltonian as a random process indexed by $\sigma \in \mathcal{S}_N$. Simple calculations allow to verify that the mean of H_N with respect to $\mathbb{P}$ vanishes for all σ, that is $\mathbb{E} H_N(\sigma) = 0, \quad \forall \sigma \in \mathcal{S}_N$, whereas the variance satisfies (for some number C depending on p only)

$$\alpha N(1 - CN^{-1}) \leq \mathbb{E} H_N(\sigma)^2 = \frac{p!}{N^{2p-2}} \sum_{\mu=1}^{M(N)} \sum_{\mathcal{I}\in\mathcal{P}_N} \leq \alpha N, \tag{1.7}$$

[5] In the sequel, we will write with slight abuse of notation $M(N) = \alpha N^{p-1}$ even for finite N.

which motivates our choice of normalization in the definition of H_N. The covariance is given as

$$\mathbb{E} H_N(\sigma) H_N(\sigma') = \frac{p!}{N^{2p-2}} \sum_{\mu=1}^{M(N)} \sum_{\mathcal{I} \in \mathcal{P}_N} \sigma_{\mathcal{I}} \sigma'_{\mathcal{I}} = \alpha N R^p(\sigma, \sigma')(1 + \mathcal{O}(N^{-1})), \tag{1.8}$$

where $R_N(\sigma, \sigma') \equiv \frac{1}{N} \sum_{i=1}^N \sigma_i \sigma'_i$ is the (normalized) *replica overlap*. Note that this covariance is in leading order and up to the factor α the same as the covariance for the p-spin SK-model ([T4]).

The normalizing factor $Z_{N,\beta}$ in (1.6) is called *partition function* and it is given by

$$Z_{N,\beta}[\omega] = \mathbb{E}_\sigma e^{-\beta H_N[\omega](s)}, \tag{1.9}$$

where $\mathbb{E}_\sigma$ is the expectation with respect to the uniform distribution on $\mathcal{S}_N$. We will call the mean of $Z_{N,\beta}$ under $\mathbb{P}$ the *annealed partition function.*

We define the *free energy* $F_{N,\beta}[\omega]$ by $F_{N,\beta}[\omega] \equiv \frac{1}{N} \ln Z_{N,\beta}[\omega]$.[6] Customarily one calls the mean of the free energy, $\mathbb{E} F_{N,\beta}$, the *quenched free energy*, while the normalized logarithm of the annealed partition function is called the *annealed free energy* $F^{\mathrm{an}}_{N,\beta} \equiv \frac{1}{N} \ln \mathbb{E} Z_{N,\beta}$. Observe that by Hölder's inequality, both the quenched free energy and the annealed free energy are convex functions of β.

Let us briefly mention a variant of the above model. On the same configuration space and with the same random variables ξ, we define macroscopic random order parameters

$$m^\mu[\omega](\sigma) \equiv \frac{1}{N} \sum_{i=1}^N \xi_i^\mu \sigma_i. \tag{1.10}$$

These parameters are considered as components of a vector in $\mathbb{R}^{M(N)}$ with $M(N)$ as in (1.4). New Hamiltonians are now defined through

$$\bar{H}_N[\omega](\sigma) - \frac{N}{s_p} \left(\|m[\omega](\sigma)\|_p^p - \mathbb{E} \|m[\omega](\sigma)\|_p^p \right), \tag{1.11}$$

where $s = s_p > 0$ is defined such that the covariance of $\bar{H}$ is in leading order in N equal to αN. The interaction $\bar{H}$ is a straightforward generalization of the usual $p = 2$ case. However, computing the resulting covariance function one sees that it decreases only quadratically with the Hamming distance. Therefore it will not share the special features of the p-spin SK model. An analysis of the high-temperature phase for $\bar{H}$ has been presented in [Ni1].

We will now state our results. They will always concern the model with Hamiltonian (1.3) and $p \geq 4$.

The first result we prove for both choices of the Hamiltonian is that for high enough temperatures (that is, low values of β), the limit of the annealed free energy exists.

Theorem 1.1: *If $\beta < e^{-2}(p!)^{\frac{1}{2}} \equiv \beta'_p$, then the annealed free energy corresponding to H satisfies*

$$F^{\mathrm{an}}_{N,\beta} = \frac{\alpha\beta^2}{2}(1 + \mathcal{O}(N^{-1})). \tag{1.12}$$

[6]Note that physicists often use a different normalization, $F_{N,\beta} = -\frac{1}{\beta N} \ln Z_{N,\beta}$. We use Talagrand's choice convention to facilitate comparison with [T4].

Note that for larger values of β, the annealed free energy diverges. Our analysis will be limited to the case when $\beta < \beta_p'$ where a comparison to the SK model is still possible. It is nice to see that this value tends to infinity with p very rapidly. Moreover, we shall see that this value becomes much larger than the critical temperature, as α gets large.

Jensen's inequality implies that the quenched free energy is less then or equal to the annealed free energy,

$$\mathbb{E} F_{N,\beta} = \frac{1}{N}\mathbb{E}\ln Z_{N,\beta} \leq \frac{1}{N}\ln \mathbb{E} Z_{N,\beta} = F^{\mathrm{an}}_{N,\beta}. \tag{1.13}$$

We define the critical temperature to be the infimum of values for which equality holds in (1.13), i.e. in terms of β,

$$\beta_p \equiv \sup\left\{\beta \geq 0 : \limsup_{N\uparrow\infty} \mathbb{E} F_{N,\beta} = \limsup_{N\uparrow\infty} F^{\mathrm{an}}_{N,\beta}\right\}. \tag{1.14}$$

Observe that in general $\lim_N \mathbb{E} F_{N,\beta}$ need not exist.

By (1.8), as a random process on $\mathcal{S}_N$, $H_N(\sigma)$ has (up to an overall factor) essentially the same covariance structure as the p-spin SK Hamiltonian. This suggest that as in that case, for p large the model should be similar to Derrida's *random energy model* (REM) [D1,D2]s Recall that in this model, $H_N(\sigma) \equiv \sqrt{N} X_\sigma$, where $\{X_\sigma\}_{\sigma\in\mathcal{S}_N}$ are i.i.d. standard normal random variables). Defining the corresponding partition function $Z^{\mathrm{REM}}_{N,\beta} = \mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma}$, one easily sees that the free energy satisfies [D2]

$$f^{REM}_\beta = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\ln Z^{\mathrm{REM}}_{N,\beta} = \begin{cases} \beta^2/2, & \text{if } \beta \leq \sqrt{2\ln 2} \\ \beta\sqrt{2\ln 2} - \ln 2, & \text{if } \beta \geq \sqrt{2\ln 2} \end{cases} \tag{1.15}$$

We will show that as p tends to infinity, $\sqrt{\alpha}\beta_p$ tends to the critical value $\sqrt{2\ln 2}$ of the REM. Moreover, pointwise in α, β,

$$\frac{1}{\beta}\lim_{p\to\infty}\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\ln Z_{N,\sqrt{\alpha}\beta} = \frac{1}{\beta} f^{REM}_\beta. \tag{1.16}$$

in analogy to the situation in the p-spin SK model [T6]. While this may not be very surprising, it is also not totally obvious and will require some non-trivial computations.

Our next two theorems make these relations precise. We will denote by $I(t)$ the *Cramér entropy* function,

$$I(t) = \frac{1}{2}(1-t)\ln(1-t) + \frac{1}{2}(1+t)\ln(1+t), \tag{1.17}$$

Theorem 1.2: *The critical value $\beta_p = \beta_p(\alpha)$ satisfies*

$$\beta_p(\alpha)^2 \geq \min\left(\frac{\beta_p'^2}{4}, \inf_{t\in[0,1]} I(t)\frac{1+t^p}{\alpha t^p}\right) \equiv \check{\beta}_p(\alpha)^2. \tag{1.18}$$

Furthermore, if $\alpha \geq \frac{e^4 2\ln 2}{p!} \equiv \alpha_p$ then

$$\beta_p(\alpha)^2 \leq \frac{2\ln 2}{\alpha} \equiv \hat{\beta}(\alpha)^2. \tag{1.19}$$

Remarks: (i) One can show that the inequality (1.19) actually strict. In [B2] it is shown that for the SK case, $\beta_p \geq \sqrt{2\ln 2}(1-c_p)$ with $c_p = 2^{-p(4+O(1/p))}$. This follows from a corresponding upper bound on the supremum of $H_N(\sigma)$ which can be obtained using standard techniques. These estimates can without doubt be carried over to our case.

(ii) The bounds on the critical temperature are essentially (up to a factor $\sqrt{\alpha}$) the same as for the p-spin SK-model ([T4], Theorem 1.1).[7]

By elementary analysis one finds that, as p tends to infinity,

$$\inf_{0\leq t\leq 1}((1+t^{-1}p)I(t))^{1/2} = \sqrt{2\ln 2}\left(1-\frac{2^{-p-1}}{\ln 2}\right) + \mathcal{O}(p^3 2^{-2p}). \tag{1.20}$$

This, together with the convexity of the free energy in β, will allow us to prove the following statement.

Theorem 1.3: *As $p\to\infty$, the lower bound $\check{\beta}_p \uparrow \hat{\beta}$. Moreover, for all $\beta \geq 0$ and $\alpha > 0$,*

$$\lim_{p\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\mathbb{E}F_{N,\beta} = f^{REM}_{\beta\alpha^{-1/2}}. \tag{1.21}$$

The basic strategy used to prove these results are rather general. In Chapter 2, we will explain them by means of the analogous calculations in the REM. For now, we just mention that the hard part is to prove the lower bound (1.18), whereas the upper bound (1.19) is comparatively easy and will follow from an estimate on the ground state energy.

An important point in the study of disordered models is the question of self-averaging of the free energy. While in many cases this follows from general principles [MS,T1] of mass concentration, due to the failure of certain convexity properties, it turns out to be surprisingly difficult to prove the following result[8]

Theorem 1.4: *For all $\beta, n, \tau, \varepsilon > 0$ there exists $C_n < \infty$ (depending only on n and β), and $\bar{N} < \infty$ such that the free energy satisfies*

$$\mathbb{P}\left[|F_{N,\beta} - \mathbb{E}F_{N,\beta}| \geq \tau\beta N^{-\frac{1}{2}+\varepsilon}\right] \leq CN^{-n} \tag{1.22}$$

for all $N \geq \bar{N}$. In particular,

$$\lim_{N\uparrow\infty}|F_{N,\beta} - \mathbb{E}F_{N,\beta}| = 0, \quad \mathbb{P}-a.s. \tag{1.23}$$

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[7] Observe that in [T4], the normalization of the Hamiltonian contains an extra factor $2^{-1/2}$.

[8] A sharper estimate can be proven with much less effort for the interaction $\bar{H}_N$, see [Ni1].

Remark: From recent results in the p-spin SK-model and the REM [BKL], one actually expects that the fluctuations in the small β region are of much lower order.

While the critical temperature is defined in terms of the behavior of the free energy, it turns out that this phase transition goes along with a change in the behavior of the replica overlap parameter, $R_N(\sigma,\sigma')$. This will eventually lead to rather detailed insight into the properties of the Gibbs measures at low temperatures.

The crucial link between the two will be provided by the next theorem.

Theorem 1.5: *Assume that $\beta < \frac{1}{2}\beta'_p$. Then the replica overlap $R_N(\sigma,\sigma')$ satisfies*

$$\mathbb{E}\,\frac{\partial F_{N,\beta}}{\partial\beta} = \alpha\beta\,(1 - \mathbb{E}\mathcal{G}_{N,\beta}\otimes\mathcal{G}_{N,\beta}[R_N(\sigma,\sigma')^p])\,(1+\mathcal{O}(N^{-1})), \tag{1.24}$$

Note that in the case of the Gaussian SK models, this relation is a trivial consequence of the *integration by parts formula*

$$\mathbb{E}[gf(g)] = \mathbb{E}[g^2]\mathbb{E}[f'(g)], \tag{1.25}$$

which holds for any centered Gaussian random variable g and any function f not growing faster than some polynomial at infinity. To establish this result without the help of this formula turns out to require a considerable effort. Similar tools are also instrumental in the proof of Theorem 1.4.

We then have the following consequence to Theorem 1.2 and Theorem 1.5.

Theorem 1.6: *Assume that $\alpha \geq \alpha_p$. If $\beta < \beta_p$, then*

$$\limsup_{N\uparrow\infty}\mathbb{E}\mathcal{G}_{N,\beta}\otimes\mathcal{G}_{N,\beta}[R_N(\sigma,\sigma')^p] = 0. \tag{1.26}$$

Conversely, if $\limsup_N \mathbb{E}\,\frac{\partial F_{N,\beta}}{\partial\beta} < \alpha\beta$, then

$$\liminf_{N\uparrow\infty}\mathbb{E}\mathcal{G}_{N,\beta}\otimes\mathcal{G}_{N,\beta}[R_N(\sigma,\sigma')^p] > 0. \tag{1.27}$$

In particular, (1.27) holds for all $\beta \in [\hat\beta, \frac{1}{2}\beta'_p)$.

Remark: It seems reasonable that (1.27) should hold for all β above the critical β_p, but there seems to be no general principal that would prohibit a *reentrant phase transition.*

Inequality (1.27) expresses in a weak way that below the critical temperature, the Gibbs measure gives some mass to a a small subset of the configuration space. This result can be strengthened. As in [T4], we show that the overlap between replicas is either very close to one, or to zero:

Theorem 1.7: *For any $\epsilon > 0$ there exists $p_0 < \infty$ such that for all $p \geq p_0$, $\alpha > \alpha_p$, and for all $0 \leq \beta < \beta'_p$*

$$\lim_{N\uparrow\infty}\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}(|R_N(\sigma,\sigma')| \in [\epsilon, 1-\epsilon]) = 0 \tag{1.28}$$

If, moreover, $\beta < \check\beta_p$, then for any $\epsilon > 0$ there exists $p_0 < \infty$ such that for all $p \geq p_0$, such that for some $\delta > 0$, for all large enough N,

$$\mathbb{E}\mathcal{G}_N^{\otimes 2}\left(|R_N(\sigma,\sigma')| \in [\epsilon,1]\right) \leq e^{-\delta N} \tag{1.29}$$

Remark: Note that we prove this result without any restriction on the temperature, while Talagrand requires some upper bound on β both in [T4] and in the announcement [T5] even though the bound in [T5] is greatly improved. We stress that the our result is also valid for the p-spin SK-model. The same applies for all subsequent results.

The information provided by Theorem's 1.6 and 1.7 allow gain considerable insight into the nature of the Gibbs measures in the low temperature phase. This observation is due to Talagrand.

In [T4] he showed that whenever (1.27) and (1.28) hold, it is possible to decompose the state space $\mathcal{S}_N$ into a collection of disjoint subsets $\mathcal{C}_k$ such that

(i)

$$\lim_{N\uparrow\infty} \mathbb{E}\mathcal{G}_N^{\otimes 2}\left(\left\{(\sigma,\sigma')\,|\,|R_N(\sigma,\sigma')| > \epsilon\right\}\setminus \cup_k\, \mathcal{C}_k \times \mathcal{C}_k\right) = 0 \tag{1.30}$$

(where the $\mathcal{C}_k$ depend both on N and on the random parameter!), and

(ii) If $\sigma,\sigma' \in \mathcal{C}_k$, then $R_N(\sigma,\sigma') \geq 1-\epsilon$.

Note that because of the global spin flip symmetry of our models with p even, these lumps necessarily appear in symmetric pairs.

In [T4] Talgrand analyzed the properties of these lumps further using the cavity method. He showed that, under a certain hypothesis that we shall discuss shortly, for β not too large this lumps correspond to what is known as "pure states". While it is very likely that this analysis can also be carried over to our models, we will leave this question open to further investigation. We find it however interesting to discuss the situation of the general hypothesis. Talgrand's hypothesis in [T4] concern the distribution of mass on the lumps. Roughly, they can be states as

Theorem 1.8: *Assume that $\frac{1}{2}\beta'_p > \beta > \beta_p$. Let $\mathcal{C}_k$ be ordered such that for all k, $\mathcal{G}_{N,\beta}(\mathcal{C}_k) \geq \mathcal{G}_{N,\beta}(\mathcal{C}_{k+1})$. Then for all $k \in \mathbb{N}$, there exists $p_k < \infty$ such that for all $p \geq p_k$,*

$$\lim_{N\uparrow\infty} \mathbb{E}\mathcal{G}_{N,\beta}\left(\cup_{l=1}^k \mathcal{C}_l\right) < 1 \tag{1.31}$$

except possibly for an exceptional set of β's of zero Lebesgue measure. Moreover, for k large, $p_k \sim \frac{2}{3}\frac{\ln k}{\ln 2}$.

In [T5] Talagrand has announced a proof of an even stronger theorem in the p-spin SK model that makes use of general identities between replica overlaps proven by Ghirlanda and Guerra [GG]. We show that at least Theorem 1.8 also holds in our model.

A final result is particular to the Hopfield model and concerns the storage properties of the model. Newman has proven in [N1] that for small α, the Hamiltonian has deep local minima in the vicinity of each pattern. Here we show a somewhat converse result, stating that if α is not too small, then small neighborhoods of the patterns have asymptotically mass zero. In other words, none of the patterns falls into one of the 'lumps'. This gives the final justification to call the phase transition we have observed a transition to a genuine *spin glass phase*.

Theorem 1.9: *Suppose that α satisfies $\alpha\beta_p(\alpha) > (p!)^{-1/2}$. Then there exists a $\delta \in (0, \frac{1}{p})$ and $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$,*

$$\mathbb{P}[\arg\sup |H_N(\sigma)| \in \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu)] \leq N^{-m}, \tag{1.32}$$

where $B_\delta(\xi^\mu)$ is the $N\delta$-ball around ξ^μ in the space $\mathbb{R}^N$ with respect to the Hamming metric. In particular, there exists an $\alpha_{sp} = \alpha_{sp}(p)$ such that (1.32) holds for all $\alpha > \alpha_{sp}$. Furthermore,

$$\arg\sup |H_N(\sigma)| \notin \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu)] \text{ eventually} \quad \mathbb{P} - a.s. \tag{1.33}$$

The proof of this result is based on the comparison between the ground state energy of the system and an estimate on the values of the Hamiltonian in the balls around the patterns. While the former increases as $N\sqrt{\alpha}$, the latter is almost constant and with high probability close to $N(p!)^{-1/2}$.

The remainder of this paper is organized as follows. In Chapter 2, we explain the ideas behind the proof of the bounds on the critical temperature by calculating the corresponding quantities in the REM. In Chapter 3, Theorem 1.1 is proved. Chapter 4 is devoted to the lower and the upper bound on the critical β (as well as the proof of Corollary 1.3). In Chapter 5 we prove Theorem 1.4 In Chapter 6 we prove the results on the distribution of the replica overlap, Theorems 1.5 to 1.8. In Chapter 8 we prove Theorem 1.9.

2. Second Moment Method: The REM

This section is meant to give a pedagogical exposition of Talagrand's truncated second moment method [T3,T4] in the context of the simplest possible setting, the random energy model. A more detailed exposition can also be found in [B2] and [T6]. Since the application of this method in our case will become rapidly somewhat technical in our case, we still find it useful to give the reader an outline in a non-technical context[9]. Moreover, the REM provides important bounds for the real model.

We will now show how this method works by using it to compute the free energy of the REM. Note first that in general,

$$\frac{\partial F_{N,\beta}}{\partial \beta} = -\frac{1}{N}\mathcal{G}_{N,\beta}[H_N] \leq \frac{1}{N}\mathbb{E}[\sup_\sigma |H_N(\sigma)|]. \tag{2.1}$$

[9]Note that of course much sharper results than those presented here can be obtained in the REM when making use of its special features. See e.g. [BKL] for a full analysis of the fluctuations of the free energy.

Moreover, since

$$\mathbb{P}\,[\sup_\sigma |H_N(\sigma)| > tN] \leq 2^N \mathbb{P}\,[|H_N(\sigma)| > tN] \leq 2^{N+1} e^{-\frac{t^2 N}{2}}. \tag{2.2}$$

from this it follows easily that

$$\begin{aligned} \frac{1}{N}\mathbb{E}\,[\sup_\sigma H_N(\sigma)] &\leq \sqrt{2\ln 2} + 2 \int_{\sqrt{2\ln 2}}^{\infty} e^{-N(\frac{t^2}{2} - \ln 2)} dt \\ &\leq \sqrt{2\ln 2} + N^{-1}\sqrt{\frac{2}{\ln 2}}. \end{aligned} \tag{2.3}$$

This is the upper bound on the derivative of the expectation of the free energy. Suppose now that $\beta > \sqrt{2\ln 2} = \beta'$. Convexity of the free energy then implies that

$$\mathbb{E}F_{N,\beta} \leq \mathbb{E}F_{N,\beta'} + (\beta - \beta')\beta' \tag{2.4}$$

and in the limit

$$\limsup_{N\uparrow\infty} \mathbb{E}F_{N,\beta} \leq -\frac{\beta'^2}{2} + \beta\beta' = \frac{\beta^2}{2} - (\beta - \beta')^2 < \frac{\beta^2}{2}, \tag{2.5}$$

which by definition means that $\beta' \geq \beta_{\text{REM}}$. In the case of the p-spin Hopfield model, the corresponding calculations will be identical to those above, except for the bounds on the extrema of the Hamiltonian, where the non Gaussian character induces somewhat more involved calculations.

