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On a model for phase separation in binary alloys driven by mechanical effects

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On a model for phase separation in binary alloys driven by mechanical effects

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Abstract. This work is concerned with the mathematical analysis of a system of partial differential equations modeling the effect of phase separation driven by mechanical actions in binary alloys like tin/lead solders. The system combines the (quasistationary) balance of linear momentum with a fourth order evolution equation of Cahn–Hilliard type for the phase separation, and it is highly nonlinearly coupled. Existence and uniqueness results are shown.

1. Introduction

In many cases binary alloys consist of two coexisting phases. If these alloys are exposed to thermo-mechanical loads, the interface boundaries are set into motion and drastic changes of the morphology in the μm (micron) range will arise. Phase field models describe the morphology by means of an order parameter that indicates the present phase at time t and at any point x of the alloy. In the binary tin/lead alloy, which was studied intensively by Dreyer and Müller (see [6–7]), the tin concentration by itself can be used as a phase field.

The phase field system that was used in the recent paper [6] to study and describe qualitatively phase seperation and coarsening processes under external thermomechanical load observed in the binary tin/lead alloy is the following.

The variables are the fields of

$$\begin{array}{ll} \mathbf{u}(x,t) & \text{(mechanical) displacement} \\ \chi(x,t) & \text{(tin) concentration.} \end{array}$$

The field equations rely on the static momentum balance and on the conservation law of the tin content. They read

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{and} \qquad \frac{\partial \chi}{\partial t} + \frac{\partial J_k}{\partial x_k} = 0$$

where the repeated index convention is in force. Let us describe the ingredients of such equations. The stress tensor is given by Hooke's law including eigenstrains that result here from different thermal expansions of the phases

$$\sigma_{ij} = C_{ijhk}(\chi)\big(\varepsilon_{hk} - \varepsilon^*_{hk}(\chi)\big) \qquad \text{with} \qquad \varepsilon_{hk} = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right).$$

For a realistic description of the tin/lead system the stiffness matrix and the eigenstrains should depend on the concentration because both phases behave differently in their elastic properties as well as in their thermal expansion coefficients. We take care for this by the representations

$$C_{ijhk}(\chi) = \Theta(\chi)\, C^{\alpha}_{ijhk} + (1 - \Theta(\chi))\, C^{\beta}_{ijhk}$$

$$\text{with the shape function} \quad \Theta(\chi) = \frac{c^{\beta} - \chi}{c^{\beta} - c^{\alpha}}\,.$$

In the above equation C^{α}_{ijhk} and C^{β}_{ijhk} denote the stiffness matrices of the cubic $\alpha-$ phase and of the tetragonal $\beta-$ phase, respectively. The concentrations c^{α} and c^{β} appearing in the shape function are the temperature dependent equilibrium concentrations of the tin/lead phase diagram.

The eigenstrains are assumed to be given by

$$\begin{aligned} &\varepsilon^*_{hk}(\chi) = \alpha_{hk}(\chi)\,(T - T_R) \\ &\text{with} \qquad \alpha_{hk}(\chi) = \Theta(\chi)\,\alpha^{\alpha}_{hk} + (1 - \Theta(\chi))\,\alpha^{\beta}_{hk}. \end{aligned}$$

The matrices of thermal expansion coefficients of the phases are denoted by α^{α}_{hk} and α^{β}_{hk}, and T and T_R are the actual temperature and the reference temperature, respectively. We assume T and T_R to be two fixed constants since our analysis is confined to the isothermal case. For details regarding data and explicit forms of these matrices, we refer the reader to [6].

Next, we consider the diffusion flux which is given by the extended Cahn–Hilliard form

$$J_i = -M_{ij}(\chi)\,\frac{\partial \widehat{w}}{\partial x_j}$$

where the potential $\widehat{w}$ is defined according to

$$\begin{aligned} \widehat{w} = {} & \frac{\partial \psi(\chi)}{\partial \chi} - a_{ij}(\chi)\,\frac{\partial^2 \chi}{\partial x_i \partial x_j} \\ & + \frac{1}{2}\frac{\partial}{\partial \chi}\Big((\varepsilon_{ij} - \varepsilon^*_{ij}(\chi))\,C_{ijhk}(\chi)\,(\varepsilon_{hk} - \varepsilon^*_{hk}(\chi))\Big). \end{aligned}$$

The function $\psi(\chi)$ is the non-convex combined free energy of the phases, the matrix $a_{ij}(\chi)$ contains the gradient coefficients that can be related to interface surface tensions and the mobility appears also as a matrix here, i.e. $M_{ij}(\chi)$, in order to reflect the anisotropy of the diffusion process. The matrices $M_{ij}(\chi)$ and $a_{ij}(\chi)$ are constructed in the same way as the stiffness matrix and the eigenstrains, namely

$$\begin{aligned} M_{ij}(\chi) &= \Theta(\chi)\,M^{\alpha}_{ij} + (1 - \Theta(\chi))\,M^{\beta}_{ij} \\ a_{ij}(\chi) &= \Theta(\chi)\,a^{\alpha}_{ij} + (1 - \Theta(\chi))\,a^{\beta}_{ij}. \end{aligned}$$

For given initial and boundary data, the system was used by Dreyer and Müller (see [6–9]) for a numerical simulation of various phase separation processes in tin/lead alloys.

However, for a rigorous mathematical treatment, the system contains too many complexities. In particular, the quadratic dependence of the potential $\widehat{w}$ with respect to the strain tensor renders the analysis difficult. Indeed, L^p-estimates for ∇u are known to hold just for p close to 2 while global L^4-estimates would be required. Therefore, in order that the problem turns out to be accessible, we have to make some simplifications concerning the dependence of the matrices M_{ij}, C_{ijhk}, and a_{ij} on the concentration field χ. Firstly, we restrict ourselves to the setting described below.

- $M_{ij}(\chi) = \delta_{ij}$, i.e. M_{ij} is the identity matrix;
- $a_{hk}(\chi) = \left(\Theta(\chi)a^\alpha + (1-\Theta(\chi))a^\beta\right)\delta_{ij} =: a(\chi)\delta_{ij}$, i.e. a_{ij} reduces to an isotropic matrix;
- the concentration field χ is forced to attain only values within the closed interval $[\underline{\chi}, \overline{\chi}]$ by including the indicator function I of the interval $[\underline{\chi}, \overline{\chi}]$ in the potential ψ;
- the potential $\widehat{w}$ is replaced by a a new variable w which contains in addition the term $\mu\partial_t\chi$, where μ is a fixed positive constant.

Note that in our framework the new constitutive relation w-χ has to be properly read as a differential inclusion. Hence, we are led to the following system

$$\partial_{x_j}\sigma_{ij} = 0 \tag{1.1}$$

$$\sigma_{ij} = C_{ijhk}(\chi)\big(\varepsilon_{hk} - \varepsilon^*_{hk}(\chi)\big) \tag{1.2}$$

$$\varepsilon_{hk} = \frac{1}{2}\left(\partial_{x_h}u_k + \partial_{x_k}u_h\right) \tag{1.3}$$

$$\partial_t\chi - \Delta w = 0 \tag{1.4}$$

$$\begin{aligned} w \in \mu\,\partial_t\chi - a(\chi)\Delta\chi + \frac{\partial\psi}{\partial\chi} + \partial I(\chi) - \sigma_{hk}\frac{\partial\varepsilon^*_{hk}}{\partial\chi} \\ + \frac{1}{2}\big(\varepsilon_{ij} - \varepsilon^*_{ij}\big)\frac{\partial C_{ijhk}}{\partial\chi}\big(\varepsilon_{hk} - \varepsilon^*_{hk}\big). \end{aligned} \tag{1.5}$$

The system has to be complemented by appropriate initial and boundary conditions that will be given explicitly in the next section. It is the aim of this paper to study the well-posedness of the initial-boundary value problem. We are going to show that existence and uniqueness results can be obtained for the case $N = 1$, and for the case $N = 2$ provided that the matrix C_{ijhk} is independent of χ.

These results have to be compared with Garcke's recent thesis [11] on the same subject: Garcke makes the simplifying assumption that the gradient matrix a_{ij} does not depend on χ; moreover, he does not consider a differential inclusion in order to guarantee the constraint $\chi \in [\underline{\chi}, \overline{\chi}]$. However, Garcke's existence results apply to the general $N-$ dimensional case, and the stiffness matrix C_{ijhk} may depend on χ. For uniqueness, he also has to assume that C_{ijhk} does not depend on χ (see the last remark at the end of this paper). We also note at this point that, owing to the presence of the smoothing term $\mu\partial_t\chi$ in (1.5), our solution has more regularity than Garcke's. On the other hand, the introduction of such a term yields a model that is in agreement with observed results in short time intervals. We also stress the fact that there is strong experimental evidence for a $\chi-$dependence of the gradient matrix, which influences the evolution of the microstructure drastically.

