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A Quadrature Algorithm for Wavelet Galerkin Methods

Andreas Rathsfeld[1]

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[1] Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstr. 39
D-10117 Berlin
Germany E-Mail: rathsfel@wias-berlin.de

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Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the wavelet Galerkin method for the solution of boundary integral equations of the first and second kind including integral operators of order $\mathbf{r}$ less than zero. This is supposed to be based on an abstract wavelet basis which spans piecewise polynomials of order d_T. For example, the bases can be chosen as the basis of tensor product interval wavelets defined over a set of parametrization patches. We define and analyze a quadrature algorithm for the wavelet Galerkin method which utilizes Smolyak quadrature rules of finite order. In particular, we prove that quadrature rules of an order larger than $2d_T - \mathbf{r}$ are sufficient to compose a quadrature algorithm for the wavelet Galerkin scheme such that the compressed and quadrature approximated method converges with the maximal order $2d_T - \mathbf{r}$ and such that the number of necessary arithmetic operations is less than $\mathcal{O}(N \log N)$ with N the number of degrees of freedom. For the estimates, a degree of smoothness greater or equal to $2[2d_T - \mathbf{r}] + 1$ is needed.

1 Introduction

It is well-known that boundary element discretizations of boundary integral equations lead to systems with fully populated matrices. In order to cope with the resulting huge matrices several algorithms have been proposed. In particular, the wavelet algorithm of Beylkin, Coifman, and Rokhlin [5] has been thoroughly investigated by e.g. Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [11, 12, 28, 29, 35] (cf. also the contributions by Alpert, Ehrich, Harten, Hu, Micchelli, Rathsfeld, Xu, Yad-Shalom [1, 22, 32, 19, 27, 23]). In the present paper, we shall apply the wavelet technique to a piecewise polynomial Galerkin scheme for two-dimensional boundary integral equations of first or second kind with an integral operator of order $\mathbf{r} < 0$. This corresponds to three-dimensional boundary value problems.

The general construction of wavelet bases over manifolds is quite difficult. We shall use an abstract basis for the space of piecewise polynomials which are continuous or smooth only over the parametrization patches (cf. (2.1)). The possible jump of our trial functions over the boundary lines of the parametrization patches, allows us to work simply with the tensor product versions of the well-established biorthogonal interval wavelets by Dahmen, Kunoth, and Urban [9] (cf. also [7]). For more information on diverse wavelet bases we refer the reader to [6, 8, 13, 14, 15, 25, 33, 37]. All our functions will be defined over grids which are uniform refinements of a coarse initial square shaped grid. Applying these wavelet basis functions of the trial space, we shall recall the well-known compression results due to Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [12, 29, 35]. Note that these wavelet methods are based on the Calderón-Zygmund estimate for the kernel function, i.e. we have to assume that the derivatives $\partial_P^\alpha \partial_Q^\beta K(P,Q)$ of the kernel function $K(P,Q)$ are continuous and bounded by $\mathcal{O}(|P-Q|^{-2-\mathbf{r}-|\alpha|-|\beta|})$.

If the boundary surface is not piecewise planar, then the Galerkin stiffness matrix cannot be computed exactly by simple analytic formulae. For a fast implementation, an efficient quadrature algorithm is needed. This is a very important issue since, for the wavelet methods as well as for the conventional boundary element methods, the assembling of the stiffness matrix takes the major part of the computing time. However, a naive approach usually leads to slow algorithms or to unacceptably large quadrature errors. Now, that the compression algorithm is well-understood, the quadrature part of the wavelet algorithm turns out to be the most difficult part. Nevertheless, good quadrature schemes are available. The first ideas in this direction go back to the starting paper of Beylkin, Coifman, and Rokhlin [5], and related results can be found e.g. in [10, 2, 38, 18, 3, 4]. Complete algorithms for boundary integral operators are presented by v.Petersdorff, Schwab, and Schneider [29, 35] (cf. also the numerical implementation by Lage and Schwab [24] and the recycling scheme of Harbrecht and Schneider [21]). These quadrature methods are h-p-methods, i.e. they are based on tensor products of Gauß type rules defined over geometrically graded meshes. The order of the Gauß rule is chosen appropriately and tends to infinity in the asymptotic convergence analysis. The corresponding error estimates rely on the Calderón-Zygmund estimate for the kernel function and on the piecewise analyticity of the solution, the kernel, and the underlying surface. Note that these assumptions are met in a lot of engineering applications of the boundary element methods.

On the other hand, since the wavelet compression scheme is a pure h-method, it is natural to ask whether it is possible to define a corresponding h-method quadrature scheme which is based on rules of a fixed order, only, and which requires a finite degree of smoothness for the solution, the kernel, and the surface. Such an algorithm can be helpful for applications to surfaces with finite degree of smoothness in problems of e.g. the geosciences. Alternative fast methods like multipole and panel clustering can treat these surfaces with low degree of smoothness as well. However, they assume that the kernel function $K(P,Q)$ is either a restriction to the boundary surface of a piecewise analytic kernel defined in the space domain or such a restriction multiplied by functions depending on one of the two variables P and Q. In contrast to multipole and panel clustering, the wavelet method together with an h-method of quadrature is capable to treat kernel functions which are not the restrictions of analytic functions in space to the boundary surface and which are of finite degree of smoothness, only. For the wavelet collocation, this fact has been established in [19, 34] (cf. also [31, 30]), and the topic of the present paper is to develop and analyze an analogous h-method quadrature scheme for the wavelet Galerkin procedure (cf. Sects. 5-7). If N is the number of degrees of freedom, then we shall show that there is a quadrature algorithm which requires $\mathcal{O}(N \log N)$ arithmetic operations and a memory capacity for $\mathcal{O}(N \log N)$ real numbers to compute an approximated linear system of equations to the Galerkin method. This system can be solved by $\mathcal{O}(N \log N)$ arithmetic operations using a diagonally preconditioned iterative method. Moreover, if $h = \mathcal{O}(N^{-1/2})$ is the step size of the uniform discretization and if $d_T - 1$ is the polynomial degree of the piecewise polynomial trial space, then the numerical solution of the quadrature approximated and

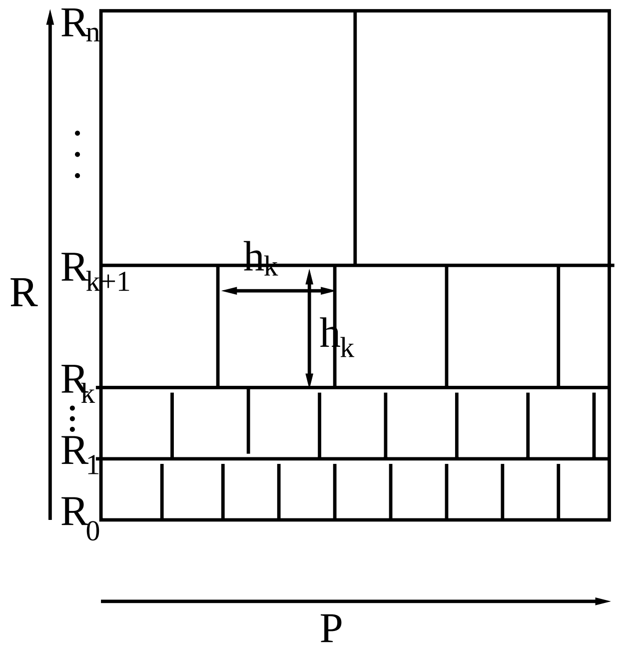

Figure 1: Mesh grading for non separated singularity: $h \sim \frac{1}{n}$, $R_i = (\frac{i}{n})^\eta$, $\mathcal{N} \sim h^{-\eta}$, $\eta > 2$, $h_i = [(\frac{i+1}{n})^\eta - (\frac{i}{n})^\eta]^{-1}$.

compressed wavelet Galerkin method converges with the same optimal order $2d_T - \mathbf{r}$ as the conventional Galerkin method with exactly computed stiffness matrix and right-hand side. Unfortunately, to get this result we have to assume the doubled order $2[2d_T - \mathbf{r}]$ for the degree of smoothness of the kernel function. To avoid a lot of logarithmic factors and to simplify the presentation, we even suppose a smoothness of degree greater or equal to $2[2d_T - \mathbf{r}] + 1$, and we use quadrature rules with an order of convergence $d_Q > 2d_T - \mathbf{r}$. Now let us describe the quadrature rules and let us heuristically indicate why the restrictive smoothness assumption cannot be avoided. First, we look at the far field, i.e. at the entries in the stiffness matrix corresponding to trial and test functions for which the distance of the supports is larger than the diameter of the supports. Due to the compression step the number of these entries is at most $\mathcal{O}(N \log N)$. Nevertheless these remaining far field entries are to be computed. For the collocation method (cf. e.g. [34]) which has almost the same compression structure, a composite quadrature rule of finite convergence order over a uniform partition is applied to the integration of the trial function. The integration over the test functional degenerates to point evaluations for the collocation. The complexity of this quadrature is proportional to the number of subdivision domains of the composite rule. Therefore, for each entry in the collocation stiffness matrix, the maximal number of subdivision domains, depending on level and distance of the corresponding test and trial wavelet function, is determined such that the sum of these numbers over all entries is less than $\mathcal{O}(N \log N)$. Fortunately, using standard quadrature estimates and standard wavelet techniques, it turns out that the resulting quadrature error for the solution of the integral equation is less or equal to the discretization error of the collocation. Now we would like to do the same for the Galerkin method. In this case, unfortunately, we have an additional integration over the test function, but no additional arithmetic operations are allowed since the numbers of quadrature knots for the collocation has already been chosen to be maximal. So the task is to set up a quadrature for the integration over test and trial function which has the same number of quadrature knots as the collocation quadrature, where the integration is performed over the support of the trial function, only. The answer due to Smolyak [36] is to use a special tensor product technique to obtain a sparse grid rule over the tensor product of the supports of test and trial function. This leads to, with the exception of additional logarithmic factors, the same number of quadrature knots as for the integration over one domain and to the same error bounds. Summarizing, for the computation of the far field part of the Galerkin stiffness matrix, we first determine the number of possible quadrature knots such that the sum over all entries is less than $\mathcal{O}(N \log N)$ and such that the number depends on level and distance of the trial and test functional, only (cf. Sect. 7). Using this bound for the knots, we define the Smolyak rule over the tensor product of the supports of test and trial function basing on composite rules of finite order over each factor space (cf. Lemma 5.3).

For the entries in the near field, i.e. for entries for which the corresponding trial and test functions have a distance of the supports less than the maximum diameter of the supports, the quadrature is to be adapted to the singularity of the kernel function $K(P,Q)$ for $P \to Q$. Here the Calderón-Zygmund estimate for the kernel function is not sufficient since unlike to a kernel of the form $K(P,Q) = |P-Q|^{-2-\mathbf{r}}$ the direction of the singularity can not be separated. Indeed, the k-th order derivatives of a Calderón-Zygmund kernel are bounded by $\mathcal{O}(|P-Q|^{-2-\mathbf{r}-k})$ independently of in which direction the derivative is taken, whereas, for $|P-Q|^{-2-\mathbf{r}}$, only the k-th order derivatives with respect to radial direction $R = P-Q$ are bounded by $\mathcal{O}(|P-Q|^{-2-\mathbf{r}-k})$, and those in the tangential direction perpendicular to R are less than $\mathcal{O}(|P-Q|^{-2-\mathbf{r}})$. A missing separation is bad since, for this case, the graded meshes are to be chosen such that the subdomains close to the singularity set $\{(P,Q) : \; P = Q\}$ are small in diameter. The number of subdomains in these graded meshes blows up with smaller step sizes. More precisely, for a graded mesh with mesh size h, the number $\mathcal{N}$ of subdomains is bounded from below by a constant multiple of $h^{-\eta}$ where the exponent η depends on the degree of the grading (cf. the univariate case in Figure 1). If we can separate the singularity by a substitution of variables $(P,Q) = (P, P+R)$ such that $|P-Q|^{-2-\mathbf{r}} = |R|^{-2-\mathbf{r}}$, then the subdomains of the graded mesh close to the singularity set $\{(P,R) : \; R = 0\}$ must be small in R direction but can be larger in the direction of P. For a graded mesh with mesh size h, the number $\mathcal{N}$ of subdomains can be bounded by $\mathcal{O}(h^{-2})$ independently of the degree of mesh grading (cf. the example in Figure 2). Hence, in accordance with this argument we assume that the kernel behaves like $|P-Q|^{-2-\mathbf{r}}$, and we separate the singularity substituting (P,Q) by (P,R) with $R := P-Q$.[1] After this we apply the Smolyak quadrature rule based on the composite rule over uniform partitions for the P-domain and on the composite rule for the Q-domain over partitions graded towards the singularity point $R = 0$ (cf. Lemma 5.2). If test and trial functions are supported on different parametrization patches, the separation of the singularity is more involved, and

[1] More precisely, this separation is to be performed with respect to the variables in the parameter domain.

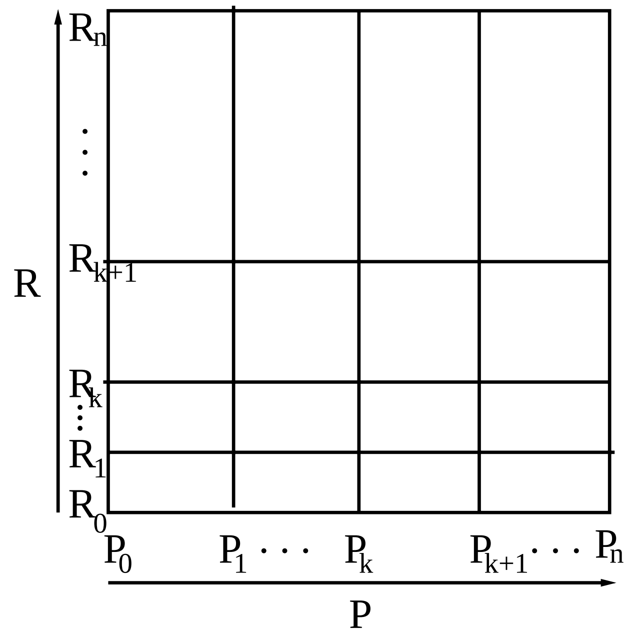

Figure 2: Mesh for separated singularity: $h \sim \frac{1}{n}$, $R_i = (\frac{i}{n})^\eta$, $P_i = \frac{i}{n}$, $\mathcal{N} \sim h^{-2}$.

slight modifications of the quadratures are needed (cf. Lemmata 5.4 and 5.5).

If we have only one parametrization mapping $\kappa :\ S \longrightarrow \Gamma$ to represent the whole boundary surface Γ and if the kernel $K(P,Q)$ takes the simple product form $K(P,Q) = k(x, y - x)|y-x|^{-2-\mathbf{r}}$ with $P = \kappa(x)$ and $Q = \kappa(y)$ and with a $2d_T - \mathbf{r}$ times differentiable function $k(x,z)$, then we almost get the optimal order of convergence $2d_T - \mathbf{r}$ for the quadrature approximated wavelet Galerkin method without supposing a doubled degree of smoothness. Here, as usual for Smolyak rules, differentiability of order $2d_T - \mathbf{r}$ means the existence of mixed derivatives $\partial_x^\alpha \partial_z^\beta k(x,z)$ with $|\alpha| \le 2d_T - \mathbf{r}$ and $|\beta| \le 2d_T - \mathbf{r}$. However, to ensure a representation $K(P,Q) = k(x,z)|z|^{-2-\mathbf{r}}$, $z = y - x$ for the function $K(P,Q) = K(\kappa(x), \kappa(x+z))$ we need the doubled order of smoothness $2[2d_T - \mathbf{r}] + 1$ for the surface and the doubled order of smoothness $2[2d_T - \mathbf{r}]$ for the kernel function $K(P,Q)$ (for more details see (2.6) and the assumptions of Sect. 2).

Finally, we emphasize that the presented quadrature algorithm is designed to get an asymptotically optimal method. So far the existence of this algorithm is more a theoretical result. The method is still to be tested numerically, and its efficiency must be improved by numerous modifications. For this purpose, we shall present the Smolyak rule, which is based on a piecewise polynomial interpolation rule, with the help of the wavelet basis due to Harten and Yad-Shalom [22] (cf. (5.1)). We believe that this is helpful for an optimization which is yet to be done.

2 The Operator Equation

We suppose that the integral equation to be solved is given on a two-dimensional closed boundary manifold $\Gamma \subset \mathbb{R}^3$ with finite degree of smoothness. More exactly, we assume that Γ is the union of m_Γ square shaped parametrization patches Γ_m, i.e.

$$\begin{aligned} \Gamma &= \cup_{m=1}^{m_\Gamma} \Gamma_m, \quad \Gamma_m := \kappa_m(S), \\ S &:= \Big\{ (s,t) \in \mathbb{R}^2 : \; 0 \le s \le 1, \; 0 \le t \le 1 \Big\}. \end{aligned} \tag{2.1}$$

Here the κ_m denote parametrization mappings from the standard square S to the manifold Γ. We assume that the κ_m extend to mappings from the larger square

$$S^e := \Big\{ (s,t) \in \mathbb{R}^2 : \; -1 \le s \le 2, \; -1 \le t \le 2 \Big\} \tag{2.2}$$

to Γ and that these extensions are d_Γ times continuously differentiable with a prescribed $d_\Gamma \ge 2$. Further we suppose that the intersection of two patches Γ_m and $\Gamma_{m'}$ is either empty or a common corner point or a common side. In the last case we suppose that there exist corner points $e_1, e_2, e_1', e_2' \in S$ such that

$$\begin{aligned} \Gamma_m \cap \Gamma_{m'} &= \Big\{ \kappa_m(e_1 + \lambda(e_2 - e_1)) : \; 0 \le \lambda \le 1 \Big\}, \\ \kappa_m\Big(e_1 + \lambda(e_2 - e_1)\Big) &= \kappa_{m'}\Big(e_1' + \lambda(e_2' - e_1')\Big), \; 0 \le \lambda \le 1. \end{aligned} \tag{2.3}$$

Since the manifold is at least continuously differentiable, for each $Q \in \Gamma$, there exists a unit vector n_Q normal to Γ at Q and pointing into the exterior domain bounded by Γ. The Sobolev spaces $H^s(\Gamma)$ over Γ can be defined in the usual way. We define the space $H^s(\Gamma_m)$ over Γ_m as the image of the Sobolev space over S, i.e. $H^s(\Gamma_m) := \{f : \; f \circ \kappa_m \in H^s(S)\}$ and the space $PH^s(\Gamma) := \oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)$. Consequently, we get

$$\begin{aligned} H^s(\Gamma) &= PH^s(\Gamma), \quad -\frac{1}{2} < s < \frac{1}{2}, \\ C\|f\|_{H^s(\Gamma)} &\ge \|f\|_{PH^s(\Gamma)} := \sqrt{\sum_{m=1}^{m_\Gamma} \|f|_{\Gamma_m}\|^2_{H^s(\Gamma_m)}}, \quad f \in H^s(\Gamma), \; -\frac{1}{2} < s. \end{aligned} \tag{2.4}$$

Over Γ we consider a pseudo-differential operator A of order $\mathbf{r} \le 0$ mapping $H^{\mathbf{r}/2}$ into $H^{-\mathbf{r}/2}$. We suppose that A is strongly elliptic, i.e. there is a positive constant c_{se}, a nonzero number $\theta \in \mathbb{C}$, and a compact operator $T : \; H^{\mathbf{r}/2}(\Gamma) \longrightarrow H^{-\mathbf{r}/2}(\Gamma)$ such that the Gårding inequality $\langle \theta(A-T)u, u \rangle_{L^2} \ge c_{se}\|u\|^2_{H^{\mathbf{r}/2}}$ holds for any $u \in H^{\mathbf{r}/2}(\Gamma)$. Moreover, we assume that A is an integral operator of the form $A = K$ for $\mathbf{r} < 0$ and $A = aI + K$ for $\mathbf{r} = 0$, where aI stands for the operator of multiplication by a function a with $\inf_\Gamma |a| > 0$ and where the integral operator K is defined by

$$Ku(P) := \int_\Gamma K(P,Q)u(Q)\,\mathrm{d}_Q\Gamma, \tag{2.5}$$

$$K(P,Q) := \begin{cases} k\left(P, Q, \frac{Q-P}{|Q-P|}\right) |Q-P|^{-2-\mathbf{r}} & \text{if } \mathbf{r} < 0 \\ k\left(P, Q, \frac{Q-P}{|Q-P|}\right) |Q-P|^{-1} & \text{if } \mathbf{r} = 0. \end{cases} \tag{2.6}$$

Thus, for $\mathbf{r} = 0$, we only consider Fredholm equations of the second kind where the compact integral operator is a pseudodifferential operator of order -1. The function k depends on the points $P, Q \in \Gamma$, and on the unit vector in the direction of the difference $Q - P$. We suppose that this function $k : \Gamma \times \Gamma \times S^2 \longrightarrow \mathbb{C}$ is d_k times continuously differentiable. More precisely, for any d_k-th order derivative ∂_P^α, $|\alpha| = d_k$ taken with respect to variable $P \in \Gamma$, for any d_k-th order derivative ∂_Q^β, $|\beta| = d_k$ taken with respect to the variable $Q \in \Gamma$, and for any d_k-th order derivative ∂_Θ^γ, $|\gamma| = d_k$ taken with respect to the variable $\Theta \in S^2$, we require that the mixed derivative $\partial_P^\alpha \partial_Q^\beta \partial_\Theta^\gamma k(P, Q, \Theta)$ is continuous. The function k need not to be a restriction to $\Gamma \times \Gamma$ of a function defined on the space $\mathbb{R}^3 \times \mathbb{R}^3$. It may depend for instance on the unit normals n_P and n_Q pointing into the exterior or on any different kind of differentiable vector field over Γ. For the operator A including the just defined integral operator K, we assume the continuity and the invertibility of the mapping

$$A : H^s(\Gamma) \longrightarrow H^{s-\mathbf{r}}(\Gamma), \quad \mathbf{r} - d_T \leq s \leq d_T. \tag{2.7}$$

For an operator A which satisfies all these assumptions, we shall solve the operator equation $Au = v$ with known right-hand side v and unknown u. To get error estimates with optimal order $2d_T - \mathbf{r}$ (cf. (3.2)), we finally assume $u \in H^{d_T}(\Gamma)$.

