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To the uniqueness problem for nonlinear parabolic equations

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Dedicated to Professor M. I. Vishik on the occasion of his eightieth birthday

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Abstract

We prove a priori estimates in $L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q)$, existence and uniqueness of solutions to Cauchy–Dirichlet problems for parabolic equations

$$\frac{\partial\sigma(u)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\Big\} + a\Big(t,x,u,\frac{\partial u}{\partial x}\Big) = 0,$$

$(t,x) \in Q = (0,T)\times\Omega$, where $\rho(u) = \frac{d}{du}\sigma(u)$. We consider solutions u such that $\rho^{\frac{1}{2}}(u)\left|\frac{\partial u}{\partial x}\right| \in L^2(0,T;L^2(\Omega))$, $\frac{\partial}{\partial t}\sigma(u) \in L^2(0,T;[\overset{\circ}{W}{}^{1,2}(\Omega)]^*)$. Our non-standard assumption is that $\log\rho(u)$ is concave. Such assumption is natural in view of drift diffusion processes for example in semiconductors and binary alloys, where u has to be interpreted as chemical potential and σ is a distribution function like $\sigma = e^u$ or $\sigma = \frac{1}{1+e^u}$.

1 Introduction

We prove a priori estimates, existence and uniqueness of weak solutions to initial boundary problems of the form

$$\frac{\partial}{\partial t}\sigma(u) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\Big\} + a\Big(t,x,u,\frac{\partial u}{\partial x}\Big) = 0, \quad (t,x)\in Q, \tag{1}$$

$$u(t,x) = f(t,x),\ (t,x) \in \Gamma = (0,T)\times\partial\Omega, \tag{2}$$

$$u(0,x) = g(x), \quad x \in \Omega, \tag{3}$$

where $\rho(u) = \frac{d}{du}\sigma(u)$, Ω is a bounded open set in $\mathbb{R}^n$ and $Q = (0,T)\times\Omega,\ T > 0$.

Cauchy–Dirichlet problems for degenerate parabolic equations have been studied extensively by many authors (see for example papers of H.W. Alt and S. Luckhaus [1], Ph. Benilan and P. Wittbold [3], F. Otto [14], H. Gajewski and K. Gröger [5]). But the structure of equation (1) is different from that one considered in papers [1], [3], [14].

Equations of the form (1) arise in mathematical models of various applied problems, for instance drift–diffusion processes in porous media, chemotaxis [8], semiconductors [5] and binary alloys [10, 9], where u and $\sigma(u)$ have to be interpreted as chemical potential and distribution function, respectively. Equation (1) may be also looked at as realization of the double nonlinear evolution equation

$$\frac{d}{dt}Eu + Au = 0, \tag{4}$$

where the operator E is gradient of a potential F with the convex conjugate functional F^*, which may have thermodynamical meaning as free energy [5, 9]. In [4] a uniqueness result for such abstract equations was obtained by showing that the map $u(0) \to u(t)$ is contractive with respect to the distance

$$d(u,v) = F^*(Eu) + F^*(Ev) - 2F^*(\frac{Eu+Ev}{2}),$$

provided the operator A is E-monotone in the sense that

$$\langle Au, u-w\rangle + \langle Av, v-w\rangle \geq 0, \quad w = E^{-1}[(Eu+Ev)/2], \quad \forall u, v.$$

In particular, if u^* is stationary solution of (4), $d(u, u^*)$ turns out to be Lyapunov functional of (4). As to the relation between (1) and (4), it was pointed out in [4] that E-monotony and hence uniqueness of solutions to (1) – (3) are consequences of following conditions: (i) $\log\rho(u)$ is concave, (ii) $a = a(x,u)$ is nonnegative and ρ is nonincreasing or $a(x,u)$ is nonpositive and ρ is nondecreasing.

The stationary variant of problem (1) – (3) was considered by the authors in [7].

We consider problem (1) – (3) under standard conditions for the functions $b_i(t,x,\xi)$ and some conditions for the function $a(t,x,u,\xi)$ to be formulated in Section 2. Our main specific assumption reads:

ρ) $\rho \in (\mathbb{R}^1 \to \mathbb{R}^1)$ with $\rho(u) > 0$, $u \in \mathbb{R}^1$, is continuous and has a piecewise continuous derivative ρ' such that $\frac{\rho'(u)}{\rho(u)}$ is nonincreasing on $\mathbb{R}^1$.