The Figure 1.1 shows the specific free energy of the model and the corresponding common tangent construction. This construction results in the sharp interface limit if the gradient coefficient is independent of the concentration. The consideration of an observed concentration dependence leads in the same limit to a modified common tangent construction that includes effective specific free energies with reduced barriers between the two minima. If additionally the gradient coefficients are formed by an

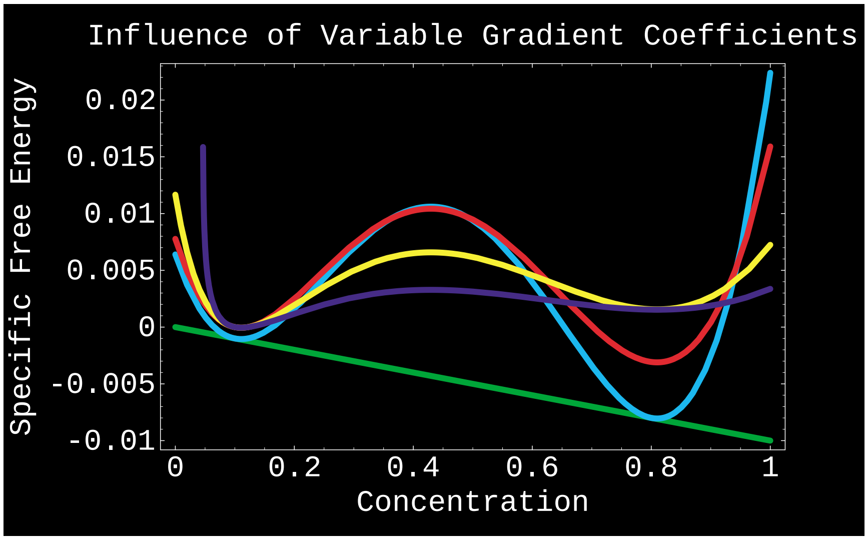

Figure 1.1

anisotropic matrix, the effective specific free energies become dependent on the interface normal. This fact is reflected by the three other graphs of Figure 1.1 which give for a diagonal matrix in 2D with different coefficients, the effective free energies in the directions (1,0), (1,1), and (0,1), respectively (see [10] for details).

2. Statement of the problem

In this section, we first rewrite system (1.1–5) in a form that is more suitable for the mathematical treatment. Then, we list the precise assumptions we need and state our results. We introduce

$$y_{ij}(\chi) := -C_{ijhk}(\chi)\,\varepsilon^*_{hk}(\chi) \qquad \text{for } \chi \in [\underline{\chi}, \overline{\chi}] \tag{2.1}$$

and present (1.1–2) as

$$-\partial_{x_j}\big(C_{ijhk}\,\varepsilon_{hk}\big) = \partial_{x_j} y_{ij}.$$

Moreover, we assume that the tensor $C = (C_{ijhk})$ is a Lipschitz function of χ, and define the tensor $C' = (C'_{ijhk})$ and the function ρ as follows

$$C'_{ijhk}(\chi) = \frac{\partial C_{ijhk}}{\partial \chi} \tag{2.2}$$

$$\rho(\chi) = \frac{\partial \psi(\chi)}{\partial \chi} + \frac{1}{2}\varepsilon^*_{ij}(\chi)\,C'_{ijhk}(\chi)\,\varepsilon^*_{hk}(\chi) \qquad \text{for } \chi \in [\underline{\chi}, \overline{\chi}]. \tag{2.3}$$

Hence, C' is bounded and equation (1.5) reads

$$w \in \mu\,\partial_t\chi - a(\chi)\Delta\chi + \partial I(\chi) + \rho(\chi) + z_{ij}(\chi)\,\varepsilon_{ij} + \frac{1}{2}\varepsilon_{ij}\,C'_{ijhk}(\chi)\,\varepsilon_{hk}$$

for suitable functions z_{ij} defined on $[\underline{\chi}, \overline{\chi}]$. More generally, we replace the sum of the subdifferential ∂I and of some monotone part ρ_M of $\rho = \rho_M + \rho_A$ by a maximal monotone graph β.

So, accounting also for the boundary and initial conditions, we can state the full problem, at least formally, as described below. To this aim, we explain our notation.

In the sequel, Ω denotes a bounded connected open set in $\mathbb{R}^N$ and $|\Omega|$ stands for its Lebesgue measure. The boundary Γ of Ω is smooth and consists of two smooth and nonempty parts Γ_u and Γ_σ. We term $\mathbf{n}$ the outward unit normal on Γ and, given a final time T, for the sake of convenience we set

$$Q := \Omega \times (0,T), \quad \Sigma := \Gamma \times (0,T), \quad \text{and} \quad \Sigma_i := \Gamma_i \times (0,T) \quad \text{for } i = u, \sigma. \tag{2.4}$$

We look for a quadruplet $(\mathbf{u}, \chi, \xi, w)$ defined in Q, where the displacement $\mathbf{u}$ is a vector valued function while χ, ξ, w are scalar valued functions, satisfying the couple of systems described below. The first one consists in the linear elasticity system for $\mathbf{u}$ with mixed boundary conditions, namely

$$\partial_{x_j}\big(C_{ijhk}(\chi)\,\varepsilon_{hk}(\mathbf{u}) + y_{ij}(\chi)\big) = 0 \qquad \text{in } Q \tag{2.5}$$

$$\mathbf{u} = 0 \qquad \text{on } \Sigma_u \tag{2.6}$$

$$\big(C_{ijhk}(\chi)\,\varepsilon_{hk}(\mathbf{u}) + y_{ij}(\chi)\big)n_j = 0 \qquad \text{on } \Sigma_\sigma \tag{2.7}$$

where the linearized strain tensor $\varepsilon(\mathbf{u}) = (\varepsilon_{hk}(\mathbf{u}))$ is defined as in (1.3), i.e.,

$$\varepsilon_{hk}(\mathbf{u}) = \frac{1}{2}\,(\partial_{x_h} u_k + \partial_{x_k} u_h) \tag{2.8}$$

and the right hand sides in (2.6–7) have been taken equal to zero just for the sake of simplicity (however, see the first remark below).

The second system is an initial–boundary value problem for a Cahn–Hilliard type equation for χ, namely

$$\partial_t \chi - \Delta w = 0 \qquad \text{in } Q \tag{2.9}$$

$$w = \mu\,\partial_t\chi - a(\chi)\Delta\chi + \xi + \gamma(\chi, \varepsilon(\mathbf{u})) \qquad \text{in } Q \tag{2.10}$$

$$\xi \in \beta(\chi) \qquad \text{in } Q \tag{2.11}$$

$$\nabla\chi \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0 \qquad \text{on } \Sigma \tag{2.12}$$

$$\chi(0) = \chi_0 \qquad \text{in } \Omega \tag{2.13}$$

where $\gamma : [\underline{\chi}, \overline{\chi}] \times \mathbb{R}^{N^2} \to \mathbb{R}$ is related to the previous functions by

$$\gamma(\chi, \varepsilon) := \rho_A(\chi) + z_{ij}(\chi)\,\varepsilon_{ij} + \frac{1}{2}\varepsilon_{ij}\,C'_{ijhk}(\chi)\,\varepsilon_{hk} \tag{2.14}$$

and χ_0 is a prescribed initial datum.