For instance, the single and double layer potential equations over smooth curves belong to our class of strongly elliptic operator equations. Indeed, for the single layer case $A = A_s$ corresponding to Laplace's equation, the order $\mathbf{r}_s$ is -1, and

$$K_s(P, Q) := \frac{1}{4\pi} \frac{1}{|P - Q|}.$$

In case of the double layer operator $A = A_d$ we get the order $\mathbf{r}_d = 0$, and the multiplication function $a_d \equiv 0.5$ is constant. The kernel of the integral operator K_d is defined by

$$K_d(P, Q, n_Q) \quad = \quad -\frac{1}{4\pi} \frac{n_Q \cdot (P - Q)}{|P - Q|^3}.$$

Note that the operator $K_d := A_d - a_d I$ is a pseudo-differential operator of order -1. Boundary integral operators for the Stokes system or for Lamè's system can be represented in a similar fashion (cf. [26]).

3 The Wavelet Galerkin Method

To solve $Au = v$ numerically, we seek an approximate solution u_L in the trial space V_L depending on the positive level $L \in \mathbb{Z}$. This is the space of all piecewise continuous trial functions $f : \Gamma \longrightarrow \mathbb{C}$ such that f is $d_T - 2$ ($d_T \geq 1$) times continuously differentiable[2] over each parametrization patch Γ_m, $m = 1, \ldots, m_\Gamma$, and such that, for each $k_1, k_2 = 1, \ldots, 2^L$, the restriction of the function $(x_1, x_2) \mapsto f(\kappa_m(x_1, x_2))$ to the square $[(k_1 - 1)2^{-L}, k_1 2^{-L}] \times$

[2]Here, $d_T = 1$ means no condition on the differentiability and on the continuity of f, and $d_T = 2$ simply means continuity.

$[(k_2-1)2^{-L}, k_2 2^{-L}]$ is polynomial of degree less than d_T with respect to both components x_1 and x_2 of $x=(x_1,x_2)$. Using the notation

$$\langle u,w\rangle \;:=\; \sum_{m=1}^{m_\Gamma}\int_S u\Big(\kappa_m(x)\Big)\overline{w\Big(\kappa_m(x)\Big)}|\kappa_m'(x)|\,\mathrm{d}_S x, \quad |\kappa_m'(x)| := |\partial_{x_1}\kappa_m(x)\times\partial_{x_2}\kappa_m(x)|$$

for the $L^2(\Gamma)$ scalar product, the Galerkin method consists in seeking $u_L\in V_L$ from

$$\langle Au_L,w\rangle \;=\; \langle v,w\rangle, \quad w\in V_L. \tag{3.1}$$

If the operator in (2.7) is bounded and invertible and if $u\in H^{d_T}(\Gamma)$, then we arrive at the standard error estimate

$$\|u-u_L\|_{H^{\mathbf{r}-d_T}(\Gamma)} \quad\le\quad C_G[2^{-L}]^{2d_T-\mathbf{r}}\;\|u\|_{H^{d_T}(\Gamma)}. \tag{3.2}$$

For the computation of u_L we utilize a representation with respect to a wavelet basis. Therefore, we introduce the grids

$$\begin{aligned}\triangle_l \;&:=\; \Big\{\kappa_m\left(k_1 2^{-l}, k_2 2^{-l}\right):\; k_1,k_2=0,\dots,2^l,\; m=1,\dots,m_\Gamma\Big\}, \quad l=0,\dots,L,\\ \nabla_l \;&:=\; \begin{cases}\triangle_0 & \text{if } l=-1\\ \triangle_{l+1}\setminus\triangle_l & \text{if } l=0,\dots,L-1,\end{cases}\end{aligned}$$

and choose a wavelet basis[3] $\{\psi_P,\; P\in\triangle_L\}$ for the space V_L. The level $l(P)$ of the point $P\in\triangle_L$ and of the corresponding wavelet ψ_P is the unique integer l with $P\in\nabla_l$. We require:

i) There is a constant $C_\psi>0$ such that, for any L and any $P=\kappa_m(k_1'2^{-l},k_2'2^{-l})\in\triangle_L$, the function $|\psi_P|$ is less than C_ψ and that the support $\operatorname{supp}\psi_P$ of ψ_P is contained in the neighbourhood $\{Q\in\Gamma_m:\; |P-Q|\le C_\psi 2^{-l(P)}\}$ of P.

ii) There is a positive integer $\tilde d_T>2d_T-\mathbf{r}$, such that ψ_P with $P\in\nabla_l$, $l=0,\dots,L-1$ has $\tilde d_T$ vanishing moments. For $\operatorname{supp}\psi_P$ contained in Γ_m, this means that the integral $\int_S[\psi_P\circ\kappa_m][p\circ\kappa_m]$ is zero for any $p:\;\Gamma\longrightarrow\mathbb{C}$ such that $p\circ\kappa_m$ is polynomial of total degree less than $\tilde d_T$.

iii) There are constants $\tilde d_T^*$ $(\max\{1,-\mathbf{r}\}<\tilde d_T^*<\tilde d_T)$ and $C_{NE}>0$ such that, for s with $-\tilde d_T^*<s<d_T-1/2$, for arbitrary $L\ge 0$, and for any sequence of coefficients $\xi_P,\; P\in\triangle_L$, the discrete norm equivalence

$$C_{NE}^{-1}\left\|\sum_{P\in\triangle_L}\xi_P\psi_P\right\|_{PH^s(\Gamma)} \;\le\; \sqrt{\sum_{P\in\triangle_L}|\xi_P|^2\,2^{2l(P)[s-1]}} \;\le\; C_{NE}\left\|\sum_{P\in\triangle_L}\xi_P\psi_P\right\|_{PH^s(\Gamma)} \tag{3.3}$$

[3] Taken as indeces, we distinguish the grid points $P=\kappa_m\left(k_1 2^{-l},k_2 2^{-l}\right)$ and $P'=\kappa_{m'}\left(k_1'2^{-l},k_2'2^{-l}\right)$ with $m\ne m'$ even if they coincide as points of $\mathbb{R}^3$. In particular, the function ψ_P will depend on the representation $P=\kappa_m\left(k_1 2^{-l},k_2 2^{-l}\right)$.

is valid. Moreover, for any $f \in PH^s(\Gamma)$ with $d_T - 1/2 \le s \le d_T$ and for the coefficients $\xi_P,\ P \in \triangle_L$ of the L^2 orthogonal projection $\sum_{P\in\triangle_L} \xi_P \psi_P$ of f, we require

$$\sqrt{\sum_{P\in\nabla_l} |\xi_P|^2\, 2^{2l[s-1]}} \ \le\ C_{NE}\, \|f\|_{PH^s(\Gamma)}\,, \quad l = -1, \dots, L-1. \tag{3.4}$$

For examples of wavelets with all these properties cf. e.g. [17, 7, 9, 8, 13, 14, 15, 16, 6].

Now the vector of coefficients $\xi := (\xi_P)_{P\in\triangle_L}$ of the Galerkin solution u_L is to be determined by solving the linear system $A_L \xi = \eta$ resulting from (3.1). Here the stiffness matrix is given by $A_L := (\langle A\psi_P, \psi_{P'}\rangle)_{P',P\in\triangle_L}$ and the right-hand side $\eta := (\eta_P)_{P\in\triangle_L}$ by $\eta_P := \langle v, \psi_P\rangle$. To represent the Galerkin equation as an operator equation, we choose a Bessel potential type operator Λ, which is continuous and invertible as an operator in $H^s \longrightarrow H^{s-\mathbf{r}/2}$, $s \le d_T + \mathbf{r}/2$ and which is selfadjoint with respect to the L^2 scalar product. We denote the L^2 orthogonal projection onto the space ΛV_L by π_L and introduce the projection $P_L := \Lambda \pi_L \Lambda^{-1}$ onto the space $\tilde{V}_L := \Lambda^2 V_L$. Clearly, we can choose a basis $\tilde{\psi}_P$ of V_L such that

$$\pi_L f \ = \ \sum_{P\in\triangle_L} \langle f, \Lambda\psi_P\rangle \Lambda\tilde{\psi}_P, \qquad P_L f \ = \ \sum_{P\in\triangle_L} \langle f, \psi_P\rangle \Lambda^2\tilde{\psi}_P.$$

The image space of the operator P_L^*, which is the adjoint to P_L with respect to the L^2 scalar product, is V_L. Hence, we can write (3.1) as

$$[P_L A|_{V_L}]\, u_L \ = \ P_L v, \quad [P_L A|_{V_L}] :\ H^{\mathbf{r}/2} \supseteq V_L \longrightarrow \tilde{V}_L := \Lambda^2 V_L \subseteq H^{-\mathbf{r}/2}. \tag{3.5}$$

Obviously, the basis $\{\Lambda^2\tilde{\psi}_P\}$ in $\tilde{V}_L$ is dual to $\{\psi_P\}$ (i.e. $\langle \psi_P, \Lambda^2\tilde{\psi}_{P'}\rangle = \delta_{P,P'},\ P, P' \in \triangle_L$), and A_L is the matrix with respect to the bases $\{\psi_P\}$ and $\{\Lambda^2\tilde{\psi}_P\}$ of the discretized Galerkin operator $P_L A|_{V_L}$ mapping V_L into $\tilde{V}_L$. In view of the one to one correspondence of operator and matrix representation, we shall denote the last operator with the same symbol A_L. In general we identify the operators from V_L to $\tilde{V}_L$ with their matrix representation taken with respect to the bases $\{\psi_P\}$ and $\{\Lambda^2\tilde{\psi}_P\}$. The discrete norm equivalence (3.3) and the following lemma allow us to reduce the operator norm estimates to bounds on the matrix norm in weighted l^2 spaces.

Lemma 3.1 *The system $\{\Lambda^2\tilde{\psi}_P\}$ forms a Riesz basis in $H^{-\mathbf{r}}$, i.e. there is a constant $C_{NE} > 0$ such that, for any $L \ge 0$ and for any sequence of coefficients $\xi_P,\ P \in \triangle_L$, the discrete norm equivalence*

$$C_{NE}^{-1} \left\| \sum_{P\in\triangle_L} \xi_P \Lambda^2\tilde{\psi}_P \right\|_{H^{-\mathbf{r}}(\Gamma)} \ \le\ \sqrt{\sum_{P\in\triangle_L} |\xi_P|^2\, 2^{2l(P)[-\mathbf{r}+1]}} \ \le\ C_{NE} \left\| \sum_{P\in\triangle_L} \xi_P \Lambda^2\tilde{\psi}_P \right\|_{H^{-\mathbf{r}}(\Gamma)} \tag{3.6}$$

is valid.

Proof. Here and in the following we denote by C a generic constant the value of which may change from instance to instance. From the definition of dual norms and from the continuous embedding $H^{-\mathbf{r}} \subseteq PH^{-\mathbf{r}}$, we infer

$$C \left\| \sum_{P\in\triangle_L} \xi_P \Lambda^2\tilde{\psi}_P \right\|_{H^{-\mathbf{r}}(\Gamma)} \ \ge\ \sup_{\|v\|_{PH^{\mathbf{r}}} \le 1} \left| \left\langle \sum_{P\in\triangle_L} \xi_P \Lambda^2\tilde{\psi}_P, v \right\rangle \right|.$$

The boundedness of the projection π_L in the Sobolev spaces of order s with $\mathbf{r}/2 - 1/2 < s < -\mathbf{r}/2 + 1/2$ implies the boundedness of P_L for $-1/2 < s < -\mathbf{r} + 1/2$, and together with (3.3) and with the duality of $\{\psi_P\}$ and $\{\Lambda^2\tilde{\psi}_P\}$ we arrive at

$$\begin{aligned} C\left\|\sum_{P\in\triangle_L}\xi_P\Lambda^2\tilde{\psi}_P\right\|_{H^{-\mathbf{r}}(\Gamma)} &\geq \sup_{\|P_L^*v\|_{PH^{\mathbf{r}}}\leq 1}\left|\left\langle\sum_{P\in\triangle_L}\xi_P\Lambda^2\tilde{\psi}_P, P_L^*v\right\rangle\right| \\ &\geq \sup_{\|\sum_{P'\in\triangle_L}\eta_{P'}\psi_{P'}\|_{PH^{\mathbf{r}}}\leq 1}\left|\left\langle\sum_{P\in\triangle_L}\xi_P\Lambda^2\tilde{\psi}_P, \sum_{P'\in\triangle_L}\eta_{P'}\psi_{P'}\right\rangle\right| \\ &\geq \sup_{\sum_{P'\in\triangle_L}|\eta_{P'}|^2 2^{2l(P')[\mathbf{r}-1]}\leq 1}\left|\sum_{P\in\triangle_L}\xi_P\overline{\eta_P}\right| \\ &\geq \sqrt{\sum_{P\in\triangle_L}|\xi_P|^2 2^{2l(P)[-\mathbf{r}+1]}}. \end{aligned}$$

To get the upper bound, we set $u_l := \sum_{P\in\nabla_l}\xi_P\Lambda^2\psi_P$ and observe that the space $\tilde{V}_L$ satisfies the usual approximation and inverse properties in the Sobolev spaces of order s with $s < \mathbf{r} + 1/2$ and that P_L is uniformly bounded in the Sobolev spaces of order s with $-1/2 < s < \mathbf{r} + 1/2$. Consequently (cf. the arguments in [8]), we conclude

$$\left\|\sum_{P\in\triangle_L}\xi_P\Lambda^2\tilde{\psi}_P\right\|^2_{H^{-\mathbf{r}}(\Gamma)} \leq C\sum_{l=-1}^{L-1}\|u_l\|^2_{H^{-\mathbf{r}}(\Gamma)}. \tag{3.7}$$

For each term in the last sum, we apply the inverse property and arguments like those used above to arrive at

$$\begin{aligned} \|u_l\|_{H^{-\mathbf{r}}(\Gamma)} &\leq 2^{-\mathbf{r}l}\left\|\sum_{P\in\nabla_l}\xi_P\Lambda^2\psi_P\right\|_{H^0(\Gamma)} \\ &\leq C2^{-\mathbf{r}l}\sup_{\|[P_l^*-P_{l-1}^*]v\|_{H^0}\leq 1}\left|\left\langle\sum_{P\in\nabla_l}\xi_P\Lambda^2\tilde{\psi}_P, \left[P_l^*-P_{l-1}^*\right]v\right\rangle\right| \\ &\leq C2^{-\mathbf{r}l}\sup_{\sum_{P'\in\nabla_l}|\eta_{P'}|^2 2^{2l[0-1]}\leq 1}\left|\sum_{P\in\nabla_l}\xi_P\overline{\eta_P}\right| \leq C2^{-\mathbf{r}l}\sqrt{\sum_{P\in\nabla_l}|\xi_P|^2 2^{2l}}. \end{aligned} \tag{3.8}$$

The estimates (3.7) and (3.8) imply the upper bound. ∎

In the wavelet algorithm the fully populated matrix A_L is approximated by the sparse compressed matrix A_L^C defined by

$$\begin{aligned} A_L^C &:= \left(a_{P',P}\right)_{P',P\in\triangle_L}, \quad a_{P',P} := \begin{cases} 0 \text{ if } \operatorname{dist}(\operatorname{supp}\psi_P, \operatorname{supp}\psi_{P'}) \geq \mathbf{m} \\ \langle A\psi_P, \psi_{P'}\rangle \text{ else,} \end{cases} \\ \mathbf{m} &:= \max\left\{2^{-l(P)}, 2^{-l(P')}, D2^{[2\mu-1]L-\mu l(P)-\mu l(P')}\right\}. \end{aligned} \tag{3.9}$$

Here $D > 1$ is an appropriate constant and $\mu < 1$ is chosen close to one. From [11, 12, 29, 35] (cf. also [5]), we infer

Theorem 3.1 *Suppose $\tilde{d}_T$ is chosen such that $\tilde{d}_T > -\mathbf{r}/2$ and $\tilde{d}_T > d_T - \mathbf{r}$. Take μ from the open interval $(a, 1)$ with $a := [\tilde{d}_T + d_T]/[2\tilde{d}_T + \mathbf{r}]$, suppose that the operator in (2.7) is bounded and invertible and that $u \in H^{d_T}(\Gamma)$. Moreover, suppose that the smoothness order d_Γ of the boundary manifold Γ is greater or equal to $\tilde{d}_T + 1$, and that the differentiation order d_k of the kernel function is greater or equal to $\tilde{d}_T$. Then there exists a constant $D_0 \geq 1$ and an integer $L_0 > 0$ such that, for all $D > D_0$ and $L \geq L_0$, the discretized operator $A_L^C : \; H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}$ is invertible and the inverse is bounded uniformly with respect to L and D. In particular, the solution $\xi^C = (\xi_P^C)_{P \in \triangle_L}$ of the compressed equation $A_L^C \xi^C = \eta$ exists at least for $L \geq L_0$, and the approximate solution $u_L^C := \sum_{P \in \triangle_L} \xi_P^C \psi_P$ obeys the estimate*

$$\|u - u_L^C\|_{H^{\mathbf{r}-d_T}(\Gamma)} \quad \leq \quad C_W [2^{-L}]^{2d_T - \mathbf{r}} \; \|u\|_{H^{d_T}(\Gamma)}. \tag{3.10}$$

If $N = \mathcal{O}(2^{2L})$ is the dimension of the trial space (number of degrees of freedom), then the number of non-zero entries in the compressed stiffness matrix A_L^C is of the size $\mathcal{O}(N \log N)$.

Note that the linear equation $A_L^C \xi^C = \eta$ admits an asymptotically optimal diagonal preconditioning (cf. (3.3) and (3.6)). Moreover, a second compression step for matrix entries corresponding to wavelet basis functions with overlapping supports is possible. This second compression reduces the number of non-zero entries in A_L^C to $\mathcal{O}(N)$ (cf. [35]). However, we shall use the compression of Theorem 3.1, only.

To prepare the estimation of the quadrature error we shortly review the derivation of the basic estimates in the proof of Theorem 3.1. First, the Lax-Milgram theorem together with standard compact perturbation arguments yields the invertibility and stability of $A_L : H^{\mathbf{r}/2} \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}/2}$. From this and from the approximation and inverse properties of the spaces V_L and $\tilde{V}_L$ we conclude the invertibility and stability of the discretized operator $A_L : \; H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}$. Now, using a decay property of the matrix entries, the norm equivalences (3.3) and (3.6), as well as a Schur lemma argument (cf. the next lemma), the difference of the Galerkin operator and the compressed Galerkin operator $A_L - A_L^C : \; H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}$ turns out to be small at least for sufficiently large D. Hence, A_L^C is invertible and stable, too. The error estimate in the Sobolev norm of order $\mathbf{r} - d_T$ follows from the Aubin-Nitsche trick

$$\begin{aligned}
\left\|u - u_L^C\right\|_{H^{-d_T+\mathbf{r}}} &\leq C \left\|A\left[u - u_L^C\right]\right\|_{H^{-d_T}} \\
&\leq C \sup_{\|v\|_{H^{d_T}} \leq 1} \left| \left\langle A\left[u - u_L^C\right], v - v_L \right\rangle + \left\langle A\left[u - u_L^C\right], v_L \right\rangle \right|, \\
&\leq C \left\|u - u_L^C\right\|_{H^0} \sup_{\substack{\|v\|_{H^{d_T}} \leq 1 \\ v_L \text{best appr. of } v}} \|v - v_L\|_{H^{\mathbf{r}}} \\
&\quad + C \sup_{\substack{\|v\|_{H^{d_T}} \leq 1 \\ v_L \text{best appr. of } v}} \left| \left\langle A\left[u - u_L^C\right], v_L \right\rangle \right|,
\end{aligned}$$

from the identity $\langle A(u - u_L^C), v_L \rangle_{L^2} = [(A_L^C - A_L)\xi^C, \zeta]$, and from a Schur lemma argument applied to the estimation of $[(A_L^C - A_L)\xi^C, \zeta]$ (cf. the next lemma). Here $[\cdot, \cdot]$ stands for the scalar product in the Euclidean space and the coordinates of ξ^C and ζ are defined by $u_L^C = \sum \xi_P^C \psi_P$ and $v_L = \sum \zeta_P \psi_P$, respectively. The function v_L is the best approximation to a function v from H^{d_T} chosen in accordance with the Aubin-Nitsche trick, and, due to the stability in H^0, the Galerkin solution u_L^C is an almost[4] best approximation to the function $u \in H^{d_T}$. Due to this the components ξ_P and ζ_P of $u_L^C = \sum \xi_P^C \psi_P$ and $v_L = \sum \zeta_P \psi_P$ satisfy the estimate (3.4) with $s = d_T$. Finally, we recall the Schur lemma which reduces the estimates of $A_L^C - A_L$ to the decay estimate of the matrix entries.