The basic idea behind Talagrand's approach to prove the lower bound (which he did for the p-spin SK-model in [T4]), is to obtain a variance estimate on the partition function. This will imply that the expectation of the logarithm behaves like the logarithm of the expectation of this quantity. In the REM, one would naively compute

$$\begin{aligned} \mathbb{E}[Z_{N,\beta}^{\text{REM}\,2}] &= \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\,e^{\beta\sqrt{N}(X_\sigma + X_{\sigma'})} \\ &= 2^{-2N}\left(\sum_{\sigma\neq\sigma'} e^{N\beta^2} + \sum_\sigma e^{2N\beta^2}\right) \\ &= e^{N\beta^2}\left[(1-2^{-N}) + 2^{-N}e^{N\beta^2}\right]. \end{aligned} \tag{2.6}$$

The second term in the brackets is exponentially small if and only if $\beta^2 < \ln 2$, and this cannot be the critical value since it violates the upper bound β' above.[10] The point is that while in the computation of $\mathbb{E}\,e^{2\beta\sqrt{N}X_\sigma}$, the dominant contribution comes from the part of the distribution of X_σ around $X_\sigma = 2\beta\sqrt{N}$, whereas in $\mathbb{E}\,Z_{N,\beta}^{\text{REM}}$ the main part is contributed by X_σ around $\beta\sqrt{N}$. One is thus led to consider the second moment of a suitably truncated version of $Z_{N,\beta}^{\text{REM}}$. Namely, for $c > 0$,

$$\tilde{Z}_{N,\beta}^{\text{REM}}(c) = \mathbb{E}_\sigma\, e^{\beta\sqrt{N}X_\sigma}\,\mathbb{1}_{\{X_\sigma < c\sqrt{N}\}}. \tag{2.7}$$

[10] This is already contained in [D2]

One then finds that (modulo irrelevant prefactors)

$$\mathbb{E}\tilde{Z}^{\mathrm{REM}}_{N,\beta}(c)=\begin{cases} e^{\frac{\beta^2 N}{2}}, & \text{if } \beta<c,\\ \frac{1}{\sqrt{N}(\beta-c)}e^{N\beta c-\frac{Nc^2}{2}}, & \text{if } \beta>c.\end{cases} \tag{2.8}$$

Moreover, for $\beta<c$,

$$\mathbb{E}\tilde{Z}_{N,\beta}(c)=\mathbb{E}Z_{N,\beta}\left(1-\frac{e^{-\frac{1}{2}(c-\beta)^2N}}{\sqrt{N}(c-\beta)}\right) \tag{2.9}$$

On the other hand,

$$\mathbb{E}\tilde{Z}_{N,\beta}(c)^2=(1-2^{-N})\left(\mathbb{E}\tilde{Z}_{N,\beta}(c)\right)^2+2^{-N}\mathbb{E}e^{2\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{X_\sigma<c\sqrt{N}\}}, \tag{2.10}$$

where the second term satisfies

$$2^{-N}\mathbb{E}e^{2\beta\sqrt{N}X_\sigma}\leq\begin{cases} 2^{-N}e^{2\beta^2N}, & \text{if } 2\beta<c\\ 2^{-N}\frac{(2\beta-c)\sqrt{N}}{e^{2c\beta N-\frac{c^2N}{2}}}, & \text{otherwise,}\end{cases} \tag{2.11}$$

and thus

$$\begin{aligned} &2^{-N}\mathbb{E}e^{2\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{X_\sigma<(1+\varepsilon)\beta\sqrt{N}\}}\\ &\leq(\mathbb{E}\tilde{Z}_{N,\beta})^2\times\begin{cases} e^{-N(\ln 2-\beta^2)}, & \text{if}\,\beta<\frac{c}{2},\\ \frac{e^{-N(c-\beta)^2-N(\ln 2-\frac{c^2}{2})}}{(2\beta-c)\sqrt{N}}, & \text{if}\,\frac{c}{2}<\beta<c,\\ e^{(c^2/2-\ln 2)N}\sqrt{N}\frac{(\beta-c)^2}{2\beta-c}, & \text{if}\,\beta>c\end{cases}\end{aligned} \tag{2.12}$$

Hence, for all $c<\sqrt{2\ln 2}$, and all $\beta\neq c$

$$\mathbb{E}\frac{(\tilde{Z}_{N,\beta}(c)-\mathbb{E}\tilde{Z}_{N,\beta}(c))^2}{\mathbb{E}[\tilde{Z}_{N,\beta}(c)^2]}\leq e^{-Ng(c,\beta)}, \tag{2.13}$$

where $g(c,\beta)>0$. Thus, by Chebyshev's inequality, it is immediate that

$$\lim_{N\uparrow\infty}\frac{1}{N}\mathbb{E}\ln\tilde{Z}_{N,\beta}(c)=\lim_{N\uparrow\infty}\frac{1}{N}\ln\mathbb{E}\tilde{Z}_{N,\beta}(c),\quad\forall c<\sqrt{2\ln 2}. \tag{2.14}$$

Since this gives a lower bound of the free energy that is as close to the upper bound as desired, we see that the upper bound gives in fact the true value.

This is a remarkable feature of the REM: the expectation of the logarithm of the partition function coincides with the log of the expectation of a suitably truncated partition function. While this is rather special to the REM, the method is general enough to provide lower bounds in the far more complicated situations, as we will see.

3. The Annealed Free Energy.

In this Section we compute the anneled free energy. Apart from the intrinsic interest this can be seen as the computation of the log-moment generating function of the Hamiltonian and this will be a basic input in the sequel. While in the SK models this is a two line computation, here even this will require a considerable effort. The idea is to use Taylor expansions and to exploit the fact that the Hamiltonian is a sum of a very large number of independent random variables. Namely

$$\begin{aligned} \mathbb{E}Z_{N,\beta} &= \mathbb{E}e^{-\beta H_N[\omega](\sigma)} = \mathbb{E}\exp\left(\beta\left(\frac{p!}{N^{2p-2}}\right)^{\frac{1}{2}}\sum_{\mu=1}^{M(N)}\sum_{\mathcal{I}\in\mathcal{P}_N}\xi_{\mathcal{I}}^{\mu}\right) \\ &= \prod_{\mu=1}^{M(N)}\left[\mathbb{E}\exp\left(\beta\left(\frac{p!}{N^{2p-2}}\right)^{\frac{1}{2}}\sum_{\mathcal{I}\in\mathcal{P}_N}\xi_{\mathcal{I}}^{\mu}\right)\right] \\ &= \left[\mathbb{E}\exp\left(\beta\left(\frac{p!}{N^{p-2}}\right)^{\frac{1}{2}}Y\right)\right]^{M(N)}, \end{aligned} \tag{3.1}$$

where we introduced the abbreviation $Y \equiv N^{-\frac{p}{2}}\sum_{\mathcal{I}\in\mathcal{P}_N}\xi_{\mathcal{I}}^{1}$. We now expand the exponential function according to the bound $\left|e^x - 1 - x - \frac{x^2}{2}\right| < |x|^3 e^{|x|}$. Thus,

$$\begin{aligned} &\left|\mathbb{E}\left[\exp\left(\beta\left(\frac{p!}{N^{p-2}}\right)^{\frac{1}{2}}Y\right)\right] - 1 - \frac{\beta^2 N^{2-p}}{2}\right| \\ &\qquad\le \mathbb{E}\left[\beta^3\left(\frac{p!}{N^{p-2}}\right)^{\frac{3}{2}}|Y|^3\exp\left(\beta\left(\frac{p!}{N^{p-2}}\right)^{\frac{1}{2}}|Y|\right)\right] + \mathcal{O}(N^{1-p}). \end{aligned} \tag{3.2}$$

Observe that the quadratic term is in fact just N^{p-1} times the variance of H_N. We will show in a moment that the expectation on the right-hand side of (3.2) is bounded by a constant times $N^{3-\frac{3p}{2}}$. Assuming this and recalling that $p \ge 4$, it is evident that

$$\begin{aligned} \ln\mathbb{E}Z_N &= M(N)\ln\left(1 + \frac{\beta^2 N^{2-p}}{2}(1 + O(N^{-1})\right) \\ &= \frac{\alpha\beta^2 N}{2}(1 + O(N^{-1})). \end{aligned} \tag{3.3}$$

which is what we want to prove. We now turn to the non-trivial part of the proof, the estimate of the remainder on the right-hand side of (3.2). To to this, we decompose the exponent into two factors, and use on one the obvious bound $|Y| \le (p!)^{-1}N^{p/2}$. This yields

$$\begin{aligned} \mathbb{E}\left[|Y|^3\exp\left(\beta(p!)^{\frac{1}{2}}N^{\frac{2-p}{2}}|Y|\right)\right] &= \mathbb{E}\left[|Y|^3\exp\left(\beta(p!)^{\frac{1}{2}}N^{\frac{2-p}{2}}|Y|^{\frac{2}{p}}|Y|^{\frac{p-2}{p}}\right)\right] \\ &\le \mathbb{E}\left[|Y|^3\exp\left(\beta(p!)^{\frac{2}{p}-\frac{1}{2}}|Y|^{\frac{2}{p}}\right)\right]. \end{aligned} \tag{3.4}$$

The point is that the term $|Y|^{2/p}$ should behave almost like the square of a Gaussian. More precisely, we have the following bound.

Lemma 3.1: *Let $\{X_i\}_{i=1,\dots,N}$ be a sequence of i.i.d. Bernoulli variables, taking values $+1$, -1 with equal probability. Then $\forall C \in (0, e^{-1})$, there exists an $\varepsilon'_C < \infty$ (depending also on p) and an $\bar{N} \in \mathbb{N}$ such that for all $\varepsilon > \varepsilon'_C$*

$$\mathbb{P}\left[\left|N^{-p/2}\sum_{\mathcal{I}\in\mathcal{P}_N}\prod_{l\in\mathcal{I}}X_l\right| > \varepsilon\right] \le 2\exp\left(-C^2\frac{(p!)^{\frac{2}{p}}\varepsilon^{\frac{2}{p}}}{2}\right). \tag{3.5}$$

Proof: The proof is surprisingly more involved than what one might at first suspect (at least, if optimal constants are desired). We shall show that $\sum_{\mathcal{I}\in\mathcal{P}_N} X_{\mathcal{I}}$ is a function of $\sum_{i=1}^N X_i$ only. Since the distribution of this latter random variable is well known, all we have to do is to find an accurate upper bound for the function relating the two quantities. And since we are only interested in the tail behavior, we can restrict our attention to large values of the sum (large meaning at least of the order of $\sqrt{N}$).

Suppose that $\sum_{i=1}^N X_i = N-2l$. Then the quantity $\sum_{\mathcal{I}\in\mathcal{P}_N} X_{\mathcal{I}}$ is given by

$$\sum_{\mathcal{I}\in\mathcal{P}_N} X_{\mathcal{I}} = \sum_{k=0}^{p}(-1)^k\binom{l}{k}\binom{N-l}{p-k} = \lfloor z^p\rfloor\left[(1+z)^{N-l}(1-z)^l\right], \tag{3.6}$$

where $\lfloor z^p\rfloor(\cdot) \equiv \frac{1}{p!}\frac{\partial^p}{\partial z^p}\cdot\Big|_{z=0}$ is the operator which extracts the coefficient of the term z^p from a formal power series. Note that it will be important to take into account that the sum in (3.6) is oscillating to get a useful estimate. To do this, we consider the polynomial on the right-hand side of (3.6) as an analytic function $\mathbb{C}\to\mathbb{C}$ and use Cauchy's integral formula to write

$$\lfloor z^p\rfloor\left[(1+z)^{N-l}(1-z)^l\right] = \frac{1}{2\pi i}\oint_C z^{-p-1}(1+z)^{N-l}(1-z)^l\,dz, \tag{3.7}$$

for any closed path $\mathcal{C}$ surrounding the origin counterclockwise. To evaluate this integral, we apply the well known saddle point method (see for instance [CH]). We choose $\mathcal{C}$ to be a circle around the origin with radius

$$r = \frac{N-2l}{2(N-p)}\left(1-\sqrt{1-\frac{4p(N-p)}{(N-2l)^2}}\right). \tag{3.8}$$

Suppose that $\frac{4p(N-p)}{(N-2l)^2} < \kappa < 1$. Then the argument of the square root is positive. Moreover, the following bounds for r hold,

$$\frac{p}{N-2l} \le r \le \frac{p}{N-2l}(1+C_1(\kappa)), \tag{3.9}$$

where C_1 increases from zero to some finite constant as κ varies from zero to 1.

Indeed, $\sqrt{1-x}$ is C^∞ for all $|x|<1$. Therefore, for all $\kappa<1$, we can find a $C>0$ such that $\sqrt{1-x}\ge 1-\frac{x}{2}-Cx^2$, for all $|x|<\kappa$. Obviously, C tends to $\frac{1}{8}$ as κ tends to zero. This implies the upper bound. On the other hand, $\sqrt{1-x}\le 1-\frac{x}{2}$, for all $x\ge -1$, which yields the lower bound.

The contour integral in (3.7) then becomes

$$\begin{aligned} I \equiv&\frac{1}{2\pi i}\oint_C z^{-p-1}(1+z)^{N-l}(1-z)^l\,dz\\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(-ip\vartheta\ln r+(N-l)\ln(1+re^{i\vartheta})+l\ln(1-re^{i\vartheta})\right)\,d\vartheta \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{g(\vartheta)}\,d\vartheta. \end{aligned} \tag{3.10}$$

As usual, we expand the function g around its maximum (which happens to lie at $\vartheta = 0$) and try to control the error. This yields

$$
\begin{aligned}
I &= \exp\left(g(0) + \frac{(2\pi)^3}{3!} \sup_{\zeta\in[-\pi,\pi)} g^{(3)}(\zeta)\right) \int_{-\pi}^{\pi} e^{\frac{\vartheta^2}{2} g^{(2)}(0)}\, d\vartheta \\
&= r^{-p}(1+r)^{N-l}(1-r)^l \exp\left(\frac{(2\pi)^3}{3!} \sup_{\zeta\in[-\pi,\pi)} g^{(3)}(\zeta)\right) \int_{-\pi}^{\pi} e^{\frac{\vartheta^2}{2} g^{(2)}(0)}\, d\vartheta
\end{aligned}
\tag{3.11}
$$

The main contribution comes from the term $r^{-p}(1+r)^{N-l}(1-r)^l$. Using (3.9), this is bounded by

$$
\begin{aligned}
r^{-p}(1+r)^{N-l}(1-r)^l &= \exp\left(-p\ln r + (N-l)\ln(1+r) + l\ln(1-r)\right) \\
&\le \exp\left(-p\ln p + p\ln(N-2l) + (N-l)r - lr\right) \\
&\le \exp\left(-p\ln p + p\ln(N-2l) + (N-2l)r\right) \\
&\le \frac{(N-2l)^p}{p!}\sqrt{p}e^{C_1(\kappa)p}.
\end{aligned}
\tag{3.12}
$$

The integral in (3.11) is explicitly

$$
\int_{-\pi}^{\pi} e^{\frac{\vartheta^2}{2} g^{(2)}(0)}\, d\vartheta \le \int_{\mathbb{R}} \exp\left(\frac{\vartheta^2}{2}\left(\frac{lr}{(1-r)^2} - \frac{(N-l)r}{(1+r)^2}\right)\right) d\vartheta = \left(\frac{\pi}{\frac{(N-l)r}{(1+r)^2} - \frac{lr}{(1-r)^2}}\right)^{1/2}, \tag{3.13}
$$

and can be bounded by (for all N large enough)

$$
\left((N-l)\frac{r}{(1+r)^2} - l\frac{r}{(1-r)^2}\right)^{-\frac{1}{2}} \le p^{-\frac{1}{2}}\left(1 - \frac{p^2}{(N-2l)^2}\right)\left(1 - \frac{2\kappa}{3}\right). \tag{3.14}
$$

Finally, we estimate the error due to the remainder in the Taylor expansion in (3.11). One shows by a straightforward computation that for all $\kappa, \delta > 0$ there exists an $\bar{N}_{\kappa,\delta} \in \mathbb{N}$ such that

$$
|g^{(3)}(\vartheta)| \le p(1 + C_1(\kappa))\left(1 + \kappa(1 + C_1(\kappa)) + \delta\right) = pC_3(\kappa,\delta), \tag{3.15}
$$

where $C_3 = 1$ for $\kappa = \delta = 0$. Hence, the error committed can be bounded as (if $N > \bar{N}_{\kappa,\delta}$)

$$
\exp\left(\frac{(2\pi)^3}{3!} \sup_{\zeta\in[-\pi,\pi)} g^{(3)}(\zeta)\right) \le \exp\left(\frac{2\pi}{3!}p(1 + C_1(\kappa))\left(1 + \kappa(1 + C_1(\kappa)) + \frac{C_2}{N-2l}\right)\right). \tag{3.16}
$$

This follows from the exact expression for $g^{(3)}$,

$$
g^{(3)}(\vartheta) = ire^{i\vartheta}\left((N-l)\frac{re^{i\vartheta} - 1}{(1+re^{i\vartheta})^3} - l\frac{1+re^{i\vartheta}}{(re^{i\vartheta}-1)^3}\right), \tag{3.17}
$$

which one gets through straightforward derivation.

Inserting the bounds (3.12), (3.14), and (3.15) into the estimate (3.11) then gives

$$
I \le \frac{(N-2l)^p}{p!} e^{(C_1(\kappa) + C_3(\kappa,\delta))p}, \tag{3.18}
$$

and thus

$$f\left(\sum_{i\in\mathcal{N}} X_i\right) \leq \frac{1}{p!} e^{p(C_1(\kappa)+C_3(\kappa,\delta))} \left(\sum_{i\in\mathcal{I}} X_i\right)^p, \quad N \geq \bar{N}_{\kappa,\delta} \tag{3.19}$$

Let $\rho(\kappa,\delta) = e^{(C_1(\kappa)+C_3(\kappa,\delta))p}$, for $\kappa \in (0,1)$ and $\delta > 0$. Then ρ is increasing in κ and bounded below by e^p. Thus, for all $C \in (0, e^{-p})$, we can find $\tilde{\kappa} \in (0,1)$ and $\tilde{\delta} > 0$ such that $C \leq \rho(\tilde{\kappa}, \tilde{\delta})^{-1}$. Let now

$$\varepsilon_{\kappa,\delta} \equiv \left(\frac{4p}{\kappa}\right)^{p/2} \frac{\rho(\kappa,\delta)}{p!}. \tag{3.20}$$

Suppose that $\varepsilon > \varepsilon_{\tilde{\kappa},\tilde{\delta}}$ and $N \geq \bar{N}_{\tilde{\kappa},\tilde{\delta}}$. Then, we have that

$$\mathbb{P}\left[N^{-1/2}\sum_{i\in\mathcal{N}} X_i > \left(\varepsilon p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{1/p}\right] \leq \exp\left(-\frac{1}{2}\left(\varepsilon p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{2/p}\right), \tag{3.21}$$

by the standard bound on sums of Bernoulli variables. On the other hand, since

$$N^{-1/2}\sum_{i\in\mathcal{N}} X_i > \left(\varepsilon p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{1/p} > \left(\varepsilon_{\tilde{\kappa},\tilde{\delta}} p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{1/p} = \left(\frac{4p}{\tilde{\kappa}}\right)^{1/2} \tag{3.22}$$

implies that

$$\frac{4pN}{(N-2l)^2} < \tilde{\kappa} < 1, \tag{3.23}$$

the condition following (3.8) is satisfied and hence the above bound on $f(\sum_{i\in\mathcal{N}} X_i)$ is valid. Thus

$$\begin{aligned}\mathbb{P}\left[N^{-1/2}\sum_{i\in\mathcal{N}} X_i > \left(\varepsilon p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{1/p}\right] &= \mathbb{P}\left[N^{-p/2}\frac{\rho(\tilde{\kappa},\tilde{\delta})}{p!}\left(\sum_{i\in\mathcal{N}} X_i\right)^p > \varepsilon\right] \\ &\geq \mathbb{P}\left[N^{-p/2} f\left(\sum_{i\in\mathcal{N}} X_i\right) > \varepsilon\right].\end{aligned} \tag{3.24}$$

Hence, by (3.21) and (3.24),

$$\mathbb{P}\left[N^{-p/2} f(\sum_{i\in\mathcal{I}} X_i) > \varepsilon\right] \leq \exp\left(-\frac{1}{2}\left(\varepsilon p!\,\rho(\tilde{\kappa},\tilde{\delta})^{-1}\right)^{2/p}\right) \leq \exp\left(-\frac{C^{2/p}}{2}(\varepsilon p!\,)^{2/p}\right). \tag{3.25}$$

Thus, we have shown that for all $C \in (0, e^{-p})$, there exists $\tilde{\varepsilon}_C = \varepsilon_{\tilde{\kappa},\tilde{\delta}}$ such that (3.25) holds for all $\varepsilon > \tilde{\varepsilon}_C$ and all N large enough. Together with the analogue bound for the negative tails, this proves the lemma. $\square$

To finish the proof of the theorem, let us go back to (3.4). To get the claimed bound, it is enough to show that the integral on the right-hand side is bounded uniformly in N. Indeed, since the variable Y satisfies the bound (3.5) of the lemma, we get for any $C' < e^{-p}$

$$\begin{aligned}\mathbb{E}\left[|Y|^3 \exp\left(\beta(p!)^{\frac{2}{p}-\frac{1}{2}}|Y|^{\frac{2}{p}}\right)\right] &\leq \sum_{l\geq 1}\mathbb{E}\left[|Y|^3 1\!\!1_{\{|Y|\in[l,l+1)\}}\exp\left(\beta(p!)^{\frac{2}{p}-\frac{1}{2}}|Y|^{\frac{2}{p}}\right)\right] \\ &\leq (l+1)^3\mathbb{P}[|Y|\geq l]\exp\left(\beta(p!)^{\frac{2}{p}-\frac{1}{2}}(l+1)^{\frac{2}{p}}\right) \\ &\leq \int_0^\infty (x+1)^3\exp\left(\beta(p!)^{\frac{2}{p}-\frac{1}{2}}(x+1)^{\frac{2}{p}} - C'^{2/p}(p!)^{\frac{2}{p}}x^{\frac{2}{p}}\right)dx \\ &\quad + (\tilde{\varepsilon}_{p,C'}+1)^3.\end{aligned} \tag{3.26}$$

By the preceding lemma, for any $\beta \leq e^{-2}(p!)^{\frac{1}{2}}$, we can find $C' < e^{-p}$ and a corresponding $\varepsilon'_{C'}$ such that the above integral is finite. Setting $C^p = C'$, this proves the theorem. $\square$

We observe that we could have equally well replaced H_N by in $-H_N$ in the proof of Theorem 1.1, without changing the result (since only the square of the Hamiltonian does enter). We therefore have readily the following result, which we state for further use.