Now, we specify our assumptions on the structure of systems (2.5–7) and (2.9–13). Although we could let some of the coefficients and functions depend also on x and t, we prefer to avoid further technicalities and assume the stronger conditions listed below, where $\alpha_0 > 0$ and $L, M, \eta \geq 0$ are constants and the corresponding inequalities hold for any $\chi, \chi' \in [\underline{\chi}, \overline{\chi}]$ and any symmetric tensors $\varepsilon, \varepsilon' \in \mathbb{R}^{N^2}$.

$$\underline{\chi}, \overline{\chi}, \mu \in \mathbb{R} \quad \text{with} \quad \underline{\chi} < \overline{\chi} \quad \text{and} \quad \mu > 0 \tag{2.15}$$

$$C = (C_{ijhk}) : [\underline{\chi}, \overline{\chi}] \to \mathbb{R}^{N^4} \quad \text{is Lipschitz continuous} \tag{2.16}$$

$$C_{jihk}(\chi) = C_{ijhk}(\chi) = C_{hkij}(\chi) \tag{2.17}$$

$$C_{ijhk}(\chi)\, \varepsilon_{hk}\, \varepsilon_{ij} \geq \alpha_0 |\varepsilon|^2 \quad \text{where} \quad |\varepsilon|^2 := \varepsilon_{ij}\, \varepsilon_{ij} \tag{2.18}$$

$$y = (y_{ij}) : [\underline{\chi}, \overline{\chi}] \to \mathbb{R}^{N^2} \quad \text{is Lipschitz continuous} \tag{2.19}$$

$$a : [\underline{\chi}, \overline{\chi}] \to \mathbb{R} \quad \text{is Lipschitz continuous and}$$
$$a_0 := \inf a - (\overline{\chi} - \underline{\chi}) \sup |a'| > 0 \tag{2.20}$$

$$\beta \quad \text{is a maximal monotone graph in } \mathbb{R} \times \mathbb{R} \text{ with } \operatorname{dom} \beta = [\underline{\chi}, \overline{\chi}] \tag{2.21}$$

$$\widehat{\beta} : \mathbb{R} \to (-\infty, +\infty] \quad \text{is convex, proper, l.s.c.} \quad \text{and} \quad \partial \widehat{\beta} = \beta \tag{2.22}$$

$$\gamma : [\underline{\chi}, \overline{\chi}] \times \mathbb{R}^{N^2} \to \mathbb{R} \quad \text{satisfies}$$
$$|\gamma(\chi, \varepsilon) - \gamma(\chi', \varepsilon')|$$
$$\leq L\big(1 + |\varepsilon| + |\varepsilon'|\big) \big(|\chi - \chi'| + \eta|\varepsilon - \varepsilon'|\big) + L|\varepsilon - \varepsilon'| \tag{2.23}$$
as well as
$$|\gamma(\chi, \varepsilon)| \leq M(1 + |\varepsilon| + \eta|\varepsilon|^2). \tag{2.24}$$

Clearly, the above assumptions on y_{ij} and γ are fulfilled if y_{ij} and γ are constructed as above, provided that the functions ψ, C_{ijhk}, and ε^*_{ij} are smooth. Moreover, note that (2.20) is satisfied if a is a positive constant; more generally, it is fulfilled whenever the variation of a is small enough. Finally, $\eta = 0$ if the tensor C' vanishes, i.e., if C does not depend on χ.

Theorem 2.1. *Assume* (2.15–24) *and either* $N = 2$ *and* $\eta = 0$ *or* $N = 1$. *Assume moreover*

$$\chi_0 \in H^1(\Omega) \qquad \textit{and} \qquad \chi_0 \in [\underline{\chi}, \overline{\chi}] \quad \textit{a.e. in } \Omega \tag{2.25}$$

$$\underline{\chi} < \chi^* < \overline{\chi}, \qquad \textit{where} \quad \chi^* := \frac{1}{|\Omega|} \int_\Omega \chi_0. \tag{2.26}$$

Then, there exists a quadruplet $(\mathbf{u}, \chi, \xi, w)$ *such that*

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^N) \tag{2.27}$$

$$\chi \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(Q) \tag{2.28}$$

$$\xi \in L^2(Q) \tag{2.29}$$

$$w \in L^2(0, T; H^2(\Omega)) \tag{2.30}$$

which solves (2.5–13) *in the following sense: equation* (2.5) *is understood in the sense of distributions,* (2.9–11) *are satisfied a.e. in* Q, *and the boundary conditions hold in the sense of the appropriate trace theorems.* ∎

As far as uniqueness is concerned, we observe that the components ξ and w of a solution would be uniquely determined by $\mathbf{u}$ and χ if β were single valued. However, this is not the case in our framework. Hence, we look for a unique pair $(\mathbf{u}, \chi)$, only.

Theorem 2.2. *Assume* (2.15–24) *and* (2.25–26) *and let* $(\mathbf{u}_i, \chi_i, \xi_i, w_i)$, $i = 1, 2$, *be two solutions to problem* (2.5–13). *Then* $\mathbf{u}_1 = \mathbf{u}_2$ *and* $\chi_1 = \chi_2$ *provided that one of the following assumptions is fulfilled:* (*i*) $N = 1$; (*ii*) $N = 2$ *and the supplementary regularity condition*

$$\mathbf{u}_i \in L^4(0, T; W^{1,4}(\Omega)^N) \tag{2.31}$$

holds for $i = 1, 2$; (*iii*) $N = 2$, $\eta = 0$ *in* (2.23–24), *and condition* (2.31) *holds for either* $i = 1$ *or* $i = 2$. ■

Remark 2.3. Concerning the interpretation of (2.7), we point out that the regularity of $\mathbf{u}$ and χ and our structure assumption together with equation (2.5) imply that, for a.a. $t \in (0, T)$, each row of the matrix

$$C_{ijhk}(\chi(t))\, \varepsilon_{hk}(\mathbf{u}(t)) + y_{ij}(\chi(t))$$

belongs to $L^2(\Omega)^N$ and its divergence is still in $L^2(\Omega)$. Hence, the left hand side of (2.7) makes sense in $H^{-1/2}(\Gamma)$ (see, e.g., [4, Thm. 1 p. 240]). As far as the couple of boundary conditions (2.6–7) is regarded, we note that minor changes in the sequel would allow us to deal with non zero right hand sides satisfying very weak regularity assumptions. Moreover, we could also consider the case $\Gamma_u = \Gamma$, i.e., Γ_σ is empty, and one forgets about (2.7). On the contrary, some modification has to be done even in the statements if one takes an empty Γ_u, since the first component $\mathbf{u}$ of the solution would be unique only up to a rigid motion. Nevertheless, this case can be treated too. ■

The next sections are devoted to the proof of our results. For the sake of convenience, we define

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \ \partial_n v = 0 \ \text{ on } \Gamma\} \tag{2.32}$$

$$\mathbf{V} := \{\mathbf{v} \in V^N : \ \mathbf{v} = 0 \ \text{ on } \Gamma_u\} \qquad \text{and} \qquad \mathcal{V} := L^2(0, T; \mathbf{V}). \tag{2.33}$$

We see H as a subspace of V' and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V. Moreover, we use the symbol $\|\cdot\|$ for the standard norm in V, while $|\cdot|_\Omega$ and $(\,\cdot\,,\,\cdot\,)_\Omega$ stand for the norm in H and for the corresponding scalar product, respectively. For the sake of simplicity, we use the same symbol for the norm (or for the scalar product) in a space and in a power of it. In particular, this holds for V and $\mathbf{V}$, which is a subspace of V^N. Next, we define

$$\mathcal{K}_R := \left\{\chi \in L^2(0, T; V) : \ \|\chi\|_{L^2(0,T;V)} \le R \quad \text{and} \quad \underline{\chi} \le \chi \le \overline{\chi} \quad \text{a.e. in } Q\right\} \tag{2.34}$$

where the radius R will be chosen later. Indeed, in order to prove our existence result, we are going to apply the Schauder fixed point theorem to the map $\mathcal{F}$ constructed on

$\mathcal{K}_R$ as follows: given χ, we solve (2.5–7) for $\mathbf{u}$ and then we solve (2.9–13) for χ. Finally, we give advice to the reader that we widely use the notation

$$Q_t := \Omega \times (0,t) \qquad \text{for } t \in (0,T) \tag{2.35}$$

and the elementary inequality

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2 \qquad \forall\, a,b \in \mathbb{R} \quad \forall\, \delta > 0 \tag{2.36}$$

and that we write always c, even in the same formula, for different constants which depend only on the constants and on the norms of the functions involved in assumptions (2.15–24), and on the final time T. On the contrary, a notation like c_δ allows the constant to depend on δ, in addition.

3. The elliptic problem

In this section, we build and study the first step of our construction, i.e., we deal with the elliptic part (2.5–7) of problem (2.5–13). We show that, for a given χ, (2.5–7) is well-posed, we derive an a priori estimate, and prove that the map

$$\mathcal{F}_1 : \mathcal{K}_R \to \mathcal{V}, \qquad \chi \mapsto \mathbf{u} \tag{3.1}$$

that gives the solution $\mathbf{u}$ for a given datum χ is continuous.

First of all, we write the variational formulation of problem (2.5–7). We choose any $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbf{V}$ and multiply (2.5) by v_i. Then, we integrate over Ω, sum over i, rearrange, and use the Green formula. Accounting for (2.17) and (2.7), we obtain

$$\int_\Omega C_{ijhk}(\chi(t))\, \varepsilon_{hk}(\mathbf{u}(t))\, \varepsilon_{ij}(\mathbf{v}) = \int_\Omega y_{ij}(\chi(t))\, \varepsilon_{ij}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{V} \tag{3.2}$$

for a.a. $t \in (0,T)$. We note that (3.2) yields a variational formulation of system (2.5–7), if we specify in advance that $\mathbf{u}$ belongs, e.g., to $\mathcal{V}$.