Lemma 3.2 *For any real number x, there hold the estimates*

$$\|A_L - A_L^C\|_{H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}} \leq \sqrt{\Sigma_1^{-\mathbf{r},0} \Sigma_2^{-\mathbf{r},0}}, \tag{3.11}$$

$$\sup_{\substack{\xi: \sum_{P \in \nabla_l} |\xi_P|^2 2^{2l(d_T-1)} \leq 1 \\ \zeta: \sum_{P \in \nabla_l} |\zeta_P|^2 2^{2l(d_T-1)} \leq 1}} \left| \left[(A_L - A_L^C)\xi, \zeta \right] \right| \leq \sqrt{\Sigma_1^{-d_T,d_T} \Sigma_2^{-d_T,d_T}}, \tag{3.12}$$

$$\Sigma_1^{t',t} := \sum_{l(P)=-1}^{L-1} \sup_{P \in \nabla_{l(P)}} \left\{ \sum_{P' \in \triangle_L} 2^{(t'+1-x)l(P)} |b_{P,P'}| 2^{(-t+1+x)l(P')} \right\},$$

$$\Sigma_2^{t',t} := \sum_{l(P')=-1}^{L-1} \sup_{P' \in \nabla_{l(P')}} \left\{ \sum_{P \in \triangle_L} 2^{(t'+1+x)l(P)} |b_{P,P'}| 2^{(-t+1-x)l(P')} \right\}.$$

Proof. For the $H^{-\mathbf{r}}$ norm of a function $v_L = \sum \zeta_P \Lambda^2 \psi_P \in \tilde{V}_L = \text{im } P_L$, we get

$$\|v_L\|_{H^{-\mathbf{r}}} = \sup_{w: \|w\|_{H^{\mathbf{r}}} \leq 1} \left\langle v_L, w \right\rangle \leq C \sup_{w_L \in V_L: \|w_L\|_{H^{\mathbf{r}}} \leq 1} \left\langle v_L, w_L \right\rangle.$$

Here we have used the uniform boundedness of P_L^* in the space $H^{\mathbf{r}}$. Setting $w_L = \sum v_P \psi_P$ and taking the discrete norm equivalences into account, we arrive at

$$\|v_L\|_{H^{-\mathbf{r}}} \leq C \sup_{v: \sum |v_P|^2 2^{2(\mathbf{r}-1)l(P)} \leq 1} \sum \zeta_P \overline{v_P} \leq \sqrt{\sum |\zeta_P|^2 2^{2(-\mathbf{r}+1)l(P)}}.$$

On the other hand, if $v_L = (A_L - A_L^C)u_L$ and if the entries of $(A_L - A_L^C)$ are the numbers $b_{P,P'}$, then

$$\begin{aligned} |\zeta_P| &= \left| \sum_{P' \in \triangle_L} b_{P,P'} \xi_{P'} \right| \leq \sqrt{\sum_{P' \in \triangle_L} |b_{P,P'}| 2^{(1+x)l(P')} \sum_{P' \in \triangle_L} |b_{P,P'}| 2^{(-1-x)l(P')} |\xi_{P'}|^2}, \\ \|v_L\|_{H^{-\mathbf{r}}}^2 &\leq \sum_{P \in \triangle_L} 2^{2(-\mathbf{r}+1)l(P)} \sum_{P' \in \triangle_L} |b_{P,P'}| 2^{(1+x)l(P')} \sum_{P' \in \triangle_L} |b_{P,P'}| 2^{(-1-x)l(P')} |\xi_{P'}|^2 \\ &\leq \Sigma_1^{-\mathbf{r},0} \sum_{P \in \triangle_L} 2^{(-\mathbf{r}+1+x)l(P)} \sum_{P' \in \triangle_L} |b_{P,P'}| 2^{(-1-x)l(P')} |\xi_{P'}|^2 \end{aligned} \tag{3.13}$$

[4] I.e. the approximation error is like that of the best approximation but with an additional constant factor

$$
\begin{aligned}
&\le \Sigma_1^{-\mathbf{r},0} \sum_{P'\in\triangle_L} \sum_{P\in\triangle_L} 2^{(-\mathbf{r}+1+x)l(P)} |b_{P,P'}| 2^{(1-x)l(P')} 2^{-2l(P')} |\xi_{P'}|^2 \\
&\le \Sigma_1^{-\mathbf{r},0} \sum_{l(P')=-1}^{L-1} \sup_{P'\in\nabla_{l(P')}} \left\{ \sum_{P\in\triangle_L} 2^{(-\mathbf{r}+1+x)l(P)} |b_{P,P'}| 2^{(1-x)l(P')} \right\} \sum_{P'\in\nabla_L} 2^{-2l(P')} |\xi_{P'}|^2 \\
&\le \Sigma_1^{-\mathbf{r},0} \Sigma_2^{-\mathbf{r},0} \sup_{l=-1,\dots,L-1} \sum_{P'\in\nabla_l} 2^{-2l(P')} |\xi_{P'}|^2 .
\end{aligned}
$$

The estimate (3.11) is proved.

To derive (3.12) we observe

$$
\sup_{\nu:\,\sum_{P\in\nabla_l} |\nu_P|^2 2^{2l(d_T-1)} \le 1} \Big| [\zeta,\nu] \Big| \;=\; \sup \Big| \sum_{P\in\triangle_L} \zeta_P \nu_P \Big| \;\le\; \sum_{l(P)=-1}^{L-1} \sqrt{ \sum_{P\in\nabla_{l(P)}} 2^{2(-d_T+1)l(P)} |\zeta_P|^2 } .
$$

Using an estimate like in the first step of (3.13) and the Cauchy-Schwarz inequality, we arrive at

$$
\begin{aligned}
&\sup_{\nu:\,\sum_{P\in\nabla_l} |\nu_P|^2 2^{2l(d_T-1)} \le 1} \Big| [\zeta,\nu] \Big| \le \sum_{l(P)=-1}^{L-1} \\
&\sqrt{ \sum_{P\in\nabla_{l(P)}} 2^{2(-d_T+1)l(P)} \sum_{P'\in\triangle_L} |b_{P,P'}| 2^{(-d_T+1+x)l(P')} \sum_{P'\in\triangle_L} |b_{P,P'}| 2^{(d_T-1-x)l(P')} |\xi_{P'}|^2 } \\
\le\; & \sum_{l(P)=-1}^{L-1} \sqrt{ \sup_{P\in\nabla_{l(P)}} 2^{(-d_T+1-x)l(P)} \sum_{P'\in\triangle_L} |b_{P,P'}| 2^{(-d_T+1+x)l(P')} } \\
&\sqrt{ \sum_{P\in\nabla_{l(P)}} \sum_{P'\in\triangle_L} 2^{(-d_T+1+x)l(P)} |b_{P,P'}| 2^{(d_T-1-x)l(P')} |\xi_{P'}|^2 } \\
\le\; & \sqrt{\Sigma_1^{-d_T,d_T}} \sqrt{ \sum_{l(P)=-1}^{L-1} \sum_{P\in\nabla_{l(P)}} \sum_{P'\in\triangle_L} 2^{(-d_T+1+x)l(P)} |b_{P,P'}| 2^{(d_T-1-x)l(P')} |\xi_{P'}|^2 } . \\
\le\; & \sqrt{\Sigma_1^{-d_T,d_T}} \sqrt{ \sum_{l(P')=-1}^{L-1} \sum_{P'\in\nabla_{l(P)}} \sum_{P\in\triangle_L} 2^{(-d_T+1+x)l(P)} |b_{P,P'}| 2^{(d_T-1-x)l(P')} |\xi_{P'}|^2 } .
\end{aligned}
$$

Applying the same arguments as in the last steps of (3.13), the estimate (3.12) follows. ∎

Now, if A_L^Q is the quadrature approximation of the matrix A_L^C and if $\eta^Q := (\eta_P^Q)_{P\in\triangle_L}$, $\eta_P^Q \sim \eta_P := \langle v, \psi_P \rangle$ is the quadrature approximated vector on the right-hand side of the Galerkin equation, then we get the same conclusions as in Theorem 3.1 for the quadrature approximated method if we can prove (compare the Strang lemmas for finite element methods)

$$
\left\| A_L^C - A_L^Q \right\|_{H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}} \quad \le \quad \mathcal{O}(L^{-1}), \tag{3.14}
$$

$$\left|\left[(A_L^C - A_L^Q)\xi, \zeta\right]\right| \quad \leq \quad C_Q[2^{-L}]^{2d_T - \mathbf{r}}, \tag{3.15}$$

$$\sqrt{\sum_{P \in \triangle_L} \left|\eta_P - \eta_P^Q\right|^2 2^{2l(P)[0+1]}} \quad \leq \quad C_Q[2^{-L}]^{2d_T - \mathbf{r}}, \tag{3.16}$$

where the components ξ_P and ζ_P of ξ and ζ are supposed to satisfy the estimate (3.4) with $s = d_T$. Note that the last condition (3.16) could be relaxed if necessary.

4 The Quadrature Rule for the Multiplication Operator and the Right-Hand Side

Since the quadrature for the integral operator will take the main part of the computing time, the quadrature for the multiplication operator and the right-hand side need not to be optimal. Therefore, we present very simple quadrature algorithms, only. The proofs of the corresponding error estimates are straightforward.

Lemma 4.1 *Suppose the right-hand side function v is d_Q times continuously differentiable over Γ with $d_Q > 2d_T - \mathbf{r}$. We split Γ into the union of the level L squares*

$$\Gamma_{m,L,k} \quad := \quad \left\{\kappa_m(x) : \; k_1 2^{-L} \leq x_1 \leq [k_1+1]2^{-L}, \; k_2 2^{-L} \leq x_2 \leq [k_2+1]2^{-L}\right\},$$
$$m = 1, \ldots, m_\Gamma, \; k_1 = 0, \ldots, 2^L - 1, \; k_2 = 0, \ldots, 2^L - 1$$

and, for each $\Gamma_{m,L,k}$ and for each polynomial $p \circ \kappa_m$ of degree less than d_T, we compute the integrals

$$\int_{k_1 2^{-L}}^{[k_1+1]2^{-L}} \int_{k_2 2^{-L}}^{[k_2+1]2^{-L}} v\Big(\kappa_m(x)\Big)\overline{p\Big(\kappa_m(x)\Big)}\,\mathrm{d}x \quad \sim \quad Q\Big(v \circ \kappa_m \, \overline{p \circ \kappa_m}\Big)$$

by applying the product quadrature rule $Q(\cdot)$ which is the integral of a polynomial interpolation to $v \circ \kappa_m$ multiplied by $\overline{p \circ \kappa_m}$ and which has the order of convergence d_Q. If the approximate values η_P^Q of $\langle v, \psi_P\rangle$, $P \in \triangle_L \cap \Gamma_m$ are determined from these values $Q(v \circ \kappa_m \, \overline{p \circ \kappa_m})$ by

$$\eta_P^Q \quad := \quad \sum_{\substack{k_1,k_2=0,\ldots,2^L-1: \\ \Gamma_{m,L,k} \cap \operatorname{supp} \psi_P \neq \emptyset}} Q(v \circ \kappa_m \, \overline{\psi_P \circ \kappa_m}), \tag{4.1}$$

then we arrive at the estimate

$$\left|\eta_P^Q - \Big\langle v, \psi_P \Big\rangle\right| \quad \leq \quad C\Big[2^{-L}\Big]^{d_Q} 2^{-2l(P)}.$$

Lemma 4.2 *Suppose the multiplication function a is d_Q times continuously differentiable over Γ with $d_Q > 2d_T - \mathbf{r}$. We split Γ into the union of the level L squares $\Gamma_{m,L,k}$ and, for* *each $\Gamma_{m,L,k}$ and for each pair of polynomials $p_1 \circ \kappa_m$ and $p_2 \circ \kappa_m$ of degree less than d_T, we compute the integrals*

$$\int_{k_1 2^{-L}}^{[k_1+1]2^{-L}} \int_{k_2 2^{-L}}^{[k_2+1]2^{-L}} a\Big(\kappa_m(x)\Big) p_1\Big(\kappa_m(x)\Big) \overline{p_2\Big(\kappa_m(x)\Big)} \,\mathrm{d}x \;\sim\; Q_{L,k,p_1,p_2},$$

$$Q_{L,k,p_1,p_2} \;:=\; Q\Big(a \circ \kappa_m \; p_1 \circ \kappa_m \; \overline{p_2 \circ \kappa_m}\Big)$$

by applying the product quadrature rule $Q(\cdot)$ which is the integral of a polynomial interpolation to $a \circ \kappa_m$ multiplied by $p_1 \circ \kappa_m \; \overline{p_2 \circ \kappa_m}$ and which has the order of convergence d_Q. Proceeding with l from $L-1$ to 0, we compute the quadrature approximations

$$\int_{k_1 2^{-l}}^{[k_1+1]2^{-l}} \int_{k_2 2^{-l}}^{[k_2+1]2^{-l}} a\Big(\kappa_m(x)\Big) p_1\Big(\kappa_m(x)\Big) \overline{p_2\Big(\kappa_m(x)\Big)} \,\mathrm{d}x \;\sim\; Q_{l,k,p_1,p_2},$$

$$Q_{l,k,p_1,p_2} \;:=\; \sum_{k_1'=2k_1}^{2k_1+1} \sum_{k_2'=2k_2}^{2k_2+1} Q_{l+1,k',p_1,p_2}. \tag{4.2}$$

If the approximate values $m^Q_{P',P}$, $P, P' \in \triangle_L \cap \Gamma_m$ of $\langle a\psi_P, \psi_{P'} \rangle$ are determined from these values Q_{l,k,p_1,p_2} by

$$m^Q_{P',P} \;:=\; \sum_{\substack{k_1,k_2=0,\ldots,2^{\max\{l(P),l(P')\}}-1: \\ \Gamma_{m,\max\{l(P),l(P')\},k} \cap \operatorname{supp} \psi_P \cap \operatorname{supp} \psi_{P'} \neq \emptyset}} Q_{\max\{l(P),l(P')\},k,\psi_P,\psi_{P'}}, \tag{4.3}$$

then we arrive at the estimate

$$\Big| m^Q_{P',P} - \Big\langle a\psi_P, \psi_{P'} \Big\rangle \Big| \;\le\; C \Big[2^{-L}\Big]^{d_Q} 2^{-2\max\{l(P),l(P')\}}.$$

5 The Wavelet Quadrature Rule for the Discretized Integral Operator

5.1. In this section we introduce a wavelet quadrature rule. This rule is an easy generalization of the tensor product rules which are known e.g. under the names Smolyak quadrature, blending, and sparse grid rule (cf. [36] and e.g. [20]). In contrast to the classical Smolyak rules a certain type of mesh refinement is involved. In particular for the integrals in our Galerkin stiffness matrix, our quadrature is almost a Smolyak tensor product of rules over uniform meshes in a direction with smooth derivatives and of rules over graded meshes in the perpendicular direction where the integrand exhibits a weakly singular behaviour.

The basic idea for our rule is to represent the piecewise polynomial interpolation projection with the help of a wavelet basis (cf. [22]), to reduce this basis adaptively to the basis functions important for the approximation of the integrand, and to define the quadrature as the corresponding interpolation rule. We first introduce the basis functions and their dual functionals for the interval. Then we define the two-dimensional tensor product wavelets and use these to construct a general adaptive Smolyak rule over a four dimensional domain. Finally, we shall derive the orders of convergence for the quadrature applied to special types of integrands arising in the computation of the Galerkin stiffness matrix.

First we fix the order of quadrature $d_Q \leq \tilde{d}_T$. Over $I := [0,1]$ we introduce the grids $\triangle_l^I := \{[k + j/(d_Q - 1)]2^{-l} : \; k = 0, 1 \ldots, 2^l - 1, \; j = 1, \ldots, d_Q - 1\} \cup \{0\}$ and $\nabla_l^I := \triangle_{l_c}^I$ for $l = l_c - 1$ and $\nabla_l^I := \triangle_{l+1}^I \setminus \triangle_l^I$ for $l = l_c, l_c + 1, \ldots$. Here l_c is a fixed level for the coarsest grid. We denote by V_l^I the space of continuous piecewise polynomial functions f over I such that, for each subinterval $I_k^l := [k2^{-l}, (k+1)2^{-l}]$, the function f coincides over I_k^l with a polynomial of degree less than d_Q. For $t = [k + j/(d_Q - 1)]2^{-l} \in \triangle_l^I$, we define the (k,j)-th scaling function φ_t^l of level l as the unique Lagrange basis function in V_l determined by $\varphi_t^l([k' + j'/(d_Q - 1)]2^{-l}) = \delta_{k,k'}\delta_{l,l'}$. A hierarchical system of basis function $\{\psi_t^I\}$ is obtained by choosing the wavelets $\psi_t^I := \varphi_t^l$ for $t \in \nabla_l^I, l \geq l_c$ and $\psi_t^I := \varphi_t^{l_c}$ for $t \in \nabla_{l_c-1}^I$. We get a dual basis of functionals by setting

$$\tilde{\psi}_t^I(f) \;\; := \;\; \begin{cases} f(t) - \sum_{\tau \in \triangle_l : \, t \in \mathrm{supp}\, \varphi_\tau^l} \varphi_\tau^l(t) f(\tau), & \text{if } t \in \nabla_l^I, \; l = l_c, \ldots \\ f(t) & \text{if } t \in \nabla_{l_c-1}^I \end{cases}$$

and arrive at $\tilde{\psi}_t^I(\psi_\tau^I) = \delta_{t,\tau}$ for all points $t, \tau \in \cup_{l=l_c}^\infty \triangle_l$. Note that each dual wavelet functional is a linear combination of a finite number of Dirac delta functionals. Since the polynomials of degree less than d_Q are in the span of the functions $\psi_x^I, \; x \in \nabla_{l_c-1}^I$, the duality implies that $\tilde{\psi}_t^I$ with $t \in \nabla_l^I, \; l \geq l_c$ vanishes over polynomials of degree less than d_Q, i.e. the dual functionals have d_Q vanishing moments. The interpolation over the uniform grid takes the form

$$I_L^I f \;\; := \;\; \sum_{t \in \triangle_L^I} f(t) \varphi_t^L \;\; = \;\; \sum_{t \in \triangle_L^I} \tilde{\psi}_t^I(f) \psi_t^I .$$

We even get $f = \sum_{t \in \cup \triangle_l^I} \tilde{\psi}_t^I(f) \psi_t^I$ for continuous functions f and for functions f with weak singularities in a finite number of points. The convergence of the last representation is to be understood in the L^1 sense and the function values in $\tilde{\psi}_t^I(f)$ which are not defined are set to zero.

To construct the tensor product wavelets on the square $S := I \times I$, we introduce the grids $\triangle_l^S := \triangle_l^I \times \triangle_l^I$ and $\nabla_l^S := \triangle_{l_c}^S$ for $l = l_c - 1$ resp. $\nabla_l^S := \triangle_{l+1}^S \setminus \triangle_l^S$ for $l = l_c, l_c + 1, \ldots$. We set $l(x) := l$ if $x \in \nabla_l^S, \; l = l_c - 1, \ldots$ and define

$$\psi_x^S(t_1, t_2) \;\; := \;\; \begin{cases} \varphi_{x_1}^0(t_1)\varphi_{x_2}^0(t_2) & \text{if } x = (x_1, x_2) \in \nabla_{l_c-1}^S \\ \psi_{x_1}^I(t_1)\varphi_{x_2}^l(t_2) & \text{if } l \geq l_c, \; x_1 \in \nabla_l^I, \text{ and } x_2 \in \triangle_l^I \\ \varphi_{x_1}^l(t_1)\psi_{x_2}^I(t_2) & \text{if } l \geq l_c, \; x_2 \in \nabla_l^I, \text{ and } x_1 \in \triangle_l^I \\ \psi_{x_1}^I(t_1)\psi_{x_2}^I(t_2) & \text{if } l \geq l_c, \; x_1 \in \nabla_l^I, \text{ and } x_2 \in \nabla_l^I, \end{cases}$$

$$
\tilde\psi^S_x(F) \;:=\; \begin{cases} F(x_1,x_2) & \text{if } x=(x_1,x_2)\in\nabla^S_{l_c-1} \\ F(x_1,x_2)-\sum_{\tau\in\triangle_l:\, x_1\in\operatorname{supp}\varphi^l_\tau}\varphi^l_\tau(x_1)F(\tau,x_2) & \text{if } x_1\in\nabla^I_l \text{ and } x_2\in\triangle^I_l \\ F(x_1,x_2)-\sum_{\tau\in\triangle_l:\, x_2\in\operatorname{supp}\varphi^l_\tau}\varphi^l_\tau(x_2)F(x_1,\tau) & \text{if } x_2\in\nabla^I_l \text{ and } x_1\in\triangle^I_l \\ \Big\{F(x_1,x_2)-\sum_{\tau\in\triangle_l:\, x_1\in\operatorname{supp}\varphi^l_\tau}\varphi^l_\tau(x_1)F(\tau,x_2)\Big\} & \\ -\sum_{\tau'\in\triangle_l:\, x_2\in\operatorname{supp}\varphi^l_{\tau'}}\varphi^l_{\tau'}(x_2)\Big\{F(x_1,\tau')- & \\ \qquad\qquad \sum_{\tau\in\triangle_l:\, x_1\in\operatorname{supp}\varphi^l_\tau}\varphi^l_\tau(t)F(\tau,\tau')\Big\} & \text{if } x_1\in\nabla^I_l \text{ and } x_2\in\nabla^I_l. \end{cases}
$$

In tensor product notation, the last case of the last formula can be written as $\tilde\psi^S_x = \tilde\psi^I_{x_1}\otimes\tilde\psi^I_{x_2}$ for $x_1\in\nabla^I_l$ and $x_2\in\nabla^I_l$ and $l\geq l_c$. Again we get the duality relation $\tilde\psi^S_x(\psi^S_y)=\delta_{x,y}$ for all points $x,y\in\cup_{l=l_c}^\infty\triangle^S_l$, the d_Q vanishing moments for the functionals $\tilde\psi^S_x,\ x\in\nabla^S_l,\ l\geq l_c$, and the representation for the interpolation

$$
I^S_L F \;:=\; \sum_{x\in\triangle^S_L}\tilde\psi^S_x(F)\psi^S_x.
$$

If we set $\triangle^S := \cup_{l=l_c}^\infty\triangle^S_l$, then we additionally get $F=\sum_{x\in\triangle^S}\tilde\psi^s_x(F)\psi^S_x$ for continuous functions F and for functions F with weak singularities along a finite number of smooth curves or with weak singularities at a finite number of points. This representation is to be defined as in the univariate case.