Corollary 3.2: *If $|\beta| < \beta'_p$, then*

$$\mathbb{E}\mathbb{E}_\sigma e^{\beta H_N} = e^{\frac{\alpha\beta^2 N}{2}(1+\mathcal{O}(N^{-1}))}. \tag{3.27}$$

Proof: Completely analogous to the proof of Theorem 1.1. $\square$

We also put a result here, that will be used in the next chapter, but whose proof is very similar to the above.

Lemma 3.3: *If $|\beta| < \frac{1}{2}\beta'_p$, then there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[e^{-\beta H_N(\sigma)-\beta H_N(\sigma')}\right] \leq e^{\alpha N\beta^2(1+R(\sigma,\sigma')^p+C)}, \tag{3.28}$$

for all N large enough.

Proof: The proof is actually almost identical to the proof of Theorem 1.1. We start by expanding the exponential up to order two, with the same error as in the proof of Theorem 1.1 (inequality (3.2)). This error term is then treated similarly, by first decoupling the terms in σ and σ' with Cauchy-Schwarz. This already shows why β has to be less than half the bound of Theorem 1.1. The linear term in the expansion vanishes, whereas the quadratic term gives us the covariance term $R(\sigma,\sigma')^p$. Indeed, if we set $Y^\mu(\sigma) = N^{-p/2}\sum_{\mathcal{I}\subset\mathcal{N}} \xi^\mu_{\mathcal{I}}\sigma_{\mathcal{I}}$, we get

$$\begin{aligned}
&\ln\mathbb{E}\left[\exp(-\beta H_N(\sigma) - \beta H_N(\sigma'))\right] \\
&\leq \sum_{\mu=1}^{M(N)} \ln\Bigg(1 + \frac{\beta^2 p!}{2}N^{2-p}\mathbb{E}\left[\left(Y^\mu(\sigma)+Y^\mu(\sigma')\right)^2\right] \\
&\quad + \frac{\beta^3(p!)^{\frac{3}{2}}}{3}N^{3-\frac{3p}{2}}\mathbb{E}\left[|Y^\mu(\sigma)+Y^\mu(\sigma')|^3\exp\left(\beta(p!)^{\frac{1}{2}}N^{1-\frac{p}{2}}|Y^\mu(\sigma)+Y^\mu(\sigma')|\right)\right]\Bigg).
\end{aligned} \tag{3.29}$$

We now apply the triangle inequality and Cauchy-Schwarz to the error term, which yields

$$\begin{aligned}
&N^{3-\frac{3p}{2}}\mathbb{E}\left[|Y^\mu(\sigma)+Y^\mu(\sigma')|^3 e^{\beta(p!)^{1/2}N^{1-p/2}|Y^\mu(\sigma)+Y^\mu(\sigma')|}\right] \\
&\leq N^{3-\frac{3p}{2}}\sum_{i=1}^{3}\left(\mathbb{E}\left[|Y^\mu(\sigma)|^{2j}\exp\left(2\beta(p!)^{\frac{1}{2}}N^{1-\frac{p}{2}}|Y^\mu(\sigma)|\right)\right]\right)^{\frac{1}{2}} \\
&\quad\times\left(\mathbb{E}\left[|Y^\mu(\sigma')|^{6-2j}\exp\left(2\beta(p!)^{\frac{1}{2}}N^{1-\frac{p}{2}}|Y^\mu(\sigma')|\right)\right]\right)^{\frac{1}{2}} \\
&\leq C_1 N^{3-\frac{3p}{2}},
\end{aligned} \tag{3.30}$$

if $\beta < \frac{1}{2}\beta_p'$ and N large enough, by the result in the proof of Theorem 1.1 (cf. the remark after (3.2)).

The quadratic term in (3.29) is evaluated easily. One obtains (observing that the covariance of H_N appears)

$$\begin{aligned}\mathbb{E}\left[(-Y^\mu(\sigma)-Y^\mu(\sigma'))^2\right] &= 2\mathbb{E}[Y^\mu(\sigma)^2]+2\mathbb{E}[Y^\mu(\sigma)Y^\mu(\sigma')]\\ &= 2N^{-p}\binom{N}{p}+2N^{-p}\sum_{\mathcal{I}\subset\mathcal{N}}\sigma_{\mathcal{I}}\sigma'_{\mathcal{I}}\\ &= \frac{2}{p!}(1+R(\sigma,\sigma')^p)+\mathcal{O}(N^{-1}).\end{aligned} \tag{3.31}$$

Hence,

$$\begin{aligned}\ln\mathbb{E}e^{-\beta(H_N(\sigma)+H_N(\sigma'))} &\le \sum_{\mu=1}^{M(N)}\ln\left(1+\frac{\beta^2}{N^{p-2}}(1+R(\sigma,\sigma')^p)+\frac{C_2}{N^{p-1}}+\frac{C_1}{N^{\frac{3p}{2}-3}}\right)\\ &\le M(N)(\beta^2N^{2-p}(1+R(\sigma,\sigma')^p)+C_3N^{1-p}),\end{aligned} \tag{3.32}$$

that is,

$$\mathbb{E}e^{-\beta H_N(\sigma)-\beta H_N(\sigma')}\le e^{\alpha\beta^2N(1+R(\sigma,\sigma')^p)+C_4}. \tag{3.33}$$

This proves the lemma. $\square$

Finally, we have as an application of Corollary 3.2.

Lemma 3.4: *The Hamiltonian satisfies*

$$\mathbb{P}\left[\sup_\sigma|H_N(\sigma)|>tN\right]\le C\begin{cases}\exp\left(-N(\frac{t^2}{2\alpha}-\ln 2)\right), & \text{if } t\le\frac{\alpha(p!)^{\frac12}}{e^2},\\ \exp\left(-N(\frac{(p!)^{\frac12}}{e^2}t-\frac{\alpha p!}{2e^4}-\ln 2)\right), & \text{otherwise.}\end{cases} \tag{3.34}$$

Proof: We start with a crude bound to extract the supremum. Standard arguments and Chebyshev's inequality in its exponential form yield

$$\mathbb{P}[\sup_\sigma|H_N(\sigma)|>tN]\le 2^N\inf_{q>0}e^{-qtN}\mathbb{E}e^{qH_N(\sigma)}+2^N\inf_{q>0}e^{-qtN}\mathbb{E}e^{-qH_N(\sigma)}. \tag{3.35}$$

We now use Theorem 1.1, respectively Corollary 3.2 to bound the two integrals and obtain

$$\begin{aligned}\mathbb{P}\left[\sup_\sigma|H_N(\sigma)|>tN\right] &\le C_12^{N+1}\inf_{q\in(0,\beta_p')}e^{-qtN}e^{\frac{\alpha q^2N}{2}}\\ &= C_2\begin{cases}\exp\left(-N(\frac{t^2}{2\alpha}-\ln 2)\right), & \text{if } t\le\frac{\alpha(p!)^{\frac12}}{e^2},\\ \exp\left(-N(\frac{(p!)^{\frac12}}{2e^2}t-\frac{\alpha p!}{2e^4}-\ln 2)\right), & \text{otherwise.}\end{cases}\end{aligned} \tag{3.36}$$

This proves the lemma. $\square$

4. Critical β and Convergence to the REM

4.1. Estimates on the Truncated Partition Function.

To get the lower bound for the critical temperature, we would like to compare $\mathbb{E}Z_{N,\beta}^2$ and $(\mathbb{E}Z_{N,\beta})^2$. However, as mentioned in the introduction and explained in Chapter 2 it is essential to do this comparison for a truncated partition function. Define therefore

$$\widetilde{Z}_{N,\beta}(c) \equiv \mathbb{E}_\sigma \left[e^{-\beta H_N[\omega](\sigma)} \mathbb{1}_{\{-H_N(\sigma)\leq c\alpha\beta N\}} \right], \tag{4.1}$$

for $c > 1$. The key observation is that the truncation has no influence on the expectation of the partition function if c is chosen appropriately. This is the content of the following lemma.

Lemma 4.1: *For all $\beta > 0$, $c > 1$ such that $\beta c < \beta_p'$ there exist $K, K' > 0$ such that*

$$\mathbb{E}\widetilde{Z}_{N,\beta}(c) \left(1 - K e^{-K'(c-1)^2 N}\right) \mathbb{E}Z_{N,\beta}. \tag{4.2}$$

Proof: Let us set $q = q(N) \equiv \alpha\beta^2 N$. Note that $\mathbb{E}Z_{N,\beta} - \mathbb{E}\widetilde{Z}_{N,\beta} = \mathbb{E}\left[e^{-\beta H_N(\sigma)} \mathbb{1}_{\{-\beta H_N(\sigma) > cq\}}\right]$ and thus by the exponential Chebyshev inequality

$$\mathbb{E}Z_{N,\beta} - \mathbb{E}\widetilde{Z}_{N,\beta} \leq \mathbb{E}_\sigma \inf_{t>0} e^{-tcq} \mathbb{E}\left[e^{-\beta(1+t)H_N(\sigma)}\right]. \tag{4.3}$$

We now use Theorem 1.1 with β replaced by $(1+t)\beta$ to estimate the expectation to get

$$\inf_{t>0} e^{-tcq} \mathbb{E}\left[e^{-\beta(1+t)H_N(\sigma)}\right] \leq \inf_{0<t\leq\beta/\beta_p'-1} e^{-tcq + \frac{(1+t)^2 q}{2} + qCN^{-1}}. \tag{4.4}$$

The exponent is minimized for $t = c - 1$. By assumption, $\beta c < \beta_p'$, so that this value falls into the interval over which the inf on the right is taken. Thus,

$$\inf_{t>0} e^{-tcq} \mathbb{E}\left[e^{-\beta(1+t)H_N(\sigma)}\right] \leq e^{-\frac{q}{2}(c-1)^2 + CqN^{-1}} e^{\frac{q}{2}} \leq e^{-\frac{q}{2}(c-1)^2 + CqN^{-1}} \mathbb{E}Z_{N,\beta}, \tag{4.5}$$

This implies the statement of the lemma. $\square$

We now turn to the square of the truncated partition function. We bound

$$\mathbb{E}\widetilde{Z}_{N,\beta}^2 \mathbb{E}e^{-\beta H(\sigma) - \beta H(\sigma')} \mathbb{1}_{\{-H_N(\sigma)\leq c\alpha\beta N\}} \mathbb{1}_{\{-H_N(\sigma')\leq c\alpha\beta N\}} \tag{4.6}$$

by two different functions. When calculating the expectation with respect to σ and σ', we use one bound for small values of the replica overlap $R(\sigma, \sigma')$, and the other for the rest. Define therefore

$$S(b) \equiv \mathbb{E}_{\sigma,\sigma'}\left[e^{-\beta(H_N(\sigma)+H_N(\sigma'))} \mathbb{1}_{\{|R(\sigma,\sigma')|<b\}}\right] \tag{4.7}$$

and

$$T(c, b, b') \equiv \mathbb{E}_{\sigma,\sigma'}\left[e^{-\beta(H_N(\sigma)+H_N(\sigma'))} \mathbb{1}_{\{|R(\sigma,\sigma')|\in[b,b']\}} \mathbb{1}_{\{-\beta(H_N(\sigma)+H_N(\sigma'))<2c\alpha\beta^2 N\}}\right]. \tag{4.8}$$

Then

$$\widetilde{Z}_{N,\beta}(c)^2 \leq S(b) + T(c,b,1), \tag{4.9}$$

for all $b \in (0,1)$. We now control each of the terms separately. We start with $S(b)$.

Lemma 4.2: *Suppose $\beta < \frac{\beta_p'}{2}$, and b is such that*

$$\gamma \equiv \alpha\beta^2 b^{p-2} < \frac{1}{2}. \tag{4.10)}$$

Then for all $\varepsilon \in (0, \frac{1}{2} - \gamma)$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $N > N_\varepsilon$,

$$\mathbb{E}S(b) \leq \frac{1}{\sqrt{1 - 2(\gamma + \varepsilon)}} e^{\alpha\beta^2 N}. \tag{4.11}$$

Proof: If β satisfies the above condition, we can apply Lemma 3.3 to the integrand of the right-hand side of (4.7). One obtains

$$\mathbb{E}\left[e^{-\beta(H_N(\sigma)+H_N(\sigma'))}\mathbb{1}_{\{|R(\sigma,\sigma')|<b\}}\right] \leq \mathbb{1}_{\{|R(\sigma,\sigma')|<b\}} e^{\alpha\beta^2 N(1+(R(\sigma,\sigma')^p)+CN^{-1})}. \tag{4.12}$$

Thus,

$$\begin{aligned} \mathbb{E}S(b) &\leq \mathbb{E}_{\sigma,\sigma'}\left[e^{\alpha\beta^2 N(1+R(\sigma,\sigma')^p+CN^{-1})}\mathbb{1}_{\{|R(\sigma,\sigma')|<b\}}\right] \\ &\leq \mathbb{E}_{\sigma,\sigma'}\left[e^{\alpha\beta^2 N(1+R(\sigma,\sigma')^2 b^{p-2}+CN^{-1})}\mathbb{1}_{\{|R(\sigma,\sigma')|<b\}}\right] \\ &= e^{\alpha\beta^2 N}\mathbb{E}_{\sigma,\sigma'}\left[e^{\alpha\beta^2 N(R(\sigma,\sigma')^2 b^{p-2}+CN^{-1})}\right] \\ &\leq \sum_{k=N/2-[bN]}^{N/2+[bN]} 2^{-N}\binom{N}{k} e^{(\gamma+\epsilon)Nk^2}. \end{aligned} \tag{4.13}$$

by assumption (4.10), for any $\epsilon > 0$, if N is large enough. Standard estimates then yield (4.11).$\square$

The next result concerns the term $T(c,b,1)$ in (4.9).

Lemma 4.3: *Let $I(t)$ be the Cramèr Entropy as defined in (1.17). Suppose that there exist $c > 1$, $d > 0$, such that*

$$\forall t \in [b,b'], \quad 2\alpha\beta^2 c\Big(1 - \frac{c}{2(1+t^p)}\Big) \leq \alpha\beta^2 + I(t) - d. \tag{4.14}$$

Then, if

$$c < \min\big(\frac{1}{2\beta}\beta_p', 1 + b^p\big), \tag{4.15}$$

there exists $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$,

$$\mathbb{E}T(c,b,1) \leq e^{\alpha\beta^2 N} e^{-\frac{Nd}{2}}. \tag{4.16}$$

Proof: By definition,

$$\mathbb{E}T(c,b,b') = \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left[e^{-\beta(H(\sigma)+H(\sigma'))}\mathbb{1}_{\{|R(\sigma,\sigma')|\in[b,b']\}}\mathbb{1}_{\{-\beta(H_N(\sigma)+H_N(\sigma'))\leq 2c\alpha\beta^2 N\}}\right]. \tag{4.17}$$

In a first step, we bound the expectation over the disorder for fixed σ, σ'. Similar to the proof of Lemma 4.1 we get (again $q \equiv \alpha\beta^2 N$), yields

$$\mathbb{E}\left[\mathbb{1}_{\{-\beta(H_N(\sigma)+H_N(\sigma'))\leq 2cq\}}e^{-\beta(H_N(\sigma)+H_N(\sigma'))}\right] \\ \leq \inf_{t>0} e^{2tcq}\mathbb{E}\left[e^{-\beta(1-t)(H_N(\sigma)+H_N(\sigma'))}\right] \tag{4.18}$$

We now use Lemma 3.3, with β replaced by $\beta(1-t)$ to obtain

$$\inf_{t>0} e^{2tcq}\mathbb{E}\left[e^{-\beta(1-t)(H_N(\sigma)+H_N(\sigma'))}\right] \leq \inf_{t>1-\beta_p'/(2\beta)} e^{2tcq}e^{(1-t)^2 q(1+R(\sigma,\sigma')^p)}e^{C_2 N^{-1}q}. \tag{4.19}$$

The infimum is attained for $t = 1 - \frac{c}{(1+R^p)} > 1 - \frac{\beta_p'}{2\beta}$ (by assumption (4.15). Thus

$$\mathbb{E}\left[e^{-\beta H_N(\sigma)-\beta H_N(\sigma')}\mathbb{1}_{\{-\beta H_N(\sigma)-\beta H_N(\sigma')\leq 2c\alpha\beta N\}}\right] \\ \leq C_3 \exp\left(2c\alpha\beta^2 N\left(1 - \frac{c}{2(1+R(\sigma,\sigma')^p)}\right)\right). \tag{4.20}$$

Finally, we integrate over all configurations σ, σ' satisfying $|R(\sigma,\sigma')| \in [b,b']$. We observe that $R(\sigma,\sigma')$ has the same distribution as $S(\sigma) = N^{-1}\sum_{i=1}^N \sigma_i$. Hence,

$$\begin{aligned} \mathbb{E}\left[T(c,b,b')\right] &\leq C_3\mathbb{E}_{\sigma,\sigma'}\left[\exp\left(2c\alpha\beta^2\left(1-\frac{c}{2(1+R(\sigma,\sigma')^p)}\right)\right)\mathbb{1}_{\{|R(\sigma,\sigma')|\in[b,b']\}}\right] \\ &= C_3\mathbb{E}_{\sigma}\left[\exp\left(2c\alpha\beta^2 N\left(1-\frac{c}{2(1+S(\sigma)^p)}\right)\right)\mathbb{1}_{\{|S(\sigma)|\in[b,b']\}}\right] \\ &\leq 2C_3 N\exp\left(N\sup_{t\in[b,b']}\left[2\alpha\beta^2 c\left(1-\frac{c}{2(1+t^p)}\right) - I(t)\right]\right) \\ &\leq 2C_3 e^{N(\alpha\beta^2+\frac{\ln N}{N}-d)} \leq e^{N(\alpha\beta^2-\frac{d}{2})}. \end{aligned} \tag{4.21}$$

The second to last inequality follows from the hypothesis of the lemma, and the observation that we sum over at most $2N$ values of $S(\sigma)$. The last inequality holds for all N larger than a certain $\bar{N} \in \mathbb{N}$. Since this estimate is uniform in b', we may choose $b' = 1$. □

From the preceding results, we now get a variance estimate for the truncated partition function.

Proposition 4.4: *Suppose that $\beta < \check{\beta}_p$. Then there exist constants $C > 0$ and $c > 1$ such that*

$$\mathbb{E}[\widetilde{Z}_{N,\beta}(c)^2] \leq C(\mathbb{E}\widetilde{Z}_{N,\beta}(c))^2. \tag{4.22}$$

and,

$$\mathbb{P}[\widetilde{Z}_{N,\beta}(c) > \frac{1}{2}\mathbb{E}\widetilde{Z}_{N,\beta}(c)] \geq \frac{3}{4C}. \tag{4.23}$$

Proof: We first prove that the hypothesis implies that the assumptions of Lemmas 4.1–4.3 are satisfied. Consider therefore $\beta < \frac{1}{2}\beta_p'$ such that

$$\beta^2 < \inf_{0\leq t\leq 1} I(t)\frac{1+t^p}{\alpha t^p}. \tag{4.24}$$

Then it is immediate that

$$2\alpha\beta^2\left(1-\frac{1}{2(1+t^p)}\right) = 2\alpha\beta^2\frac{1+2t^p}{2(1+t^p)} < \alpha\beta^2 + I(t), \tag{4.25}$$

for all $t\in[0,1]$. By continuity, there exist $c^* > 1$ and $d^* > 0$ such that $\forall c \in (1,c^*)$ and $d\in(0,d^*)$

$$2c\alpha\beta^2\left(1-\frac{c}{2(1+t^p)}\right) < \alpha\beta^2 + I(t) - d, \quad \forall t\in[0,1]. \tag{4.26}$$

This implies the hypothesis of Lemma 4.3.