In order to study (3.2), we apply [5, Thm. 3.3, p. 115], which combines the Korn inequality ([5, p. 110]) with a property of the subspace of the rigid motions, and obtain the inequality

$$\int_\Omega C_{ijhk}(\chi(t))\, \varepsilon_{hk}(\mathbf{v})\, \varepsilon_{ij}(\mathbf{v}) \ge \alpha \, \|\mathbf{v}\|^2 \qquad \forall\, \mathbf{v} \in \mathbf{V} \tag{3.3}$$

where α is a positive constant, related to α_0 (see (2.18)) and independent of $\mathbf{v}$, χ, and t. Hence, we apply the Lax–Milgram theorem and obtain, for a.a. $t \in (0,T)$, a unique solution $\mathbf{u}(t) \in \mathbf{V}$ to (2.5–7).

Basic a priori estimate. We choose $\mathbf{v} = \mathbf{u}(t)$ in (3.2) and use (3.3). We obtain

$$\|\mathbf{u}(t)\| \le c \max_{i,j} |y_{ij}(\chi(t))|_\Omega \le \widehat{C}$$

with a constant $\widehat{C}$ which depends only on α, Ω, and on the maximum norm of y_{ij} (see (2.19)). In particular, $\widehat{C}$ does not depend on χ. We write the previous inequality as

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \le \widehat{C}. \tag{3.4}$$

The next step is studying the continuity of $\mathcal{F}_1$.

Lemma 3.1. *Under the assumptions of Theorem 2.1, the map $\mathcal{F}_1 : \mathcal{K}_R \to \mathcal{V}$ is continuous, if $\mathcal{K}_R$ is endowed with the topology induced by $L^2(0,T;V)$.* ∎

Proof. Assuming that $\chi_n, \chi_* \in \mathcal{K}_R$ and that $\chi_n \to \chi_*$ in $L^2(0,T;V)$ strongly. We set $\mathbf{u}_n := \mathcal{F}_1(\chi_n)$ and $\mathbf{u}_* := \mathcal{F}_1(\chi_*)$ and show that $\mathbf{u}_n \to \mathbf{u}_*$ in $\mathcal{V}$ strongly. For any $\mathbf{v} \in \mathbf{V}$ and for a.a. $t \in (0,T)$, we have

$$\begin{aligned}
&\int_\Omega C_{ijhk}(\chi_n(t))\big(\varepsilon_{hk}(\mathbf{u}_n(t)) - \varepsilon_{hk}(\mathbf{u}_*(t))\big)\,\varepsilon_{ij}(\mathbf{v})\\
&= \int_\Omega C_{ijhk}(\chi_n(t))\,\varepsilon_{hk}(\mathbf{u}_n(t))\,\varepsilon_{ij}(\mathbf{v}) - \int_\Omega C_{ijhk}(\chi_*(t))\,\varepsilon_{hk}(\mathbf{u}_*(t))\,\varepsilon_{ij}(\mathbf{v})\\
&\quad - \int_\Omega \big(C_{ijhk}(\chi_n(t)) - C_{ijhk}(\chi_*(t))\big)\,\varepsilon_{hk}(\mathbf{u}_*(t))\,\varepsilon_{ij}(\mathbf{v})\\
&= \int_\Omega \big(y_{ij}(\chi_n(t)) - y_{ij}(\chi_*(t))\big)\,\varepsilon_{ij}(\mathbf{v})\\
&\quad - \int_\Omega \big(C_{ijhk}(\chi_n(t)) - C_{ijhk}(\chi_*(t))\big)\,\varepsilon_{hk}(\mathbf{u}_*(t))\,\varepsilon_{ij}(\mathbf{v}).
\end{aligned}$$

Now, we choose $\mathbf{v} = \mathbf{u}_n(t) - \mathbf{u}_*(t)$ and integrate over $(0,T)$. In view of (3.3), we easily get

$$\begin{aligned}
\alpha\,\|\mathbf{u}_n - \mathbf{u}_*\|^2_{\mathcal{V}} \le &\int_Q \big(y_{ij}(\chi_n) - y_{ij}(\chi_*)\big)\big(\varepsilon_{ij}(\mathbf{u}_n) - \varepsilon_{ij}(\mathbf{u}_*)\big)\\
&- \int_Q \big(C_{ijhk}(\chi_n) - C_{ijhk}(\chi_*)\big)\,\varepsilon_{hk}(\mathbf{u}_*)\,\big(\varepsilon_{ij}(\mathbf{u}_n) - \varepsilon_{ij}(\mathbf{u}_*)\big).
\end{aligned}$$

Thanks to (2.19), we deduce that $y_{ij}(\chi_n) \to y_{ij}(\chi_*)$ strongly in $L^2(Q)$ and immediately see that the first integral on the right hand side tends to 0 owing to (3.4). The second one can be treated in the same way, provided we prove that the product of the first two factors of the integrand converges to 0 strongly in $L^2(Q)$. The first factor tends to 0 strongly in $L^p(Q)$ for any $p < \infty$, since $\{C_{ijhk}(\chi_n)\}$ is bounded in $L^\infty(Q)$ and converges to $C_{ijhk}(\chi_*)$ in $L^2(Q)$ by (2.16). Hence, we can conclude once we know that $\varepsilon_{hk}(\mathbf{u}_*) \in L^q(Q)$ for some $q > 2$. But this regularity result for the solution to problem (2.5–7) follows, for instance, by [15, Thm. 2.6, p. 192]. ∎

Remark 3.2. In the one-dimensional case, system (2.5–7) can be solved by an explicit formula and one easily sees that (3.4) can be improved. Indeed, for any $p \in [1,\infty]$ the following estimate holds

$$\|\mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} \le \widehat{C}_p \tag{3.5}$$

for any solution $\mathbf{u}$ to (2.5–7), where the constant $\widehat{C}_p$ depends on p, in addition. If instead $N > 1$, we can only say that (3.5) holds for some $p > 2$ as a consequence of [15]. Unfortunately, this value of p is close to 2 in general and we cannot ensure that (3.5) holds with $p = 4$. Moreover, even a weaker inequality like

$$\|\mathbf{u}\|_{L^4(0,T;W^{1,4}(\Omega))} \le \widehat{C}_4$$

(cf. (2.31)) is not known, and that is why we have to assume $\eta = 0$ somewhere in our statements. This fact will be clear in the next sections.

4. The Cahn–Hilliard system

In this section, we build and study the second step of our construction, i.e., the map

$$\mathcal{F}_2 : \operatorname{dom} \mathcal{F}_2 := \left\{ \mathbf{u} \in \mathcal{V} : \ \|\mathbf{u}\|_{L^\infty(0,T;\mathbf{V})} \le \widehat{C} \right\} \to \mathcal{K}_R, \qquad \mathbf{u} \mapsto \chi \tag{4.1}$$

where χ is the first component of the solution (χ, ξ, w) to (2.9–13) corresponding to the given $\mathbf{u}$. The choice of the domain of $\mathcal{F}_2$ prescribes that every given $\mathbf{u}$ fulfills the basic estimate (3.4). So, we have first to show that (2.9–13) has a unique solution and to estimate the norm of that solution in order to fix the parameter R suitably. Then, we prove continuity for $\mathcal{F}_2$ and relative compactness for its range by means of a number of a priori estimates.