Now consider a function $f:\ S\times S\longrightarrow\mathbb{C}$, which is the product $f=gh$ of a piecewise polynomial function h and another function g. If the factorization $f=gh$ is not given explicitely, then we assume $h\equiv 1$ and $f\equiv g$. We define two coarsest levels, l^x_c for the wavelets with respect to the variable x and l^y_c for those with respect to y. Then, for any prescribed $\varepsilon>0$, the integral $I(f)$ of f over a subdomain $D\subseteq S\times S$ can be approximated by

$$
I(f) \;:=\; \iint_D f(x',y')\,\mathrm{d}y'\,\mathrm{d}x' \quad\sim\quad Q_\varepsilon(f) \;:=\; \sum_{x\in\triangle^S,\,y\in\triangle^S:\,|\sigma_{f,x,y}|\geq\varepsilon}\sigma_{f,x,y}, \tag{5.1}
$$

$$
\sigma_{f,x,y} \;:=\; \left[\tilde\psi^S_x\otimes\tilde\psi^S_y\right](g)\iint_D h(x',y')\psi^S_x(x')\psi^S_y(y')\,\mathrm{d}y'\,\mathrm{d}x'.
$$

The quadrature weights $\iint_D h\psi^S_x\psi^S_y$ are integrals of piecewise polynomial functions and can be computed easily at least for simple domains D. Note that our quadrature rule possibly involves function evaluations outside of the domain of integration D. Thus we always assume that our function can be extended to these points. For tensor product subdomains D, we could modify our interpolation easily to an interpolation inside of the domain. This leads to classical quadrature rules. In particular, the Smolyak quadrature for smooth functions f corresponds to a summation in (5.1) taken over all $x,y\in\triangle^S_L$ such that $x\in\nabla^S_l$, $y\in\nabla^S_{l'}$, and $l+l'<L$. Obviously, the error of quadrature and the number of arithmetic operations N_ε can be estimated as

$$
|I(f)-Q_\varepsilon(f)| \;\leq\; \sum_{\substack{x,y\in\triangle^S:\\ |\sigma_{f,x,y}|<\varepsilon}}|\sigma_{f,x,y}|, \qquad N_\varepsilon \;\leq\; C\sum_{\substack{x,y\in\triangle^S:\\ |\sigma_{f,x,y}|\geq\varepsilon}} 1. \tag{5.2}
$$

If we have upper bounds $|I(f)-Q_\varepsilon(f)| \le C\varepsilon^\alpha$ and $N_\varepsilon \le C\varepsilon^{-\beta}$, then we obtain $\varepsilon \le CN_\varepsilon^{-1/\beta}$ and the complexity estimate $|I(f)-Q_\varepsilon(f)| \le CN_\varepsilon^{-\alpha/\beta}$. For the case that the underlying domain D is $S \times S$, we can choose $l_c^x = l_c^x = -1$. For domains $D = D^x \times D^y$ with $D_x = \operatorname{supp}[\psi_P \circ \kappa_m]$ and $D_y = \operatorname{supp}[\psi_{P'} \circ \kappa_m]$, we choose the coarsest levels l_c^x and l_c^y such that the wavelets of level $l_c^x - 1$ and $l_c^y - 1$ (i.e. the scaling functions of level l_c^x and l_c^y) are polynomials over $\operatorname{supp}[\psi_P \circ \kappa_m]$ and $\operatorname{supp}[\psi_{P'} \circ \kappa_m]$, respectively. Then, for the choice $h(x,y) = \psi_P(\kappa_m(x))\psi_{P'}(\kappa_m(y))$, the quadrature weights $\iint_D h\psi_x^S\psi_y^S$ vanish due to the vanishing moments of $\psi_P \circ \kappa_m$ and $\psi_{P'} \circ \kappa_m$. In other words, the quadrature rule Q_ε contains only terms $\sigma_{f,x,y}$ for which $\tilde{\psi}_x^S$ and $\tilde{\psi}_y^S$ are wavelets of level greater or equal to l_c^x and l_c^y, respectively. These wavelets have d_Q vanishing moments.

5.2. To apply the quadrature (5.1) to the computation of the stiffness matrix, we observe the following separation of the singularity direction in the singular kernel function defined over $S^e \times S^e$ (cf. (2.2)).

Lemma 5.1 *i) For $m = 1, \ldots, m_\Gamma$, $\mathbf{r} < 0$ and for any two-dimensional multiindices α and β with $|\alpha|, |\beta| \le \frac{1}{2}\min\{d_k, d_\Gamma - 1\}$, there exist a constant $C_{\alpha,\beta}$ such that*

$$\left|\partial_x^\alpha \partial_z^\beta K\Big(\kappa_m(x), \kappa_m(x+z)\Big)\right| \le C_{\alpha,\beta}|z|^{-2-\mathbf{r}-|\beta|}, \quad x, x+z \in S^e, \tag{5.3}$$

$$\left|\partial_z^\alpha \partial_y^\beta K\Big(\kappa_m(y+z), \kappa_m(y)\Big)\right| \le C_{\alpha,\beta}|z|^{-2-\mathbf{r}-|\beta|}, \quad y, y+z \in S^e. \tag{5.4}$$

ii) Suppose that $1 \le m, m' \le m_\Gamma$ and that $\mathbf{r} < 0$. For the sake of definiteness and in accordance with the assumption (2.3), we assume that the intersection of $\Gamma_m \cap \Gamma_{m'}$ is the common side $\{\kappa_m((x_1,0)) : 0 \le x_1 \le 1\}$ with the property $\kappa_m((x_1,0)) = \kappa_{m'}((x_1,0))$, $0 \le x_1 \le 1$. Then, for any multiindices α and β with $|\alpha| + |\beta| \le \min\{d_k, d_\Gamma - 1\}$ and for any $(y_1+z_2, z_1), (y_1, z_3) \in S^e$, there exist a constant $C_{\alpha,\beta}$ such that

$$\left|\partial_{y_1}^\alpha \partial_z^\beta K\Big(\kappa_m(y_1+z_2, z_1), \kappa_{m'}(y_1, z_3)\Big)\right| \le C_{\alpha,\beta}|z|^{-2-\mathbf{r}-|\beta|}. \tag{5.5}$$

iii) Suppose that $1 \le m, m' \le m_\Gamma$, that the intersection of Γ_m and $\Gamma_{m'}$ consists of one point, only, and that $\mathbf{r} < 0$. For the sake of definiteness, we suppose $\kappa_m(0,0) = \kappa_{m'}(0,0)$. Then, for any two-dimensional multiindices α and β with $|\alpha| + |\beta| \le \min\{d_k, d_\Gamma - 1\}$ and for any $x, y \in S$, there exist a constant $C_{\alpha,\beta}$ such that

$$\left|\partial_x^\alpha \partial_y^\beta K\Big(\kappa_m(x), \kappa_{m'}(y)\Big)\right| \le C_{\alpha,\beta}\left[\sqrt{|x|^2 + |y|^2}\right]^{-2-\mathbf{r}-|\alpha|-|\beta|}. \tag{5.6}$$

iv) If $\mathbf{r} = 0$, then the preceding estimates hold with $\mathbf{r}$ replaced by -1.

Proof. Indeed, we may suppose $\mathbf{r} < 0$ without loss of generality. Using (2.6) and the Newton-Leibniz formula, we conclude

$$K\left(\kappa_m(x), \kappa_m(x+z)\right) = k\left(\kappa_m(x), \kappa_m(x+z), \frac{\kappa_m(x+z) - \kappa_m(x)}{|\kappa_m(x+z) - \kappa_m(x)|}\right).$$

$$
\begin{aligned}
& \left|\frac{\kappa_m(x+z)-\kappa_m(x)}{z}\right|^{-2-\mathbf{r}} |z|^{-2-\mathbf{r}} \\
= \; & k\left(\kappa_m(x), \kappa_m(x+z), \frac{\int_0^1 \kappa_m'(x+\lambda z)\,\mathrm{d}\lambda \cdot \frac{z}{|z|}}{\left|\int_0^1 \kappa_m'(x+\lambda z)\,\mathrm{d}\lambda \cdot \frac{z}{|z|}\right|}\right) \cdot \\
& \left|\int_0^1 \kappa_m'(x+\lambda z)\,\mathrm{d}\lambda \cdot \frac{z}{|z|}\right|^{-2-\mathbf{r}} |z|^{-2-\mathbf{r}}.
\end{aligned}
$$

From this representation the estimate (5.3) follows easily. The second estimate (5.4) can be derived analogously.

On the other hand, using the identity $\kappa_m(x_1,0)=\kappa_{m'}(x_1,0)$, we get

$$
\begin{aligned}
\kappa_m(y_1+z_2,z_1)-\kappa_{m'}(y_1,z_3) &= \Big[\kappa_m(y_1+z_2,z_1)-\kappa_m(y_1+z_2,0)\Big] + \\
& \quad \Big[\kappa_m(y_1+z_2,0)-\kappa_m(y_1,0)\Big] - \Big[\kappa_{m'}(y_1,z_3)-\kappa_{m'}(y_1,0)\Big] \\
\kappa_m(y_1+z_2,z_1)-\kappa_{m'}(y_1,z_3) &= F(y_1,z)\cdot z \;:=\; \Big(F_1(y_1,z), F_2(y_1,z), F_3(y_1,z)\Big)\cdot z, \\
F_1(y_1,z) &:= \int_0^1 \kappa_{m'}'(y_1+z_2,\lambda z_1)\,\mathrm{d}\lambda, \\
F_2(y_1,z) &:= \int_0^1 \kappa_m'(y_1+\lambda z_2,0)\,\mathrm{d}\lambda, \\
F_3(y_1,z) &:= -\int_0^1 \kappa_m'(y_1,\lambda z_3)\,\mathrm{d}\lambda.
\end{aligned}
$$

We arrive at

$$
\begin{aligned}
K\Big(\kappa_m(y_1+z_2,z_1), \kappa_{m'}(y_1,z_3)\Big) &= k\left(\kappa_m(y_1+z_2,z_1), \kappa_{m'}(y_1,z_3), \frac{F(y_1,z)\cdot\frac{z}{|z|}}{|F(y_1,z)\cdot\frac{z}{|z|}|}\right) \cdot \\
& \quad \left|F(y_1,z)\cdot\frac{z}{|z|}\right|^{-2-\mathbf{r}} |z|^{-2-\mathbf{r}}. \qquad (5.7)
\end{aligned}
$$

From this representation the estimate (5.5) follows easily. The assertion of part iii) follows analogously. ∎

Hence, to define efficient quadrature rules we distinguish four cases. First we consider integrands of the form (compare Lemma 5.1)

$$
\begin{aligned}
f_1(x,y) &= K\Big(\kappa_m(x),\kappa_m(x+y)\Big)|\kappa_m'(x+y)||\kappa_m'(x)|\psi_P(\kappa_m(x))\psi_{P'}(\kappa_m(x+y)), \\
& \qquad\qquad l(P)\ge l(P'), \qquad (5.8) \\
f_2(x,y) &= K\Big(\kappa_m(y+x),\kappa_m(x)\Big)|\kappa_m'(y+x)||\kappa_m'(y)|\psi_P(\kappa_m(y+x))\psi_{P'}(\kappa_m(y)), \\
& \qquad\qquad l(P')\ge l(P),
\end{aligned}
$$

in case that the supports of ψ_P and $\psi_{P'}$ are contained in the same parametrization patch Γ_m and that the distance dist between the supports is less than $\Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$, where Υ is a prescribed positive constant. Second we consider

$$f_3(x,y) \;=\; K\Big(\kappa_m(x), \kappa_{m'}(y)\Big)|\kappa'_{m'}(y)||\kappa'_m(x)|\psi_P(\kappa_m(x))\psi_{P'}(\kappa_{m'}(y)), \tag{5.9}$$

for $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ contained in possibly different patches Γ_m and $\Gamma_{m'}$ such that dist is greater or equal to $\Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$. Third we analyze (5.9) for disjoint $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ contained in different patches Γ_m and $\Gamma_{m'}$ such that dist is less than the value $\Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$. Finally, we consider $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ contained in different patches Γ_m and $\Gamma_{m'}$ but with dist $= 0$. In this case we split the domain into smaller subdomains and treat the arising disjoint pairs of two-dimensional domains like in the previous case. The non-disjoint pairs are treated by a transform like in (5.5), by Duffy's transform, and by an additional potential transform. This way we have to integrate functions of the form

$$\begin{aligned}
f_4(x,y) \;=\; & K\bigg(\kappa_m\Big(2^{-l}(y_1+x_2^\alpha x_1), 2^{-l}x_2^\alpha\Big), \kappa_{m'}\Big(2^{-l}y_1, 2^{-l}(x_2^\alpha y_2)\Big)\bigg) \times \\
& \Big|\kappa'_{m'}\Big(2^{-l}y_1, 2^{-l}(x_2^\alpha y_2)\Big)\Big| \Big|\kappa'_m\Big(2^{-l}(y_1+x_2^\alpha x_1), 2^{-l}x_2\Big)\Big| \times \qquad (5.10) \\
& \psi_P\Big(\kappa_m\Big(2^{-l}(y_1+x_2^\alpha x_1), 2^{-l}x_2^\alpha\Big)\Big)\psi_{P'}\Big(\kappa_{m'}\Big(2^{-l}y_1, 2^{-l}(x_2^\alpha y_2)\Big)\Big)[x_2^\alpha]^2\alpha x_2^{\alpha-1}, \\
f_5(x,y) \;=\; & K\bigg(\kappa_m\Big(2^{-l}x_1, 2^{-l}(y_2^\alpha x_2)\Big), \kappa_{m'}\Big(2^{-l}(x_1+y_2^\alpha y_1), 2^{-l}y_2^\alpha\Big)\bigg) \times \\
& \Big|\kappa'_{m'}\Big(2^{-l}(x_1+y_2^\alpha y_1), 2^{-l}y_2^\alpha\Big)\Big| \Big|\kappa'_m\Big(2^{-l}x_1, 2^{-l}(y_2^\alpha x_2)\Big)\Big| \times \qquad (5.11) \\
& \psi_P\Big(\kappa_m\Big(2^{-l}x_1, 2^{-l}(y_2^\alpha x_2)\Big)\Big)\psi_{P'}\Big(\kappa_{m'}\Big(2^{-l}(x_1+y_2^\alpha y_1), 2^{-l}y_2^\alpha\Big)\Big)[y_2^\alpha]^2\alpha y_2^{\alpha-1}2^{-4l}, \\
f_6(x,y) \;=\; & K\bigg(\kappa_m\Big(2^{-l}x_1, 2^{-l}(y_1^\alpha x_2)\Big), \kappa_{m'}\Big(2^{-l}(x_1+y_1^\alpha), 2^{-l}(y_1^\alpha y_2)\Big)\bigg) \times \\
& \Big|\kappa'_{m'}\Big(2^{-l}(x_1+y_1^\alpha), 2^{-l}(y_1^\alpha y_2)\Big)\Big| \Big|\kappa'_m\Big(2^{-l}x_1, 2^{-l}(y_1^\alpha x_2)\Big)\Big| \times \qquad (5.12) \\
& \psi_P\Big(\kappa_m\Big(2^{-l}x_1, 2^{-l}(y_1^\alpha x_2)\Big)\Big)\psi_{P'}\Big(\kappa_{m'}\Big(2^{-l}(x_1+y_1^\alpha), 2^{-l}(y_1^\alpha y_2)\Big)\Big)[y_1^\alpha]^2\alpha y_1^{\alpha-1}2^{-4l}, \\
f_7(x,y) \;=\; & K\bigg(\kappa_m\Big(2^{-l}x_1^\alpha, 2^{-l}(x_1^\alpha x_2)\Big), \kappa_{m'}\Big(2^{-l}(x_1^\alpha y_1), 2^{-l}(x_1^\alpha y_2)),\Big)\bigg) \times \\
& \Big|\kappa'_{m'}\Big(2^{-l}(x_1^\alpha y_1), 2^{-l}(x_1^\alpha y_2)\Big)\Big| \Big|\kappa'_m\Big(2^{-l}x_1^\alpha, 2^{-l}(x_1^\alpha x_2)\Big)\Big| \times \qquad (5.13) \\
& \psi_P\Big(\kappa_m\Big(2^{-l}x_1^\alpha, 2^{-l}(x_1^\alpha x_2)\Big)\Big)\psi_{P'}\Big(2^{-l}(x_1^\alpha y_1), 2^{-l}(x_1^\alpha y_2)\Big)\Big)[x_1^\alpha]^3\alpha x_1^{\alpha-1}2^{-4l},
\end{aligned}$$

where $l = l(P)$ and $\alpha > 1$ is a sufficiently large integer. Without loss of generality, we always suppose $\mathbf{r} < 0$ in (5.8) -(5.13). The case $\mathbf{r} = 0$ follows from the case $\mathbf{r} = -1$ since by definition the singularity of the integral operator is like that of an operator of order -1.

5.3. First we suppose that $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ are contained in the same parametrization patch Γ_m. We consider f_1 and f_2 (cf. (5.8)) and assume that these are singular or almost singular, i.e. that the distance between the supports $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ is less than $\Upsilon\max\{2^{-l(P)},2^{-l(P')}\}$. Since f_2 can be treated analogously to f_1, we may restrict our consideration to $\max\{2^{-l(P)},2^{-l(P')}\}=2^{-l(P')}$ and to f_1 . The integrand f_1 is to be integrated over the support $\operatorname{supp}[\psi_P\circ\kappa_m]$ times the support $\operatorname{supp}[\psi_{P'}\circ\kappa_m]$ shifted by x.