We now show that $(\mathbb{E}[\widetilde{Z}_N])^2$ is of the order of $\mathbb{E}[\widetilde{Z}_N{}^2]$. We start by fixing the free parameters b, b', and c. Choose first b such that $\gamma(b) = \frac{1}{4}$ (or any other constant less than one half). Then choose c such that

$$c < \min\left(c^*, \frac{\beta_p'}{2\beta}, 1+b^p\right). \tag{4.27}$$

Then the hypotheses of all preceding lemmas are fulfilled. Finally, choose $b' = 1$. By Lemmas 4.2 and 4.3, we then have

$$\mathbb{E}\left[\widetilde{Z}_N{}^2\right] \leq \mathbb{E}[S(b)+T(c,b,1)] \leq (C_1 + e^{-Nd/2})e^{\alpha\beta^2 N}. \tag{4.28}$$

The right-hand side is by Theorem 1.1 bounded by

$$(C_1 + e^{-Nd/2})e^{\alpha\beta^2 N} \leq 2C_2\Big(\mathbb{E}[Z_N]\Big)^2, \tag{4.29}$$

which in turn is of the order of $(\mathbb{E}[\widetilde{Z}_N])^2$ by Lemma 4.1, so that

$$(C_1 + e^{-Nd/2})e^{\alpha\beta^2 N} \leq C_3\Big(\mathbb{E}[\widetilde{Z}_N]\Big)^2. \tag{4.30}$$

This implies (4.22). The second assertion of the proposition follows from the Paley-Zygmund inequality, which states that for a positive random variable Y and any positive constant g,

$$\mathbb{P}\Big[Y \geq g\mathbb{E}Y\Big] \geq (1-g)^2\frac{(\mathbb{E}Y)^2}{\mathbb{E}[Y^2]}. \tag{4.31}$$

This relation gives us a lower bound on the probability that $\widetilde{Z}_N \geq g\mathbb{E}[\widetilde{Z}_N]$, which is strictly greater than zero and uniform in N. Indeed, if we set $g = \frac{1}{2}$ in (4.31), then, by (4.22), we get

$$\mathbb{P}[\widetilde{Z}_N \geq \frac{1}{2}\mathbb{E}\widetilde{Z}_N] \geq \frac{1}{2C_3}. \tag{4.32}$$

This concludes the proof of the proposition. □

4.2. Proof of the Lower Bound.

We will now proof the lower bound assuming that Theorem 1.4 holds. This is by now quite standard [T1,T2,T3], but we repeat the argument for the reader's convenience. Note that by Lemma 4.1 for N large enough, for any $\delta > 0$,

$$\mathbb{P}\big[\widetilde{Z}_N \geq \frac{1}{2}\mathbb{E}\widetilde{Z}_N\big] \leq \mathbb{P}\left[Z_N \geq \frac{1}{2}(1-\delta)\mathbb{E}Z_N\right]. \tag{4.33}$$

But

$$\mathbb{P}\left[Z_N \geq \frac{1}{2}(1-\delta)\mathbb{E}Z_N\right] = \mathbb{P}\left[F_N - \mathbb{E}F_N \geq N^{-1}(\ln \mathbb{E}Z_N - \mathbb{E}\ln Z_N - \ln\left(\frac{1}{2}(1-\delta)\right)\right]. \tag{4.34}$$

But Theorem 1.4 implies that this quantity is smaller than B^{-n}, if

$$N^{-1}(\ln \mathbb{E}Z_N - \mathbb{E}\ln Z_N) \geq N^{-1/2+\epsilon} \tag{4.35}$$

in contradiction to the lower bound (4.32). This proves that for $\beta < \check{\beta}$,

$$\lim_{N\uparrow\infty} N^{-1}(\ln \mathbb{E}Z_N - \mathbb{E}\ln Z_N) = 0 \tag{4.36}$$

proving the lower bound on β_p. $\square$

Remark: It should be noted that the above argument requires only an upper deviation inequality for the free energy. Such an inequality can be obtained in a much simpler way than Theorem 1.4 (in that it does not require the results of Section 5) on the basis of a result of Ledoux [Le]. The reason is that while the free energy is not a convex function of all the disorder variables, it is separately convex in each ξ_i^μ. This suffices to apply Ledoux's theorem. A proof of the corresponding one-sided inequality can be found in [Ni].

4.3. Upper Bound on the Critical β.

The proof of the upper bound in Theorem 1.2 is considerably simpler than the lower bound. By (2.1), $\mathbb{E}\,\frac{\partial F_N}{\partial\beta} \leq N^{-1}\mathbb{E}\sup_\sigma |H_N(\sigma)|$, while Lemma 3.4 yields immediately (see the argument leading to (2.3)) that:

Lemma 4.5: *There exists $C < \infty$, such that: If $\alpha \geq \frac{8\ln 2}{p!}$, then*

$$\mathbb{E}\sup_\sigma |H_N(\sigma)| \leq NB_\alpha + C \tag{4.37}$$

where

$$B_\alpha = \begin{cases} \sqrt{2\alpha\ln 2}, & if \alpha \geq \frac{e^4 2\ln 2}{p!} \\ \frac{\alpha\sqrt{p!}}{2e^2} + \frac{e^2\ln 2}{\sqrt{p!}}, & if 0 \leq \alpha \leq \frac{e^4 2\ln 2}{p!} \end{cases} \tag{4.38}$$

Let $\beta_\infty \equiv B_\alpha/\alpha$ and assume that $\alpha \geq \alpha_p$. Now assume that $\beta_p > \beta_\infty$. Then for $\beta_\infty < \beta < \beta_p$, we have that

$$
\begin{aligned}
\limsup_{N\uparrow\infty} \mathbb{E}F_N(\beta) &\leq \limsup_{N\uparrow\infty} \mathbb{E}F_N(\beta_\infty) + +(\beta-\beta_\infty)B_\alpha \\
&= \frac{\alpha\beta_\infty^2}{2} - (\beta-\beta_\infty)\alpha\beta_\infty = \frac{\alpha\beta^2}{2} - \frac{(\beta-\beta_\infty)^2}{2} < \frac{\alpha\beta^2}{2}
\end{aligned}
\tag{4.39}
$$

in contradiction to the assumption that $\beta < \beta_p$. Thus $\beta_p \leq \beta_\infty$ which proves the upper bound (1.19). $\square$

4.4. Convergence to the REM: Proof of Theorem 1.3.

The convergence of the free energy as $p \uparrow \infty$ follows now from a simple convexity argument. Note that for all $\beta < \check{\beta}_p$, $\lim_{N\uparrow\infty} \mathbb{E}F_{N,\beta} = f_b^{REM}$, while for all $\beta > \hat{\beta}_p$, by convexity of $F_{N,\beta}$,

$$
\liminf_{N\uparrow\infty} \mathbb{E}F_{N,\beta} \geq \liminf_{N\uparrow\infty} \mathbb{E}F_{N,\hat{\beta}} + \alpha\hat{\beta}_p(\beta-\hat{\beta}_p) = \frac{\alpha\hat{\beta}^2}{2} + \alpha\hat{\beta}(\beta-\hat{\beta})
\tag{4.40}
$$

while on the other hand

$$
\limsup_{N\uparrow\infty} \mathbb{E}F_{N,\beta} \leq \liminf_{N\uparrow\infty} \mathbb{E}F_{N,\hat{\beta}} + \alpha\check{\beta}_p(\beta-\hat{\beta}_p) = \frac{\alpha\hat{\beta}^2}{2} + \alpha\check{\beta}_p(\beta-\hat{\beta}_p)
\tag{4.41}
$$

provided p is large enough such that $\alpha > \alpha_p$. But since $\lim_{p\uparrow\infty} \check{\beta}_p = \lim_{p\uparrow\infty} \hat{\beta}_p$, the two bounds above both converge to f_β^{REM}, as $p \uparrow \infty$, for any $a > 0$. This proves Theorem 1.2.$\square$

5. Fluctuations: Proof of Theorem 1.4

The main line of reasoning of the proof of the fluctuation theorem is as follows. First, for each N we define a set whose complement has a very small probability (of the order of N^{-n}). On this set, we prove the estimates on the deviation with the so-called Yurinskii martingale method [Yu]. On the complement, we simply use that the free energy is bounded by a polynomial function. This approach was first used in the context of the mean field model in [PS,ST] for variance estimates and in [BGP2,B1] for exponential inequalities, but has later been made obsolete by new concentration of measure inequalities provided by Talagrand in [T1]. Unfortunately, these require convexity of the level sets of the random functions considered which in the current situation do not appear to hold. Although, as remarked at the end of Section 4, the hypotheses of Ledoux's inequalities from [Le] do hold, these provide only one-sided deviation estimates which will not be sufficient for our later purposes. In this situation the return to Yurinskii's method appears to be the only way out.

Define the decreasing sequence of σ-algebras $\mathcal{F}_k = \sigma(\{\xi_i^\mu\}_{i\geq k}^{\mu\in\mathbb{N}}$. Furthermore, for $c,\gamma > 0$ and $k \in \mathcal{N}$, let

$$
\mathcal{A}_k = \mathcal{A}_{k,c,\gamma,N} \equiv \left\{\omega \in \Omega : |\mathcal{G}_{N,\beta}\left[H_k^\mu(\sigma)\right]| < cN^{-1+\gamma}\right\}
\tag{5.1}
$$

where

$$H_k^\mu(\sigma) \equiv -\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\mu=1}^{M(N)} \sum_{\substack{\mathcal{I}\ni k\\ |\mathcal{I}|=p}} \xi_{\mathcal{I}}^\mu \sigma_{\mathcal{I}}. \tag{5.2}$$

We put and $\mathcal{A} \equiv \mathcal{A}_{c,\gamma,N} \equiv \bigcap_{k=1}^N \mathcal{A}_k$. The set $\mathcal{A}$ will be our 'good' set. We first show that its measure is large.

Lemma 5.1: *For all $\gamma, c, m > 0$, there exists $C > 0$, such that*

$$\mathbb{P}[\mathcal{A}_{c,\gamma,N}] \geq 1 - CN^{-m}. \tag{5.3}$$

Proof: Since $\mathbb{P}[\mathcal{A}^c] \leq \sum_{k=1}^N \mathbb{P}[\mathcal{A}_k{}^c]$ we only need to show that for each k, $\mathbb{P}[\mathcal{A}_k{}^c] \leq CN^{-m}$, for any m. By the definition of the sets $\mathcal{A}_k$, Chebyshev's inequality and Jensen's inequality, we have, for any $l \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}[\mathcal{A}_k{}^c] = \mathbb{P}\left[\left|\mathcal{G}\Big[\sum_{\mu=1}^{M(N)} H_k^\mu\Big]\right| \geq cN^\gamma\right] &\leq (cN^\gamma)^{-2l}\mathbb{E}\left(\mathcal{G}\Big[\sum_{\mu=1}^{M(N)} H_k^\mu\Big]\right)^{2l} \\ &\leq (cN^\gamma)^{-2l}\mathbb{E}\mathcal{G}\left[\Big(\sum_{\mu=1}^{M(N)} H_k^\mu\Big)^{2l}\right].\end{aligned} \tag{5.4}$$

If we can show that the expectation on the right-hand side is bounded by some N-independent constant, (5.4) will prove the lemma.

Expanding the power in the integrand yields, with the usual multi-index notation,

$$\mathbb{E}\mathcal{G}\left[\Big(\sum_{\mu=1}^{M(N)} H_k^\mu\Big)^{2l}\right] = \sum_{r:|r|=2l} c_{2l,r}\mathbb{E}\mathcal{G}\left[\prod_{\mu=1}^{M(N)} (H_k^\mu)^{r_\mu}\right], \tag{5.5}$$

where r is a multi-index and the numbers $c_{2l,r}$ are the multinomial coefficients. The main point in what follows is the realisation that the difficult terms are those which have at least one μ with $r_\mu = 1$. This is due to the following observation, which is a simple consequence of a result proven in [Ni2].

Lemma 5.2: *There exist constants $c, K > 0$ such that for all N large enough,*

$$\sup_{\sigma\in\mathcal{S}_N} \sum_{\mu=1}^{M(N)} (H_k^\mu(\sigma))^2 \leq c \tag{5.6}$$

with probability at least $1 - e^{-KN^{1/4}}$.

Proof: We write the left-hand side of (5.6) as

$$\sum_{\mu=1}^{M(N)} (H_k^\mu(\sigma))^2 = \frac{p!}{N^{2p-2}} \sum_{\mu=1}^{M(N)} \sum_{\mathcal{I},\mathcal{J}\ni k} \sigma_{\mathcal{I}}\xi_{\mathcal{I}}^\mu\xi_{\mathcal{J}}^\mu\sigma_{\mathcal{J}} = \frac{\alpha p!}{N^{p-1}} \sum_{\mathcal{I},\mathcal{J}\ni k} \sigma_{\mathcal{I}} \sum_\mu \frac{\xi_{\mathcal{I}}^\mu\xi_{\mathcal{J}}^\mu}{\alpha N^{p-1}}\sigma_{\mathcal{J}}. \tag{5.7}$$

Consider σ as a vector in an $\binom{N-1}{p-1}$ dimensional space, and $\alpha^{-1}N^{1-p}\sum_\mu \xi^\mu_{\mathcal{I}}\xi^\mu_{\mathcal{J}}$ as the coefficients of a matrix P representing a map from this space onto itself. Then, denoting by λ_{max} the operator norm of P, uniformly in σ,

$$\sum_{\mu=1}^{M(N)} (H^\mu_k(\sigma))^2 = \frac{\alpha p!}{N^{1-p}}(\sigma, P\sigma) \leq \frac{\alpha p!}{N^{1-p}}\|\sigma\|_2^2\lambda_{max} = \frac{\alpha p!}{N^{1-p}}\binom{N-1}{p-1}\lambda_{max} \leq \alpha p \lambda_{max}. \tag{5.8}$$

In [Ni2, Theorem 2] it is shown that λ_{max} is bounded by a constant with probability at least $1 - e^{-KN^l}$ with $l \in (0, \frac{1}{3})$. This proves the lemma. $\square$

Returning to (5.5), we will try to get only terms of the form bounded by the lemma above, the idea being that we do not really want to integrate, but rather use a uniform bound for the integrands. We therefore single out those μ's for which $r_\mu = 1$. We obtain

$$\mathbb{E}\mathcal{G}\Big[\Big(\sum_{\mu=1}^{M(N)} H^\mu_k\Big)^{2l}\Big] = \sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\ |\mathcal{J}|\leq 2l}} \sum_{\substack{r: r\lhd\mathcal{J}\\ |r|=2l}} c_{2l,r}\mathbb{E}\mathcal{G}\left[\prod_{\mu\in\mathcal{J}} H^\mu_k \prod_{\mu\in\mathcal{M}\setminus\mathcal{J}} (H^\mu_k)^{r_\mu}\right], \tag{5.9}$$

where the compatibility relation $r \lhd \mathcal{J}$ means that for all $\mu \in \mathcal{J}$, $r_\mu = 1$. Since the $\mu \in \mathcal{M}\setminus\mathcal{J}$ will not enter in any of the calculations that follow, we write (the relation $r \prec \mathcal{J}$ now denotes the condition that $\forall\mu \in \mathcal{J}$, $r_\mu = 0$)

$$\begin{aligned} I = \mathbb{E}\mathcal{G}\Big[\Big(\sum_{\mu=1}^{M(N)} H^\mu_k\Big)^{2l}\Big] &= \sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\ |\mathcal{J}|\leq 2l}} \sum_{\substack{r: r\prec\mathcal{J}\\ |r|=2l-|\mathcal{J}|}} c_{2l,q,|\mathcal{J}|}\mathbb{E}\mathcal{G}\left[\prod_{\mu\in\mathcal{J}} H^\mu_k \prod_{\mu\in\mathcal{M}\setminus\mathcal{J}} (H^\mu_k)^{r_\mu}\right] \\ &= \sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\ |\mathcal{J}|\leq 2l}} \mathbb{E}\mathcal{G}\Big[\prod_{\mu\in\mathcal{J}} H^\mu_k \underbrace{\sum_{\substack{r: r\prec\mathcal{J}\\ |r|=2l-|\mathcal{J}|}} c_{2l,q,|\mathcal{J}|} \prod_{\mu\in\mathcal{M}\setminus\mathcal{J}} (H^\mu_k)^{r_\mu}}_{\mathcal{L}_{\mathcal{J}}(\sigma)}\Big]. \end{aligned} \tag{5.10}$$

At this point, we expand recursively the Boltzmann weights with respect to the terms H^μ_k, $\mu \in \mathcal{J}$. This will generate new terms which are slightly more complicated than the term we started with. The procedure stops when no H^μ_k is left to expand in. In particular, since $|\mathcal{J}|$ does not depend on N, this will ensure that none of the appearing factors will depend on N.[11]

We use the following notation. We order the set $\mathcal{J}$ in the canonical way, i.e. $\mathcal{J} = \{\mu_1, \dots, \mu_n\}$, with $i < j \Rightarrow \mu_i < \mu_j$. Then, we define interpolating Hamiltonians (they will reappear later)

$$H^{\mu_1,\dots,\mu_n}_{u_1,\dots,u_n}(\sigma) = H(\sigma) - \sum_{i=1}^n (1-u_i)H^{\mu_i}_k(\sigma). \tag{5.11}$$

In particular, $H = H^{\mu_1,\dots,\mu_n}_{1,\dots,1}$, and if $u_j = 0$, then $H^{\mu_1,\dots,\mu_n}_{u_1,\dots,u_n}$ is independent of $\xi^{\mu_j}_k$. The associated Gibbs measures and partition functions will be denoted by $\mathcal{G}^{\mu_1,\dots,\mu_n}_{u_1,\dots,u_n}$, respectively $Z^{\mu_1,\dots,\mu_n}_{u_1,\dots,u_n}$.

[11] One may ask why we do not expand jointly in all the patterns $\mu \in \mathcal{J}$ at once. It turns out that one needs a similar recursive scheme since there will always be error terms which cannot be treated by Lemma 5.2.

The terms that will appear are of the form

$$\mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_{n'}\otimes q}_{u_1,\dots,u_{n'}}\left[\prod_{i=n'+1}^{n} H_k^{\mu_i}(\sigma^1)\prod_{i=1}^{n'} H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi_{n'}(i)})\mathcal{L}_{\mathcal{J}}(\sigma^1)\right], \tag{5.12}$$

where $q \le n'$, and the π_j, $j=1,\dots,n$ are functions from $\{1,\dots,n'\}$ to $\{1,\dots,n'\}$. They appear because the expansion of the denominator (the partition function) will introduce new copies of the measure (hence the power q).

The first product in the integrand above contains the H_k^μ with respect to which the expansion has not yet been done. The second corresponds to those which have been used.

The initial expressions on the right of (5.10) correspond to the case $q=1$, $n'=0$, $u_i=1, \forall i$, that is,

$$\mathbb{E}\mathcal{G}\left[\prod_{\mu\in\mathcal{J}} H_k^\mu \mathcal{L}_{\mathcal{J}}(\sigma)\right] = \mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_n}_{1,\dots,1}\left[\prod_{i=1}^{n} H_k^\mu \mathcal{L}_{\mathcal{J}}(\sigma^1)\right]. \tag{5.13}$$

The following provides the basic recursion relation.

Lemma 5.3: *For all numbers $n'\in\{0,\dots,n-1\}$, $q\in\mathbb{N}$, and $u_1,\dots,u_{n'}$, and functions $\pi_{n'}$, there exist functions $(\pi^j_{n'+1})_{j=1,\dots,q+1}$, and a number $u_{n'+1}\in[0,1]$ such that*

$$\begin{aligned}
&\mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_{n'}\otimes q}_{u_1,\dots,u_{n'}}\left[\prod_{i=n'+1}^{n} H_k^{\mu_i}(\sigma^1)\prod_{i=1}^{n'} H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi_{n'}(i)})\mathcal{L}_{\mathcal{J}}(\sigma^1)\right]\\
&\quad= -\beta\sum_{j=1}^{q}\mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_{n'+1}\otimes q}_{u_1,\dots,u_{n'+1}}\left[\prod_{i=n'+2}^{n} H_k^{\mu_i}(\sigma^1)\prod_{i=1}^{n'+1} H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi^j_{n'+1}(i)})\mathcal{L}_{\mathcal{J}}(\sigma^1)\right]\\
&\qquad+\frac{\beta}{q}\mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_{n'+1}\otimes q+1}_{u_1,\dots,u_{n'+1}}\left[\prod_{i=n'+2}^{n} H_k^{\mu_i}(\sigma^1)\prod_{i=1}^{n'+1} H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi^{q+1}_{n'+1}(i)})\mathcal{L}_{\mathcal{J}}(\sigma^1)\right]
\end{aligned}\tag{5.14}$$

The functions $(\pi^j_{n'+1})_{j=1,\dots,q+1}$ satisfy

$$\pi^j_{n'+1}(i)=\begin{cases}\pi_{n'}(i), & \text{if } i\le n';\\ j, & \text{if } i=n'+1.\end{cases}\tag{5.15}$$

Proof: We expand the Boltzmann weight of the Gibbs measure on the left-hand side of (5.14) in the pattern $\mu_{n'+1}$. Since $H^{\mu_1,\dots,\mu_{n'}}_{u_1,\dots,u_{n'}} = H^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}}|_{u_{n'+1}=1}$, expanding in the variable $u_{n'+1}$

about 0 to zero order with remainder of order 1 yields

$$\begin{aligned}\frac{\exp\Big(-\beta\sum_{j'=1}^{q}H^{\mu_1,\dots,\mu_{n'}}_{u_1,\dots,u_{n'}}(\sigma^{j'})\Big)}{(Z^{\mu_1,\dots,\mu_{n'}}_{u_1,\dots,u_{n'}})^q}&=\frac{\exp\Big(-\beta\sum_{j'=1}^{q}H^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},0}(\sigma^{j'})\Big)}{(Z^{\mu_1,\dots,\mu_{n'+1}}_{u_1,\dots,u_{n'},0})^q}\\&\quad-\beta\sum_{j=1}^{q}\frac{\exp\Big(-\beta\sum_{j'=1}^{q}H^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}}(\sigma^{j'})\Big)}{(Z^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}})^q}H_k^{\mu_{n'+1}}(\sigma^j)\\&\quad+\frac{\beta\exp\Big(-\beta\sum_{j'=1}^{q}H^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}}(\sigma^{j'})\Big)}{q(Z^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}})^{q+1}}\\&\quad\times\mathbb{E}_{\sigma^{q+1}}\exp\Big(-\beta H^{\mu_1,\dots,\mu_{n'},\mu_{n'+1}}_{u_1,\dots,u_{n'},u_{n'+1}}(\sigma^{q+1})\Big)H_k^{\mu_{n'+1}}(\sigma^{q+1}),\end{aligned}\tag{5.16}$$

for some $u_{n'+1}\in[0,1]$.