To this aim, we introduce some operators and present (2.9–13) in an abstract form. First, we denote by A the operator from V to V' defined by

$$\langle Au, v \rangle := \int_\Omega \nabla u \cdot \nabla v \qquad \forall\, u, v \in V. \tag{4.2}$$

Note that $u \in W$ and $Au = -\Delta u$ whenever $Au \in H$. Let us also introduce the following subspaces, characterized by the zero mean value condition,

$$V_0 := \{ v \in V : \ \langle v, 1 \rangle = 0 \} \qquad \text{and} \qquad V_0' := \{ u \in V' : \ \langle u, 1 \rangle = 0 \}. \tag{4.3}$$

We remark that our assumptions on Ω imply that A maps V onto V_0' and that the kernel of A is the subspace of all constant functions. Therefore, the restriction of A to V_0 maps V_0 onto V_0' isomorphically and we can define $\mathcal{N}$ by the conditions

$$\mathcal{N} : V_0' \to V_0 \qquad \text{and} \qquad A\mathcal{N} v = v \quad \forall\, v \in V_0' \tag{4.4}$$

i.e., $\mathcal{N} v$ is the solution with zero mean value of a generalized Neumann problem with right hand side v. Hence, the following relations hold

$$\langle Au, \mathcal{N} v \rangle = \langle v, u \rangle \qquad \forall\, u \in V, \quad \forall\, v \in V_0' \tag{4.5}$$

$$\langle u, \mathcal{N} v \rangle = \langle v, \mathcal{N} u \rangle = \int_\Omega (\nabla \mathcal{N} u) \cdot (\nabla \mathcal{N} v) \qquad \forall\, u, v \in V_0' \tag{4.6}$$

$$\int_0^t \langle v'(s), \mathcal{N} v(s) \rangle \, ds = \frac{1}{2} \|v(t)\|_*^2 - \frac{1}{2} \|v(0)\|_*^2 \qquad \forall\, v \in H^1(0,T;V_0'), \quad \forall\, t \in [0,T] \tag{4.7}$$

where $\|\cdot\|_*$ is a norm on V_0' which is equivalent to the usual one. Indeed

$$\|v\|_*^2 = \langle v, \mathcal{N}v\rangle = |\nabla\mathcal{N}v|_\Omega^2 \qquad \forall\, v \in V_0' \tag{4.8}$$

and the Poincaré inequality holds in V_0.

Now, we are ready to write the abstract version of problem (2.9–13), where we take $\mu = 1$ without loss of generality. Under the assumptions of Theorem 2.1, given $\mathbf{u} \in \mathcal{V}$ satisfying estimate (3.4), we look for a triplet $(\chi, \xi\, w)$ which fulfils

$$\chi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W) \cap L^\infty(Q) \tag{4.9}$$

$$\xi \in L^2(Q) \tag{4.10}$$

$$w \in L^2(0,T;W) \tag{4.11}$$

and solves the following system

$$\partial_t\chi(t) + Aw(t) = 0 \qquad \text{in } V', \quad \text{for a.a. } t \in (0,T) \tag{4.12}$$

$$w = \partial_t\chi + a(\chi)A\chi + \xi + \gamma(\chi, \varepsilon(\mathbf{u})) \qquad \text{a.e. in } Q \tag{4.13}$$

$$\xi \in \beta(\chi) \qquad \text{a.e. in } Q \tag{4.14}$$

$$\chi(0) = \chi_0. \tag{4.15}$$

In the sequel, we assume $N = 2$ and $\eta = 0$ in (2.23–24), while we make just some remarks to include the case $N = 1$ and arbitrary η. First of all, we observe that (4.12) immediately implies

$$\partial_t \int_\Omega \chi(t) = \int_\Omega \partial_t\chi(t) = 0 \qquad \text{for a.a. } t \in (0,T)$$

i.e., $\partial_t\chi(t)$ always belongs to the domain of $\mathcal{N}$. Moreover, we also have

$$\frac{1}{|\Omega|}\int_\Omega \chi(t) = \frac{1}{|\Omega|}\int_\Omega \chi_0(t) = \chi^* \qquad \forall\, t \in [0,T]$$

and the same remarks hold for the regularized problems we are going to introduce. Indeed, we are going to solve (4.12–15) by approximating that system and passing to the limit with the help or suitable a priori estimates. We use the Yosida regularization β_λ of β (see, e.g., [2, p. 28]).

As β_λ is defined everywhere, the constraints $\underline{\chi} \le \chi \le \overline{\chi}$ included in (4.14) are lost in the regularized problem. Hence, we have first to extend the definitions of a and γ and allow any value of χ in their arguments. Clearly, this can be done in a way that preserves the boundedness, Lipschitz continuity, growth, and ellipticity properties prescribed in assumptions (2.16–20) and (2.23–24). As far as (2.20) is concerned, we have to extend a by setting $a(\chi) = a(\underline{\chi})$ and $a(\chi) = a(\overline{\chi})$ for $\chi < \underline{\chi}$ and $\chi > \overline{\chi}$, respectively. For the sake of simplicity, we do not write the analogous inequalities for the extended functions and still refer to (2.16–20) and (2.23–24).

Here is the regularized problem. We look for a pair $(\chi_\lambda, w_\lambda)$, satisfying regularity requirements analogous to (4.9) and (4.11) but the boundedness of the first component, such that

$$\partial_t \chi_\lambda(t) + A w_\lambda(t) = 0 \qquad \text{in } V', \quad \text{for a.a. } t \in (0,T) \tag{4.16}$$

$$w_\lambda = \partial_t \chi_\lambda + a(\chi_\lambda) A \chi_\lambda + \beta_\lambda(\chi_\lambda) + \gamma(\chi_\lambda, \varepsilon(\mathbf{u})) \qquad \text{a.e. in } Q \tag{4.17}$$

$$\chi_\lambda(0) = \chi_0. \tag{4.18}$$

The existence of a solution to the above problem can be shown by using, e.g., a Galerkin scheme. However, we avoid this proof, since the discretization procedure is standard and the a priori estimates we are going to derive give also the outline of the convergence of the discrete solution to the solution to the approximating problem (4.16–18). So, we start with a solution $(\chi_\lambda, w_\lambda)$ to (4.16–18), directly. We remark instead that the existence of such a solution is ensured provided that $\gamma(\,\cdot\,, \varepsilon(\mathbf{u}))$ maps $L^2(0,T;V)$ into $L^2(0,T;H)$ and is Lipschitz continuous. This condition follows from (2.23–24) with $\eta = 0$. Indeed, in this case, $\gamma(\,\cdot\,, \varepsilon(\mathbf{u}))$ maps $L^2(0,T;H)$ into itself and is Lipschitz continuous.

Lemma 4.1. *Under the assumptions of Theorem 2.1 with $N = 2$ and $\eta = 0$ in inequalities* (2.23–24)*, fix any $\mathbf{u} \in \operatorname{dom} \mathcal{F}_2$. Then, problem* (4.12–15) *has at least a solution satisfying the regularity requirements* (2.28–30) *and the estimate*

$$\|\chi\|_{L^2(0,T;V)} \leq R \tag{4.19}$$

where R depends only on the constants in assumptions (2.15–24)*, on the domain Ω, on the final time T, and on the initial datum χ_0.* ■

Proof. Our argument relies on a number of a priori estimates.

First a priori estimate. We test (4.16) by $\mathcal{N}(\chi_\lambda - \chi^*)$ and (4.17) by $\chi_\lambda - \chi^*$. Then, we subtract the resulting equalities to each other and integrate over $(0,t)$, where $t \in (0,T)$ is arbitrary. In view of (4.5), two terms cancel and we obtain, with the help of (4.7),

$$\begin{aligned} &\frac{1}{2} \|\chi_\lambda(t) - \chi^*\|_*^2 + \frac{1}{2} |\chi_\lambda(t) - \chi^*|_\Omega^2 \\ &\qquad + \int_{Q_t} \nabla \chi_\lambda \cdot \nabla \big(a(\chi_\lambda)(\chi_\lambda - \chi^*) \big) + \int_{Q_t} \big(\beta_\lambda(\chi_\lambda) - \beta_\lambda(\chi^*) \big)(\chi_\lambda - \chi^*) \\ &= \frac{1}{2} \|\chi_0 - \chi^*\|_*^2 + \frac{1}{2} |\chi_0 - \chi^*|_\Omega^2 - \int_{Q_t} \gamma(\chi_\lambda, \varepsilon(\mathbf{u}))(\chi_\lambda - \chi^*) \end{aligned} \tag{4.20}$$

since $\beta_\lambda(\chi^*)$ is a constant. Now, we have to treat the three integrals. As $a' = 0$ in $\mathbb{R} \setminus [\underline{\chi}, \overline{\chi}]$, the use of (2.20) leads to

$$\begin{aligned} &\int_{Q_t} \nabla \chi_\lambda \cdot \nabla \big(a(\chi_\lambda)(\chi_\lambda - \chi^*) \big) = \int_{Q_t} \big(a(\chi_\lambda) + (\chi_\lambda - \chi^*) a'(\chi_\lambda) \big) |\nabla \chi_\lambda|^2 \\ &\geq a_0 \int_{Q_t} |\nabla \chi_\lambda|^2. \end{aligned}$$

The last integral on the left hand side is nonnegative, since β_λ is monotone. Moreover, due to (2.24), we have that

$$
\begin{aligned}
&-\int_{Q_t} \gamma(\chi_\lambda, \varepsilon(\mathbf{u}))(\chi_\lambda - \chi^*) \le M \int_{Q_t} (1 + |\varepsilon(\mathbf{u})|)\, |\chi_\lambda - \chi^*| \\
&\le 1 + \widehat{C}^2 + c \int_0^t |\chi_\lambda(s) - \chi^*|_\Omega^2 \, ds
\end{aligned}
\tag{4.21}
$$

where $\widehat{C}$ is given by (3.4). Finally, we apply the Gronwall lemma and get

$$
\|\chi_\lambda\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \le R \tag{4.22}
$$

where R is chosen with the dependences specified in the statement.