Lemma 5.2 *Choose a parameter ϱ from the interval $(0,1)$ and suppose the relations (cf. Lemma 5.1)*

$$\max\left\{0,1+\frac{\mathbf{r}}{d_Q}\right\}<\varrho,\quad 0<-\mathbf{r}<d_Q\le\frac{1}{2}\min\{d_k,d_\Gamma-1\}$$

are fulfilled. We consider the domain of integration

$$D:=\left\{(x,y)\in\mathbb{R}^2:\ x\in\operatorname{supp}\,[\psi_P\circ\kappa_m]\,,\ y+x\in S_y\right\},$$

where $S_y\subseteq\operatorname{supp}[\psi_{P'}\circ\kappa_m]$ is a square of size $2^{-l(P')}$ such that $\psi_{P'}\circ\kappa_m$ is polynomial over S_y. Furthermore, we consider the integrand function $f=f_1$ from (5.8) over D. However, we extend f replacing the wavelet function $(x,y)\mapsto\psi_{P'}(\kappa_m(x+y))$ by the polynomial $p(x,y)$ which coincides with $\psi_{P'}(\kappa_m(x+y))$ over D. For our quadrature formula, we introduce the auxiliary functions $h(x,y)=\psi_P(\kappa_m(x))$ and

$$g(x,y)=K\Big(\kappa_m(x),\kappa_m(x+y)\Big)|\kappa_m'(x+y)||\kappa_m'(x)|\begin{cases}\psi_{P'}\Big(\kappa_m(x+y)\Big) & \text{if } (x,y)\in D\\ p(x,y) & \text{else .}\end{cases}$$

If $\sigma_{f_1,x,y}$ is defined by (5.1), then the Smolyak rule

$$\begin{aligned}Q(f_1)\ &:=\ \sum_{\substack{x\in\triangle^S,\,y\in\triangle^S:\\ [l(x)-l(P)]+(1-\varrho)[l(y)-l(P')]+\varrho\log_2\left(2^{l(y)}(|y_1|+|y_2|-\mathrm{dist})\right)\le l_0}}\sigma_{f_1,x,y},\\ l_0\ &:=\ \frac{1}{2}\log_2 N-\frac{1}{2}\log_2\log_2 N,\quad \mathrm{dist}\ :=\ \mathrm{dist}\Big(\operatorname{supp}[\psi_P\circ\kappa_m],\operatorname{supp}[\psi_{P'}\circ\kappa_m]\Big)\end{aligned}$$

includes no more than $\mathcal{O}(N)$ terms and quadrature knots. It requires no more than $\mathcal{O}(N)$ arithmetic operations, and the corresponding quadrature error satisfies

$$|I(f_1)-Q(f_1)|\ \le\ C\Upsilon^{d_Q}N^{-d_Q/2}\left[\log_2 N\right]^{[d_Q+2]/2}2^{-[d_Q+2]l(P)}2^{l(P')\mathbf{r}}.$$

Remark 5.1 *Note that for $\varrho=0$ we have the classical Smolyak rule. The parameter $\varrho\neq0$ leads to a mesh grading in the y domain toward the direction of* $\operatorname{supp}[\psi_P\circ\kappa_m]$. *This mesh grading exhibits the same approximation properties like the mesh with the grading in Fig. 3 with parameter $\alpha=1/[1-\varrho]$ (In this picture we first have divided the square of* $\operatorname{supp}[\psi_{P'}\circ\kappa_m]$ *into the 2^l strips Str_k, $k=0,\dots,2^l-1$ of all points with distance between $[(k+1)/2^l]^\alpha$ and $[k/2^l]^\alpha$ to the corner point closest to* $\operatorname{supp}[\psi_P\circ\kappa_m]$. *Second we have divided each strip Str_k uniformly into squares of side length equal to the width $h=[(k+1)/2^l]^\alpha-[k/2^l]^\alpha$ of the strip.). Thus the rule of Lemma 5.2 is a Smolyak quadrature based on a tensor product of a uniform mesh in x direction and a graded mesh in y direction.*

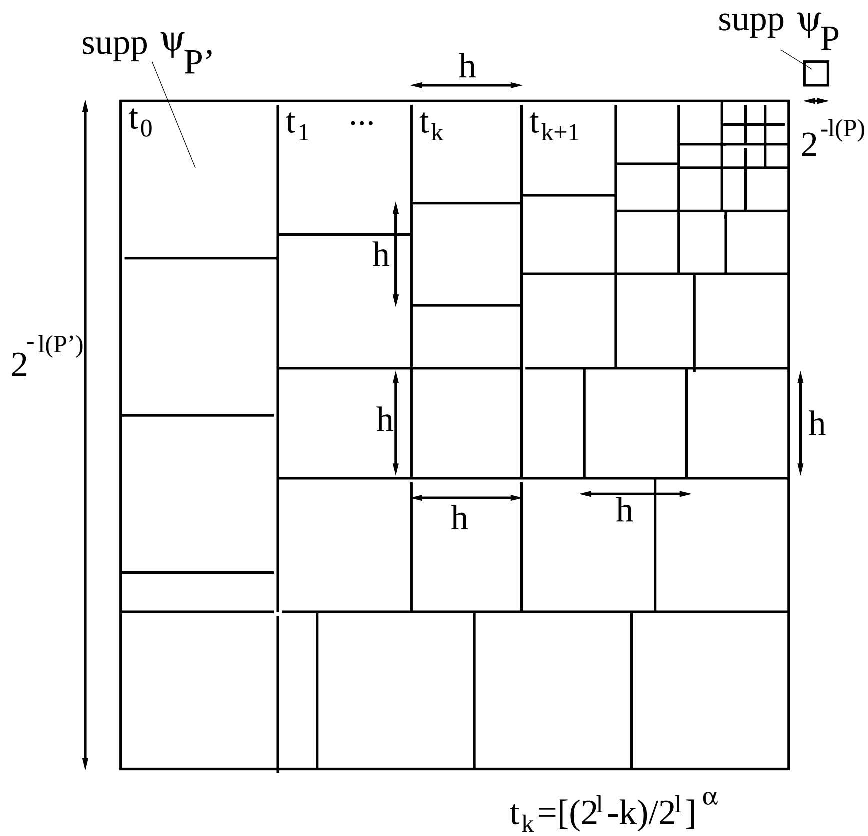

Figure 3: Mesh grading in y domain.

Proof. Without loss of generality we consider the most critical case of intersecting supports $\operatorname{supp}[\psi_P \circ \kappa_m]$ and $\operatorname{supp}[\psi_{P'} \circ \kappa_m]$ (dist $= 0$), and, for simplicity of notation, we suppose $\operatorname{supp}[\psi_P \circ \kappa_m] = [0, 2^{-l(P)}]^2$ and that the support $\operatorname{supp}[\psi_{P'} \circ \kappa_m]$ shifted by x is $[0, 2^{-l(P')}]^2$. We get the estimate

$$|\sigma_{f_1,x,y}| \;\leq\; C \begin{cases} 2^{-[2+d_Q]l(x)} 2^{-[2+d_Q]l(y)} |y|^{-2-\mathbf{r}-d_Q} \Upsilon^{d_Q} & \text{if } |y| \geq [d_Q+1]2^{-l(y)} \\ 2^{-[2+d_Q]l(x)} 2^{-2l(y)} |2^{-l(y)}|^{-2-\mathbf{r}} & \text{if } |y| < [d_Q+1]2^{-l(y)}, \end{cases} \tag{5.14}$$

$$|\sigma_{f_1,x,y}| \;\leq\; C \Upsilon^{d_Q} 2^{-[2+d_Q]l(x)} 2^{\mathbf{r}l(y)} \Big[1 + [|y_1| + |y_2|] 2^{l(y)}\Big]^{-2-\mathbf{r}-d_Q}. \tag{5.15}$$

Indeed, if $|y| \geq [d_Q+1]2^{-l(y)}$ and $l(x) \geq l_c^x$, $l(y) \geq l_c^y$, then the kernel function is non-singular over $\operatorname{supp}\psi_x^S \times \operatorname{supp}\psi_y^S$, the vanishing moments of the quadrature functionals admit the estimate

$$|\tilde{\psi}_x^S \otimes \tilde{\psi}_y^S(f)| \leq C 2^{-d_Q l(x)} 2^{-d_Q l(y)} \sup_{|\alpha| \leq d_Q,\, |\beta| \leq d_Q} |\partial_x^\alpha \partial_y^\beta f|,$$

the modulus of the integral $\iint h \psi_x^S \otimes \psi_y^S$ is less than $2^{-2l(x)} 2^{-2l(y)}$, the d_Q-th order derivatives of $\psi_{P'} \circ \kappa_m$ are bounded by $2^{d_Q l(P')}$, and the derivatives of the kernel can be estimated using Lemma 5.1, i). Due to the "almost singular" location of the supports $\operatorname{supp}[\psi_P \circ \kappa_m]$ and $\operatorname{supp}[\psi_{P'} \circ \kappa_m]$ the factor $2^{d_Q l(P')}$ is less than $C|y|^{-d_Q} \Upsilon^{d_Q}$. All these facts lead to the first estimate in (5.14). If $|y| < [d_Q+1]2^{-l(y)}$ and $l(x) \ge l_c^x,\ l(y) \ge l_c^y$, then, beside the Dirac delta functionals δ_z with grid points z, $|z| \ge 2^{-l(y)}$, the functional $\tilde{\psi}_y^S$ may also contain the Dirac delta functional δ_z at $z=0$. The corresponding infinite function values are set to zero by definition, i.e. the functional $\tilde{\psi}_y^S$ applied to f is modified into a linear combination of Dirac delta functionals δ_z with $|z| \ge 2^{-l(y)}$. Thus the vanishing moment property is lost, and we arrive at the lower line in (5.14). If $l(x) \ge l_c^x,\ l(y) = l_c^y - 1$, then $|y| \le C2^{-l(y)}$ and, again, without using any vanishing moments we arrive at the lower line in (5.14). Finally, if $l(x) = l_c^x - 1$, then the quadrature term vanishes due to the vanishing moments of $h(x,y) = \psi_P(\kappa_m(x))$.

To get the quadrature error estimate, we determine the error $|I(f_1) - Q(f_1)|$ and prove that the number of arithmetic operations is $\mathcal{O}(N)$. Since the case $\Upsilon > 1$ can be treated like the case $\Upsilon = 1$, we may assume $\Upsilon = 1$ without loss of generality. Setting

$$
\begin{aligned}
l_0' &:= l_0'(l) &&:= \frac{l_0 - [l - l(P)]}{1-\varrho},\\
l_1' &:= l_1'(l) &&:= l_0 - [l - l(P)],\\
m_0 &:= m_0(l,l') &&:= 2^{\{l_0 - [l-l(P)] - [l'-l(P')]\}/\varrho + [l'-l(P')]},
\end{aligned}
$$

we obtain that each of the following three inequalities $l(x) > l_0 + l(P)$, $l(y) > l_0' + l(P')$, and $|1+[|y_1|+|y_2|]2^{l(y)}| > m_0$ imply $[l(x)-l(P)] + (1-\varrho)[l(y)-l(P')] + \varrho \log_2\left(2^{l(y)}(|y_1|+|y_2|)\right) > l_0$, which excludes the term $\sigma_{f_1,x,y}$ from the sum in $Q(f_1)$. On the other hand, for a fixed $l(y)$, the maximal value for $|1+[|y_1|+|y_2|]2^{l(y)}|$ with $\operatorname{supp}\psi_y^S \cap \operatorname{supp}\psi_{P'} \neq \emptyset$ is $\mathcal{O}(2^{l(y)-l(P')})$. Hence, for $l(y) \le l_1' + l(P')$, all $\sigma_{f_1,x,y}$ with $\operatorname{supp}\psi_y^S \cap \operatorname{supp}\psi_{P'} \neq \emptyset$ belong to the sum in $Q(f_1)$. Obviously, there are at most $\mathcal{O}(2^{2(l-l(P))})$ points x of level l in $\operatorname{supp}\psi_P$ and there are at most $\mathcal{O}(m)$ points y of level l' with $|1+[|y_1|+|y_2|]2^{l(y)}| = m$ in $\operatorname{supp}\psi_{P'}$. Choosing $l = l(x)$ and $l' = l(y)$, and transforming these variables to $l = l - l(P)$ and $l' = l' - l(P')$, respectively, we get the estimate

$$
\begin{aligned}
|I(f_1) - Q_\varepsilon(f_1)| \;\le\; & \sum_{l=l_0+l(P)}^{\infty} 2^{2(l-l(P))} \sum_{l'=l(P')}^{\infty} \sum_{m=1}^{\infty} Cm2^{-[d_Q+2]l} 2^{\mathbf{r}l'} m^{-[d_Q+\mathbf{r}+2]} \qquad (5.16)\\
& + \sum_{l=l(P)}^{l_0+l(P)} 2^{2(l-l(P))} \sum_{l'=l_0'+l(P')}^{\infty} \sum_{m=1}^{\infty} Cm2^{-[d_Q+2]l} 2^{\mathbf{r}l'} m^{-[d_Q+\mathbf{r}+2]}\\
& + \sum_{l=l(P)}^{l_0+l(P)} 2^{2(l-l(P))} \sum_{l'=l_1'+l(P')}^{l_0'+l(P')} \sum_{m=m_0}^{\infty} Cm2^{-[d_Q+2]l} 2^{\mathbf{r}l'} m^{-[d_Q+\mathbf{r}+2]}\\
\le\; & C2^{-[d_Q+2]l(P)} 2^{\mathbf{r}l(P')} \Bigg\{ \sum_{l=l_0+l(P)}^{\infty} 2^{-d_Q[l-l(P)]} \sum_{l'=l(P')}^{\infty} 2^{\mathbf{r}[l'-l(P')]} +\\
& \sum_{l=l(P)}^{l_0+l(P)} 2^{-d_Q[l-l(P)]} \sum_{l'=l_0'+l(P')}^{\infty} 2^{\mathbf{r}[l'-l(P')]} +\\
& \sum_{l=l(P)}^{l_0+l(P)} 2^{-d_Q[l-l(P)]} \sum_{l'=l_1'+l(P')}^{l_0'+l(P')} 2^{\mathbf{r}[l'-l(P')]} \sum_{m=m_0}^{\infty} m^{-[d_Q+\mathbf{r}+1]} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}\Bigg\{\sum_{l=l_0}^{\infty}2^{-d_Q l}\sum_{l'=0}^{\infty}2^{\mathbf{r}l'}+\sum_{l=0}^{l_0}2^{-d_Q l}\sum_{l'=l_0'}^{\infty}2^{\mathbf{r}l'}+\\
&\qquad\sum_{l=0}^{l_0}2^{-d_Q l}\sum_{l'=l_1'}^{l_0'}2^{\mathbf{r}l'}\left[2^{\{l_0-l-l'\}/\varrho+l'}\right]^{-[d_Q+\mathbf{r}]}\Bigg\}\\
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}\Bigg\{2^{-d_Q l_0}+\sum_{l=0}^{l_0}2^{-d_Q l}2^{\mathbf{r}[l_0-l]/[1-\varrho]}+\\
&\qquad 2^{-l_0[d_Q+\mathbf{r}]/\varrho}\sum_{l=0}^{l_0}2^{l\{-d_Q+[d_Q+\mathbf{r}]/\varrho\}}\sum_{l'=l_1'}^{l_0'}2^{l'\{\mathbf{r}-[d_Q+\mathbf{r}][1-1/\varrho]\}}\Bigg\}\\
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}\Bigg\{2^{-d_Q l_0}+2^{\mathbf{r}l_0/[1-\varrho]}\sum_{l=0}^{l_0}2^{l\{-\mathbf{r}/[1-\varrho]-d_Q\}}+\\
&\qquad 2^{-l_0[d_Q+\mathbf{r}]/\varrho}\sum_{l=0}^{l_0}2^{l\{-d_Q+[d_Q+\mathbf{r}]/\varrho\}}2^{l_1'\{\mathbf{r}-[d_Q+\mathbf{r}][1-1/\varrho]\}}\Bigg\}\\
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}\Bigg\{2^{-d_Q l_0}+2^{\mathbf{r}l_0/[1-\varrho]}2^{l_0\{-\mathbf{r}/[1-\varrho]-d_Q\}}+2^{-l_0 d_Q}l_0\Bigg\}\\
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}2^{-d_Q l_0}l_0\\
&\leq C2^{-[d_Q+2]l(P)}2^{\mathbf{r}l(P')}N^{-d_Q/2}\left[\log_2 N\right]^{[d_Q+2]/2}.
\end{aligned}
$$

On the other hand, the number $\mathcal{N}$ of arithmetic operations can be estimated as

$$
\begin{aligned}
\mathcal{N}&\leq\sum_{l=l(P)}^{l_0+l(P)}2^{2[l-l(P)]}\Bigg\{\sum_{l'=l(P')}^{l_1'+l(P')}2^{2[l'-l(P')]}C+\sum_{l'=l_1'+l(P')}^{l_0'+l(P')}\sum_{m=1}^{m_0}Cm\Bigg\}\\
&\leq C\sum_{l=0}^{l_0}2^{2l}\Bigg\{\sum_{l'=0}^{l_1'}2^{2l'}+\sum_{l'=l_1'}^{l_0'}\left[2^{\{l_0-l-l'\}\}/\varrho+l'}\right]^2\Bigg\}\\
&\leq C2^{2l_0}l_0+C2^{2l_0/\varrho}\sum_{l=0}^{l_0}2^{2l[1-1/\varrho]}\sum_{l'=l_1'}^{l_0'}2^{2l'[1-1/\varrho]}\\
&\leq C2^{2l_0}l_0+C2^{2l_0/\varrho}\sum_{l=0}^{l_0}2^{2l[1-1/\varrho]}2^{2l_1'[1-1/\varrho]}\ \leq\ C2^{2l_0}l_0\ \leq\ CN.
\end{aligned}
$$

∎

5.4. Next we consider the function f_3 (cf. (5.9)) under the assumption that the support $\operatorname{supp}\psi_P$ is contained in the boundary patch Γ_m and $\operatorname{supp}\psi_{P'}$ in $\Gamma_{m'}$. Moreover, we assume that the distance dist $:= \operatorname{dist}(\operatorname{supp}\psi_P,\operatorname{supp}\psi_{P'})$ is greater or equal to the expression $\Upsilon\max\{2^{-l(P)},2^{-l(P')}\}$. Since two parametrizations κ_m and $\kappa_{m'}$ are involved for possibly different m and m', we are not able to separate the singularity direction in the kernel function. Hence, we get a factor $\operatorname{dist}^{-\alpha}$ in the estimate of each α-th order derivative of the kernel independent of whether we differentiate with respect to x or y. We arrive at

Lemma 5.3 *Suppose the relations $-d_Q < \mathbf{r} < 0$, $d_Q \le \frac{1}{2}\min\{d_k, d_\Gamma - 1\}$ are valid. We consider the domain of integration $D = \operatorname{supp}[\psi_P \circ \kappa_m] \times \operatorname{supp}[\psi_{P'} \circ \kappa_{m'}]$. Furthermore suppose our integrand $f = f_3$ is the function from (5.9). For our quadrature, we introduce the auxiliary functions g and h by $g(x,y) = K(\kappa_m(x), \kappa_{m'}(y))|\kappa'_{m'}(y)||\kappa'_m(x)|$ as well as $h(x,y) = [\psi_P \circ \kappa_m(x)][\psi_{P'} \circ \kappa_m(y)]$. If $\sigma_{f_3,x,y}$ is given by (5.1), then the Smolyak rule*

$$Q(f_3) := \sum_{\substack{x\in\triangle^S, y\in\triangle^S: \\ [l(x)-l(P)]+[l(y)-l(P')]\le l_0}} \sigma_{f_3,x,y}, \quad l_0 := \frac{1}{2}\log_2 N - \frac{1}{2}\log_2\log_2 N \tag{5.17}$$

includes no more than $\mathcal{O}(N)$ terms and quadrature knots. It requires no more than $\mathcal{O}(N)$ arithmetic operations, and the corresponding quadrature error satisfies

$$|I(f_3) - Q(f_3)| \le C 2^{-[d_Q+2]l(P)} 2^{-[d_Q+2]l(P')} N^{-\frac{d_Q}{2}} [\log_2 N]^{\frac{d_Q+2}{2}} \operatorname{dist}^{-[2d_Q+\mathbf{r}+2]}. \tag{5.18}$$

Proof. Clearly, the number $\mathcal{N}$ of arithmetic operations can be estimated as

$$\begin{aligned} \mathcal{N} &\le C \sum_{l=l(P)}^{l_0+l(P)} 2^{2(l-l(P))} \sum_{l'=l(P')}^{l_0+l(P')-[l-l(P)]} 2^{2(l'-l(P'))} \le C \sum_{l=0}^{l_0} 2^{2l} \sum_{l'=0}^{l_0-l} 2^{2l'} \\ &\le C \sum_{l=0}^{l_0} 2^{2l} 2^{2(l_0-l)} \le C 2^{2l_0} l_0 \le CN. \end{aligned} \tag{5.19}$$

Similarly to (5.15), there holds

$$|\sigma_{f_3,x,y}| \le C 2^{-[2+d_Q]l(x)} 2^{-[2+d_Q]l(y)} \operatorname{dist}^{-[2d_Q+\mathbf{r}+2]}. \tag{5.20}$$

For the quadrature error, we obtain

$$\begin{aligned} |I(f_3) - Q_\varepsilon(f_3)| &\le C \sum_{l=l_0+l(P)}^{\infty} 2^{2(l-l(P))} \sum_{l'=l(P')}^{\infty} 2^{2(l'-l(P'))} \frac{2^{-[2+d_Q]l} 2^{-[2+d_Q]l'}}{\operatorname{dist}^{2d_Q+\mathbf{r}+2}} \\ &\quad + C \sum_{l=l(P)}^{l_0+l(P)} 2^{2(l-l(P))} \sum_{l'=l_0-[l-l(P)]+l(P')}^{\infty} 2^{2(l'-l(P'))} \frac{2^{-[2+d_Q]l} 2^{-[2+d_Q]l'}}{\operatorname{dist}^{2d_Q+\mathbf{r}+2}} \\ &\le C \frac{2^{-2l(P)} 2^{-2l(P')}}{\operatorname{dist}^{2d_Q+\mathbf{r}+2}} \Bigg[\sum_{l=l_0+l(P)}^{\infty} 2^{-d_Q l} \sum_{l'=l(P')}^{\infty} 2^{-d_Q l'} + \\ &\qquad \sum_{l=l(P)}^{l_0+l(P)} 2^{-d_Q l} \sum_{l'=l_0-[l-l(P)]+l(P')}^{\infty} 2^{-d_Q l'} \Bigg] \\ &\le C \frac{2^{-2l(P)} 2^{-2l(P')}}{\operatorname{dist}^{2d_Q+\mathbf{r}+2}} \Bigg[2^{-d_Q[l_0+l(P)]} 2^{-d_Q l(P')} + \\ &\qquad \sum_{l=l(P)}^{l_0+l(P)} 2^{-d_Q l} 2^{-d_Q\{l_0-[l-l(P)]+l(P')\}} \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{2^{-2l(P)}2^{-2l(P')}}{\mathrm{dist}^{2d_Q+\mathbf{r}+2}}\left[2^{-d_Q l(P')}2^{-d_Q[l_0+l(P)]}l_0\right]\\
&\leq C2^{-[d_Q+2]l(P)}2^{-[d_Q+2]l(P')}\mathrm{dist}^{-[2d_Q+\mathbf{r}+2]}N^{-d_Q/2}\left[\log_2 N\right]^{[d_Q+2]/2}.
\end{aligned}
$$

■

5.5. Next we consider the function f_3 under the assumption that the support $\operatorname{supp}\psi_P$ is contained in the boundary patch Γ_m and $\operatorname{supp}\psi_{P'}$ in $\Gamma_{m'}$. Moreover, we assume that the distance $\mathrm{dist} := \mathrm{dist}(\operatorname{supp}\psi_P, \operatorname{supp}\psi_{P'})$ satisfies $0 < \mathrm{dist} < \Upsilon\max\{2^{-l(P)}, 2^{-l(P')}\}$. In this case dist is automatically greater or equal to the expression $c_\Gamma^{-1}\min\{2^{-l(P)}, 2^{-l(P')}\}$, where c_Γ is the Lipschitz constant for the inverse parametrization mappings κ_m^{-1}, $m = 1, \ldots, m_\Gamma$. We arrive at

Lemma 5.4 *Suppose the relations $-d_Q < \mathbf{r} < 0$, $d_Q \leq \frac{1}{2}\min\{d_k, d_\Gamma - 1\}$ are valid and assume $0 < \mathrm{dist} < \max\{2^{-l(P)}, 2^{-l(P')}\}$. For the sake of definiteness, we first assume $\max\{2^{-l(P)}, 2^{-l(P')}\} = 2^{-l(P')}$. We consider the domain of integration given by $D = \operatorname{supp}[\psi_P\circ\kappa_m]\times\operatorname{supp}[\psi_{P'}\circ\kappa_{m'}]$. For our quadrature, we split the larger support $\operatorname{supp}[\psi_{P'}\circ\kappa_{m'}]$ into the union of dyadic subsquares S_y^i, $i = 1, \ldots, i_d$ such that:*

i) The wavelet $\psi_{P'}\circ\kappa_{m'}$ is polynomial over each S_y^i.

ii) The side length $2^{-l_y^i}$ of S_y^i is less than the distance d from $\kappa_m(S_y^i)$ to $\operatorname{supp}\psi_P$ and larger than $0.25d$.

iii) The minimal side length of a S_y^i is 2^{-l_d} with l_d the largest integer such that $2^{-l_d} \geq \mathrm{dist}$.

iv) The number of subdomains for a fixed side length 2^{-l} is less than 8.