Remark 1 *Condition ρ) means that $\log\rho$ is concave. Examples for such functions are*

$$\rho(u) = e^u \quad (\sigma(u) = e^u), \qquad \rho(u) = \frac{e^u}{(1+e^u)^2} \quad (\sigma(u) = \frac{1}{1+e^u})$$

and Fermi integrals

$$\rho(u) = \mathcal{F}_\gamma(u) = \frac{1}{\Gamma(\gamma+1)}\int_0^\infty \frac{s^\gamma\, ds}{1+\exp(s-u)}, \quad \gamma > -1, \quad (\sigma = \mathcal{F}_{\gamma+1}).$$

Evidently, with $\rho(u)$ also $\rho(-u)$ satisfies condition ρ).

Remark 2 *On the first glance it seems to be convenient to introduce $v = \sigma(u)$ as a new function to get rid the nonlinearity under the time derivation in (1). But in fact this substitution destroys the structure of the problem resulting from the thermodynamically motivated assumption that the gradient $\frac{\partial u}{\partial x}$ of (the chemical potential)* *u is driving force for mass flux rather than the gradient of (the density) $v = \sigma(u)$. Moreover, this transformation would favour the functional*

$$F(v) = \frac{1}{2}\int_\Omega v^2 \, dx$$

as energy instead of the physically and geometrically [15] adequate expression

$$F^*(v) = \int_\Omega \int_1^v \sigma^{-1}(s) \, ds \, dx, \tag{5}$$

which in the special case (of Boltzmann statistics) $\sigma = \exp$ coincides with the familiar free energy

$$F^*(v) = \int_\Omega v(\log v - 1) \, dx = \int_\Omega e^u(u-1) \, dx. \tag{6}$$

In the case of no flux boundary conditions and vanishing coefficient a it is easy to see that (5) is Lyapunov functional for (1).

We consider problem (1) – (3) with boundary resp. initial functions f, g satisfying

$$f \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(Q), \ \frac{\partial f}{\partial t} \in L^2(Q), \tag{7}$$

$$g \in L^\infty(\Omega). \tag{8}$$

Definition 1 *A function $u \in L^2(0,T; W^{1,2}(\Omega))$ is called solution of problem (1) – (3), if following conditions are fulfilled:*

i) $\sigma(u) \in L^1_{loc}(Q)$,

$$\int_Q \rho(u)\left|\frac{\partial u}{\partial x}\right|^2 dx \, dt < \infty, \tag{9}$$

the derivative $\frac{\partial \sigma(u)}{\partial t}$ in the sense of distributions satisfies

$$\frac{\partial \sigma(u)}{\partial t} \in L^2(0,T,[\mathring{W}^{1,2}(\Omega)]^*) \tag{10}$$

and the integral identity

$$\begin{aligned} &\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \varphi > \, dt \\ &+ \int_0^\tau \int_\Omega \Big\{ \sum_{i=1}^n \rho(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} + a\Big(t,x,u,\frac{\partial u}{\partial x}\Big)\varphi \Big\} \, dx \, dt = 0 \end{aligned} \tag{11}$$

holds for arbitrary function $\varphi \in C^\infty(\overline{Q})$ vanishing near Γ and arbitrary $\tau \in (0,T)$;

ii) the boundary condition holds such that

$$u - f \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(\Omega)); \tag{12}$$

iii) for function φ, as in (11) and satisfying additionally $\varphi(\tau,x)=0$ for $x\in\Omega$, the equality

$$\int_0^\tau < \frac{\partial\sigma(u)}{\partial t}, \varphi > \, dt + \int_0^\tau \int_\Omega [\sigma(u)-\sigma(g)] \frac{\partial\varphi}{\partial t}\, dx\, dt = 0 \tag{13}$$

holds for $\tau \in (0,T)$.

In Section 2 we shall justify the integrals in (11), (13) by suitable conditions on the functions b_i, a. We formulate our assumptions and main results in Section 2. Some auxiliary lemmas are proved in Section 3. A priori estimates for solutions u in terms of $L^2(0,T;W^{1,2}(\Omega))$ and $L^\infty(Q)$ norms are given in Section 4. Using these estimates we establish in Section 5 the solvability of problem (1) – (3). Our main result, uniqueness of solutions, is proved in Section 6.

The key role in our paper play special test functions ((44), (66), (83), (93)) which allow us to analyze the behaviour of solutions u on subsets of Q where $\rho(u)$ could tend to zero. Remark also that for regular coefficients and smooth solutions uniqueness for problem (1) – (3) can be proved using maximum principle.