Second a priori estimate. We test (4.16) by χ_λ and (4.17) by $A\chi_\lambda$. Then, we subtract the resulting equalities to each other and integrate as before. We obtain

$$
\begin{aligned}
&\frac{1}{2}\, |\chi_\lambda(t)|_\Omega^2 + \frac{1}{2}\, |\nabla \chi_\lambda(t)|_\Omega^2 + \int_{Q_t} a(\chi_\lambda)(\Delta \chi_\lambda)^2 + \int_{Q_t} \beta_\lambda'(\chi_\lambda) |\nabla \chi_\lambda|^2 \\
&= \frac{1}{2}\, |\chi_0|_\Omega^2 + \frac{1}{2}\, |\nabla \chi_0|_\Omega^2 + \int_{Q_t} \gamma(\chi_\lambda, \varepsilon(\mathbf{u}))\, \Delta \chi_\lambda .
\end{aligned}
$$

Now we use (2.20), (2.24–25), the monotonicity of β_λ, and well-known regularity results on elliptic homogeneous Neumann problems with data in $L^2(\Omega)$. Thus, we easily deduce

$$
\|\chi_\lambda\|_{L^\infty(0,T;V) \cap L^2(0,T;W)} \le c. \tag{4.23}
$$

Third a priori estimate. We test (4.16) by $\mathcal{N}\chi_\lambda'$ and (4.17) by χ_λ'. Then, we subtract and integrate as before. Two terms cancel again. Hence, if $\widehat{\beta}_\lambda$ is the Yosida regularization of $\widehat{\beta}$, owing to (4.8), we have

$$
\begin{aligned}
&\int_0^t \|\chi_\lambda'(s)\|_*^2 \, ds + \int_0^t |\chi_\lambda'(s)|_\Omega^2 \, ds + \int_\Omega \widehat{\beta}_\lambda(\chi_\lambda(t)) \\
&= \int_\Omega \widehat{\beta}_\lambda(\chi_0) - \int_{Q_t} a(\chi_\lambda)(A\chi_\lambda)\chi_\lambda' - \int_{Q_t} \gamma(\chi_\lambda, \varepsilon(\mathbf{u}))\chi_\lambda' .
\end{aligned}
$$

Now, using the boundedness of a, (2.24), the inequality (stated in [2, Prop. 2.11, p. 39])

$$
\widehat{\beta}_\lambda(r) \le \widehat{\beta}(r) \qquad \forall\, r \in \mathbb{R}
$$

along with (2.25), and estimate (4.23) already proved, we easily deduce that

$$
\|\partial_t \chi_\lambda\|_{L^2(0,T;H)} + \|\widehat{\beta}_\lambda(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{4.24}
$$

Fourth a priori estimate. We introduce a notation. We set

$$
\beta_\lambda^*(t) := \frac{1}{|\Omega|} \int_\Omega \beta_\lambda(\chi_\lambda(t)) \qquad \text{for a.a. } t \in (0,T)
$$

and treat $\beta^*_\lambda(t)$ as a function on Ω as well. As $v := \beta_\lambda(\mathcal{X}) - \beta^*_\lambda \in H \cap V_0'$, the function $\mathcal{N}v$ is well defined and we can test (4.16) by $\mathcal{N}v$ and (4.17) by v. Then, integrating in time and subtracting as before, with the help of the estimates already proved and (4.6) it is straightforward to obtain

$$\|\beta_\lambda(\mathcal{X}_\lambda) - \beta^*_\lambda\|_{L^2(0,T;H)} \le c.$$

At this point, in order to get the useful estimate

$$\|\beta_\lambda(\mathcal{X}_\lambda)\|_{L^2(0,T;H)} \le c \tag{4.25}$$

we have to find an upper bound for the norm of β^*_λ. This can be done using (2.26) and following [13, Lemma 5.2] (see also [3, third a priori estimate in the proof of Thm. 2.1] for a detailed application).

Fifth a priori estimate. Clearly, by applying the previous estimates to the right hand side of (4.17), we deduce an estimate for w_λ in $L^2(0,T;H)$. Hence, by comparison in (4.16) and using (4.24), we infer that

$$\|w_\lambda\|_{L^2(0,T;W)} \le c. \tag{4.26}$$

Conclusion of the proof. Collecting (4.22–26), we see that the generalized sequence $\{(\mathcal{X}_\lambda, \beta_\lambda(\mathcal{X}_\lambda), w_\lambda)\}$ converges, at least for a subsequence (still denoted by the same symbol for simplicity). More precisely, the three components converge weakly or weakly star in the spaces

$$H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad L^2(0,T;H), \quad L^2(0,T;W)$$

respectively, thanks to well-known weak and weak star compactess results. We term $(\mathcal{X}, \xi, w)$ the corresponding limit. Clearly, the regularity conditions (2.28–30) hold but the boundedness of $\mathcal{X}$, and we have to show that the triplet $(\mathcal{X}, \xi, w)$ solves problem (4.12–15), i.e., that we can pass to the limit in the nonlinear terms and in the Cauchy condition. This will imply also the boundedness of $\mathcal{X}$ as a consequence of (4.14) and (2.21).

First of all, note that $\{\mathcal{X}_\lambda\}$ converges strongly in $C^0([0,T];H)$ due to estimates (4.23–24) and the compact embedding of V into H. As far as (4.13) is concerned, we can use the Lipschitz continuity and boundedness of a and check that

$$a(\mathcal{X}_\lambda)A(\mathcal{X}_\lambda) \to a(\mathcal{X})A(\mathcal{X}) \quad \text{weakly in } L^p(Q) \text{ for any } p < 2.$$

In addition, we see that (2.23) yields $\gamma(\mathcal{X}, \varepsilon(\mathbf{u})) \to \gamma(\mathcal{X}, \varepsilon(\mathbf{u}))$ strongly in $L^\infty(0,T;L^1(\Omega))$ (even better). Equation (4.14) follows immediately since $\{\beta_\lambda(\mathcal{X}_\lambda)\}$ is weakly convergent in $L^2(Q)$ and we can apply, e.g., [1, Prop. 1.1, p. 42]. Finally, (4.19) follows by (4.22) with the same R. ∎

Remark 4.2. If $N = 1$ we can allow any positive η in (2.23–24). Actually, we have to modify (4.1) by including $\|\mathbf{u}\|_{L^\infty(0,T;W^{1,4}(\Omega))} \le \widehat{C}_4$ in the definition of $\operatorname{dom}\mathcal{F}_2$. Here the constant $\widehat{C}_4$ is given by estimate (3.5), which holds in the one-dimensional case. We just discuss the crucial estimate (4.21), which involves γ. This inequality can be replaced by

$$- \int_{Q_t} \gamma(\chi_\lambda, \varepsilon(\mathbf{u}))(\chi_\lambda - \chi^*) \leq M \int_{Q_t} (1 + |\varepsilon(\mathbf{u})| + \eta |\varepsilon(\mathbf{u})|^2) \, |\chi_\lambda - \chi^*|$$
$$\leq 1 + (\widehat{C}_4)^4 + c \int_0^t |\chi_\lambda(s) - \chi^*|_\Omega^2 \, ds.$$

The same remark applies to the related estimates we have proved. We also note that no upper bound on N is needed in the above proof if $\eta = 0$. We will exploit the assumption $N = 2$ to show that the solution is unique. ∎

Remark 4.3. The above proof also yields

$$\|\chi\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\xi\|_{L^2(0,T;H)} + \|w\|_{L^2(0,T;W)} \leq c \tag{4.27}$$

where c depends on $\widehat{C}$ but is independent of $\mathbf{u} \in \operatorname{dom} \mathcal{F}_2$. ∎

Lemma 4.4. *Under the assumptions of Theorem 2.1 with $N = 2$ and $\eta = 0$ in inequalities* (2.23–24)*, fix any $\mathbf{u} \in \mathcal{V}$ satisfying* (3.4)*. Then, the component χ of the solution given by Lemma 4.1 is unique.* ∎