We consider the integrand $f = f_3$ from (5.9) over $\operatorname{supp}[\psi_P\circ\kappa_m]\times S_y^i$. However, we extend f replacing the wavelet function $(x,y)\mapsto\psi_{P'}(\kappa_{m'}(y))$ by the polynomial $p(x,y)$ which coincides with $\psi_{P'}(\kappa_{m'}(y))$ over D. Over each domain $\operatorname{supp}[\psi_P\circ\kappa_m]\times S_y^i$ we apply the quadrature rule of Lemma 5.3. More precisely, we introduce the auxiliary functions $h(x,y) = [\psi_P\circ\kappa_m(x)]$ and

$$
g(x,y) = K\Big(\kappa_m(x), \kappa_{m'}(y)\Big)|\kappa'_{m'}(y)||\kappa'_m(x)|\left\{\begin{array}{ll}\psi_{P'}(\kappa_m(y)) & \text{if } y\in S_y^i\\ p(x,y) & \text{else}.\end{array}\right.
$$

If $\sigma_{f_3,x,y}$ is give by (5.1), then the composite Smolyak rule

$$
\begin{aligned}
Q(f_3) &:= \sum_{i=1}^{i_d}\sum_{\substack{x\in\triangle^S,\, y\in\triangle^S\cap S_y^i:\\ [l(x)-l(P)]+[l(y)-l_y^i]\leq l_0}}\sigma_{f_3,x,y}, \qquad (5.21)\\
l_0 &:= \frac{1}{2}\log_2 N - \frac{1}{2}\log_2\log_2 N - \frac{1}{2}\log_2[l(P)+1-l(P')]
\end{aligned}
$$

includes no more than $\mathcal{O}(N)$ terms and quadrature knots. It requires no more than $\mathcal{O}(N)$ arithmetic operations, and the corresponding quadrature error satisfies

$$
|I(f_3) - Q(f_3)| \leq C\frac{2^{-[d_Q+2]l(P)}}{\mathrm{dist}^{[d_Q+\mathbf{r}]}}N^{-\frac{d_Q}{2}}[\log_2 N]^{\frac{d_Q+2}{2}}[l(P)+1-l(P')]^{\frac{d_Q}{2}}.
$$

Under the same assumption but with $0 < \mathrm{dist} < \max\{2^{-l(P)}, 2^{-l(P')}\} = 2^{-l(P)}$*, an analogous quadrature with the roles of* ψ_P *and* $\psi_{P'}$ *interchanged leads to the estimate*

$$|I(f_3) - Q(f_3)| \;\leq\; C\frac{2^{-[d_Q+2]l(P')}}{\mathrm{dist}^{[d_Q+\mathbf{r}]}}N^{-\frac{d_Q}{2}}[\log_2 N]^{\frac{d_Q+2}{2}}[l(P') + 1 - l(P)]^{\frac{d_Q}{2}}.$$

Proof. Suppose $l(P) > l(P')$. For a fixed i and $\mathrm{supp}\,[\psi_P \circ \kappa_m] \times S_y^i$, the proof of Lemma 5.3 implies the bound $C2^{2l_0}l_0$ for the number of knots and the bound

$$C2^{-[d_Q+2]l(P)}2^{-[d_Q+2]l_y^i}\left[2^{-l_y^i}\right]^{-[2d_Q+\mathbf{r}+2]}2^{-d_Q l_0}l_0 \tag{5.22}$$

for the corresponding quadrature error. Note that the Smolyak rule for $\mathrm{supp}\,[\psi_P \circ \kappa_m] \times S_y^i$ is defined with a coarsest level l_c^y adjusted to l_y^i and not to $l(P')$. Summing up over i we arrive at

$$\begin{aligned}
\mathcal{N} \;&\leq\; \sum_{i=1}^{i_d} C2^{2l_0}l_0 \;\leq\; C2^{2l_0}l_0 i_d \;\leq\; CN,\\
|I(f_3) - Q(f_3)| \;&\leq\; \sum_{i=1}^{i_d} C2^{-[d_Q+2]l(P)}2^{-[d_Q+2]l_y^i}\left[2^{-l_y^i}\right]^{-[2d_Q+\mathbf{r}+2]}2^{-d_Q l_0}l_0\\
&\leq\; C\sum_{l=l(P')-\log\Upsilon}^{l_d} 2^{-[d_Q+2]l(P)}2^{-[d_Q+2]l}\left[2^{-l}\right]^{-[2d_Q+\mathbf{r}+2]}2^{-d_Q l_0}l_0\\
&\leq\; C2^{-[d_Q+2]l(P)}\sum_{l=l(P')-\log\Upsilon}^{l_d} 2^{[d_Q+\mathbf{r}]l}2^{-d_Q l_0}l_0\\
&\leq\; C2^{-[d_Q+2]l(P)}\mathrm{dist}^{-[d_Q+\mathbf{r}]}2^{-d_Q l_0}l_0\\
&\leq\; C\frac{2^{-[d_Q+2]l(P)}}{\mathrm{dist}^{[d_Q+\mathbf{r}]}}N^{-\frac{d_Q}{2}}[\log_2 N]^{\frac{d_Q+2}{2}}[l(P) + 1 - l(P')]^{\frac{d_Q}{2}}.
\end{aligned}$$

∎

Remark 5.2 *If the levels* $l(P)$ *and* $l(P')$ *coincide, then we can split both, the support of the trial function* $\psi_P \circ \kappa_m$ *and the support of the test function* $\psi_{P'} \circ \kappa_{m'}$ *into the squares* S_x^i *and* S_y^j *of level* $l(P)$*, and we can set* $h(x, y) \equiv 1$ *in the quadrature of Lemma 5.4. Indeed, the derivatives of order* k *to the polynomials* $\psi_P \circ \kappa_m$ *and* $\psi_{P'} \circ \kappa_{m'}$ *are bounded by* $2^{kl(P)}$*, which is less than* dist^{-k}*, and (5.22) remains valid.*

5.6. Finally, we consider the case that the support $\mathrm{supp}\,\psi_P$ is contained in the boundary patch Γ_m and $\mathrm{supp}\,\psi_{P'}$ in $\Gamma_{m'}$, $m \neq m'$ and that these supports are not disjoint. Without loss of generality suppose $\max\{2^{-l(P)}, 2^{-l(P')}\} = 2^{-l(P')}$. In this case we split the domain of integration $D = \mathrm{supp}\,[\psi_P \circ \kappa_m] \times \mathrm{supp}\,[\psi_{P'} \circ \kappa_{m'}]$ into two parts $D = D_1 \cup D_2$. Here, by D_1 we denote the union of all direct products $S_x \times S_y$ such that $S_x \subseteq \mathrm{supp}\,[\psi_P \circ \kappa_m]$ and $S_y \subseteq \mathrm{supp}\,[\psi_{P'} \circ \kappa_{m'}]$ holds, that the curved squares $\kappa_m(S_x)$ and $\kappa_{m'}(S_y)$ have at least one point in common, and that S_x and S_y are squares of level $l(P)$, i.e. squares from the uniform partition of the parameter domain S into $2^{2l(P)}$ equal parts. By D_2 we denote the remainder. Now we can apply the quadrature of Lemma 5.3 and Remark 5.2 to the domain D_2. For the subdomains $S_x \times S_y \subseteq D_1$ with a common edge, we use

Lemma 5.5 *Without loss of generality we assume that the intersection of $\Gamma_m \cap \Gamma_{m'}$ is the common side $\{\kappa_m((x_1,0)) : \ 0 \le x_1 \le 1\}$ with the additional property $\kappa_m((x_1,0)) = \kappa_{m'}((x_1,0))$, $0 \le x_1 \le 1$. Moreover, to simplify the notation, we assume that S_x and S_y coincide with the square $\{(x_1,x_2) : \ 0 \le x_1, x_2 \le 2^{-l(P)}\}$. We split the domain of integration $D' = S_x \times S_y$ into the three parts*

$$\begin{aligned} D_4' &:= \Big\{(x',y') \in D' : \ x_2' \ge y_2', \ x_2' \ge |x_1' - y_1'|\Big\}, \\ D_5' &:= \Big\{(x',y') \in D' : \ y_2' \ge x_2', \ y_2' \ge |x_1' - y_1'|\Big\}, \\ D_6' &:= \Big\{(x',y') \in D' : \ |x_1' - y_1'| \ge x_2', \ |x_1' - y_1'| \ge y_2'\Big\}, \end{aligned}$$

and apply a transform similar to that of Duffy (cf. the formulae (5.10)-(5.12))

$$\begin{aligned} D_4' \ni \Big((x_1',x_2'),(y_1',y_2')\Big) &= 2^{-l(P)}\Big((y_1 + x_2^\alpha x_1, x_2^\alpha),(y_1, x_2^\alpha y_2)\Big), \ \Big((x_1,x_2),(y_1,y_2)\Big) \in D_4'', \\ D_5' \ni \Big((x_1',x_2'),(y_1',y_2')\Big) &= 2^{-l(P)}\Big((x_1, y_2^\alpha x_2),(x_1 + y_2^\alpha y_1, y_2^\alpha)\Big), \ \Big((x_1,x_2),(y_1,y_2)\Big) \in D_5'', \\ D_6' \ni \Big((x_1',x_2'),(y_1',y_2')\Big) &= 2^{-l(P)}\Big((x_1, y_1^\alpha x_2),(x_1 + y_1^\alpha, y_1^\alpha y_2)\Big), \ \Big((x_1,x_2),(y_1,y_2)\Big) \in D_6'', \end{aligned}$$

where α stands for a positive integer with $\alpha > [d_Q+1]/[-\mathbf{r}]$. Over the domains D_i'' we have to integrate the function $f = f_{i+3}$ (cf. (5.10)-(5.12)). If $\psi_P \circ \kappa_m(x')$ coincides with the polynomial $p(x')$ for $x' \in S_x$ and if $\psi_{P'} \circ \kappa_m(y')$ coincides with the polynomial $\tilde{p}(y')$ for $y' \in S_y$, then we set

$$\begin{aligned} g(x,y) &:= K\Bigg(\kappa_m\Big(2^{-l(P)}(y_1 + x_2^\alpha x_1), 2^{-l(P)} x_2^\alpha\Big), \kappa_{m'}\Big(2^{-l(P)} y_1, 2^{-l(P)}(x_2^\alpha y_2)\Big)\Bigg) \times \\ &\quad \Big|\kappa_{m'}'\Big(2^{-l(P)} y_1, 2^{-l(P)}(x_2^\alpha y_2)\Big)\Big| \Big|\kappa_m'\Big(2^{-l(P)}(y_1 + x_2^\alpha x_1), 2^{-l(P)} x_2\Big)\Big| \times \\ &\quad p\Big(2^{-l(P)}(y_1 + x_2^\alpha x_1), 2^{-l(P)} x_2^\alpha\Big) \tilde{p}\Big(2^{-l(P)} y_1, 2^{-l(P)}(x_2^\alpha y_2)\Big) [x_2^\alpha]^2 \alpha x_2^{\alpha-1} \end{aligned}$$

for $f = f_4$, define g for $f = f_5$ and $f = f_6$ analogously, choose $h \equiv 1$, and retain the definition of $\sigma_{f,x,y} = \sigma_{f_{i+3},x,y}$ from (5.1). Then the Smolyak rule

$$Q(f_{i+3}) := \sum_{\substack{x \in \triangle^S, y \in \triangle^S: \\ [l(x)-l(P)]+[l(y)-l(P')]+ \le l_0}} \sigma_{f_{i+3},x,y}, \qquad l_0 := \frac{1}{2}\log_2 N - \frac{1}{2}\log_2\log_2 N$$

includes no more than $\mathcal{O}(N)$ terms and quadrature knots. It requires no more than $\mathcal{O}(N)$ arithmetic operations, and the corresponding quadrature error satisfies

$$|I(f_{i+3}) - Q(f_{i+3})| \le CN^{-d_Q/2} [\log_2 N]^{[d_Q+2]/2} 2^{-[2-\mathbf{r}]l(P)}.$$

The quadrature weights $\iint_{D_{i+3}''} \psi_x^S \otimes \psi_y^S$ can still be computed analytically.

Proof. For definiteness sake, we restrict our consideration to the quadrature over D_4''. From (5.7) we infer

$$\left[K\left(\kappa_m\left(2^{-l(P)}(y_1+x_2^\alpha x_1),2^{-l(P)}(x_2^\alpha)\right),\kappa_{m'}\left(2^{-l(P)}y_1,2^{-l(P)}(x_2^\alpha y_2)\right)\right)\right][x_2^\alpha]^2\alpha x_2^{\alpha-1}2^{-4l(P)}$$

$$= k\left(\kappa_m\left(2^{-l(P)}(y_1+x_2^\alpha x_1),2^{-l(P)}(x_2^\alpha)\right)\ ,\ \kappa_{m'}\left(2^{-l(P)}y_1,2^{-l(P)}(x_2^\alpha y_2)\right)\ ,\right.$$

$$\left.\frac{F\left(2^{-l(P)}y_1,2^{-l(P)}(x_2^\alpha,x_2^\alpha x_1,x_2^\alpha y_2)\right)\cdot\dfrac{(1,x_1,y_2)}{|(1,x_1,y_2)|}}{\left|F\left(2^{-l(P)}y_1,2^{-l(P)}(x_2^\alpha,x_2^\alpha x_1,x_2^\alpha y_2)\right)\cdot\dfrac{(1,x_1,y_2)}{|(1,x_1,y_2)|}\right|}\right)\cdot$$

$$\left|F\left(2^{-l(P)}y_1,2^{-l(P)}(x_2^\alpha,x_2^\alpha x_1,x_2^\alpha y_2)\right)\cdot\frac{(1,x_1,y_2)}{|(1,x_1,y_2)|}\right|^{-2-\mathbf{r}}\frac{x_2^{-\mathbf{r}\alpha-1}}{|(1,x_1,y_2)|^{2+\mathbf{r}}}2^{[-2+\mathbf{r}]l(P)}.$$

The only singular factor in the last expression is $x_2^{-\mathbf{r}\alpha-1}$. Differentiation with respect to x_1, y_1, and y_2 does not lead to higher order singular factors. Only the differentiation with respect to x_2 decreases the singularity exponent in the power of x_2 by one. Consequently, the kernel function including the substituted variables and multiplied by the Jacobian of the transformation mapping has continuous derivatives up to order d_Q, and the partial derivatives of f_4 taken up to order d_Q are uniformly bounded by $C2^{[-2+\mathbf{r}]l(P)}$. This fact and the proof to Lemma 5.3 imply the estimate of the present lemma.

The quadrature weights $\iint_{D_4''}\psi_x^S\otimes\psi_y^S$ can still be computed analytically. Indeed, the boundary of the transformed domain is determined by simple rational functions, and the integration of polynomials in such a domain requires the primitives of simple rational functions, only. ∎

The case $\kappa_m(S_x)\cap\kappa_m(S_y)=\{Q\}$ can be treated similarly. For simplicity of notation, we assume that S_x and S_y coincide with $\{(x_1,x_2):\ 0\le x_1,x_2\le 2^{-l(P)}\}$ and that $\Gamma_m\cap\Gamma_{m'}=\{Q\}$ and $Q=\kappa_m(0,0)=\kappa_{m'}(0,0)$. The domain D' is to be split into four parts according to which of the four coordinates x_1, x_2, y_1, y_2 is the largest. For definiteness, we consider the part where x_1 is the largest. The transformation can be chosen by

$$D_1' \ := \ \left\{(x',y')\in D':\ x_1'=\max\{x_1,x_2,y_1,y_2\}\right\},$$
$$D_1'\ni\left((x_1',x_2'),(y_1',y_2')\right) \ = \ 2^{-l(P)}\left((x_1^\alpha,x_1^\alpha x_2),(x_1^\alpha y_1,x_1^\alpha y_2)\right),\ \left((x_1,x_2),(y_1,y_2)\right)\in D_1''.$$

Over the domains D_i'' we have to integrate the function $f=f_7$ (cf. (5.13)). We choose $h\equiv 1$ and retain the definition of $\sigma_{f,x,y}=\sigma_{f_{i+3},x,y}$ from (5.1). Then the assertions of the last lemma remain valid for the Smolyak rule applied to f_7, and the proof is the same. Even more, the condition on α can be relaxed to $\alpha>[d_Q+1]/[2-\mathbf{r}]$.

6 The Quadrature Algorithm and its Complexity

Using the quadrature formulas of Sect. 4, the algorithm for the assembling of the stiffness matrix corresponding to the operator of multiplication and the computation of the right-hand side is defined. To set up the integrals $\langle K\psi_{P'}, \psi_P\rangle$, we apply the Lemmata 5.2-5.5, where the number of quadrature knots $N = N_{P,P'}$ is still to be given. We choose parameters β slightly less than two, γ slightly larger than two, γ' slightly larger than one, β_m slightly less than zero, and β_M slightly less than $1+\gamma'$. We introduce the numbers $\alpha := 2\beta - \gamma$ and $\alpha' := 1 + \beta_M + \beta_m - \gamma'$, and set (compare the compression strategy in (3.9))

$$N_{P,P'} \quad := \quad \begin{cases} \max\left\{1, \dfrac{2^{\alpha L - \beta l(P) - \beta l(P')}}{\mathrm{Dist}^{\gamma}}\right\} & \text{if } \operatorname{supp}\psi_P, \operatorname{supp}\psi_{P'} \subseteq \Gamma_m \text{ or if} \\ & \mathrm{dist} \geq \Upsilon \max\left\{2^{-l(P)}, 2^{-l(P')}\right\} \\ \max\left\{1, \dfrac{2^{\alpha' L - \beta_m \min\{l(P),l(P')\} - \beta_M \max\{l(P),l(P')\}}}{\Upsilon \mathrm{dist}^{\gamma'}}\right\} & \\ & \text{if } 0 < \mathrm{dist} < \Upsilon \max\left\{2^{-l(P)}, 2^{-l(P')}\right\} \text{ and} \\ & \operatorname{supp}\psi_P \cup \operatorname{supp}\psi_{P'} \text{ not contained in a} \\ & \text{single } \Gamma_m \end{cases} \tag{6.1}$$

Here the symbol Dist stands for the sum of the distance between the points P and P' plus the minimum $\min\{2^{-l(P)}, 2^{-l(P')}\}$. Hence, we have dist $< C$Dist and dist $\sim$ Dist if dist $\geq \max\{2^{-l(P)}, 2^{-l(P')}\}$. It turns out that the choice of the quadrature leads to an algorithm of almost optimal complexity.