The first term on the right does not depend on $\xi_k^{\mu_{n'+1}}$ (see the remark after (5.11)). Hence, when multiplied by the products of the H_k^μ, this disorder variable appears exactly once, so that integration with respect to it yields zero.

The second and third term above give the new terms on the right in (5.14). The relations for the functions $\pi^j_{n'+1}$ are easily verified. □

Applying this recursion relation n times yields the following decomposition.

Lemma 5.4: *Let $\mathcal{J}=\{\mu_1,\dots,\mu_n\}$, $n\le 2l$. Then there exist numbers $u_1,\dots,u_n\in[0,1]$ such that*

$$\mathbb{E}\mathcal{G}\Big[\prod_{i=1}^n H_k^{\mu_i}(\sigma^1)\mathcal{L}_{\mathcal{J}}(\sigma)\Big]=\sum_{q=1}^n\sum_{\pi\sim q}c_{\pi,q}\beta^n\mathcal{G}^{\mu_1,\dots,\mu_n\otimes q}_{u_1,\dots,u_n}\Big[\prod_{i=1}^n H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\mathcal{L}_{\mathcal{J}}(\sigma^1)\Big],\tag{5.17}$$

where the functions π permute the indices $i\in\{1,\dots,n\}$, and the relation $\pi\sim q$ describes the condition that $|\{i\in\{1,\dots,n\}:\pi(i)\ne 1\}|=q$. The number of such functions π is thus independent of N.

Proof: The proof follows by applying the recursion relation from Lemma 5.3 n times. Observing that each step adds at most one other replica implies that $q\le n$. □

We finally sum over the sets $\mathcal{J}\subset\mathcal{M}$ on the right of (5.10). We obtain

$$|I|\le\sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\|\mathcal{J}|=2l}}\sum_{q=1}^{|\mathcal{J}|}\sum_{\pi\sim q}c_{\pi,q}\beta^n\mathbb{E}\mathcal{G}^{\mu_1,\dots,\mu_n\otimes q}_{u_1,\dots,u_n}\Big[\prod_{i=1}^n\Big|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\Big|\Big|\mathcal{L}_{\mathcal{J}}(\sigma^1)\Big|\Big].\tag{5.18}$$

First, we observe that since $|H_k^\mu|<1$,

$$|\mathcal{L}_{\mathcal{J}}|\le\sum_{\substack{r:r\prec\mathcal{J}\\|r|=2l-|\mathcal{J}|}}c_{2l,r,|\mathcal{J}|}\prod_{\mu\in\mathcal{M}\setminus\mathcal{J}}|H_k^\mu|^{r_\mu}\le\sum_{\substack{r:r\prec\mathcal{J}\\|r|=2l-|\mathcal{J}|}}c_{2l,r,|\mathcal{J}|}\prod_{\mu\in\mathcal{M}\setminus\mathcal{J}}|H_k^\mu|^{2\delta_{r_\mu,2}},\tag{5.19}$$

where $\delta_{a,b} = 1$, if $a \geq b$ and zero otherwise.

For any multi-index r, denote by $\#r$ the number of r_μ which are not zero. Hence, the products on the right-hand side of the above inequality are just the completely off-diagonal terms of the form $\left(\sum_{\mu\in\mathcal{M}\setminus\mathcal{J}}(H_k^\mu)^2\right)^{\#r}$. Then, adding the terms which have at least two indices equal (and which are obviously positive), yields uniformly in σ

$$|\mathcal{L}_\mathcal{J}| \leq \sum_{j=1}^{2l-|\mathcal{J}|} c_{j,2l,|\mathcal{J}|}\Big(\sum_{\mu\in\mathcal{M}\setminus\mathcal{J}}(H_k^\mu)^2\Big)^j \leq \sum_{j=1}^{2l-|\mathcal{J}|} c_{j,2l,|\mathcal{J}|}\Big(\sum_{\mu\in\mathcal{M}}(H_k^\mu)^2\Big)^j \leq C, \tag{5.20}$$

on a set $\mathcal{B}$ of measure at least $1 - e^{-KN^{1/4}}$ by Lemma 5.2. Using this in (5.18), we bound $I' = \mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}[(\sum_\mu H_k^\mu)^{2l}]$ by

$$|I'| \leq \sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\|\mathcal{J}|\leq 2l}}\sum_{q=1}^{|\mathcal{J}|}\sum_{\pi\sim q} c_{\pi,q,|\mathcal{J}|,\beta}\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}^{\mu_1,\dots,\mu_n\otimes q}_{u_1,\dots,u_n}\Big[\prod_{i=1}^n\Big|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\Big|\Big]. \tag{5.21}$$

Since the integrand is non-negative and $|H_k^\mu| < 1$, we can change the Boltzmann weights back to the original ones (that is, setting all $u_i = 1$), and committing at most an error of $e^{\beta n}$. Furthermore, the functions π depend only on the size of $\mathcal{J}$. Hence, adding again positive terms in the third step below (and observing that $|\mathcal{J}|$ is even),

$$\begin{aligned}
|I'| &\leq C\sum_{\substack{n=0\\\text{even}}}^{2l}\sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\|\mathcal{J}|=n}}\sum_{\pi\sim q}\sum_{q=1}^{n} c_{\pi,q,\beta}\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}^{\otimes q}\Big[\prod_{i=1}^n\Big|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\Big|\Big]\\
&\leq C\sum_{\substack{n=0\\\text{even}}}^{2l}\sum_{\pi\sim q}\sum_{\substack{\mathcal{J}\subset\mathcal{M}:\\|\mathcal{J}|=n}}\sum_{q=1}^{n} c_{\pi,q,\beta}\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}^{\otimes q}\Big[\prod_{i=1}^n\Big|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\Big|\Big]\\
&\leq C\sum_{\substack{n=0\\\text{even}}}^{2l}\sum_{\pi\sim q}\frac{1}{n!}\sum_{\mu_1,\dots,\mu_n=1}^{M}\sum_{q=1}^{n} c_{\pi,q,\beta}\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}^{\otimes q}\Big[\prod_{i=1}^n\Big|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})\Big|\Big]\\
&\leq C\sum_{\substack{n=0\\\text{even}}}^{2l}\sum_{\pi\sim q}\frac{1}{n!}\sum_{q=1}^{n} c_{\pi,q,\beta}\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}^{\otimes q}\Big[\prod_{i=1}^n\Big(\sum_{\mu_i=1}^{M}|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})|\Big)\Big].
\end{aligned} \tag{5.22}$$

Finally, we apply Cauchy-Schwarz to get rid of the absolute value in the sum over μ_i,

$$\sum_{\mu=1}^{M}|H_k^{\mu_i}(\sigma^1)H_k^{\mu_i}(\sigma^{\pi(i)})| \leq \Big(\sum_{\mu_i=1}^{M}(H_k^{\mu_i}(\sigma^1))^2\Big)^{\frac{1}{2}}\Big(\sum_{\mu_i=1}^{M}(H_k^{\mu_i}(\sigma^{\pi(i)}))^2\Big)^{\frac{1}{2}} \leq C \tag{5.23}$$

on $\mathcal{B}$ by Lemma 5.2. Inserting the above in (5.22) shows that $\mathbb{E}\mathbb{I}_\mathcal{B}\mathcal{G}[\sum_\mu H_k^\mu]$ is bounded by a number independent of N, since all the remaining sums are over finite sets whose sizes do not depend on N.

Since $(\sum_\mu H_k^\mu)^{2l}$ is polynomially bounded in N, uniformly in ω, the remaining part $I - I'$, (that is, the integral on the set $\mathcal{B}^c$), is obviously bounded by an exponentially small number in $N^{1/5}$ (e.g.), and is thus also smaller than a constant.

We use this in (5.4) which shows that

$$\mathbb{P}[\mathcal{A}_k{}^c] \leq c_l c^{-2l} N^{-2\gamma l}. \tag{5.24}$$

Thus for all $\gamma, m > 0$, there exist l and $C_{l,m}$ such that

$$\mathbb{P}[\mathcal{A}_k{}^c] \leq C_{l,m} N^{-m-1}. \tag{5.25}$$

Summing over all $k = 1, \ldots, N$ shows that indeed $\mathbb{P}[\mathcal{A}^c] \leq C_{l,m} N^{-m}$. $\square$

We now bound the fluctuations of the free energy on the set $\mathcal{A}$.

Proposition 5.5: *Let $\tilde{F}_N = N^{-1} \ln Z_N \mathbb{1}_{\mathcal{A}_{c,\gamma,N}}$. Then, for all β, all $\tau > 0$ and all $\varepsilon > \gamma$, there exists $\bar{N} < \infty$ such that for all $N > \bar{N}$,*

$$\mathbb{P}\left[|\tilde{F}_N - \mathbb{E}\tilde{F}_N| > \tau\beta N^{-\frac{1}{2}+\varepsilon}\right] \leq 3e^{-N^{\varepsilon/2}}. \tag{5.26}$$

Proof: In the sequel, N, β, γ, c will be fixed, and we will therefore frequently drop the corresponding indices. The approach to the proof follows the general idea of [BGP2,B1]. Define a decreasing sequence of σ-algebras $\{\hat{\mathcal{F}}_k\}_{k\in\mathbb{N}}$ by

$$\hat{\mathcal{F}}_k = \sigma\left(\{\xi_i^\mu\}_{i\geq k}^{\mu\in\mathbb{N}}\right) \vee \mathcal{A}_{c,\gamma,N}. \tag{5.27}$$

This allows to introduce a martingale difference sequence (see [Yu])

$$\tilde{F}^k \equiv \mathbb{E}[\tilde{F}|\hat{\mathcal{F}}_k] - \mathbb{E}[\tilde{F}|\hat{\mathcal{F}}_{k+1}]. \tag{5.28}$$

By the definition of conditional expectations

$$\tilde{F} - \mathbb{E}\tilde{F} = \sum_{k=1}^N \tilde{F}^k \mathbb{P}[\mathcal{A}]. \tag{5.29}$$

The factor $\mathbb{P}[\mathcal{A}]$ tends to one as $N \uparrow \infty$ by Lemma 5.1 (even polynomially as fast as we want). It is therefore enough to control the sum $\sum_{k=1}^N \tilde{F}^k$. We observe that

$$\begin{aligned}\mathbb{P}[|\sum_{k=1}^N \tilde{F}^k| > z(\mathbb{P}[\mathcal{A}])^{-1}] &\leq 2\inf_{t>0} e^{-tz(\mathbb{P}[\mathcal{A}])^{-1}} \mathbb{E} e^{t\sum_{k=1}^N \tilde{F}^k} \\ &= 2\inf_{t>0} e^{-tz} \mathbb{E}[\mathbb{E}[\ldots\mathbb{E}[\mathbb{E}[e^{t\tilde{F}^1}|\hat{\mathcal{F}}_2]e^{t\tilde{F}^2}|\hat{\mathcal{F}}_3]\ldots]e^{t\tilde{F}^N}|\hat{\mathcal{F}}_{N+1}].\end{aligned} \tag{5.30}$$

To make use of this inequality, we need bounds on the conditional Laplace transforms, that is, we want to show that for some $\mathcal{L}^k(t)$,

$$\ln \mathbb{E}[e^{t\tilde{F}^k}|\hat{\mathcal{F}}_{k+1}] \leq \mathcal{L}^k(t), \tag{5.31}$$

uniformly in $\hat{\mathcal{F}}_{k+1}$. Using a standard second order bound for the exponential function, we get

$$\mathbb{E}[e^{t\tilde{F}^k}|\hat{\mathcal{F}}_{k+1}] \leq 1 + \frac{t^2}{2}\mathbb{E}[(\tilde{F}^k)^2 e^{|t\tilde{F}^k|}|\hat{\mathcal{F}}_{k+1}]. \tag{5.32}$$

To make use of this we need to bound $|\tilde{F}^k|$. A conventional strategy is to introduce a family of Hamiltonians $\tilde{H}^k(\sigma, u)$, defined by

$$\tilde{H}^k(\sigma, u) = H(\sigma) + (1-u)\frac{(p!)^{1/2}}{N^{p-1}} \sum_{\mu=1}^{M(N)} \sum_{\substack{\mathcal{I}\ni k \\ |\mathcal{I}|=p}} \xi_{\mathcal{I}}^{\mu}\sigma_{\mathcal{I}}. \tag{5.33}$$

This new Hamiltonian is equal to the original one for $u = 1$, and independent of $\{\xi_k^\mu\}^{\mu=1,\dots,M}$ for $u = 0$. Denote by $\tilde{Z}^k(u)$ and $\mathcal{G}^k(u)$ the partition function, respectively the Gibbs measure associated to this Hamiltonian. Observe that the condition on being on the set $\mathcal{A}$ is stable against the change in parameter $u \in [0,1]$, that is

$$\mathcal{G}^k(u)\Big[N^{-p}\sum_{\mu=1}^{M(N)}\sum_{\mathcal{I}\ni k}\xi_{\mathcal{I}}^{\mu}\sigma_{\mathcal{I}}\Big] \in [-c, c], \quad \forall u \in [0,1], \tag{5.34}$$

on the set $\mathcal{A}$. Indeed, the derivative of the left-hand side with respect to u is non-negative, since it is the variance of the integrand with respect to the measure $\mathcal{G}(u)$. Moreover, for $u = 0$, the Boltzmann weight does not contain σ_k, whence the left is zero for $u = 0$. The absolute value of the left-hand side thus assumes its maximal value for $u = 1$.

Define

$$g^k(u) = \frac{1}{N}\mathbb{I}_{\mathcal{A}} \ln \tilde{Z}^k(u) - \frac{1}{N}\mathbb{I}_{\mathcal{A}} \ln \tilde{Z}^k(0). \tag{5.35}$$

Since $\tilde{Z}^k(0)$ is independent of σ_k, this quantity relates to $\tilde{F}^k$ via

$$\tilde{F}^k = \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_k] - \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}] \tag{5.36}$$

Observe that $g^k(u)$ is convex in u, since its derivative is equal to the expectation of the left-hand side of (5.34), whose derivative is the variance of a random variable with respect to the measure $\mathcal{G}$. Since by its definition $g^k(0) = 0$, and therefore $|g^k(1)| \leq \max(|(g^k)'(1)|, |(g^k)'(0)|)$, where the prime denotes the derivative with respect to u. Moreover, since $\tilde{H}^k(\sigma, u = 0)$ does not depend on σ_k, it follows that $(g^k)'(0) = 0$, and hence we can use $|g^k(1)| \leq |(g^k)'(1)|$. Explicitly, this is

$$|g^k(1)| \leq |(g^k)'(1)| = \left|\beta\frac{(p!)^{1/2}}{N^p}\mathcal{G}_{N,\beta}[\omega]\left(\sum_{\mu=1}^{M(N)}\sum_{\substack{\mathcal{I}\ni k \\ |\mathcal{I}|=p}}\xi_{\mathcal{I}}^{\mu}\sigma_{\mathcal{I}}\right)\right|\mathbb{I}_{\mathcal{A}} \leq cN^{-1+\gamma}. \tag{5.37}$$

Inserting this bound into the exponent on the right-hand side of (5.32) gives

$$\begin{aligned} 1 + \frac{t^2}{2}\mathbb{E}[(\tilde{F}^k)^2 e^{|t\tilde{F}^k|}|\hat{\mathcal{F}}_{k+1}] &\leq 1 + \frac{t^2}{2}\mathbb{E}[(\tilde{F}^k)^2 e^{|tg^k(1)|}|\hat{\mathcal{F}}_{k+1}] \\ &\leq 1 + \frac{t^2}{2}e^{2ctN^{-1+\gamma}}\mathbb{E}[(\tilde{F}^k)^2|\hat{\mathcal{F}}_{k+1}]. \end{aligned} \tag{5.38}$$

To treat the quadratic term, we observe that by (5.36), the properties of conditional expectations, and Jensen's inequality (see also [B] and [BGP]),

$$\begin{aligned}
\mathbb{E}[(\tilde{F}^k)^2|\hat{\mathcal{F}}_{k+1}] &= \mathbb{E}\left[\left(\mathbb{E}[g^k(1)|\hat{\mathcal{F}}_k] - \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}]\right)^2\middle|\hat{\mathcal{F}}_{k+1}\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}[g^k(1) - \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}]|\hat{\mathcal{F}}_k]\right)^2\middle|\hat{\mathcal{F}}_{k+1}\right] \\
&\leq \mathbb{E}\left[\mathbb{E}[(g^k(1) - \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}])^2|\hat{\mathcal{F}}_k]\middle|\hat{\mathcal{F}}_{k+1}\right] \\
&= \mathbb{E}\left[(g^k(1) - \mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}])^2\middle|\hat{\mathcal{F}}_{k+1}\right] \\
&= \mathbb{E}[(g^k(1))^2|\hat{\mathcal{F}}_{k+1}] - \left(\mathbb{E}[g^k(1)|\hat{\mathcal{F}}_{k+1}]\right)^2 \\
&\leq \mathbb{E}[(g^k(1))^2|\hat{\mathcal{F}}_{k+1}] \leq \mathbb{E}[(g^k(1))'^2|\hat{\mathcal{F}}_{k+1}].
\end{aligned} \tag{5.39}$$

The last term is bounded since we are in the set $\mathcal{A}_k$. Indeed,

$$\mathbb{E}\left[(g^k(1))'^2|\hat{\mathcal{F}}_{k+1}\right] = \frac{p!}{N^{2p}}\beta^2\mathbb{E}\left[\left(\mathbb{1}_{\mathcal{A}}\mathcal{G}\Big[\sum_{\mu=1}^{M(N)}\sum_{\mathcal{I}\ni k}\xi_{\mathcal{I}}^{\mu}\sigma_{\mathcal{I}}\Big]\right)^2\middle|\hat{\mathcal{F}}_{k+1}\right] \leq \beta^2 C N^{2\gamma-2}, \tag{5.40}$$

Thus, using the bound (5.40) in (5.38),

$$1 + \frac{t^2}{2}\mathbb{E}[(\tilde{F}^k)^2 e^{|t\tilde{F}^k|}|\hat{\mathcal{F}}_{k+1}] \leq 1 + \frac{t^2}{2}e^{2c\beta t N^{-1+\gamma}}C\beta^2 N^{2\gamma-2} \leq \exp\left(\frac{t^2}{2}e^{2c\beta t N^{-1+\gamma}}C\beta^2 N^{2\gamma-2}\right). \tag{5.41}$$

Inserting this in (5.30) yields

$$\mathbb{P}[|\sum_{k=1}^{N}\tilde{F}^k| > z(\mathbb{P}[\mathcal{A}])^{-1}] \leq 2\inf_{t>0}\exp\left(-tz + \frac{t^2}{2}e^{2c\beta t N^{-1+\gamma}}C\beta^2 N^{2\gamma-1}\right). \tag{5.42}$$

We choose $z = \tau\beta N^{-1/2+\varepsilon}$, and $t = \frac{1}{z}N^{\frac{\varepsilon}{2}} = \frac{1}{\tau\beta}N^{\frac{1-\varepsilon}{2}}$. This implies that

$$\mathbb{P}[|\sum_{k=1}^{N}\tilde{F}^k| > \beta\tau N^{-\frac{1}{2}+\varepsilon}(\mathbb{P}[\mathcal{A}])^{-1}] \leq 2\exp\left(-N^{\frac{\varepsilon}{2}} + C\tau^{-2}N^{2\gamma-\varepsilon}e^{2c\tau^{-1}N^{-1/2+\gamma-\varepsilon/2}}\right). \tag{5.43}$$

Choose $\gamma < \varepsilon/2$. Then for any $\tau > 0$, and N large enough, the right hand side of (5.43) is bounded by $3e^{-N^{\epsilon/2}}$. Since $\mathbb{P}[\mathcal{A}]$ tends to 1 as $1 - N^{-m}$, the claimed estimate follows. $\square$

Proof of Theorem 4: The assertion is now an immediate consequence of Lemma 5.1 and Proposition 5.5. Indeed,

$$|F_N - \mathbb{E}F_N| \leq |F_N - \tilde{F}_N| + |\tilde{F}_N - \mathbb{E}\tilde{F}_N| + |\mathbb{E}\tilde{F}_N - \mathbb{E}F_N|. \tag{5.44}$$

The first term is non zero only on $\mathcal{A}^c$. Also, the last summand is bounded by $\mathbb{P}[\mathcal{A}^c]\sup F_N \leq CN^p\mathbb{P}[\mathcal{A}^c]$. If we choose m in Lemma 5.1 larger than $p + n + 1$, then this term is eventually less than N^{-2}, and thus also less than $z = \tau N^{-1/2+\varepsilon}$. Thus, for all $n, \tau, \varepsilon > 0$, and N large enough,

$$\begin{aligned}
\mathbb{P}[|F_N - \mathbb{E}F_N| > z] &\leq \mathbb{P}[|F_N - \tilde{F}_N| > \frac{z}{3}] + \mathbb{P}[|\tilde{F}_N - \mathbb{E}\tilde{F}_N| > \frac{z}{3}] \\
&\leq CN^p \mathbb{P}[\mathcal{A}^c] + \mathbb{P}[|\tilde{F}_N - \mathbb{E}\tilde{F}_N| > \frac{z}{3}] \\
&\leq CN^{-n-1} + e^{-N^\varepsilon} < N^{-n}.
\end{aligned} \tag{5.45}$$

This concludes the proof of the theorem. □

6. Results on the Replica Overlap.

In this section, we prove the results on the replica overlap, Theorems 1.5, 1.9, and 1.7.