We are planing in forthcoming papers to apply our approach to problem (1) – (3) with unbounded boundary and initial functions and to systems of equations describing electro–reaction–diffusion and phase separation processes.

2 Formulation of assumptions and main results

Let Ω be a bounded open set in $\mathbb{R}^n$ and $Q=(0,T)\times\Omega,\ T>0$. We shall assume that $n>2$. For $n\le 2$ it is necessary to make simple changes in our conditions that are connected with Sobolev's embedding theorem. Let the coefficients b_i, a from (1) satisfy following assumptions:

i) *$a(t,x,u,\xi),\ b_i(t,x,\xi),\ i=1,\dots,n$, are measurable functions with respect to t,x for every $u\in\mathbb{R}^1,\ \xi\in\mathbb{R}^n$ and continuous with respect to $u\in\mathbb{R}^1,\ \xi\in\mathbb{R}^n$, for almost every $(t,x)\in Q$;*

ii) *there exist positive constants $\nu_1,\ \nu_2$ such that for arbitrary $(t,x,u,\xi)\in Q\times\mathbb{R}^1\times\mathbb{R}^n$ following inequalities hold*

ii)$_1$ $\sum_{i=1}^n b_i(t,x,\xi)\xi_i \ge \nu_1|\xi|^2$,

ii)$_2$ $|b_i(t,x,\xi)| \le \nu_2(|\xi|+1),\ i=1,\dots,n;$

iii) $a(t,x,u,\xi) = a_0(t,x,u,\xi) + a_1(t,x,u,\xi)$
and there exists a nonnegative function $\alpha \in L^p(Q), p > \frac{n+2}{2}$, *such that*

iii)$_1$ $\frac{a_0(t,x,u,\xi)}{\rho(u)}$ *is nondecreasing in* u *for arbitrary* $(t,x) \in Q,\ \xi \in \mathbb{R}^n$;

iii)$_2$ *for arbitrary* $(t,x,u,\xi) \in Q \times \mathbb{R}^1 \times \mathbb{R}^n$ *following inequalities hold*
$|a_0(t,x,u,\xi)| \le \nu_2\{[\rho(u)+1]|\xi| + |\sigma(u)| + \alpha(t,x)\}$,
$|a_1(t,x,u,\xi)| \le \nu_2 \cdot \rho(u)\{|\xi| + \gamma(u)\}$,
where $\gamma(u) = \min\{\frac{|\sigma(u)|}{\rho(u)}, |u|\}$.

We note some simple consequences from the condition ρ). Let

$$\alpha_\pm = \lim_{u \to \pm\infty} \rho(u). \tag{14}$$

Then for nonconstant function ρ at least one of the numbers $\alpha_-,\ \alpha_+$ is zero. If $\alpha_- = 0$, then

$$\rho(u) \le R_1 \exp(\lambda_1 u) \quad \text{for} \quad u \le 0 \tag{15}$$

holds with positive numbers R_1, λ_1. Analogously, if $\alpha_+ = 0$, then

$$\rho(u) \le R_2 \exp(-\lambda_2 u) \quad \text{for} \quad u \ge 0 \tag{16}$$

holds with positive numbers R_2, λ_2. Finally,

$$\rho(u) \le R_3 \exp(\lambda_3 |u|)$$

holds with positive numbers R_3, λ_3 for all $u \in \mathbb{R}^1$.

In order to justify Definition 1 we have to show firstly that the integral identity (11) is well defined. From (15), (16) we get

$$|\int_0^{\pm\infty} \rho^{\frac{1}{2}}(s)\, ds| \le R_4 \quad \text{if} \quad \alpha_\pm = 0.$$

Using this inequality, inequality (9) and the conditions (7), (12), we obtain

$$\int_0^u \rho^{\frac{1}{2}}(s)\, ds \in L^2(0,T; W^{1,2}(\Omega)). \tag{17}$$

From condition ρ) we infer

$$\rho^{1/2}(u) = \rho^{1/2}(0) + 2\int_0^u \frac{\rho'(s)}{\rho(s)} \rho^{1/2}(s)\, ds \le \rho^{1/2}(0) + 2\frac{\rho'(0)}{\rho(0)} \int_0^u \rho^{1/2}(s)\, ds$$

for arbitrary $u \in \mathbb{R}^1$. Hence Sobolev's embedding theorem and (17) imply

$$\rho^{\frac{1}{2}}(u) \in L^2\big(0,T; L^{\frac{2n}{n-2}}(\Omega)\big). \tag{18}$$

Now (18), (9) and the conditions ii), iii) show that the integral in (11) is well defined for $\varphi \in C^\infty([0,T]; C_0^\infty(\Omega))$.