Proof. Let (χ_i, ξ_i, w_i), $i = 1, 2$, be two solutions and term (χ, ξ, w) their difference. Then, writing a_i instead of $a(\chi_i)$ for simplicity, we have

$$\partial_t \chi(t) + A w(t) = 0 \qquad \text{in } V', \quad \text{for a.a. } t \in (0, T) \tag{4.28}$$
$$w = \partial_t \chi + a_1 A \chi_1 - a_2 A \chi_2 + \xi + \gamma(\chi_1, \varepsilon(\mathbf{u})) - \gamma(\chi_2, \varepsilon(\mathbf{u})) \qquad \text{a.e. in } Q. \tag{4.29}$$

Noting that $\chi(t)$ has zero mean value for every $t \in [0, T]$, we test (4.28) and (4.29) by $\mathcal{N}\chi$ and χ, respectively. Then we integrate in time, take the difference of the resulting equalities, and obtain

$$\frac{1}{2} \|\chi(t)\|_*^2 + \frac{1}{2} |\chi(t)|_\Omega^2 + \int_{Q_t} \nabla \chi \cdot \nabla (a_1 \chi) + \int_{Q_t} \xi \chi$$
$$= - \int_{Q_t} \big(\gamma(\chi_1, \varepsilon(\mathbf{u})) - \gamma(\chi_2, \varepsilon(\mathbf{u}))\big) \chi + \int_{Q_t} (\Delta \chi_2)(a_1 - a_2) \chi. \tag{4.30}$$

The integral containing ξ is nonnegative, since $\xi_i \in \beta(\chi_i)$ a.e. in Q for $i = 1, 2$ and β is monotone. Owing to (2.20), the other integral on the left hand side is treated this way

$$\int_{Q_t} \nabla \chi \cdot \nabla (a_1 \chi) = \int_{Q_t} a_1 |\nabla \chi|^2 + \int_{Q_t} \chi a'(\chi_1) \nabla \chi_1 \cdot \nabla \chi$$
$$\geq a_0 \int_{Q_t} |\nabla \chi|^2 - c \int_{Q_t} |\chi| \, |\nabla \chi_1| \, |\nabla \chi|.$$

So, we have to estimate the last integral from above, and this is rather delicate. In order to handle this term, we take advantage of the following Gagliardo–Nirenberg inequalities (surely holding in the two-dimensional case)

$$\|v\|^2_{L^4(\Omega)} \le c\,|v|_\Omega\,|\nabla v|_\Omega \qquad \forall\, v \in V \tag{4.31}$$

$$\|\nabla v\|^2_{L^4(\Omega)} \le c\,\|v\|_{H^2(\Omega)}\,\|v\|_{L^\infty(\Omega)} \qquad \forall\, v \in H^2(\Omega). \tag{4.32}$$

Using first the Hölder inequality, we have for any $\delta > 0$

$$\begin{aligned}
&\int_{Q_t} |\chi|\,|\nabla\chi_1|\,|\nabla\chi| \le \int_0^t \|\chi(s)\|_{L^4(\Omega)}\,\|\nabla\chi_1(s)\|_{L^4(\Omega)}\,|\nabla\chi(s)|_\Omega\,ds\\
&\le \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \|\nabla\chi_1(s)\|^2_{L^4(\Omega)}\,\|\chi(s)\|^2_{L^4(\Omega)}\,ds\\
&\le \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \|\chi_1(s)\|_W\,\|\chi_1(s)\|_{L^\infty(\Omega)}\,|\chi(s)|_\Omega\,|\nabla\chi(s)|_\Omega\,ds\\
&\le 2\delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\,\|\chi_1\|^2_{L^\infty(Q)}\int_0^t \|\chi_1(s)\|^2_W\,|\chi(s)|^2_\Omega\,ds\\
&\le 2\delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \|\chi_1(s)\|^2_W\,|\chi(s)|^2_\Omega\,ds
\end{aligned}$$

also on account of (4.14). Now, we deal with the right hand side of (4.30). Thanks to the Lipschitz continuity of a and to (2.23), the whole sum is bounded by

$$\begin{aligned}
&c\int_{Q_t}\big(1+|\varepsilon(\mathbf{u})|+|\Delta\chi_2|\big)|\chi|^2\\
&\quad\le c\int_0^t\Big(1+\|\varepsilon(\mathbf{u}(s))\|_{L^2(\Omega)}+\|\Delta\chi_2(s)\|_{L^2(\Omega)}\Big)\,\|\chi(s)\|^2_{L^4(\Omega)}\,ds.
\end{aligned}$$

Using (4.31) again, we obtain

$$\begin{aligned}
&c\int_{Q_t}\big(1+|\varepsilon(\mathbf{u})|+|\Delta\chi_2|\big)|\chi|^2\\
&\quad\le c\int_0^t\big(1+|\varepsilon(\mathbf{u}(s))|_\Omega+|\Delta\chi_2(s)|_\Omega\big)\,|\chi(s)|_\Omega\,|\nabla\chi(s)|_\Omega\,ds\\
&\quad\le \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t\Big(1+|\varepsilon(\mathbf{u}(s))|^2_\Omega+\|\chi_2(s)\|^2_W\Big)\,|\chi(s)|^2_\Omega\,ds.
\end{aligned}$$

Collecting all the above inequalities, choosing $\delta = a_0/4$, and neglecting some of the positive terms on the left hand side, we get from (4.30)

$$|\chi(t)|^2_\Omega \le c\int_0^t\Big(1+|\varepsilon(\mathbf{u}(s))|^2_\Omega+\|\chi_1(s)\|^2_W+\|\chi_2(s)\|^2_W\Big)\,|\chi(s)|^2_\Omega\,ds \quad \forall\, t\in[0,T].$$

As the function in brackets belongs to $L^1(0,T)$, we can apply the generalized Gronwall lemma and conclude that χ identically vanishes. ∎

Remark 4.5. The one-dimensional case can be treated with minor changes in using the Gagliardo–Nirenberg inequalities. Moreover, in this case, we can allow any η in (2.23–24). Indeed, the above proof does not use (2.24) explicitely and just requires that the integrals involving γ are convergent. This fact is ensured by (2.24) and Remark 3.2.

5. Proof of Theorem 2.1

Now, we are ready to construct rigorously the function $\mathcal{F}$ whose fixed points are the solutions to problem (2.5–13). First, we choose the radius R given by (4.19) in Lemma 4.1 and recall the constant $\widehat{C}$ introduced in (3.4). Then, we define $\mathcal{F}_1$ and $\mathcal{F}_2$ according to (2.34), (3.1), and (4.1). Thanks to the above results, the range of $\mathcal{F}_1$ is contained in the domain of $\mathcal{F}_2$ and the range of $\mathcal{F}_2$ is contained in the domain of $\mathcal{F}_1$. Hence, the composed map

$$\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1 \tag{5.1}$$

is well defined and maps $\mathcal{K}_R$ into itself. Clearly, a quadruplet $(\mathbf{u}, \chi, \xi, w)$ satisfying the regularity requirements (2.27–30) is a solution to (2.5–13) if and only if χ is a fixed point of $\mathcal{F}$.

Hence, we just have to verify that the assumptions of the Schauder fixed point theorem are fulfilled. Clearly, $\mathcal{K}_R$ is a nonempty closed convex bounded subset of $L^2(0,T;V)$. So, accounting for Lemma 3.1, it remains to show that $\mathcal{F}_2$ is continuous and that its range is relatively compact in $L^2(0,T;V)$. The latter sentence is clear from (4.27), since the embedding

$$H^1(0,T;H) \cap L^2(0,T;W) \subset L^2(0,T;V)$$

is compact by the Aubin lemma (cf., e.g., [14, p. 58]), while the former one follows from the next lemma.