6.1. Proof of Theorem 1.5.

By the definition of the free energy,

$$\mathbb{E}\,\frac{\partial F_N}{\partial \beta} = -\frac{\beta}{N}\mathbb{E}\mathcal{G}_{N,\beta}[H] = -\beta \sum_{\mu=1}^{M(N)} \mathbb{E}\mathcal{G}_{N,\beta}[H^\mu(\sigma)], \tag{6.1}$$

where

$$H^\mu(\sigma) = -\frac{(p!)^{1/2}}{N^{p-1}} \sum_{\substack{\mathcal{I}\subset\mathcal{N} \\ |\mathcal{I}|=p}} \xi^\mu_{\mathcal{I}} \sigma_{\mathcal{I}}, \tag{6.2}$$

is the contribution to the Hamiltonian from pattern μ. We introduce the following notation. For any $u \in [0,1]$, we let $\bar{H}^\mu_u$ be an interpolating Hamiltonian of the form

$$\bar{H}^\mu_u = H - (1-u)H^\mu. \tag{6.3}$$

Observe that for $u = 0$, this quantity is independent of the pattern μ, and for $u = 1$, is equal to the original Hamiltonian. The notations $\bar{\mathcal{G}}^\mu_u$ and $\bar{Z}^\mu_u$ refer to the corresponding Gibbs measures and partition functions (dropping reference to N and β for sake of clarity). We now write the Gibbs ectation on the right of (6.1) as

$$\mathcal{G}_{N,\beta}[H^\mu(\sigma)] = \mathbb{E}_\sigma \left[\frac{e^{-\beta \bar{H}^\mu_u}}{\mathbb{E}_{\sigma'}\left[e^{-\beta \bar{H}^\mu_u}\right]} H^\mu \right] \Bigg|_{u=1}. \tag{6.4}$$

Developping the Boltzmann weights in u about 0 with second order remainder, we obtain for each term in the sum on the right-hand side of (6.1) (for some $u \in [0,1]$)

$$
\begin{aligned}
\mathcal{G}_{N,\beta}[H^\mu] &= \mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} H_0^\mu(\sigma)\right] - \beta\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} H^\mu(\sigma)^2\right] \\
&\quad + \beta\mathbb{E}_{\sigma,\sigma'}\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)-\beta \bar{H}_0^\mu(\sigma')}}{\bar{Z}_0^{\mu 2}} H^\mu(\sigma)H^\mu(\sigma')\right] + \frac{\beta^2}{2}\underbrace{\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_u^\mu(\sigma)}}{\mathbb{E}_\sigma[e^{-\beta \bar{H}_u^\mu(\sigma)}]} H^\mu(\sigma)^3\right]}_{R_1} \\
&\quad - \frac{3\beta^2}{2}\underbrace{\mathbb{E}_{\sigma,\sigma'}\left[\frac{e^{-\beta \bar{H}_u^\mu(\sigma)-\beta \bar{H}_u^\mu(\sigma')}}{(\mathbb{E}_\sigma[e^{-\beta \bar{H}_u^\mu(\sigma)}])^2} H^\mu(\sigma)^2 H^\mu(\sigma')\right]}_{R_2} \\
&\quad + \frac{\beta^2}{2}\underbrace{\mathbb{E}_{\sigma,\sigma',\sigma''}\left[\frac{e^{-\beta \bar{H}_u^\mu(\sigma)-\beta \bar{H}_u^\mu(\sigma')-\beta \bar{H}_u^\mu(\sigma'')}}{(\mathbb{E}_\sigma[e^{-\beta \bar{H}_u^\mu(\sigma)}])^3} H^\mu(\sigma)H^\mu(\sigma')H^\mu(\sigma'')\right]}_{R_3}.
\end{aligned}
\tag{6.5}
$$

As remarked above, neither $\bar{H}_0^\mu$ nor $\bar{Z}_0^\mu$ contain any of the variables $\{\xi_i^\mu\}_{i\in\mathcal{N}}$. Integration with respect to them (denoted by $\mathbb{E}_\mu$) thus yields for the linear term,

$$
\mathbb{E}\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} H^\mu(\sigma)\right] = \frac{(p!)^{1/2}}{N^{p-1}} \sum_{\substack{\mathcal{I}\subset\mathcal{N}\\ |\mathcal{I}|=p}} \mathbb{E}'\,\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} \mathbb{E}_\mu \xi_{\mathcal{I}}^\mu \sigma_{\mathcal{I}}\right] = 0, \tag{6.6}
$$

and for the second order contribution

$$
\begin{aligned}
\mathbb{E}\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} H^\mu(\sigma)^2\right] &= \frac{p!}{N^{2p-2}}\mathbb{E}'\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} \sum_{\mathcal{I},\mathcal{J}} \mathbb{E}_\mu \xi_{\mathcal{I}}^\mu \xi_{\mathcal{J}}^\mu \sigma_{\mathcal{I}}\sigma_{\mathcal{J}}\right] \\
&= \frac{p!}{N^{2p-2}}\mathbb{E}'\mathbb{E}_\sigma\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)}}{\bar{Z}_0^\mu} \sum_{\mathcal{I}} 1\right] = N^{2-2p}(1+\mathcal{O}(N^{-1})),
\end{aligned}
\tag{6.7}
$$

respectively,

$$
\mathbb{E}\mathbb{E}_{\sigma,\sigma'}\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)-\beta \bar{H}_0^\mu(\sigma')}}{\bar{Z}_0^{\mu 2}} H^\mu(\sigma)H^\mu(\sigma')\right] = \frac{p!}{N^{2p-2}}\mathbb{E}\mathbb{E}_{\sigma,\sigma'}\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)-\beta \bar{H}_0^\mu(\sigma')}}{\bar{Z}_0^{\mu 2}} \sum_{\mathcal{I}} \sigma_{\mathcal{I}}\sigma'_{\mathcal{I}}\right]. \tag{6.8}
$$

The latter sum is

$$
\begin{aligned}
\sum_{\mathcal{I}\subset\mathcal{N}} \sigma_{\mathcal{I}}\sigma'_{\mathcal{I}} &= \frac{1}{p!} \sum_{\substack{i_1,\dots,i_p=1\\ \text{all different}}}^{N} \prod_{l=1}^{p} \sigma_{i_l}\sigma'_{i_l} = \frac{1}{p!}\left(\sum_{i=1}^{N}\sigma_i\sigma'_i\right)^p (1+\mathcal{O}(N^{-1})) \\
&= \frac{1}{p!}N^p R(\sigma,\sigma')^p(1+\mathcal{O}(N^{-1})),
\end{aligned}
\tag{6.9}
$$

whence,

$$
\mathbb{E}\mathbb{E}_{\sigma,\sigma'}\left[\frac{e^{-\beta \bar{H}_0^\mu(\sigma)-\beta \bar{H}_0^\mu(\sigma')}}{\bar{Z}_0^{\mu 2}} H^\mu(\sigma)H^\mu(\sigma')\right] = N^{2-2p} E\,\bar{\mathcal{G}}_0^{\mu\otimes 2}\left[R(\sigma,\sigma')^p(1+\mathcal{O}(N^{-1}))\right]. \tag{6.10}
$$

We now show that the remainder terms in (6.5) are at least one order (in N) less than the two leading contributions above. We start with a result that shows that the perturbed partition function $\bar{Z}_u^\mu = \mathbb{E}_\sigma[e^{-\beta \bar{H}_u^\mu}]$ is bounded from below by a constant times the partition function $\bar{Z} = \bar{Z}_0^\mu$ (that is, the one not containing any of the $\{\xi_i^\mu\}_i$).

Lemma 6.1: *For all $\beta \geq 0$ there exists a constant $c > 0$ such that for all $u \in [0,1]$,*

$$\bar{Z}_u^\mu \geq c\bar{Z}_0^\mu = c\,\mathbb{E}_\sigma[e^{-\beta \bar{H}_0^\mu}]. \tag{6.11}$$

Proof: The proof is an immediate consequence of the following result.

Lemma 6.2: *Let $\{X_i\}_{i=1,\dots,N}$ be a familiy of variables taking values -1 and 1. Let $\Gamma_{p,N} = N^{-p}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}}$, and $m = N^{-1}\sum_i X_i$. Then for each even p there exist constants $c_{p,q}$ such that*

$$\Gamma_{p,N} = \sum_{q=0}^{\frac{p}{2}} c_{p,2q} m^{2q} N^{q-\frac{p}{2}}(1+\mathcal{O}(N^{-1})). \tag{6.12}$$

Moreover, $c_{p,p}$ is positive for all p.

Proof: By induction. For $p=2$, we have

$$\begin{aligned} \Gamma_{2,N} = N^{-2}\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} X_i X_j &= N^{-2}\frac{1}{2}\sum_{i,j=1}^{N} X_i X_j - N^{-2}\frac{1}{2}\sum_{i=1}^{N} 1 \\ &= \frac{1}{2}m^2 - N^{-1}, \end{aligned} \tag{6.13}$$

which is of the form claimed in (6.12).

Suppose the result is true for all even values $q \leq p$. Then,

$$\begin{aligned} \Gamma_{p+2,N} = N^{-p-2}\sum_{\mathcal{I}:|\mathcal{I}|=p+2} X_{\mathcal{I}} &= \frac{1}{\binom{p+2}{2}N^{p+2}}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}} \sum_{\substack{\mathcal{J}:|\mathcal{J}|=2\\ \mathcal{I}\cap\mathcal{J}=\emptyset}} X_{\mathcal{J}} \\ &= c_p N^{-p-2}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}} \sum_{\mathcal{J}:|\mathcal{J}|=2} X_{\mathcal{J}} - c_p N^{-p-2}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}} \sum_{\substack{\mathcal{J}:|\mathcal{J}|=2\\ \mathcal{J}\cap\mathcal{I}\neq\emptyset}} X_{\mathcal{J}}. \end{aligned} \tag{6.14}$$

By the induction hypothesis, the first term on the right-hand side is

$$\begin{aligned} & c_p N^{-p-2}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}} \sum_{\mathcal{J}:|\mathcal{J}|=2} X_{\mathcal{J}} = c_p \Gamma_{p,N}\Gamma_{2,N} \\ &\quad = c_p\left(\sum_{q=0}^{\frac{p}{2}} c_{p,2q} m^{2q} N^{q-\frac{p}{2}}(1+\mathcal{O}(N^{-1}))\right)\left(\sum_{q=0}^{1} c_{2,2q} m^{2q} N^{q-1}(1+\mathcal{O}(N^{-1}))\right) \\ &\quad = \sum_{q=0}^{\frac{p}{2}+1} c_{p,2q} m^{2q} N^{q-\frac{p}{2}-1}(1+\mathcal{O}(N^{-1})). \end{aligned} \tag{6.15}$$

The remaining term in (6.14) is

$$\begin{aligned}\sum_{\mathcal{I}:|\mathcal{I}|=p}\sum_{\substack{\mathcal{J}:|\mathcal{J}|=2\\ \mathcal{J}\cap\mathcal{I}\neq\emptyset}} X_{\mathcal{I}}X_{\mathcal{J}} &= \sum_{\substack{\mathcal{I},\mathcal{J}\\ |\mathcal{J}\cap\mathcal{I}|=1}} X_{\mathcal{I}}X_{\mathcal{J}} + \sum_{\substack{\mathcal{I},\mathcal{J}\\ |\mathcal{J}\cap\mathcal{I}|=2}} X_{\mathcal{I}}X_{\mathcal{J}}\\ &= \sum_{\mathcal{I}:|\mathcal{I}|=p}\sum_{i\in\mathcal{N}\setminus\mathcal{I}}\sum_{j\in\mathcal{I}} X_{\mathcal{I}}X_iX_j + \sum_{\mathcal{I}:|\mathcal{I}|=p}\sum_{i,j\in\mathcal{I}} X_{\mathcal{I}}X_iX_j\\ &= \sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}}\sum_{i\in\mathcal{N}\setminus\mathcal{I}} X_i^2 + \sum_{\mathcal{I}:|\mathcal{I}|=p-2} X_{\mathcal{I}}\sum_{i,j\in\mathcal{N}\setminus\mathcal{I}} X_i^2X_j^2\\ &= (N-p)N^p\Gamma_{p,N} + \binom{N-p}{2}N^{p-2}\Gamma_{p-2,N},\end{aligned} \tag{6.16}$$

and hence

$$N^{-p-2}\sum_{\mathcal{I}:|\mathcal{I}|=p} X_{\mathcal{I}}\sum_{\substack{\mathcal{J}:|\mathcal{J}|=2\\ \mathcal{J}\cap\mathcal{I}\neq\emptyset}} X_{\mathcal{J}} = N^{-1}\Gamma_{p,N}(1+\mathcal{O}(N^{-1})) + N^{-2}\Gamma_{p-2,N}(1+\mathcal{O}(N^{-1})). \tag{6.17}$$

Applying the induction hypothesis to (6.17) shows the decomposition (6.12). Positivity of $c_{p,p}$ follows from (6.14). $\square$

From this one concludes that uniformly in σ, ξ, and for all N large enough,

$$-H^\mu \geq -c. \tag{6.18}$$

Indeed, by the preceding result (setting $X_i = \xi_i^\mu\sigma_i$),

$$\begin{aligned}-H^\mu(\sigma) &= \frac{(p!)^{1/2}}{N^p}N\sum_{\mathcal{I}:|\mathcal{I}|=p}\xi_{\mathcal{I}}^\mu\sigma_{\mathcal{I}}\\ &= (p!)^{1/2}N\sum_{q=0}^{\frac{p}{2}}\left(\frac{1}{N}\sum_{i=1}^N\xi_i^\mu\sigma_i\right)^{2q} N^{q-\frac{p}{2}}(1+\mathcal{O}\sum_{\mathcal{I}:|\mathcal{I}|=p}(N^{-1}))\\ &= N\sum_{q=0}^{\frac{p}{2}} c_{p,2q}(m^\mu)^{2q}N^{q-\frac{p}{2}}(1+\mathcal{O}(N^{-1})).\end{aligned} \tag{6.19}$$

We distinguish two cases. If m^μ is large, we show that $-H^\mu(\sigma)$ is positive. Suppose therefore that $|m^\mu(\sigma)| > N^{-1/2+\delta}$ for some $\delta > 0$. Then,

$$\begin{aligned}-N^{-1}H^\mu(\sigma) &\geq c_{p,p}(m^\mu)^p - \sum_{q=0}^{p/2-1}|c_{p,q}|{m^\mu}^{2q}N^{q-\frac{p}{2}}(1+\mathcal{O}(N^{-1}))\\ &\geq c'_{p,p}N^{-\frac{p}{2}+p\delta} - \sum_{q=0}^{\frac{p}{2}-1}c'_{p,q}N^{-\frac{p}{2}+2q\delta}\\ &\geq N^{-\frac{p}{2}+p\delta}(c'_{p,p} - c''\sum_{q=0}^{\frac{p}{2}-1}N^{\delta(2q-p)}),\end{aligned} \tag{6.20}$$

which is obviously positive for all N large enough and δ less than $\frac{1}{2}$.

On the other hand, if m^μ is less than $N^{-1/2+\delta}$, then,

$$|N^{-1}H^\mu(\sigma)| \leq \sum_{q=0}^{\frac{p}{2}} c'_{p,q} N^{2q(\delta-\frac{1}{2})} N^{q-\frac{p}{2}} = \sum_{q=0}^{\frac{p}{2}} c'_{p,q} N^{-\frac{p}{2}+p\delta}. \tag{6.21}$$

Thus, if $\delta < \frac{1}{2} - \frac{1}{p}$, then $|H^\mu| = o(1)$, so that the bound (6.18) is in fact a gross underestimate.

To prove Lemma 6.1, we observe that

$$\bar{Z}_u^\mu = \mathbb{E}_\sigma[e^{-\beta \bar{H}_0^\mu(\sigma) - \beta u H^\mu(\sigma)}] \geq \mathbb{E}_\sigma[e^{-\beta \bar{H}_0^\mu(\sigma) - \beta u \sup H^\mu}] \geq \mathbb{E}_\sigma[e^{-\beta \bar{H}_0^\mu(\sigma) - \beta u \delta}] \geq c_\beta \bar{Z}_0^\mu. \tag{6.22}$$

This proves the (6.11). □

We apply this result to the error terms in the development (6.5). We start with R_1. By Jensen's inequality,

$$|R_1| = \left| \mathbb{E}_\sigma \left[\frac{e^{-\beta \bar{H}_u^\mu}}{\bar{Z}_u^\mu} H^{\mu 3} \right] \right| = \left| \bar{\mathcal{G}}_u^\mu [H^{\mu 3}] \right| \leq \bar{\mathcal{G}}_u^\mu [|H^\mu|^3] = \mathbb{E}_\sigma \left[\frac{e^{-\beta \bar{H}_u^\mu}}{\bar{Z}_u^\mu} |H^\mu|^3 \right]. \tag{6.23}$$

Since the integrand is a positive function, we may bound the expectation using Lemma 6.2 in the denominator. We obtain, noting that $\bar{H}_u^\mu = \bar{H}_0^\mu + uH^\mu$,

$$|R_1| \leq c\mathbb{E}_\sigma \left[\frac{e^{-\beta \bar{H}_u^\mu}}{\bar{Z}_0^\mu} |H^\mu|^3 \right] = c\,\bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} |H^\mu|^3]. \tag{6.24}$$

We observe that the last Gibbs measure does not depend on the pattern μ. We may therefore integrate with respect to $\{\xi_i^\mu\}_i$ "inside". In complete analogy with Chapter 3 (the result about the error term), we get

$$\mathbb{E}_\mu[e^{-\beta u H^\mu} |H^\mu|^3] \leq \mathbb{E}_\mu[e^{\beta u |H^\mu|} |H^\mu|^3] \leq c N^{3-\frac{3p}{2}}, \tag{6.25}$$

whenever $\beta u < \beta'_p$. Since $u \in [0,1]$, this condition is satisfied if $\beta < \beta'_p$.

The remainder R_3 gets essentially the same treatment. By Jensen's inequality,

$$|R_3| = \left| \bar{\mathcal{G}}_u^{\mu \otimes 3} [H^\mu(\sigma) H^\mu(\sigma') H^\mu(\sigma'')] \right| = \left| \bar{\mathcal{G}}_u^\mu [H^\mu(\sigma)] \right|^3 \leq \bar{\mathcal{G}}_u^\mu [|H^\mu|^3] = |R_1|. \tag{6.26}$$

Hence,

$$\mathbb{E}|R_3| \leq c N^{3-\frac{3p}{2}}.$$

Finally, the term R_2. By Lemma 6.1,

$$\begin{aligned} |R_2| &= |\bar{\mathcal{G}}_u^\mu [H^{\mu 2}]\, \bar{\mathcal{G}}_u^\mu [H^\mu]| \leq \bar{\mathcal{G}}_u^\mu [H^{\mu 2}]\, \bar{\mathcal{G}}_u^\mu [|H^\mu|] \\ &\leq c\,\bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} H^{\mu 2}]\, \bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} |H^\mu|]. \end{aligned} \tag{6.27}$$

Thus, by Cauchy-Schwarz and Jensen,

$$\begin{aligned} \mathbb{E}|R_2| &\leq \left| \mathbb{E} \left[\bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} H^{\mu 2}] \, \bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} |H^\mu|] \right] \right| \\ &\leq \left(\mathbb{E} \left[(\bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} H^{\mu 2}])^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[(\bar{\mathcal{G}}_0^\mu [e^{-\beta u H^\mu} H^\mu])^2 \right] \right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E} \left[\bar{\mathcal{G}}_0^\mu [e^{-2\beta u H^\mu} H^{\mu 4}] \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\bar{\mathcal{G}}_0^\mu [e^{-2\beta u H^\mu} H^{\mu 2}] \right] \right)^{\frac{1}{2}}. \end{aligned} \tag{6.28}$$

Both factors are now treated as R_1. Since the integrability of R_1 did not depend on the power of H^μ, but merely on the exponential factor (this is apparent from the estimate (3.26)), we get that whenever $2\beta u < \beta'_p$,

$$\mathbb{E}|R_2| \leq (cN^{4-2p})^{\frac{1}{2}} (cN^{2-p})^{\frac{1}{2}} = c' N^{3-\frac{3p}{2}}. \tag{6.29}$$

The above condition is always satisfied if $\beta < \frac{1}{2}\beta'_p$.

The results above almost prove the theorem. What remains to show is that in the leading terms, we can replace without harm the Gibbs measure $\bar{\mathcal{G}}_0^\mu$ by $\mathcal{G}$. More precisely, we claim that

$$\left| \mathbb{E} \bar{\mathcal{G}}_0^{\mu \otimes 2} [R^p] - \mathbb{E} \mathcal{G}^{\otimes 2} [R^p] \right| \leq c N^{1-\frac{p}{2}}, \tag{6.30}$$

for some constant c.

The proof of this claim is done exactly as before, namely by expanding the Boltzmann factors, this time, however, only to zero order. We get

$$\mathcal{G}^{\otimes 2}[R^p] = \bar{\mathcal{G}}_0^{\mu \otimes 2}[R^p] + \bar{\mathcal{G}}_u^{\mu \otimes 2}[R(\sigma,\sigma')^p (H^\mu(\sigma) + H^\mu(\sigma'))] + \bar{\mathcal{G}}_u^{\mu \otimes 3}[R(\sigma,\sigma')^p H^\mu(\sigma'')]. \tag{6.31)}$$

Since $R^p \in [0,1]$, the second term on the right is bounded by

$$|\bar{\mathcal{G}}_u^{\mu \otimes 2}[R(\sigma,\sigma')^p (H^\mu(\sigma) + H^\mu(\sigma'))]| = 2|\bar{\mathcal{G}}_u^{\mu \otimes 2}[R(\sigma,\sigma')^p H^\mu(\sigma)]| \leq 2\bar{\mathcal{G}}_u^\mu[|H(\sigma)|]. \tag{6.32}$$

Proceding as above we get,

$$|\mathbb{E}\bar{\mathcal{G}}_u^{\mu \otimes 2}[R(\sigma,\sigma')^p (H^\mu(\sigma) + H^\mu(\sigma'))]| \leq 2\mathbb{E}\bar{\mathcal{G}}_u^\mu[|H(\sigma)|] \leq 2c\mathbb{E}, \bar{\mathcal{G}}_0^\mu[e^{-\beta u H^\mu}|H^\mu|] \leq 2c' N^{1-\frac{p}{2}}. \tag{6.33}$$

The third term on the right of (6.31) is bounded by the same order. Indeed,

$$|\bar{\mathcal{G}}_u^{\mu \otimes 3}[R(\sigma,\sigma')^p H^\mu(\sigma'')]| \leq \bar{\mathcal{G}}_u^\mu[|H^\mu(\sigma)|], \tag{6.34}$$

from which the bound follows again by integration. This proves the claim (6.30).