Remark 3 *Since $C_0^\infty(\Omega)$ lies densely in $\overset{\circ}{W}{}^{1,2}(\Omega,\rho)$, the identity (11) holds for all $\varphi \in L^2(0,T;\overset{\circ}{W}{}^{1,2}(\Omega)) \cap L^\infty(\Omega)$ such that*

$$\int_Q \rho(u)\Big|\frac{\partial\varphi}{\partial x}\Big|^2 \, dx\; dt < \infty.$$

From condition $\rho)$ we have also the estimate

$$|\sigma(u)| \le c\Big\{\rho^{\frac{1}{2}}(u)\Big|\int_0^u \rho^{\frac{1}{2}}(s)\; ds\Big| + 1\Big\} \tag{19}$$

with some constant c independent on $u \in \mathbb{R}^1$. The inclusions (17), (18) and inequality (19) imply that $\sigma(u) \in L^1(Q)$ and consequently the integral in (13) is well defined for $\varphi \in C^\infty([0,T],\; C_0^\infty(\Omega))$.

Besides of (1) we consider the regularized equation

$$\frac{\partial\sigma_\delta(u)}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big[\rho_\delta(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\Big] + a\Big(t,x,u,\frac{\partial u}{\partial x}\Big) = 0 \tag{20}$$

with

$$\rho_\delta(u) = \delta + \rho(u),\;\; \sigma_\delta(u) = \delta u + \sigma(u),\;\; \delta \in [0,1]. \tag{21}$$

We understand solutions of the auxiliary problems (20), (2), (3) analogously to Definition 1 by replacing $\rho(u)$, $\sigma(u)$ in (9) – (11), (13) by $\rho_\delta(u)$, $\sigma_\delta(u)$. For such solutions we shall now prove a priori estimates in order to prepare the existence proof for solutions to problem (1) – (3).

In what follows we understand as known parameters all numbers from the conditions ii), iii), norms of functions f, g, α in respective spaces and numbers that depend only on n, Ω, ρ.

Theorem 1 *Let the conditions i) – iii), ρ), (7), (8) be satisfied. Then there exists a constant M_1 depending only on known parameters and independent of $\delta \in [0,1]$ such that each solution u of the problem (20), (2), (3) satisfies*

$$\|u\|^2_{V^2(Q)} := \operatorname*{ess\;sup}_{t\in(0,T)} \int_\Omega u^2(t,x)\; dx + \int_Q \Big|\frac{\partial u(t,x)}{\partial x}\Big|^2 dt\; dx \le M_1. \tag{22}$$

Theorem 2 *Let the conditions i) – iii), ρ), (7), (8) be satisfied. Then there exists a constant M_0 depending only on known parameters and independent of δ such that each solution u of the problem (20), (2), (3) satisfies*

$$\textit{ess sup}\{|u(t,x)| : (t,x) \in Q\} \le M_0. \tag{23}$$

For proving existence of a solution to problem (1) – (3) we need a monotonicity condition in addition to $\mathrm{ii})_1$. In view of our uniqueness result we assume even strong monotonicity, more than needed for existence only:

ii)* *condition ii) holds with*

ii)$_1^*$ $\sum_{i=1}^n \big[b_i(t,x,\xi) - b_i(t,x,\eta)\big](\xi_i - \eta_i) \geq \nu_1|\xi-\eta|^2$,
$b_i(t,x,0) = 0, \quad i = 1,\dots,n, \quad (t,x) \in Q, \ \xi, \ \eta \in \mathbb{R}^n$,
instead of ii) $_1$.

Theorem 3 *Let the conditions i), ii)*, iii), ρ), (7), (8) be satisfied. Then the initial-boundary value problem (1) - (3) has at least one solution $u \in L^2(0,T;W^{1,2}(\Omega))$.*