Theorem 6.1 *Suppose the assumptions of Theorem 3.1 are fulfilled. If $N = \mathcal{O}(2^{2L})$ is the dimension of the trial space (number of degrees of freedom), then the number $\mathcal{N}$ of all quadrature knots and the number of all arithmetic operations necessary for the assembling of the stiffness matrix and the computation of the right-hand side is $\mathcal{O}(N \log N)$. In particular, the last complexity bound does not depend on the parameter Υ.*

Proof. To get the right-hand side $(\eta_P^Q)_P$, we have to compute the approximate integrals $Q(v \circ \kappa_m\, \overline{p \circ \kappa_m})$ over the level L subdomains. This step requires $\mathcal{O}(N)$ operations. Next we have to add these values to form (4.1). Since each $\Gamma_{m,L,k}$ is contained in no more than $\mathcal{O}(L)$ supports $\operatorname{supp}\psi_P$, we need no more than $\mathcal{O}(N \log N)$ operations for the right-hand side. To obtain the approximate multiplication operator, we first compute the values Q_{L,k,p_1,p_2} which requires no more than $\mathcal{O}(N)$ operations. Even the next step, the determination of the Q_{l,k,p_1,p_2} with $l < L$ can be established with no more than $\mathcal{O}(N)$ operations. Finally, the summation in (4.3) cost no more than $\mathcal{O}(N \log N)$ operations, since each curved triangle $\Gamma_{m,l,k}$ is contained in at most $\mathcal{O}(L)$ supports $\operatorname{supp}\psi_P$ of level $l(P') \leq l$. In other words, the computation of the discretized multiplication operator requires no more than $\mathcal{O}(N \log N)$ operations, too.

Next we count the operations for the assembling of the discretized integral operator. Obviously, the number of all necessary arithmetic operations is less than a constant multiple of the number of quadrature knots. Thus we only have to count the knots. We split $\mathcal{N}$ into the sum $\sum_{i=1}^{2} \mathcal{N}_i$, where the $\mathcal{N}_i$ are the counts for the knots in the two special cases appearing in the definition (6.1). First we consider the case that $\operatorname{supp}\psi_P, \operatorname{supp}\psi_{P'} \subseteq \Gamma_m$ or dist $\geq \Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$. From (6.1) and the estimate for the non-zero entries in

Theorem 3.1, we conclude

$$
\begin{aligned}
\mathcal{N}_1 &\leq \sum_{P,P'\in\triangle_L:\, a_{P,P'}\neq 0} N_{P,P'} \leq \sum_{P,P'\in\triangle_L:\, a_{P,P'}\neq 0} 1 + 2 \sum_{P,P'\in\triangle_L:\, a_{P,P'}\neq 0,\, l(P)\leq l(P')} \frac{2^{\alpha L-\beta l(P)-\beta l(P')}}{\mathrm{Dist}^\gamma} \\
&\leq CN\log N + C2^{\alpha L} \sum_{l(P)=-1}^{L-1} \sum_{P\in\nabla_{l(P)}} 2^{-\beta l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta]l(P')} 2^{-2l(P')} \sum_{P'\in\nabla_{l(P')}} \mathrm{Dist}^{-\gamma} \\
&\leq CN\log N + C2^{\alpha L} \sum_{l(P)=-1}^{L-1} 2^{[2-\beta]l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta]l(P')} \int_{P\in\mathbb{R}^2:\,|P|>2^{-l(P')}} \frac{\mathrm{d}P}{|P|^\gamma} \\
&\leq CN\log N + C2^{\alpha L} \sum_{l(P)=-1}^{L-1} 2^{[2-\beta]l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta]l(P')} 2^{-l(P')[2-\gamma]} \\
&\leq CN\log N + C2^{\alpha L} 2^{[2-\beta]L} 2^{[\gamma-\beta]L} \leq CN\log N.
\end{aligned}
$$

In the second case

$$
\begin{aligned}
\mathcal{N}_2 &\leq \sum_{P,P'\in\triangle_L:\, a_{P,P'}\neq 0} N_{P,P'} \leq \sum_{P,P'\in\triangle_L:\, a_{P,P'}\neq 0} 1 + 2 \sum_{\substack{P,P'\in\triangle_L:\\ a_{P,P'}\neq 0,\, l(P)\leq l(P')}} \frac{2^{\alpha' L-\beta_m l(P)-\beta_M l(P')}}{\Upsilon \mathrm{dist}^{\gamma'}} \\
&\leq CN\log N + C\frac{2^{\alpha' L}}{\Upsilon} \sum_{l(P)=-1}^{L-1} \sum_{P\in\nabla_{l(P)}} 2^{-\beta_m l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta_M]l(P')} 2^{-2l(P')} \sum_{\substack{P'\in\nabla_{l(P')}\\ \mathrm{dist}\geq c_\Gamma^{-1}2^{-l(P')}}} \mathrm{dist}^{-\gamma'}
\end{aligned}
$$

Clearly, the number of all ψ_P with $P\in\nabla_{l(P)}$ such that there is a grid point P' with $P'\in\nabla_{l(P')}$, $l(P')\geq l(P)$, with $\mathrm{dist}\leq \Upsilon 2^{-l(P)}$, and with $\operatorname{supp}\psi_{P'}$ not contained in the same Γ_m, is less than $\mathcal{O}(\Upsilon 2^{l(P)})$. Using this fact, we continue

$$
\begin{aligned}
\mathcal{N}_2 &\leq CN\log N + C\frac{2^{\alpha' L}}{\Upsilon} \sum_{l(P)=-1}^{L-1} \Upsilon 2^{[1-\beta_m]l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta_M]l(P')} \int_0^{2^{-l(P)}} \int_{2^{-l(P')}}^{\Upsilon 2^{-l(P)}} y^{-\gamma'}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq CN\log N + C2^{\alpha' L} \sum_{l(P)=-1}^{L-1} 2^{[1-\beta_m]l(P)} \sum_{l(P')=l(P)}^{L-1} 2^{[2-\beta_M]l(P')} 2^{-l(P')[1-\gamma']} 2^{-l(P)} \\
&\leq CN\log N + C2^{\alpha' L} 2^{-\beta_m L} 2^{[1+\gamma'-\beta_M]L} \leq CN\log N.
\end{aligned}
$$

∎

7 The Estimate of the Quadrature Algorithm

Theorem 7.1 *Suppose that the assumptions of Theorem 3.1 are fulfilled and that the quadrature order d_Q and the number of vanishing moments $\tilde{d}_T$ satisfy $2d_T - \mathbf{r} < d_Q \leq \tilde{d}_T$.*

Moreover, suppose that $d_Q \le \frac{1}{2}\min\{d_k, d_\Gamma - 1\}$, that $D > D_0$, $L \ge L_0$ (cf. Theorem 3.1) and that the compressed matrix $A_L^Q \approx A_L^C$ together with the right-hand side $\eta^Q = (\eta_P^Q)_{P\in\triangle_L}$ is computed by the quadrature algorithm of the previous sections. Finally, suppose that the constant Υ is chosen as $\Upsilon := L^{0.5[d_Q+4]/[2d_Q+\mathbf{r}-\gamma d_Q/2]}$. Then the discretized operator $A_L^Q : H^0 \supseteq V_L \longrightarrow \tilde{V}_L \subseteq H^{-\mathbf{r}}$ is invertible and the inverse is bounded uniformly with respect to L and D. In particular, the solution $\xi^Q = (\xi_P^Q)_{P\in\triangle_L}$ of the compressed and quadrature approximated equation $A_L^Q\xi^Q = \eta^Q$ exists, and the approximate solution $u_L^Q := \sum_{P\in\triangle_L}\xi_P^Q\psi_P$ converges to the exact solution according to

$$\|u - u_L^Q\|_{H^{\mathbf{r}-d_T}(\Gamma)} \quad \le \quad C[2^{-L}]^{2d_T-\mathbf{r}} \; \|u\|_{H^{d_T}(\Gamma)}. \tag{7.1}$$

Proof. i) First we consider the error due to the quadrature applied to the computation of the terms $\langle K\psi_{P'}, \psi_P\rangle$ in the entries of the stiffness matrix. Without loss of generality we suppose $\mathbf{r} < 0$. Indeed, the case of a Fredholm integral equation of the second kind with order $\mathbf{r} = 0$ can be treated analogously to the case $\mathbf{r} < 0$. We only have to replace $\mathbf{r}$ in the kernel singularity estimates by -1 and to keep $\mathbf{r} = 0$ in the places where really the order of the operator is needed.

ii) In this point of the proof, we analyze the error caused by the quadrature for the term $\langle K\psi_{P'}, \psi_P\rangle$ in the entries with dist larger than $\Upsilon\max\{2^{-l(P)}, 2^{-l(P')}\}$. We denote the absolute values of the corresponding terms in the entries of $A_L^C - A_L^Q$ by $b_{P,P'}$. From the definition of the $N_{P,P'}$ and Lemma 5.3 we infer

$$\begin{aligned} b_{P,P'} \quad &\le \quad C2^{-[d_Q+2]l(P)}2^{-[d_Q+2]l(P')}\left[\frac{2^{\alpha L-\beta l(P)-\beta l(P')}}{\mathrm{dist}^\gamma}\right]^{-\frac{d_Q}{2}} L^{\frac{d_Q+2}{2}}\mathrm{dist}^{-[2d_Q+\mathbf{r}+2]} \\ &\le \quad CL^{\frac{d_Q+2}{2}}2^{-L\alpha\frac{d_Q}{2}}2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}\mathrm{dist}^{-[2d_Q+\mathbf{r}+2]+\gamma\frac{d_Q}{2}}. \end{aligned}$$

To derive the missing consistency estimates (cf. (3.14)-(3.15)), we use Lemma 3.2. Setting $x = -1$, we obtain

$$\begin{aligned} \Sigma_1^{t',t} \quad &\le \quad C\sum_{l(P)=-1}^{L-1}\sup_{P\in\nabla_{l(P)}}\Bigg\{2^{[t'+2]l(P)}\sum_{P'\in\triangle_L}2^{-tl(P')}L^{\frac{d_Q+2}{2}}2^{-L\alpha\frac{d_Q}{2}}\times \\ &\qquad\qquad 2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}\mathrm{dist}^{-[2d_Q+\mathbf{r}+2]+\gamma\frac{d_Q}{2}}\Bigg\} \\ &\le \quad CL^{\frac{d_Q+2}{2}}2^{-L\alpha\frac{d_Q}{2}}\sum_{l(P)}\sup_P\Bigg\{2^{l(P)\{t'+2+\beta\frac{d_Q}{2}-[d_Q+2]\}}\sum_{l(P')=-1}^{L-1}2^{l(P')\{-t+\beta\frac{d_Q}{2}-[d_Q+2]\}}\times \\ &\qquad\qquad \sum_{\substack{P'\in\nabla_{l(P')}:\\ \Upsilon\max\{2^{-l(P)},2^{-l(P')}\}\le\mathrm{dist}}}\mathrm{dist}^{-[2d_Q+\mathbf{r}+2]+\gamma\frac{d_Q}{2}}\Bigg\} \end{aligned}$$

Clearly, the last sum over $P' \in \nabla_{l(P')}$ multiplied by $2^{-2l(P')}$ is less than the bounded integral $\int_{R\in\mathbb{R}^2:|R|>\max}|R|^{-\xi}\,\mathrm{d}R$ with $\xi := [2d_Q+\mathbf{r}+2] - \gamma\frac{d_Q}{2} > 2$ and $\max := \Upsilon\max\{2^{-l(P)}, 2^{-l(P')}\}$.

Thus we continue

$$
\begin{aligned}
\Sigma_1^{t',t} &\leq CL^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} \sum_{l(P)} \left\{ 2^{l(P)\{t'+\beta\frac{d_Q}{2}-d_Q\}} \sum_{l(P')=-1}^{L-1} 2^{l(P')\{-t+\beta\frac{d_Q}{2}-d_Q\}} \max{}^{2-\xi} \right\} \\
&\leq C\, \frac{L^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}}}{\Upsilon^{[2d_Q+\mathbf{r}]-\gamma\frac{d_Q}{2}}} \sum_{l(P)} \Bigg\{ 2^{l(P)\{t'+\beta\frac{d_Q}{2}-d_Q\}} \sum_{l(P')=-1}^{l(P)-1} 2^{l(P')\{-t+[\beta-\gamma]\frac{d_Q}{2}+d_Q+\mathbf{r}\}} + \\
&\qquad\qquad 2^{l(P)\{t'+[\beta-\gamma]\frac{d_Q}{2}+d_Q+\mathbf{r}\}} \sum_{l(P')=l(P)}^{L-1} 2^{l(P')\{-t+\beta\frac{d_Q}{2}-d_Q\}} \Bigg\} \\
&\leq C\Upsilon^{-[2d_Q+\mathbf{r}]+\gamma\frac{d_Q}{2}} L^{\frac{d_Q+2}{2}} 2^{L\left\{-\alpha\frac{d_Q}{2}+[t'-t+\mathbf{r}+\alpha\frac{d_Q}{2}]_+\right\}} \\
&\leq C\Upsilon^{-[2d_Q+\mathbf{r}]+\gamma\frac{d_Q}{2}} \begin{cases} L^{\frac{d_Q+2}{2}} & \text{if } t'=-\mathbf{r},\ t=0 \\ L^{\frac{d_Q+2}{2}} 2^{-L[2d_T-\mathbf{r}]} & \text{if } t'=-d_T,\ t=d_T. \end{cases}
\end{aligned} \tag{7.2}
$$

Here, by the bracket $[\,\cdot\,]_+$ with lower index $+$, we have denoted the positive part of the expression inside of the bracket. Analogously, for the second sum in the Schur estimate, we arrive at

$$
\begin{aligned}
\Sigma_2^{t',t} &\leq C \sum_{l(P')=-1}^{L-1} \sup_{P'\in\nabla_{l(P')}} \Bigg\{ \sum_{P\in\triangle_L} 2^{t'l(P)} 2^{[-t+2]l(P')} L^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} \times \\
&\qquad 2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}} 2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}} \mathrm{dist}^{-[2d_Q+\mathbf{r}+2]+\gamma\frac{d_Q}{2}} \Bigg\} \\
&\leq CL^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} \sum_{l(P')=-1}^{L-1} 2^{l(P')\{-t+2+\beta\frac{d_Q}{2}-[d_Q+2]\}} \times \\
&\qquad \sup_{P'\in\nabla_{l(P')}} \Bigg\{ \sum_{l(P)=-1}^{L-1} 2^{l(P)\{t'+\beta\frac{d_Q}{2}-[d_Q+2]\}} \sum_{\substack{P\in\nabla_{l(P)}: \\ \Upsilon\max\{2^{-l(P)},2^{-l(P')}\}\leq \mathrm{dist}}} \mathrm{dist}^{-[2d_Q+\mathbf{r}+2]+\gamma\frac{d_Q}{2}} \Bigg\} \\
&\leq CL^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} \sum_{l(P')=-1}^{L-1} 2^{l(P')\{-t+\beta\frac{d_Q}{2}-d_Q\}} \sum_{l(P)=-1}^{L-1} 2^{l(P)\{t'+\beta\frac{d_Q}{2}-d_Q\}} \max{}^{2-\xi} \\
&\leq C\Upsilon^{-[2d_Q+\mathbf{r}]+\gamma\frac{d_Q}{2}} \begin{cases} L^{\frac{d_Q+2}{2}} & \text{if } t'=-\mathbf{r},\ t=0 \\ L^{\frac{d_Q+2}{2}} 2^{-L[2d_T-\mathbf{r}]} & \text{if } t'=-d_T,\ t=d_T. \end{cases}
\end{aligned}
$$

This estimate together with (7.2), with our choice of Υ, and with Lemma 3.2 implies the assertions of the theorem for the quadrature errors in the case that dist is larger than $\Upsilon\max\{2^{-l(P)}, 2^{-l(P')}\}$.

iii) Next we estimate the error caused by the quadrature for $\langle K\psi_{P'}, \psi_P\rangle$ in the entries with dist less than $\Upsilon\max\{2^{-l(P)}, 2^{-l(P')}\}$ but with supports $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ contained in a single parametrization patch Γ_m. We denote the absolute values of the corresponding terms in the entries of $A_L^C - A_L^Q$ by $b_{P,P'}$. From the definition of the $N_{P,P'}$, Lemma 5.2, and the analogous result for the estimation of f_2, we infer

$$
\begin{aligned}
b_{P,P'} &\leq C\Upsilon^{d_Q}\left[\frac{2^{\alpha L-\beta l(P)-\beta l(P')}}{\mathrm{Dist}^{\gamma}}\right]^{-\frac{d_Q}{2}} L^{\frac{d_Q+2}{2}} 2^{-[d_Q+2]l(P)} 2^{l(P')\mathbf{r}} \\
&\leq C\Upsilon^{d_Q} L^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} 2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}} 2^{l(P')\{\mathbf{r}+\beta\frac{d_Q}{2}\}} \mathrm{Dist}^{\gamma\frac{d_Q}{2}}, \quad l(P') < l(P), \\
b_{P,P'} &\leq C\Upsilon^{d_Q} L^{\frac{d_Q+2}{2}} 2^{-L\alpha\frac{d_Q}{2}} 2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}} 2^{l(P)\{\mathbf{r}+\beta\frac{d_Q}{2}\}} \mathrm{Dist}^{\gamma\frac{d_Q}{2}}, \quad l(P) \leq l(P').
\end{aligned}
$$

To derive the missing consistency estimates (cf. (3.14)-(3.15)), we use Lemma 3.2. Setting $x=-2$ for $l(P)\leq l(P')$ and $x=0$ for $l(P)>l(P')$, we obtain

$$
\begin{aligned}
\Sigma_1^{t',t} &\leq C\sum_{l(P)=-1}^{L-1}\sup_{P\in\nabla_{l(P)}}\left\{2^{[t'+3]l(P)}\sum_{\substack{P'\in\triangle_L:\\ l(P')\geq l(P)}} 2^{[-t-1]l(P')}\frac{2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P)\{\mathbf{r}+\beta\frac{d_Q}{2}\}}}{\Upsilon^{-d_Q}L^{-\frac{d_Q+2}{2}}2^{L\alpha\frac{d_Q}{2}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\}+ \\
&\quad C\sum_{l(P)=-1}^{L-1}\sup_{P\in\nabla_{l(P)}}\left\{2^{[t'+1]l(P)}\sum_{\substack{P'\in\triangle_L:\\ l(P')\leq l(P)}} 2^{[-t+1]l(P')}\frac{2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P')\{\mathbf{r}+\beta\frac{d_Q}{2}\}}}{\Upsilon^{-d_Q}L^{-\frac{d_Q+2}{2}}2^{L\alpha\frac{d_Q}{2}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\} \\
&\leq C\,\frac{\Upsilon^{d_Q}L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\sum_{l(P)}\sup_{P}\left\{2^{l(P)\{t'+3+\mathbf{r}+\beta\frac{d_Q}{2}\}}\sum_{l(P')=l(P)}^{L-1}2^{l(P')\{-[d_Q+3]-t+\beta\frac{d_Q}{2}\}}\sum_{\substack{P'\in\nabla_{l(P')}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P)}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\}+ \\
&\quad C\,\frac{\Upsilon^{d_Q}L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\sum_{l(P)}\sup_{P}\left\{2^{l(P)\{t'-[d_Q+1]+\beta\frac{d_Q}{2}\}}\sum_{l(P')=-1}^{l(P)-1}2^{l(P')\{-t+1+\mathbf{r}+\beta\frac{d_Q}{2}\}}\sum_{\substack{P'\in\nabla_{l(P')}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P')}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\}
\end{aligned}
$$

Here, using an integral as bound resp. a simple majorant for a sum of one item, we get the following estimates for the sums over P'.

$$
\begin{aligned}
2^{-2l(P')}\sum_{\substack{P'\in\nabla_{l(P')}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P)},\ l(P)\leq l(P')}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}} &\leq C\left[\Upsilon 2^{-l(P)}\right]^{\gamma\frac{d_Q}{2}+2}, \\
\sum_{\substack{P'\in\nabla_{l(P')}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P')},\ l(P')<l(P)}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}} &\leq C\left[2^{-l(P)}\right]^{\gamma\frac{d_Q}{2}}.
\end{aligned}
$$

Inserting these formulas, we arrive at

$$
\Sigma_1^{t',t} \leq C\Upsilon^{d_Q+\gamma\frac{d_Q}{2}+2}\,\frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\sum_{l(P)}\left\{2^{l(P)\{1+t'+\mathbf{r}+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}\sum_{l(P')=l(P)}^{L-1}2^{l(P')\{-[d_Q+1]-t+\beta\frac{d_Q}{2}\}}\right\}
$$

$$
\begin{aligned}
&+C\Upsilon^{d_Q}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P)}\left\{2^{l(P)\{t'-[d_Q+1]+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}\sum_{l(P')=-1}^{l(P)-1}2^{l(P')\{-t+1+\mathbf{r}+\beta\frac{d_Q}{2}\}}\right\}\\
\leq\ & C\Upsilon^{d_Q+\gamma\frac{d_Q}{2}+2}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P)}\left\{2^{l(P)\{1+t'+\mathbf{r}+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}2^{l(P)\{-[d_Q+1]-t+\beta\frac{d_Q}{2}\}}\right\}\\
&+C\Upsilon^{d_Q}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P)}\left\{2^{l(P)\{t'-[d_Q+1]+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}2^{l(P)[-t+1+\mathbf{r}+\beta\frac{d_Q}{2}]_+}\right\}\\
\leq\ & C\left\{\begin{array}{ll}\mathcal{O}(L^{-1}) & \text{if } t'=-\mathbf{r},\ t=0\\ 2^{-L[2d_T-\mathbf{r}]} & \text{if } t'=-d_T,\ t=d_T.\end{array}\right.
\end{aligned}
\tag{7.3}
$$