Lemma 5.1. *Under the assumption of Theorem 2.1, the map $\mathcal{F}_2 : \operatorname{dom}\mathcal{F}_2 \to \mathcal{K}_R$ is continuous, if $\operatorname{dom}\mathcal{F}_2$ is endowed with the topology induced by $\mathcal{V}$.* ∎

Proof. Take $\mathbf{u}_n, \mathbf{u} \in \operatorname{dom}\mathcal{F}_2$ and assume that the sequence $\{\mathbf{u}_n\}$ converges to $\mathbf{u}$ in $\mathcal{V}$. To prove the lemma, we verify that the sequence $\{\chi_n\}$ of the corresponding solutions to (4.12–15) converges in $L^2(0,T;V)$ to the solution corresponding to $\mathbf{u}$. As the desired limit is decided a priori, it is enough to prove the convergence for a subsequence. By Remark 4.3, the estimate

$$\|\chi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\xi_n\|_{L^2(0,T;H)} + \|w_n\|_{L^2(0,T;W)} \le c$$

holds true, where c does not depend on n. By weak and weak star compactness, we can assume that the three sequences $\{\chi_n\}$, $\{\xi_n\}$, and $\{w_n\}$ are convergent weakly or weakly star in the appropriate spaces, without loss of generality. Hence, everything reduces to show that the limit triplet (χ, ξ, w) solves (4.12–15), and this can be done following the same arguments used in the proof of Lemma 4.1, with one more observation regarding the term that involves $\mathbf{u}_n$ explicitely, namely $\gamma(\chi_n, \varepsilon(\mathbf{u}_n))$. It suffices to prove that it converges to $\gamma(\chi, \varepsilon(\mathbf{u}))$ in any reasonable topology, e.g., in $L^1(Q)$. But this follows immediately from (2.23), since $\{\varepsilon_{ij}(\mathbf{u}_n\})$ converges to $\varepsilon_{ij}(\mathbf{u})$ and $\{\chi_n\}$ converges to χ strongly in $L^2(Q)$. Note that this argument holds even if $\eta > 0$ in (2.23–24). ∎

6. Proof of Theorem 2.2

We treat both the cases *(ii)* and *(iii)*. Hence, we use (2.31) when necessary. Minor changes are needed to deal with the case *(i)*.

Our argument follows the outline of the proof of Lemma 4.4. Indeed, we have just to add the contribution due to the elliptic equation and to modify the estimate of the integral involving γ. So, we consider two solutions and keep the notation of the mentioned proof. We also introduce

$$\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$$

where $\mathbf{u}_i$ is the first component of the solution $(\mathbf{u}_i, \chi_i, \xi_i, w_i)$. We test (4.28) and (4.29) by $\mathcal{N}\chi$ and χ, respectively. Then, we integrate in time and take the difference of the resulting equalities. Moreover, we write (3.2) for both solutions, take the difference, use $\mathbf{v} = \mathbf{u}(t)$ as test function, and integrate in time. Adding the resulting expression to the previous one, we obtain a relation that generalizes (4.30), namely

$$\begin{aligned}
&\frac{1}{2}\, \|\chi(t)\|_*^2 + \frac{1}{2}\, |\chi(t)|_\Omega^2 + \int_{Q_t} \nabla\chi \cdot \nabla(a_1\chi) + \int_{Q_t} \xi\chi + \int_{Q_t} C_{ijhk}(\chi_1)\varepsilon_{hk}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{u}) \\
&= -\int_{Q_t} \big(\gamma(\chi_1, \varepsilon(\mathbf{u}_1)) - \gamma(\chi_2, \varepsilon(\mathbf{u}_2))\big)\chi + \int_{Q_t} (\Delta\chi_2)(a_1 - a_2)\chi. \\
&\quad + \int_{Q_t} \big(C_{ijhk}(\chi_2) - C_{ijhk}(\chi_1)\big)\varepsilon_{hk}(\mathbf{u}_2)\, \varepsilon_{ij}(\mathbf{u}) \\
&\quad + \int_{Q_t} \big(y_{ij}(\chi_1) - y_{ij}(\chi_2)\big)\varepsilon_{ij}(\mathbf{u}).
\end{aligned} \tag{6.1}$$

Hence, we just detail how to modify the proof of Lemma 4.4 and conclude. On the left hand side, we use to (3.3) and get a good bound from below for the last term, namely

$$\int_{Q_t} C_{ijhk}(\chi_1)\varepsilon_{hk}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{u}) \geq \alpha \int_0^t \|\mathbf{u}(s)\|^2 \, ds.$$

We treat the fourth integral on the right hand side with the help of (2.19) and obtain

$$\int_{Q_t} \big(y_{ij}(\chi_1) - y_{ij}(\chi_2)\big)\varepsilon_{ij}(\mathbf{u}) \leq \delta \int_0^t \|\mathbf{u}(s)\|^2 \, ds + c_\delta \int_0^t |\chi(s)|_\Omega^2 \, ds.$$

To handle the third integral on the right hand side of (6.1), we invoke (2.16) and (4.31). We have

$$\begin{aligned}
&\int_{Q_t} \big(C_{ijhk}(\chi_2) - C_{ijhk}(\chi_1)\big)\varepsilon_{hk}(\mathbf{u}_2)\,\varepsilon_{ij}(\mathbf{u}) \\
&\leq c\int_0^t \|\chi(s)\|_{L^4(\Omega)}\,\|\varepsilon(\mathbf{u}_2(s))\|_{L^4(\Omega)}\,|\varepsilon(\mathbf{u}(s))|_\Omega\,ds \\
&\leq \delta\int_0^t \|\mathbf{u}(s)\|^2\,ds + c_\delta\int_0^t \|\varepsilon(\mathbf{u}_2(s))\|^2_{L^4(\Omega)}\,\|\chi(s)\|^2_{L^4(\Omega)}\,ds \\
&\leq \delta\int_0^t \|\mathbf{u}(s)\|^2\,ds + c_\delta\int_0^t \|\varepsilon(\mathbf{u}_2(s))\|^2_{L^4(\Omega)}\,|\chi(s)|_\Omega\,|\nabla\chi(s)|_\Omega\,ds \\
&\leq \delta\int_0^t \|\mathbf{u}(s)\|^2\,ds + \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \|\mathbf{u}_2(s)\|^4_{W^{1,4}(\Omega)}\,|\chi(s)|^2_\Omega\,ds.
\end{aligned}$$

Finally, the last integral we have to deal with is the first one on the right hand side of (6.1). Owing to (2.23), we proceed as follows

$$\begin{aligned}
&-\int_{Q_t}\big(\gamma(\chi_1,\varepsilon(\mathbf{u}_1)) - \gamma(\chi_2,\varepsilon(\mathbf{u}_2))\big)\chi \\
&\leq L\int_{Q_t}\big(1+|\varepsilon(\mathbf{u}_1)|+|\varepsilon(\mathbf{u}_2)|\big)\Big(|\chi|+\eta|\varepsilon(\mathbf{u})|\Big)|\chi| + L\int_{Q_t}|\varepsilon(\mathbf{u})|\,|\chi| \\
&\leq c_\delta\int_0^t\Big(1+\|\mathbf{u}_1(s)\|+\|\mathbf{u}_2(s)\|\Big)\|\chi(s)\|^2_{L^4(\Omega)}\,ds + \delta\int_0^t\|\mathbf{u}(s)\|^2\,ds \\
&\qquad + \eta^2 c_\delta\int_0^t\Big(1+\|\mathbf{u}_1(s)\|^2_{W^{1,4}(\Omega)}+\|\mathbf{u}_2(s)\|^2_{W^{1,4}(\Omega)}\Big)\|\chi(s)\|^2_{L^4(\Omega)}\,ds.
\end{aligned}$$

Therefore, the argument dealing with the term involving γ differs from that of the proof of Lemma 4.4 just for the last integral in the above chain. Applying (4.31) again, we see that

$$\begin{aligned}
&\eta^2 c_\delta\int_0^t\Big(1+\|\mathbf{u}_1(s)\|^2_{W^{1,4}(\Omega)}+\|\mathbf{u}_2(s)\|^2_{W^{1,4}(\Omega)}\Big)\|\chi(s)\|^2_{L^4(\Omega)}\,ds \\
&\leq \delta\int_{Q_t}|\nabla\chi|^2 + \eta^4 c_\delta\int_0^t\Big(1+\|\mathbf{u}_1(s)\|^4_{W^{1,4}(\Omega)}+\|\mathbf{u}_2(s)\|^4_{W^{1,4}(\Omega)}\Big)|\chi(s)|^2_\Omega\,ds.
\end{aligned}$$

Therefore, arguing as in the proof of Lemma 4.4, we arrive at

$$|\chi(t)|^2_\Omega + \int_0^t\|\mathbf{u}(s)\|^2\,ds \leq c\int_0^t\varphi(s)\,|\chi(s)|^2_\Omega\,ds \quad \forall\, t\in[0,T]$$

where we have set

$$\varphi(s) := 1 + \|\chi_1(s)\|^2_W + \|\chi_2(s)\|^2_W + \eta^4\,\|\mathbf{u}_1(s)\|^4_{W^{1,4}(\Omega)} + (1+\eta^4)\,\|\mathbf{u}_2(s)\|^4_{W^{1,4}(\Omega)}\,.$$

Hence, we apply the generalized Gronwall lemma and conclude.

Remark 6.1. By a direct check in (6.1) (see alse the proof of Lemma 4.4), it is straightforward to verify that our uniqueness proof works in any space dimension if the data C and a do not depend on χ. This complies with the uniqueness result proved in [11].

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