With respect to our uniqueness result we assume the functions a_0, a_1 to be locally Lipschitz continuous in the following sense:

iv) *there exist a positive nondecreasing function $\mu \in (\mathbb{R}^1 \to \mathbb{R}^1)$ and nonnegative functions $c_1 \in L^p(Q)$, $c_2 \in L^{2p}(Q)$, $p > \frac{n+2}{2}$, such that for arbitrary $N > 0$, $(t,x) \in Q, |u|, |\nu| \leq N, \xi, \eta \in \mathbb{R}^n$, one from the two following assumptions is satisfied*

iv)$_1$ *next inequalities hold for $i = 0, 1$:*

$$|a_i(t,x,u,\xi) - a_i(t,x,v,\xi)| \leq \mu(N)[c_1(t,x) + |\xi|^{\frac{2}{p}}]|u-v|, \tag{24}$$
$$|a_i(t,x,u,\xi) - a_i(t,x,u,\eta)| \leq \mu(N)c_2(t,x)|\xi-\eta|,$$
$$|a_1(t,x,u,\xi)| \leq \mu(N)[c_1(t,x) + |\xi|^{\frac{2}{p}}];$$

iv)$_2$ *all inequalities of condition iv)$_1$ are satisfied except of inequality (24) for $i = 0$ and additionally following inequality holds*

$$sign\ \rho'(u)a_0(t,x,u,\xi) \geq -\mu(N)[c_1(t,x) + |\xi|^{\frac{2}{p}}] \quad \text{if} \quad \rho'(u) \neq 0.$$

Theorem 4 *Let the conditions i), ii)*, iii), iv), ρ), (7), (8) be satisfied. Then the initial-boundary value problem (1) - (3) has a unique solution $u \in L^2(0,T;W^{1,2}(\Omega))$ in sense of Definition 1.*

Proofs of Theorem 1, 2 are given in Section 4. Proofs of Theorem 3, 4 are given in Sections 5, 6, respectively. We formulate below a counterpart to Theorem 4 for the equation

$$\frac{\partial \sigma(u)}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left\{\rho(u)b_i\left(t,x,\frac{\partial u}{\partial x}\right)\right\} + a(t,x,u) = 0. \tag{25}$$

Theorem 5 *Assume the conditions ρ), (7), (8). Let the functions $b_i(t,x,\xi)$ satisfy the conditions i) and ii)*, $i = 1,\dots,n$. Let the function $a(t,x,u)$ be measurable with respect to t,x, continuous with respect to u and let following conditions be satisfied for $(t,x) \in Q, \ u \in \mathbb{R}^1$:*

1. $|a(t,x,u)| \leq \nu(|\sigma(u)| + |u|^r) + \alpha(t,x)$ *with positive constant ν and nonnegative function $\alpha(t,x) \in L^p(Q)$, $p > \frac{n+2}{2}$, $0 \leq r < 1 + \frac{2}{n}$;*

2. $\frac{a(t,x,u)}{\rho(u)}$ *is nondecreasing with respect to* u.

Then the initial–boundary value problem (25), (2), (3) has a unique solution $u \in L^2(0,T,W^{1,2}(\Omega))$ *in the sense of Definition 1.*

Theorem 5 can be proved essentially in the same way as Theorem 4. We need only to make small changes in the proof of Lemma 4 accounting for another growth condition of the function a.

3 Auxiliary Lemmas

The key for proving the theorems formulated above is a proper choice of test functions φ in identity (11). In following lemmas we justify formulas for partial integration of scalar products between time derivatives in distribution sense and such test functions.

Let $\lambda \in (\mathbb{R}^1 \to \mathbb{R}^1)$ be a nondecreasing function with piecewise continuous derivative $\lambda' \in L^\infty(\mathbb{R}^1)$ and $\lambda(0) = 0$. We introduce the function

$$\Lambda_\delta(u) = \int_0^u \rho_\delta(s)\lambda(s)\,ds = \sigma_\delta(u)\lambda(u) - \int_0^u \sigma_\delta(s)\lambda'(s)\,ds, \tag{26}$$

where $\rho_\delta, \sigma_\delta$ are defined by (21). Remark that for arbitrary $u_1, u_2 \in \mathbb{R}^1$ the inequality

$$\begin{aligned} \Lambda_\delta(u_2) - \Lambda_\delta(u_1) &= \big[\sigma_\delta(u_2) - \sigma_\delta(u_1)\big]\lambda(u_1) \\ &+ \int_{u_1}^{u_2} \big[\sigma(u_2) - \sigma(s)\big]\lambda'(s)\,ds \geq \big[\sigma_\delta(u_2) - \sigma_\delta(u_1)\big]\lambda(u_1) \end{aligned} \tag{27}$$