To finish the proof of the Theorem, we sum the contributions we have obtained. Relation (6.1) implies that

$$\begin{aligned} \left| \beta \mathbb{E} \frac{\partial F_N}{\partial \beta} - \alpha\beta^2 (1 - \mathbb{E}\mathcal{G}^{\otimes 2}[R^p]) \right| &= \beta \left| -\frac{1}{N} \mathbb{E}\mathcal{G}[H] - \sum_{\mu=1}^{M(N)} \beta N^{1-p} + \sum_{\mu=1}^{M(N)} \beta N^{1-p} \mathcal{G}^{\otimes 2}[R^p]) \right| \\ &\leq \beta \left| -\frac{1}{N} \mathbb{E}\mathcal{G}[H] - \sum_{\mu=1}^{M(N)} \beta N^{1-p} + \sum_{\mu=1}^{M(N)} \beta N^{1-p} \mathbb{E} \bar{\mathcal{G}}_0^{\mu \otimes 2}[R^p] \right| \\ &\quad + \beta^2 \sum_{\mu=1}^{M(N)} N^{1-p} \left| (\mathbb{E}\bar{\mathcal{G}}^{\otimes 2}[R^p] - \mathbb{E}\bar{\mathcal{G}}_0^{\mu \otimes 2}[R^p]) \right| \end{aligned} \tag{6.35}$$

Using the decomposition (6.5), and the results (6.6), (6.7) and (6.10) in the first term, and the bound (6.31) in the second, we get

$$\begin{aligned}\left|\beta\mathbb{E}\,\frac{\partial F}{\partial\beta}-\alpha\beta^2(1-\mathbb{E}\mathcal{G}^{\otimes 2}[R^p])\right| &\leq c\beta\left|\sum_{\mu=1}^{M(N)}(\beta N^{1-p}\mathcal{O}(N^{-1})+R_1^\mu+R_2^\mu+R_3^\mu)\right| \\ &+c'\beta^2\sum_{\mu=1}^{M(N)}N^{1-p}N^{1-\frac{p}{2}}.\end{aligned}\tag{6.36}$$

We finally insert the bounds (6.25), (6.26) and (6.29) on the errors R_i, which are valid if $\beta<\frac{1}{2}\beta_p'$. This yields

$$\left|\beta\mathbb{E}\,\frac{\partial F}{\partial\beta}-\alpha\beta^2(1-\mathbb{E}\mathcal{G}^{\otimes 2}[R^p])\right|\leq c_\beta N^{-1}+c_\beta' N^{2-\frac{p}{2}}\leq C_\beta.\tag{6.37}$$

This proves Theorem 1.5. □

6.2. Condensation: Proof of Theorem 1.6.

Theorem 1.6 follows now just as the analogous result in [T3] from the convexity of the free energy. Suppose that $\beta<\beta_p$. Since we always assume that $\alpha\geq\alpha_p$, then

$$\limsup_{N\uparrow\infty}\mathbb{E}F_N=\frac{\alpha\beta^2}{2}\tag{6.38}$$

by the definition of β_p. As remarked after their definition in Chapter 2, $\mathbb{E}F_N$ is convex for all N. It then follows from a standard result in convex analysis ([Ro], Theorem 25.7) that

$$\limsup_{N\uparrow\infty}\mathbb{E}\,\frac{\partial F_N}{\partial\beta}=\frac{\partial}{\partial\beta}\limsup_{N\uparrow\infty}\mathbb{E}F_N=\alpha\beta.\tag{6.39}$$

Hence, from Theorem 1.5,

$$\mathbb{E}\mathcal{G}^{\otimes 2}[R^p]+\mathbb{E}\,\frac{\partial F_N}{\partial\beta}=\alpha\beta+\mathcal{O}(N^{-1}),\tag{6.40}$$

and thus, passing to the limit,

$$\limsup_{N\uparrow\infty}\mathbb{E}\mathcal{G}^{\otimes 2}[R^p]+\alpha\beta=\alpha\beta,\tag{6.41}$$

which in turn implies that

$$\limsup_{N\uparrow\infty}\mathbb{E}\mathcal{G}_N^{\otimes 2}[R^p]=0.\tag{6.42}$$

Suppose now that

$$\limsup_{N\uparrow\infty}\mathbb{E}\,\frac{\partial F_N}{\partial\beta}<\alpha\beta.\tag{6.43}$$

Then it follows immediately from Theorem 1.5 that

$$\liminf_{N\uparrow\infty}\mathbb{E}\mathcal{G}^{\otimes 2}[R^p]=\alpha\beta-\limsup_{N\uparrow\infty}\mathbb{E}\,\frac{\partial F_N}{\partial\beta}>\alpha\beta-\alpha\beta=0.\tag{6.44}$$

This proves (1.27). To see where the condition (6.43) actually holds, we observe first that by Lemma 4.5, it is satisfied for all

$$\frac{1}{2}\beta_p'\beta > \hat{\beta}_p = \sqrt{\frac{2\ln 2}{\alpha}}. \tag{6.45}$$

This concludes the proof of the Theorem. □

Remark: Of course one would expect (6.43) starts to hold right after the critical temperature. In fact, a weak version of this can be proven. Namely, Theorem 5.5 in [Ro] implies that the function

$$f(\beta) = \limsup_{N\uparrow\infty} \mathbb{E} F_N \tag{6.46}$$

is a convex, bounded function on $\mathcal{U} = [0, \beta_p')$. By Theorem 25.3 in [Ro] it is thus differentiable on an open set $\mathcal{D} \subset \mathcal{U}$ which contains all but perhaps countably many points of $\mathcal{U}$, and its derivative f' is bounded on $\mathcal{D}$. Lebesgue's integrability criterion then implies that

$$f(\beta) = f(\beta_p) + \int_{\beta_p}^{\beta} f'(u)du, \quad \forall \beta > \beta_p. \tag{6.47}$$

Now it is immediate that for all $\beta > \beta_p$ there must exist a set $I \subset (\beta_p, \beta)$ with strictly positive Lebesgue measure, on which f' is strictly less than $\alpha\beta$. Indeed, were this not the case, then $f \geq \frac{\alpha\beta^2}{2}$, which contradicts the definition of β_p. Since β was arbitrary, the relevant condition (6.43) is satisfied on sets of positive Lebesgue measure arbitrarily close to β_p.

6.3 Proof of Theorem 1.7.

We have shown that in the low temperature phase, the replica overlap is not concentrated on zero. We will now show that its distribution is concentrated on a neighborhood of zero and 1.

Proof of Theorem 1.7: Let C_N^+, C_N^- be such that

$$\mathbb{P}\left[\sup_\sigma(-H_N(\sigma)) \notin [NC_N^-, NC_N^+]\right] = p_N = o(1) \tag{6.48}$$

Then

$$\begin{aligned} \mathbb{E}\mathcal{G}_N^{\otimes 2}(R_N(\sigma,\sigma') \in I) &\leq \mathbb{E}\mathbb{1}_{\sup_\sigma |H_N(\sigma)| \leq NC_N} \frac{\mathbb{E}_{\sigma,\sigma'} e^{-\beta(H_N(\sigma)+H_N(\sigma'))}\mathbb{1}_{R_N(\sigma,s')\in I}}{2^{-2N}e^{N2C_N^-}} + p_N \\ &= \frac{\mathbb{E}\mathbb{E}_{\sigma,\sigma'} e^{-\beta(H_N(\sigma)+H_N(\sigma'))}\mathbb{1}_{-\beta(H_N(\sigma)+H(\sigma'))\leq NC_N^+\beta}\mathbb{1}_{R_N(\sigma,\sigma')\in I}}{2^{-2N}e^{\beta N2C_N^-}} + p_N \end{aligned} \tag{6.49}$$

The numerator has been estimated in (4.21). Using this, we get

$$\begin{aligned} \mathbb{E}\mathcal{G}_N^{\otimes 2}(R_N(\sigma,\sigma') \in I) &\leq \sum_{t\in I} C_3 \frac{\exp\left(N\left[2\beta C_N^+\left(1 - \frac{C_N^+}{2\alpha\beta(1+t^p)}\right) - I(t)\right]\right)}{e^{\beta N2C_N^- - 2\ln 2}} + p_N \\ &= \sum_{t\in I} C_3 \exp\left(N2\beta(C_N^+ - C_N^-) + N\left(2\ln 2 - \frac{(C_N^+)^2}{\alpha(1+t^p)} - I(t)\right)\right) + p_N \end{aligned} \tag{6.50}$$

Let us note first that from (6.50) it is obvious that if we can choose $|C_N^+ - C_N^-| \leq N^{-\epsilon}$, then the result cannot depend on β. An obvious candidate for these numbers is thus $N^{-1}\mathbb{E}\sup_\sigma(-H_N(\sigma)) \pm \epsilon$. Indeed we have

Lemma 6.3: *For any $\epsilon > 0$, and for all N large enough,*

$$\mathbb{P}\left[|\frac{1}{N}\sup_\sigma(-H_N(\sigma)) - \mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma))| > \epsilon\right] \leq N^{-2} \tag{6.51}$$

Proof: Note first that

$$2^{-N} \leq Z_N(\beta)e^{\beta\sup_\sigma(-H_N(\sigma))} \leq 1 \tag{6.52}$$

and therefore

$$|\frac{1}{\beta}F_N(\beta) - \frac{1}{N}\sup_\sigma(-H_N(\sigma))| \leq \frac{\ln 2}{\beta} \tag{6.53}$$

Therefore, for any $\beta < \infty$,

$$\begin{aligned}
&\left|\frac{1}{N}\sup_\sigma(-H_N(\sigma)) - \mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma))\right| \\
&= \left|\frac{1}{N}\sup_\sigma(-H_N(\sigma)) - \frac{1}{\beta}F_N(\beta) + \frac{1}{\beta}F_N(\beta) - \mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma)) + \mathbb{E}\frac{1}{\beta}F_N(\beta) - \mathbb{E}\frac{1}{\beta}F_N(\beta)\right| \\
&\leq \left|\frac{1}{\beta}F_N(\beta) - \mathbb{E}\frac{1}{\beta}F_N(\beta)\right| + \frac{2\ln 2}{\beta}
\end{aligned} \tag{6.54}$$

By Proposition 6.2,

$$\mathbb{P}\left[\left|\frac{1}{\beta}F_N(\beta) - \mathbb{E}\frac{1}{\beta}F_N(\beta)\right| > N^{-1/2+\epsilon}\right] \leq CN^{-n} \tag{6.55}$$

from which the claimed result follows by choosing e.g. $\beta = \epsilon^{-1}4\ln 2$. $\square$

Using this result, and setting $C_N \equiv \mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma))$, we get that

$$\mathbb{E}\mathcal{G}_N^{\otimes 2}(R_N(\sigma,\sigma') \in I) \leq \sum_{t\in I} C_3 \exp\left(N4\beta\epsilon + N\left(2\ln 2 - \frac{(C_N+\epsilon)^2}{\alpha(1+t^p)} - I(t)\right)\right) \tag{6.56}$$

Since ϵ can be chosen as small as we like, e.g. $\delta\beta^{-1}$, we already see that our result will be uniform in β.

It remains to estimate $\mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma))$. We will only consider the case $\alpha > \frac{\ln 2}{2p!}$. In that case it follows from Lemma 3.4 that $C_N \leq \sqrt{2\alpha\ln 2} + C/N$ from a bound completely analogous to (2.3). For a lower bound, note that for any β,

$$\mathbb{E}\frac{\partial}{\partial\beta}F_N(\beta) = \frac{1}{N}\mathbb{E}\mathcal{G}_N(-H_N(\sigma)) \leq \mathbb{E}\frac{1}{N}\sup_\sigma(-H_N(\sigma) \tag{6.57}$$

But we know that for all $\beta \leq \beta_p'$, $\lim_{N\uparrow\infty}\mathbb{E}F_N(\beta) = \frac{\alpha\beta^2}{2}$, and therefore by standard results $\lim_{N\uparrow\infty}\mathbb{E}\frac{\partial}{\partial\beta}F_N(\beta) = \alpha\beta$. Thus chosing β as large as possible we see that we see that

$$C_N \geq \alpha\beta_p' - \delta_N \tag{6.58}$$

where $\delta_N \downarrow 0$, as $N \uparrow \infty$. But Theorem 1.2 and the estimate (1.20) show that

$$C_N \geq \sqrt{2\alpha \ln 2} - \frac{2^{-p-1}\sqrt{\alpha}}{\sqrt{2\ln 2}} - \delta_N \tag{6.59}$$

Therefore we have that for any $\delta > 0$, and for p large,

$$\begin{aligned}\mathbb{E}\mathcal{G}_N^{\otimes 2}(R_N(\sigma,\sigma') \in I) \leq \sum_{t\in I} C_3 \exp\Bigg(N\big(\delta + \frac{2^{-p} + 2\delta_N/\sqrt{\alpha} + O(2^{-2p})}{1+t^p}\big)+ \\ N\Big(2\ln 2 - \frac{2\ln 2}{1+t^p} - I(t)\Big)\Bigg) + p_N \\ \leq \sum_{t\in I} C_3 \exp\Bigg(N(\delta + 2^{-p}) + N\Big(\frac{2\ln 2 t^p}{1+t^p} - I(t)\Big)\Bigg) + p_N\end{aligned} \tag{6.60}$$

The function $\frac{2\ln 2t^p}{(1+t^p)} - I(t)$ vanishes at zero and at one, and is negative everywhere in the interval $(0, 1-z_p)$, where $z_p \sim 2^{-p}$. This implies the main conclusion of Theorem 1.7, (6.48). Note that since $I(t) \sim t^2$ for small t, we can chose the interval I more precisely of the form $I_p = (C2^{-p/2}, 1 - C2^p)$, with C a constant of order 1.

To proof the estimate (1.29) in the high-temperature case is considerably simpler. Since we already have the estimate $\mathbb{E}T(c,b,1) \leq e^{\alpha\beta^2 N - dN/2}$ for some positive d, it remains to show that with suffcently large probability, $Z_N^2(\beta) \geq e^{\alpha\beta^2 N - dN/4}$. To do so, we use the Paley-Zygmund inequality (4.31):

$$\mathbb{P}\left[Z_N \geq e^{-dN/8}\mathbb{E}\tilde{Z}_N\right] \geq \mathbb{P}\left[\tilde{Z}_N \geq e^{-dN/8}\mathbb{E}\tilde{Z}_N\right] \geq (1 - e^{-dN/4})C_3 \tag{6.61}$$

Given that by Lemma 4.1 and Theorem 1.1 $\mathbb{E}\tilde{Z}_N \geq Ce^{N\alpha\beta^2/2}$, (1.29) follows immediately. This completes the proof of Theorem 1.7. $\square$

6.4. Ghirlanda-Guerra identities and lump masses.

The techniques used to prove Theorem 1.5 can also be used to derive the Ghirlanda-Guerra identities [GG] (see also [AC]) that provide relations between distributions of overlaps of a larger number of replicas. This observation is due to Talagrand [T5]. Note that he annopunced more far-reaching results than those we will prove here.

The basic input is the following slight generalization of Theorem 1.5.

Proposition 6.4: *Assume that $\beta \leq \frac{1}{2}\beta_p'$. Let f denote any bounded function of n spins. Then, for any $k \in \{1,\dots,n\}$,*

$$\begin{aligned}\Big|\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n}\left(N^{-1}H_N(\sigma^k)f(\sigma^1,\dots,\sigma^n)\right) \\ - \alpha\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n+1}\left(f(\sigma^1,\dots,\sigma^n)\sum_{l=1}^n R_N^p(\sigma^k,\sigma^l) - nR_N^p(\sigma^k,\sigma^{n+1})\right)\Big| \leq CN^{-1}\end{aligned} \tag{6.62}$$

Proof: The proof of this proposition is an exact rerun of the inequalities (6.36), except for the computation of the leading terms which is however straightforward. We will not repeat the details. $\square$

As in [GG] it then follows from the concentration result Theorem 1.4 and standard arguments that for any bounded function f,

$$\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta \left|\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n}\left(N^{-1}H_N(\sigma^k)f(\sigma^1,\dots,\sigma^n)\right) - \mathbb{E}\mathcal{G}_{N,\beta}\left(N^{-1}H_N(\sigma)\right)\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n}\left(f(\sigma^1,\dots,\sigma^n)\right)\right| = 0 \tag{6.63}$$

for any $\beta' < \beta''$. Combining (6.62) and (6.63) with the bounds (6.62), we arrive at the identity

$$\begin{aligned}&\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n+1}\left[f(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^{n} R_N^p(\sigma^k,\sigma^l) - nR_N^p(\sigma^k,\sigma^{n+1}) + \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}\left(R_N^p(\sigma^1,\sigma^2)\right)\right)\right]\\ &\quad = 0\end{aligned} \tag{6.64}$$

which is the analogue of (16) of [GG]. Note that this can be written as

$$\begin{aligned}&\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n+1}\left[f(\sigma^1,\dots,\sigma^n)R_N^p(\sigma^k,\sigma^{n+1})\right]\\ &= \frac{1}{n}\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n}\left[f(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^{n} R_N^p(\sigma^k,\sigma^l) + \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}\left(R_N^p(\sigma^1,\sigma^2)\right)\right)\right]\end{aligned} \tag{6.65}$$

and choosing f to be the indicator function

$$f(\sigma^1,\dots,\sigma^n) = \mathbb{I}_{\forall_{k\neq l} R_N(\sigma^k,\sigma^l)=q_{kl}} \tag{6.66}$$

This implies that

$$\begin{aligned}&\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes n+1}\left[R_N^p(\sigma^k,\sigma^{n+1})\middle|\forall_{k\neq l}R_N(\sigma^k,\sigma_l) = q_{kl}\right]\\ &= \frac{1}{n}\sum_{l\neq k}^{n} q_{kl}^p + \frac{1}{n}\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}\left(R_N^p(\sigma^1,\sigma^2)\right)\end{aligned} \tag{6.67}$$

which is the relation (17) of [GG].

Remark: While [GG] claim to obtain the same relations also for all other moments of the replica overlaps, it needs to be said that they tacitly assume the continuity of the Gibbs measures with respect to certain random perturbations of the Hamiltonian that is not only not proven but is certain to be false in the generality they are announced. Otherwise, the argument below could be considerably sharpened and simplified.

The main use of the identities (6.67) is that they allow to draw conclusions about the distribution of the masses of the Gibbs measures on the so-called ' 'Talagrand-lumps'.

Proof of Theorem 1.8: The starting point of the argument is that Theorem 1.5 together with Theorem 1.9 in fact imply that the distribution of the replica overlaps has positive mass both near zero and near one. Let us set

$$\begin{aligned} p_0 &\equiv \mathbb{E}\mathcal{G}_N^{\otimes 2}\left(|R_N(\cdot,\cdot)| \leq \epsilon_0\right) \\ p_1 &\equiv \mathbb{E}\mathcal{G}_N^{\otimes 2}\left(|R_N(\cdot,\cdot)| \geq 1-\epsilon_1\right) \end{aligned} \tag{6.68}$$

Since by convexity (see (6.39)) for all $\beta \geq \beta_p$, except possibly for a countable number of exceptional points

$$\alpha\beta_p \leq \liminf_N \mathbb{E}\frac{\partial}{\partial\beta}F_N(\beta) \leq \liminf_N \mathbb{E}\frac{\partial}{\partial\beta}F_N(\beta) \leq \sqrt{\alpha 2\ln 2} \tag{6.69}$$

we have on the one hand that

$$\limsup_N p_0 \leq \frac{\sqrt{\alpha 2\ln 2}}{\alpha\beta(1-\epsilon_0)} \tag{6.70}$$

and

$$\limsup_N p_1 \leq \frac{\beta-\beta_p}{\beta(1-\epsilon_1)^p} \tag{6.71}$$

Since we know that $\lim_N(p_0+p_1)=1$, and this implies what we want for β somewhat larger than β_p. Recall that $\epsilon_0 \sim 2^{-p/2}$ and $\epsilon_1 \sim 2^{-p}$.

This result shows first of all that it is not possible that the mass of one single (pair of) lump(s) can be almost equal to one, since in that case p_0 would be close to zero (which is impossible by (6.71)).