For the second sum in Schur's lemma, we analogously get

$$
\begin{aligned}
\Sigma_2^{t',t}\ \leq\ & C\sum_{l(P')=-1}^{L-1}\ \sup_{P'\in\nabla_{l(P')}}\left\{\sum_{\substack{P\in\triangle_L:\\ l(P)>l(P')}}2^{[t'-1]l(P)}2^{[-t+3]l(P')}\frac{2^{l(P)\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P')\{\mathbf{r}+\beta\frac{d_Q}{2}\}}}{\Upsilon^{-d_Q}L^{-\frac{d_Q+2}{2}}2^{L\alpha\frac{d_Q}{2}}\mathrm{Dist}^{-\gamma\frac{d_Q}{2}}}\right\}+\\
& C\sum_{l(P')=-1}^{L-1}\ \sup_{P'\in\nabla_{l(P')}}\left\{\sum_{\substack{P\in\triangle_L:\\ l(P)\leq l(P')}}2^{[t'+1]l(P)}2^{[-t+1]l(P')}\frac{2^{l(P')\{-[d_Q+2]+\beta\frac{d_Q}{2}\}}2^{l(P)\{\mathbf{r}+\beta\frac{d_Q}{2}\}}}{\Upsilon^{-d_Q}L^{-\frac{d_Q+2}{2}}2^{L\alpha\frac{d_Q}{2}}\mathrm{Dist}^{-\gamma\frac{d_Q}{2}}}\right\}\\
\leq\ & C\Upsilon^{d_Q}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P')=-1}^{L-1}2^{l(P')\{-t+3+\mathbf{r}+\beta\frac{d_Q}{2}\}}\ \sup_{P'\in\nabla_{l(P')}}\left\{\sum_{l(P)=l(P')+1}^{L-1}2^{l(P)\{t'-[d_Q+3]+\beta\frac{d_Q}{2}\}}\right.\\
&\times\left.\sum_{\substack{P\in\nabla_{l(P)}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P')}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\}+C\Upsilon^{d_Q}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{-L\alpha\frac{d_Q}{2}}}\ \sum_{l(P')=-1}^{L-1}2^{l(P')\{-[d_Q+1]-t+\beta\frac{d_Q}{2}\}}\times\\
&\sup_{P'\in\nabla_{l(P')}}\left\{\sum_{l(P)=-1}^{l(P')}2^{l(P)\{1+t'+\mathbf{r}+\beta\frac{d_Q}{2}\}}\sum_{\substack{P\in\nabla_{l(P)}:\\ \mathrm{Dist}\leq\Upsilon 2^{-l(P)}}}\mathrm{Dist}^{\gamma\frac{d_Q}{2}}\right\}.
\end{aligned}
$$

Estimating the sum over P' by an integral resp. by a simple upper bound for one item, we get

$$
\begin{aligned}
\Sigma_2^{t',t}\ \leq\ & C\Upsilon^{d_Q+\gamma\frac{d_Q}{2}+2}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P')=-1}^{L-1}2^{l(P')\{1-t+\mathbf{r}+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}\sum_{l(P)=l(P')+1}^{L-1}2^{l(P)\{-[d_Q+1]+t'+\beta\frac{d_Q}{2}\}}\\
&+C\Upsilon^{d_Q}\ \frac{L^{\frac{d_Q+2}{2}}}{2^{L\alpha\frac{d_Q}{2}}}\ \sum_{l(P')=-1}^{L-1}2^{l(P')\{-[d_Q+1]-t+\beta\frac{d_Q}{2}-\gamma\frac{d_Q}{2}\}}\sum_{l(P)=-1}^{l(P')}2^{l(P)\{1+t'+\mathbf{r}+\beta\frac{d_Q}{2}\}}\\
\leq\ & C\left\{\begin{array}{ll}\mathcal{O}(L^{-1}) & \text{if } t'=-\mathbf{r},\ t=0\\ 2^{-L[2d_T-\mathbf{r}]} & \text{if } t'=-d_T,\ t=d_T.\end{array}\right.
\end{aligned}
$$

This estimate together with (7.3) and with Lemma 3.2 implies the assertions of the theorem for the quadrature errors in the case that the entries have a distance dist less than $\Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$ and supports $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ contained in a single parametrization patch Γ_m.

iv) Next we estimate the error caused by the quadrature for $\langle K\psi_{P'}, \psi_P\rangle$ in the entries with $0 < \mathrm{dist} < \Upsilon \max\{2^{-l(P)}, 2^{-l(P')}\}$ but with supports $\operatorname{supp}\psi_P$ and $\operatorname{supp}\psi_{P'}$ not contained both in the same parametrization patch Γ_m. We denote the absolute values of the corresponding terms in the entries of $A_L^C - A_L^Q$ by $b_{P,P'}$ and set $l_m := \min\{l(P), l(P')\}$ as well as $l_M := \max\{l(P), l(P')\}$. From the definition of the $N_{P,P'}$ and Lemma 5.4 we infer

$$\begin{aligned} b_{P,P'} &\le C 2^{-[d_Q+2]l_M} \left[\frac{2^{\alpha' L - \beta_m l_m - \beta_M l_M}}{\Upsilon \mathrm{dist}^{\gamma'}}\right]^{-\frac{d_Q}{2}} L^{d_Q+1} \mathrm{dist}^{-[d_Q+\mathbf{r}]} \\ &\le C \Upsilon^{d_Q/2} L^{d_Q+1} 2^{-L\alpha'\frac{d_Q}{2}} 2^{l_M\{-[d_Q+2]+\beta_M\frac{d_Q}{2}\}} 2^{l_m\beta_m\frac{d_Q}{2}} \mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}} \end{aligned}$$

To derive the missing consistency estimates (cf. (3.14)-(3.15)), we use Lemma 3.2. Setting $x = -1$, we obtain

$$\begin{aligned} \Sigma_1^{t',t} &\le C \sum_{l(P)=-1}^{L-1} \sup_{P\in\nabla_{l(P)}} \Bigg\{ 2^{[t'+2]l(P)} \sum_{P'\in\triangle_L} 2^{-tl(P')} \Upsilon^{d_Q/2} L^{d_Q+1} 2^{-L\alpha'\frac{d_Q}{2}} \times \\ &\qquad\qquad 2^{l_m\beta_m\frac{d_Q}{2}} 2^{l_M\{-[d_Q+2]+\beta_M\frac{d_Q}{2}\}} \mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}} \Bigg\} \\ &\le C \Upsilon^{d_Q/2} L^{d_Q+1} 2^{-L\alpha'\frac{d_Q}{2}} \sum_{l(P)} \sup_P \Bigg\{ 2^{l(P)\{t'+2+\beta_m\frac{d_Q}{2}\}} \sum_{l(P')=l(P)}^{L-1} 2^{l(P')\{-t+\beta_M\frac{d_Q}{2}-[d_Q+2]\}} \\ &\qquad \times \sum_{\substack{P'\in\nabla_{l(P')}:\\ 0<\mathrm{dist}\le\Upsilon 2^{-l(P)}}} \mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}} + 2^{l(P)\{t'+2+\beta_M\frac{d_Q}{2}-[d_Q+2]\}} \times \\ &\qquad \sum_{l(P')=-1}^{l(P)-1} 2^{l(P')\{-t+\beta_m\frac{d_Q}{2}\}} \sum_{\substack{P'\in\nabla_{l(P')}:\\ 0<\mathrm{dist}\le\Upsilon 2^{-l(P')}}} \mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}} \Bigg\} \end{aligned}$$

In the last right-hand side the first sum over P' multiplied by $2^{-2l(P')}$ is less than the integral $\int_0^{2^{-l(P)}} \int_{2^{-l(P')}}^{\Upsilon 2^{-l(P)}} y^{-[d_Q+\mathbf{r}]+\gamma' d_Q/2}\, \mathrm{d}y\, \mathrm{d}x$, since the relation $0 < \mathrm{dist}$ implies the lower estimate $c_\Gamma^{-1} \min\{2^{-l(P)}, 2^{-l(P')}\} < \mathrm{dist}$, where c_Γ is the Lipschitz constant for the inverse parametrization mappings κ_m^{-1}, $m = 1, \ldots, m_\Gamma$. The second sum is over one item, and the two upper bounds for these two sums over P' depend on whether or not we have $-[d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2} < -1$ and $-[d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2} < 0$, respectively. For the sake of definiteness, we assume $0 > 1 - [d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2}$. The cases $-[d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2} < 0 < 1 - [d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2}$ and $0 < -[d_Q+\mathbf{r}] + \gamma'\frac{d_Q}{2}$ can be treated similarly. We continue

$$\Sigma_1^{t',t} \le C \Upsilon^{d_Q/2} L^{d_Q+1} 2^{-L\alpha'\frac{d_Q}{2}} \sum_{l(P)} \Bigg\{ 2^{l(P)\{t'+2+\beta_m\frac{d_Q}{2}\}} \sum_{l(P')=l(P)}^{L-1} 2^{l(P')\{-t+\beta_M\frac{d_Q}{2}-[d_Q+2]\}}$$

$$\times 2^{2l(P')}2^{-l(P)}2^{-l(P')\{1-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}\}} + 2^{l(P)\{t'+2+\beta_M\frac{d_Q}{2}-[d_Q+2]\}} \times$$
$$\sum_{l(P')=-1}^{l(P)-1} 2^{l(P')\{-t+\beta_m\frac{d_Q}{2}\}}2^{-l(P)\{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}\}}\Bigg\}$$
$$\begin{aligned}
&\leq C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}\sum_{l(P)}\Bigg\{2^{l(P)\{t'+1+\beta_m\frac{d_Q}{2}\}}\sum_{l(P')=l(P)}^{L-1}2^{l(P')\{-t+[\mathbf{r}-1]+[\beta_M-\gamma']\frac{d_Q}{2}\}} \\
&\qquad +2^{l(P)\{t'+\mathbf{r}+[\beta_M-\gamma']\frac{d_Q}{2}\}}\sum_{l(P')=-1}^{l(P)-1}2^{l(P')\{-t+\beta_m\frac{d_Q}{2}\}}\Bigg\} \\
&\leq C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}2^{L\{[d_T-t]+[d_T+t']\}}\times \\
&\quad \sum_{l(P)}\Bigg\{2^{l(P)\{-d_T+1+\beta_m\frac{d_Q}{2}\}}\sum_{l(P')=l(P)}^{L-1}2^{l(P')\{-d_T+[\mathbf{r}-1]+[\beta_M-\gamma']\frac{d_Q}{2}\}} \\
&\qquad +2^{l(P)\{-d_T+\mathbf{r}+[\beta_M-\gamma']\frac{d_Q}{2}\}}\sum_{l(P')=-1}^{l(P)-1}2^{l(P')\{-d_T+\beta_m\frac{d_Q}{2}\}}\Bigg\}
\end{aligned}$$

We assume $d_Q > 2d_T - \mathbf{r}$ in Theorem 7.1. Furthermore, without loss of generality we assume $d_Q < 2d_T - 2\mathbf{r}$. Note, if $d_Q < 2d_T - 2\mathbf{r}$ does not hold, then we can replace the parameter d_Q by a d'_Q with the additional property $d'_Q < 2d_T - 2\mathbf{r}$, and Lemma 5.4 remains true with d_Q replaced by d'_Q. Thus we have $-d_T+\mathbf{r}+[\beta_M-\gamma']\frac{d_Q}{2} < 0$ and $-d_T+\beta_m\frac{d_Q}{2} < 0$. We conclude

$$\begin{aligned}
\Sigma_1^{t',t} &\leq C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}2^{L\{[d_T-t]+[d_T+t']\}}\times \\
&\quad \sum_{l(P)}\Bigg\{2^{l(P)\{-2d_T+\mathbf{r}+[\alpha'-1]\frac{d_Q}{2}\}}+2^{l(P)\{-d_T+\mathbf{r}+[\beta_M-\gamma']\frac{d_Q}{2}\}}\Bigg\} \\
&\leq \begin{cases} \mathcal{O}(L^{-1}) & \text{if } t'=-\mathbf{r},\ t=0 \\ C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}} & \text{if } t'=-d_T,\ t=d_T. \end{cases}
\end{aligned} \tag{7.4}$$

Analogously, for the second sum in the Schur estimate, we arrive at

$$\begin{aligned}
\Sigma_2^{t',t} &\leq C\sum_{l(P')=-1}^{L-1}\sup_{P'\in\nabla_{l(P')}}\Bigg\{\sum_{P\in\triangle_L}2^{t'l(P)}2^{[-t+2]l(P')}\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}\times \\
&\qquad 2^{l_m\beta_m\frac{d_Q}{2}}2^{l_M\{-[d_Q+2]+\beta_M\frac{d_Q}{2}\}}\mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}}\Bigg\} \\
&\leq C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}\sum_{l(P')=-1}^{L-1}\Bigg\{2^{l(P')\{-t+2+\beta_M\frac{d_Q}{2}-[d_Q+2]\}}\times \\
&\quad \sup_{P'\in\nabla_{l(P')}}\Bigg[\sum_{l(P)=-1}^{l(P')-1}2^{l(P)\{t'+\beta_m\frac{d_Q}{2}\}}\sum_{\substack{P\in\nabla_{l(P)}:\\ 0<\mathrm{dist}<\Upsilon 2^{-l(P)}}}\mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}}\Bigg]
\end{aligned}$$

$$+2^{l(P')\{-t+2+\beta_m\frac{d_Q}{2}\}}\times$$

$$\sup_{P'\in\nabla_{l(P')}}\Big[\sum_{l(P)=l(P')}^{L-1}2^{l(P)\{t'+\beta_M\frac{d_Q}{2}-[d_Q+2]\}}\sum_{\substack{P\in\nabla_{l(P)}:\\0<\mathrm{dist}<\Upsilon 2^{-l(P')}}}\mathrm{dist}^{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}}\Big]\Big\}$$

Estimating the sum over P by an integral resp. by a simple upper bound for one item, we get

$$\begin{aligned}
\Sigma_2^{t',t} \leq\ & C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}\sum_{l(P')=-1}^{L-1}\Big\{2^{l(P')\{-t+2+\beta_M\frac{d_Q}{2}-[d_Q+2]\}}\times\\
&\Big[\sum_{l(P)=-1}^{l(P')-1}2^{l(P)\{t'+\beta_m\frac{d_Q}{2}\}}2^{-l(P')\{-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}\}}\Big]\\
&+2^{l(P')\{-t+2+\beta_m\frac{d_Q}{2}\}}\times\\
&\Big[\sum_{l(P)=l(P')}^{L-1}2^{l(P)\{t'+\beta_M\frac{d_Q}{2}-[d_Q+2]\}}2^{2l(P)}2^{-l(P')}2^{-l(P)\{1-[d_Q+\mathbf{r}]+\gamma'\frac{d_Q}{2}\}}\Big]\Big\}\\
\leq\ & C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}}\times\\
&\sum_{l(P')=-1}^{L-1}\Big\{2^{l(P')\{-t+\mathbf{r}+[\beta_M-\gamma']\frac{d_Q}{2}\}}\sum_{l(P)=-1}^{l(P')-1}2^{l(P)\{t'+\beta_m\frac{d_Q}{2}\}}\\
&+2^{l(P')\{-t+1+\beta_m\frac{d_Q}{2}\}}\sum_{l(P)=l(P')}^{L-1}2^{l(P)\{t'+[\mathbf{r}-1]+[\beta_M-\gamma']\frac{d_Q}{2}\}}\Big\}\\
\leq\ & \begin{cases}\mathcal{O}(L^{-1}) & \text{if } t'=-\mathbf{r},\ t=0\\ C\Upsilon^{d_Q/2}L^{d_Q+1}2^{-L\alpha'\frac{d_Q}{2}} & \text{if } t'=-d_T,\ t=d_T.\end{cases}
\end{aligned}$$

This together with (7.4) and with Lemma 3.2 implies the assertions of the theorem for the quadrature errors in the case that dist is positive and less than $\Upsilon\max\{2^{-l(P)},2^{-l(P')}\}$.

v) Next we estimate the error caused by the quadrature for $\langle K\psi_{P'},\psi_P\rangle$ in the entries with $0=\mathrm{dist}$ but with supports $\mathrm{supp}\,\psi_P$ and $\mathrm{supp}\,\psi_{P'}$ not contained both in the same parametrization patch Γ_m. However, in view of the Lemmata 5.4 and 5.5 we have the same estimates as for the case that the wavelets are supported on different parametrization patches and $0<\mathrm{dist}\sim\min\{2^{-l(P)},2^{-l(P')}\}$. Therefore the technique from the part iv) of the present proof applies also to the case $0=\mathrm{dist}$.

vi) Finally, the estimate (3.16) is an immediate consequence of the estimate in Lemma 4.1, i.e. the error due to the quadrature of the right-hand side is bounded by $\mathcal{O}(2^{-L[2d_T-\mathbf{r}]})$. To estimate the error due to the discretized multiplication, we set $x=-1$ and apply the

Lemmata 3.2 and 4.2. We define $\xi := 0$ for $t > 0$ and $\xi := 1$ for $t = 0$, and get

$$\begin{aligned}
\Sigma_1^{t',t} &\leq C \sum_{l(P)=-1}^{L-1} \sup_{P\in\nabla_{l(P)}} \Bigg\{ 2^{[t'+2]l(P)} \sum_{\substack{P'\in\triangle_L \\ \operatorname{supp}\psi_P\cap\operatorname{supp}\psi_{P'}\neq\emptyset}} 2^{-tl(P')} \Big[2^{-L}\Big]^{d_Q} 2^{-2\max\{l(P),l(P')\}} \Bigg\} \\
&\leq C\Big[2^{-L}\Big]^{d_Q} \sum_{l(P)} \sup_{P} 2^{[t'+2]l(P)} \Bigg\{ \sum_{l(P')=-1}^{l(P)} 2^{-tl(P')} 2^{-2l(P)} \\
&\qquad + \sum_{l(P')=l(P)+1}^{L-1} 2^{-tl(P')} 2^{-2l(P')} \sum_{\substack{P'\in\triangle_L \\ \operatorname{supp}\psi_P\cap\operatorname{supp}\psi_{P'}\neq\emptyset}} 1 \Bigg\} \\
&\leq C2^{-d_Q L} \sum_{l(P)} 2^{\{t'+[-t]_+\}l(P)} L^{\xi} + C2^{-d_Q L} \sum_{l(P)} 2^{t'l(P)} \sum_{l(P')=l(P)+1}^{L-1} 2^{-tl(P')} \\
&\leq C \left\{ \begin{array}{ll} \mathcal{O}(L^{-1}) & \text{if } t' = -\mathbf{r},\ t = 0 \\ 2^{-Ld_Q} & \text{if } t' = -d_T,\ t = d_T. \end{array} \right.
\end{aligned}$$

On the other hand, setting $\xi := 0$ for $t' < 0$ and $\xi := 1$ for $t' = 0$, we get

$$\begin{aligned}
\Sigma_2^{t',t} &\leq C \sum_{l(P')=-1}^{L-1} \sup_{P'\in\nabla_{l(P')}} \Bigg\{ \sum_{\substack{P\in\triangle_L \\ \operatorname{supp}\psi_P\cap\operatorname{supp}\psi_{P'}\neq\emptyset}} \frac{2^{t'l(P)} 2^{[-t+2]l(P')} 2^{-2\max\{l(P),l(P')\}}}{\Big[2^{L}\Big]^{d_Q}} \Bigg\} \\
&\leq C\Big[2^{-L}\Big]^{d_Q} \sum_{l(P')=-1}^{L-1} \sup_{P'\in\nabla_{l(P')}} 2^{[-t+2]l(P')} \Bigg\{ \sum_{l(P)=-1}^{l(P')} 2^{t'l(P)} 2^{-2l(P')} \\
&\qquad + \sum_{l(P)=l(P')+1}^{L-1} 2^{t'l(P)} 2^{-2l(P)} \sum_{\substack{P\in\triangle_L \\ \operatorname{supp}\psi_P\cap\operatorname{supp}\psi_{P'}\neq\emptyset}} 1 \Bigg\} \\
&\leq C2^{-d_Q L} \sum_{l(P')=-1}^{L-1} 2^{\{-t+[t']_+\}l(P')} L^{\xi} + C2^{-d_Q L} 2^{-tl(P')} \sum_{l(P)=l(P')+1}^{L-1} 2^{t'l(P)} \\
&\leq C \left\{ \begin{array}{ll} \mathcal{O}(L^{-1}) & \text{if } t' = -\mathbf{r},\ t = 0 \\ 2^{-Ld_Q} & \text{if } t' = -d_T,\ t = d_T. \end{array} \right.
\end{aligned}$$

Thus the theorem is proved. ∎

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