Now assume that the assertion of Theorem 1.8 fails. Then there exists a first instance k^* such that

$$\lim_{N\uparrow\infty} \mathbb{E}\mathcal{G}_N\left(\cup_{l=1}^{k^*}\mathcal{C}_l\right) = 1 \tag{6.72}$$

Now define events $\mathcal{Q}_{\epsilon_0}^{(n)} \in \mathcal{B}_n$ by

$$\mathcal{Q}_{\epsilon}^{(n)} \equiv \left\{\underline{R} \in [-1,1]^{n(n-1)/2} | \forall_{1\leq l<k\leq n} |R_{lk}| \leq \epsilon_0\right\} \tag{6.73}$$

The important obervation is that if $\{R_N(\sigma_l,\sigma_k)\}_{1\leq l<k\leq k^*} \in \mathcal{Q}_\epsilon^{(k^*)}$, then there exists some permutation $\pi \in S_{k^*}$ such that with probability one $\sigma^k \in C_{\pi(k)}$ for all $k \leq k^*$. In particular

$$\begin{aligned} &\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\{R_N(\sigma^l,\sigma_m)\}_{1\leq l<\leq k^*}\in\mathcal{Q}_{\epsilon_0}^{(k^*)}}\right] \\ &= \lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\exists_\pi\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \end{aligned} \tag{6.74}$$

But

$$\begin{aligned}
\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\exists_\pi \forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] &= \sum_{\pi\in S_{k^*}} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&= \sum_{\pi\in S_{k^*}}\sum_{j=1}^{k^*} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(j)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&= \sum_{\pi\in S_{k^*}}\sum_{j\neq k}^{k^*} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{j})\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(j)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&+ \sum_{\pi\in S_{k^*}} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(k)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right]
\end{aligned} \tag{6.75}$$

where we used the symmetry betwen replicas in the terms $j\neq k$ to exchange σ^{k^*+1} with σ^j. Note that for the first term we have the obvious (though not very good) bound

$$\begin{aligned}
0 &\leq \sum_{\pi\in S_{k^*}}\sum_{j\neq k}^{k^*} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{j})\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(j)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&\leq \epsilon_0^p \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*}\left[\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&= \epsilon_0^p \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*}\left[\mathcal{Q}_\epsilon^{k^*}\right]
\end{aligned} \tag{6.76}$$

while the second satisfies

$$\begin{aligned}
&\sum_{\pi\in S_{k^*}} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(k)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&\qquad \geq (1-\epsilon)^p \sum_{\pi\in S_{k^*}} \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[\mathbb{1}_{\sigma_{k^*+1}\in\mathcal{C}_{\pi(k)}}\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&\qquad = \frac{1}{k^*}(1-\epsilon_1)^p \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[\mathbb{1}_{\forall_{l=1}^{k^*}\sigma^l\in\mathcal{C}_{\pi(l)}}\right] \\
&\qquad = \frac{1}{k^*}(1-\epsilon_1)^p \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*}\left[\mathcal{Q}_{\epsilon_0}^{k^*}\right]
\end{aligned} \tag{6.77}$$

where we used the obvious permutation symmetry among the first k^* replicas. Let us now use (6.65) with f the indicator function of the event $\mathcal{Q}_{\epsilon_0}^{(k^*)}$. clearly we get

$$\begin{aligned}
&\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[R_N^p(\sigma^k,\sigma^{k^*+1})\mathbb{1}_{\{R_N(\sigma^l,\sigma_m)\}_{1\leq l<\leq k^*}\in\mathcal{Q}_{\epsilon_0}^{(k^*)}}\right] \\
&\qquad \leq \frac{1}{k^*}\lim_{N\uparrow\infty}\int_{\beta'}^{b''} d\beta \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes k^*+1}\left[\mathbb{1}_{\{R_N(\sigma^l,\sigma_m)\}_{1\leq l<\leq k^*}\in\mathcal{Q}_{\epsilon_0}^{(k^*)}}\right]\left((k^*-1)\epsilon_0^p + \mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}R^p(\sigma,\sigma')\right)
\end{aligned} \tag{6.78}$$

Comparing (6.76), (6.77) to (6.78) we see that

$$(1-\epsilon_1)^p \leq (k^*-1)\epsilon_0^p + \lim_{N\uparrow\infty}\mathbb{E}\mathcal{G}_{N,\beta}^{\otimes 2}R^p(\sigma,\sigma') \leq (k^*-1+p_0)\epsilon^p + p_1 \tag{6.79}$$

This implies the lower bound

$$k^* \geq \frac{(1-\epsilon_1)^p - p_1}{\epsilon_0^p} \tag{6.80}$$

Quantitatively, this estimate can be refined to

$$k^* \geq C^{-1} 2^{3p/2} ((1 - C2^{-p})^p - p_1) = 2^p p_0 (1 - O(2^{-2p}) \tag{6.81}$$

This proves the theorem. $\square$

8. Spin Glass Phase: Proof of Theorem 1.9

Having established the existence of an infinity of lumps that carry the Gibbs measure in the low temperature phase, one would like to know whether these are in any way related to the original patterns. Recall that in the standard Hopfield model at small α the Gibbs measure concentrates on small balls around the patterns ξ^μ. Of course the reader will expect that this will not be the case here. To prove this fact, we first obtain two estimate the value of the Hamiltonian in the vicinity of each pattern.

Lemma 8.1: *The Hamiltonian evaluated at the patterns satisfies*

$$\mathbb{P}\left[|H_N(\sigma = \xi^\mu)| \geq \frac{N}{(p!)^{\frac{1}{2}}} + zN \right] \leq C \begin{cases} e^{-\frac{z^2 N}{2\alpha}}, & \text{if } z \leq \beta'_p, \\ e^{-\beta'_p (z - \frac{\alpha \beta'_p}{2}) N}, & \text{otherwise.} \end{cases} \tag{8.1}$$

Proof: The Hamiltonian at the pattern ξ^μ is given by

$$\begin{aligned} H(\sigma = \xi^\mu) &= -\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\mathcal{I}} \xi^\mu_{\mathcal{I}} \xi^\mu_{\mathcal{I}} - \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu \neq \mu} \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \xi^\mu_{\mathcal{I}} \\ &= -\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \binom{N}{p} - \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu \neq \mu} \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \xi^\mu_{\mathcal{I}}, \end{aligned} \tag{8.2}$$

which implies that

$$-H_N(\xi^\mu) \leq \frac{N}{(p!)^{\frac{1}{2}}} + \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu \neq \mu} \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \xi^\mu_{\mathcal{I}}. \tag{8.3}$$

We estimate the random part in (8.3) by the same method used in the proof of Theorem 1.1. By Chebyshev's exponential inequality, conditional independence of $\sum_{i=1}^N \xi_i^\nu \xi_i^\mu$ and $\sum_{i=1}^N \xi_i^{\nu'} \xi_i^\mu$ (for $\nu \neq \mu$), and expansion of the exponential, we get for $z > 0$

$$\begin{aligned} \mathbb{P}\left[\left| \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu \neq \mu} \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \xi^\mu_{\mathcal{I}} \right| \geq zN \right] &\leq \inf_{t>0} e^{-tzN} \prod_{\nu \neq \mu} \mathbb{E}\left[\exp\left(t \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \xi^\mu_{\mathcal{I}} \right) \right] \\ &\leq \inf_{t>0} e^{-tzN} \prod_{\nu \neq \mu} \left(1 + \frac{t^2 p!}{2N^{2p-2}} \binom{N}{p} \right. \\ &\qquad \left. + \frac{t^3 (p!)^{\frac{3}{2}}}{3!\, N^{3p-3}} \mathbb{E}\left[\left| \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \right|^3 e^{t \frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \left| \sum_{\mathcal{I}} \xi^\nu_{\mathcal{I}} \right|} \right] \right). \end{aligned} \tag{8.4}$$

The error term can be written as

$$\frac{1}{N^{3p-3}}\mathbb{E}\left[\left|\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}\right|^3 e^{t\frac{(p!)^{\frac{1}{2}}}{N^{p-1}}\left|\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}\right|}\right] = \frac{1}{N^{\frac{3p}{2}-3}}\mathbb{E}\left[\left|N^{-\frac{p}{2}}\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}\right|^3 e^{t\frac{(p!)^{\frac{1}{2}}}{N^{\frac{p}{2}-1}}|N^{-\frac{p}{2}}\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}|}\right] \tag{8.5}$$

This latter term is exactly the same as in (3.2) (with β replaced by t). Hence, we get (compare (3.3))

$$\mathbb{P}\left[\frac{(p!)^{\frac{1}{2}}}{N^{p-1}}\sum_{\nu\neq\mu}\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}\xi^\mu_{\mathcal{I}} \geq zN\right] \leq \inf_{t\in(0,\beta'_p)} e^{-tzN+\frac{\alpha t^2 N}{2}+C_1}. \tag{8.6}$$

Minimizing the exponent yields

$$\begin{aligned}\mathbb{P}\left[-H_N(\sigma=\xi^\mu) \geq \frac{N}{(p!)^{\frac{1}{2}}}+zN\right] &\leq \mathbb{P}\left[\frac{(p!)^{\frac{1}{2}}}{N^{p-1}}\sum_{\nu\neq\mu}\sum_{\mathcal{I}}\xi^\nu_{\mathcal{I}}\xi^\mu_{\mathcal{I}} > zN\right]\\ &\leq C_2\begin{cases} e^{-\frac{z^2}{2\alpha}N}, & \text{if } 0<z\leq\alpha\beta'_p,\\ e^{-\beta'_p(z-\frac{\alpha\beta'_p}{2})N}, & \text{otherwise.}\end{cases}\end{aligned} \tag{8.7}$$

This proves the claim. □

The next result shows that the Hamiltonian does not fluctuate much around a pattern. This result was already proven by Newman [N1] for the Hamiltonian $\bar{H}$. In our case this is even simpler. Define $B_\delta(\sigma)$ to be the $(N\delta)$-ball around the configuration σ in the Hamming distance. Then we have the following

Lemma 8.2: *If $\delta<\frac{1}{p}$, then there exists a constant $C>0$ such that*

$$\mathbb{P}\Big[\exists\sigma\in B_\delta(\xi^\mu):|H_N(\sigma)-H_N(\xi^\mu)|\geq(2^{p-1}(p!)^{-\frac{1}{2}}\delta+z)N\Big] \leq Ce^{-N(f_\delta(z)+\delta\ln\delta+(1-\delta)\ln(1-\delta))}, \tag{8.8}$$

where

$$f_\delta(z)=\begin{cases}\frac{z^2}{2^p\alpha\delta}, & \text{if } z\leq 2^{p-1}\alpha\delta\beta'_p;\\ e^{-\beta'_p N(z-\frac{\alpha\beta'_p}{2(p-1)!})}, & \text{otherwise.}\end{cases} \tag{8.9}$$

Proof: By standard arguments (see also [N1], in particular inequality (2.3) and surrounding comments),

$$\begin{aligned}&\mathbb{P}\Big[\exists\sigma\in B_\delta(\xi^\mu):|H_N(\sigma)-H_N(\xi^\mu)|\geq(\delta+z)N\Big]\\ &\qquad\leq\sum_{q=1}^{\lfloor\delta N\rfloor}\binom{N}{q}\mathbb{P}[|H_N(\zeta^q)-H_N(\xi^\mu)|\geq(\delta+z)N],\end{aligned} \tag{8.10}$$

where

$$\zeta^q_i=\begin{cases}-\xi^\mu_i, & \text{if } i\leq q;\\ \xi^\mu_i, & \text{if } i\geq q+1.\end{cases} \tag{8.11}$$

We start by calculating the difference $|H(\zeta^q) - H(\xi^\mu)|$. Let $\mathcal{J} = \mathcal{J}_q = \{1, \ldots, q\}$. One obtains

$$
\begin{aligned}
H(\zeta^q) - H(\xi^\mu) &= -\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu=1}^{M(N)} \sum_{\mathcal{I}} (\zeta^q_\mathcal{I}\xi^\nu_\mathcal{I} - \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}) \\
&= -\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu=1}^{M(N)} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} (\zeta^q_\mathcal{I}\xi^\nu_\mathcal{I} - \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}) \\
&= 2\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu=1}^{M(N)} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I} \\
&= 2\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{odd}} 1 + 2\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu\neq\mu} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}
\end{aligned}
\tag{8.12}
$$

Explicitly, this is

$$
H(\zeta^q) - H(\xi^\mu) = 2\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{r=1,\text{ odd}}^{p-1} \binom{N-q}{p-r}\binom{q}{r} + 2\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu\neq\mu} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I} \tag{8.13}
$$

Let us treat the random term in (8.13) first. By the usual procedure, we get

$$
\begin{aligned}
&\mathbb{P}\left[\left|\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu\neq\mu} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}\right| \geq zN\right] \\
&\quad\leq 2\inf_{t>0} e^{-tzN} \prod_{\nu\neq\mu} \Bigg\{1 + \frac{t^2 p!}{2N^{2p-2}} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} 1 \\
&\qquad + \frac{t^3(p!)^{\frac{3}{2}}}{3!\,N^{3p-3}} \mathbb{E}\left[\left|\sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}\right|^3 \exp\left(\frac{t(p!)^{\frac{1}{2}}}{N^{p-1}}\left|\sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}\right|\right)\right]\Bigg\} \\
&\quad\leq 2\inf_{t\in(0,\beta'_p)} e^{-tzN} \prod_{\nu\neq\mu} \left\{1 + \frac{t^2 p!}{2N^{2p-2}} \sum_{r=1\text{ odd}}^{p-1} \binom{N-q}{p-r}\binom{q}{r} + C_1 N^{3-\frac{3p}{2}}\right\}.
\end{aligned}
\tag{8.14}
$$

The last line follows from the usual bound on the error term (see the proof of Theorem 1.1 in Chapter 3; in fact, t can even be chosen somewhat larger than β'_p, since the sum over sets $\mathcal{I}$ contains fewer terms than we had there).

To treat products of binomial coefficients in last expression, observe that if $q \leq \lfloor\delta N\rfloor < \frac{N}{2}$, then the following inequality holds,

$$
\begin{aligned}
p! \sum_{r=1,\text{ odd}}^{p-1} \binom{N-q}{p-r}\binom{q}{r} &\leq \sum_{r=1,\text{ odd}}^{p-1} \binom{p}{r}(N-q)^{p-r}q^r \\
&\leq (N-q)^{p-1} q \sum_{r=1,\text{ odd}}^{p-1} \binom{p}{r} = 2^{p-1}(N-q)^{p-1}q.
\end{aligned}
\tag{8.15}
$$

Using (8.15) in (8.14) yields

$$
\begin{aligned}
&\mathbb{P}\left[\left|\frac{(p!)^{\frac{1}{2}}}{N^{p-1}} \sum_{\nu\neq\mu} \sum_{\mathcal{I}:|\mathcal{I}\cap\mathcal{J}|\text{ odd}} \xi^\mu_\mathcal{I}\xi^\nu_\mathcal{I}\right| \geq zN\right] \\
&\qquad\leq 2\inf_{t\in(0,\beta'_p)} e^{-tzN} \exp\left(\frac{\alpha t^2}{2N^{p-1}} 2^{p-1}(N-q)^{p-1}q + C_1\right).
\end{aligned}
\tag{8.16}
$$

The deterministic term in (8.13) is given by (again using (8.15))

$$\begin{aligned}\frac{(p!)^{\frac12}}{N^{p-1}}\sum_{r=1,\,\mathrm{odd}}^{p-1}\binom{N-q}{p-r}\binom{q}{r} &\le \frac{1}{(p!)^{\frac12}N^{p-1}}\sum_{r=1,\,\mathrm{odd}}^{p-1}\binom{p}{r}(N-q)^{p-r}q^r \\ &\le \frac{2^{p-1}}{(p!)^{\frac12}N^{p-1}}(N-q)^{p-1}q.\end{aligned} \tag{8.17}$$

If $\delta < \frac{1}{p}$, then the last line is bounded by the term for the maximum q. That is

$$\frac{(p!)^{\frac12}}{N^{p-1}}\sum_{r=1,\,\mathrm{odd}}^{p-1}\binom{N-q}{p-r}\binom{q}{r} \le \frac{2^{p-1}}{(p!)^{\frac12}N^{p-1}}(N-\lfloor\delta N\rfloor)^{p-1}\lfloor\delta N\rfloor \le \frac{2^{p-1}}{(p!)^{\frac12}}N\delta. \tag{8.18}$$

Collecting (8.16) and (8.18), we get

$$\begin{aligned}&\mathbb{P}\big[|H(\zeta^q)-H(\xi^\mu)| \ge \frac{2^{p-1}}{(p!)^{\frac12}}\delta N + zN\big] \\ &\qquad\le 2\inf_{t\in(0,\beta_p')} e^{-tzN}\exp\left(\frac{\alpha t^2}{2N^{p-1}}2^{p-1}(N-q)^{p-1}q^r + C_1\right).\end{aligned} \tag{8.19}$$

Plugging this into (8.10) gives

$$\begin{aligned}&\mathbb{P}[\exists\sigma\in B_\delta(\xi^\mu) : |H_N(\sigma)-H_N(\xi^\mu)| \ge (\frac{2^{p-1}}{(p!)^{\frac12}}\delta + z)N] \\ &\qquad\le 2\sum_{q=1}^{\lfloor\delta N\rfloor}\binom{N}{q}\inf_{t\in(0,\beta_p')} e^{-tzN}\exp\left(\frac{\alpha t^2}{2N^{p-1}}2^{p-1}(N-q)^{p-1}q + C_1\right).\end{aligned} \tag{8.20}$$

It is straightforward to check that under our assumptions on δ and for fixed t, the ratio between two consecutive terms in the above sum is larger than 2, and therefore the whole sum is at most twice the maximum term,

$$\begin{aligned}&\mathbb{P}[\exists\sigma\in B_\delta(\xi^\mu) : |H_N(\sigma)-H_N(\xi^\mu)| > (\frac{2^{p-1}}{(p!)^{\frac12}}\delta + z)N] \\ &\qquad\le 4\binom{N}{\lfloor\delta N\rfloor}\inf_{t\in(0,\beta_p')} e^{-tzN}\exp\left(\frac{2^{p-1}\alpha t^2}{2}N\delta + C_1\right).\end{aligned} \tag{8.21}$$

Minimizing with respect to t and using Stirling's formula for the binomial factor concludes the proof of Lemma 8.2. $\square$

Proof of Theorem 1.9: We observe the following elementary fact. By the definition of the free energy

$$F_N(\beta) \le \frac{\beta}{N}\sup_\sigma |H_N(\sigma)|. \tag{8.22}$$

Hence, by Theorem 1.4, for any $\beta, m, z > 0$ there exists $\bar N \in \mathbb{N}$ such that

$$\mathbb{P}[\frac{1}{N}\sup_\sigma |H_N(\sigma)| < \frac{1}{\beta}\mathbb{E}F_N(\beta) - z] \le \mathbb{P}[F_N(\beta) < \mathbb{E}F_N(\beta) - z] \le CN^{-m}, \tag{8.23}$$

for all $N \geq \bar{N}$. Suppose that $\alpha\beta_p(\alpha) > \frac{1}{(p!)^{\frac{1}{2}}}$. Then there exists $\beta > 0$ such that

$$\alpha > \frac{1}{(p!)^{\frac{1}{2}}(\beta_p - \frac{\beta_p^2}{2\beta})}, \tag{8.24}$$

which is equivalent to

$$\frac{1}{(p!)^{\frac{1}{2}}} < \frac{1}{\beta}(\alpha\beta\beta_p - \frac{\alpha\beta_p^2}{2}) \leq \frac{1}{\beta}\mathbb{E}F_N(\beta) + C_1 N^{-1}. \tag{8.25}$$

The second inequality follows from the convexity of $F_N(\beta)$ and the definition of β_p. But then we can find $\delta \in (0, \frac{1}{p})$ and $z > 0$ such that (for all N sufficiently large)

$$\frac{2^{p-1}}{(p!)^{\frac{1}{2}}}\delta + 3z < \frac{1}{\beta}\mathbb{E}F_N(\beta) - \frac{1}{(p!)^{\frac{1}{2}}}, \tag{8.26}$$

and (with the definition of f_δ from Lemma 8.2)

$$f_\delta(z) + \delta\ln\delta + (1-\delta)\ln(1-\delta) > 0. \tag{8.27}$$

By Lemma 8.1, resp. 8.2, for any $m > 0$, we can find an $\bar{N} \in \mathbb{N}$ such that for all $N \geq \bar{N}$

$$\begin{aligned} &\mathbb{P}[\exists\sigma \in \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu) : |H_N(\sigma)| \geq N(\frac{1}{(p!)^{\frac{1}{2}}} + \delta + 2z)] \\ &\qquad \leq \mathbb{P}[\exists\sigma \in \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu) : |H_N(\sigma) - H_N(\xi^\mu)| \geq N(\delta + z)] \\ &\qquad\quad + \mathbb{P}[\sup_\mu |H_N(\xi^\mu)| \geq N(\frac{1}{(p!)^{\frac{1}{2}}} + z)] \\ &\qquad \leq N^{-m}. \end{aligned} \tag{8.28}$$

On the other hand, the inequality (8.23) implies that

$$\mathbb{P}[\sup_\sigma |H_N(\sigma)| \leq N\frac{\mathbb{E}F_N(\beta)}{\beta} - zN] \leq N^{-m}, \tag{8.29}$$

for all N large enough, so that finally, by standard arguments,

$$\begin{aligned} \mathbb{P}[\arg\sup|H_N(\sigma)| \in \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu)] &\leq \mathbb{P}[\exists\sigma \in \bigcup_{\mu=1}^{M(N)} B_\delta(\xi^\mu) : |H_N(\sigma)| \geq N(\frac{1}{(p!)^{\frac{1}{2}}} + \delta + 2z)] \\ &\quad + \mathbb{P}[\sup_\sigma |H_N(\sigma)| \leq N\frac{\mathbb{E}F_N(\beta)}{\beta} - zN] \\ &\leq N^{-m}, \end{aligned} \tag{8.30}$$

for all N larger than some $\bar{N} \in \mathbb{N}$.

To show the existence of an α_{sp}, we observe that the bounds (1.18) and (1.19) on the critical β imply that the quantity $\alpha\beta_p(\alpha) \sim \sqrt{\alpha}$ and is thus eventually larger than any fixed number. This concludes the proof of Theorem 1.9. $\square$

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