holds.
Denote

$$m_0 = \max\{||f||_{L^\infty(Q)}, ||g||_{L^\infty(\Omega)}\} + 1. \tag{28}$$

Lemma 1 *Let the conditions* ρ), *(7), (8) be satisfied. Suppose that the function* λ *fulfils the conditions formulated above and vanishes on the interval* $[-m_0, m_0]$. *Then for arbitrary function* $u \in L^2(0,T,W^{1,2}(\Omega))$ *satisfying the conditions (10), (12), (13),* $\sigma(u) \in L^1(Q)$ *we have*

$$\Lambda_\delta(u(t,x)) \in L^\infty(0,T;L^1(\Omega)). \tag{29}$$

Moreover, for almost all τ

$$\int_0^\tau < \frac{\partial\sigma_\delta(u)}{\partial t}, \lambda(u) > dt = \int_\Omega \Lambda_\delta(u(\tau,x))dx. \tag{30}$$

Proof. Since the proof is independent on δ, we shall drop this subscript. Setting $u(t,x)=g(x)$ for $t<0$, we obtain from (27) for $t\in(0,T),\ h>0,\ x\in\Omega$

$$\Lambda(u(t,x))-\Lambda(u(t-h,x))\le[\sigma(u(t,x))-\sigma(u(t-h,x))]\lambda(u(t,x)), \tag{31}$$

$$\Lambda(u(t,x))-\Lambda(u(t-h,x))\ge[\sigma(u(t,x))-\sigma(u(t-h,x))]\lambda(u(t-h,x)). \tag{32}$$

From (10), (13) we get for $\varphi(x)\in C_0^\infty(\Omega)$ and almost all $\tau\in(0,T)$

$$\int_\Omega \sigma(u(\tau,x))\ \varphi(x)dx=\int_0^\tau <\frac{\partial\sigma(u)}{\partial t},\ \varphi>dt+\int_\Omega \sigma(g(x))\ \varphi(x)\ dx.$$

Hence $\sigma_t(u)\in\left[\overset{\circ}{W}{}^{1,2}(\Omega)\right]^*$ exists for almost all t such that

$$<\sigma_t(u),\ \varphi>=\int_\Omega \sigma(u(t,x))\ \varphi(x)\ dx.$$

Introduce for $N>m_0$ a function $\chi_N:Q\to\mathbb{R}^1$ such that $\chi_N(t,x)=1$ if $|\lambda(u(t,x))|\le N$ and $\chi_N(t,x)=\frac{N}{|\lambda(u(t,x))|}$ if $|\lambda(u(t,x))|>N$. Multiplying (31) by $\chi_N(t,x)$, (32) by $\chi_N(t-h,x)$ and integrating both inequalities over Ω, we obtain

$$\int_\Omega[\Lambda(u(t,x))-\Lambda(u(t-h,x))]\chi_N(t,x)\ dx\le<\sigma_t(u)-\sigma_{t-h}(u),\ \chi_{N,t}\ \lambda(u_t)>, \tag{33}$$

$$\begin{aligned}\int_\Omega[\Lambda(u(t,x))-\Lambda(u(t-h,x))]\chi_N(t-h,x)\ dx\\ \ge<\sigma_t(u)-\sigma_{t-h}(u),\ \chi_{N,t-h}\ \lambda(u_{t-h})>,\end{aligned} \tag{34}$$

where we used the notation $u_t(x)=u(t,x)$ and analogously for $\chi_{N,t}$. Letting $N\to\infty$ in (33) with $h>T$, we obtain that $\int_\Omega\Lambda(u(t,x))\ dx$ is finite for almost all t. Now we can pass to the limit as $N\to\infty$ in (33), (34). Integrating (33) resp. (34) with respect to t from 0 to $\tau\in(h,T)$ resp. from h to τ, we get

$$\frac{1}{h}\int_{\tau-h}^\tau\int_\Omega\Lambda(u(t,x))\ dx\ dt\le\int_0^\tau<\frac{\sigma_t(u)-\sigma_{t-h}(u)}{h},\ \lambda(u_t)>dt, \tag{35}$$

$$\begin{aligned}\frac{1}{h}\int_{\tau-h}^\tau\int_\Omega\Lambda(u(t,x))\ dx\ dt\ge\int_0^{\tau-h}<\frac{\sigma_{t+h}(u)-\sigma_t(u)}{h},\ \lambda(u_t)>dt\\ +\frac{1}{h}\int_0^h\int_\Omega\Lambda(u(t,x))\ dx\ dt.\end{aligned} \tag{36}$$

Taking the limit as $h\to0$ in (35), we obtain for almost all τ

$$\int_\Omega\Lambda(u(\tau,x))\ dx\le\int_0^\tau<\frac{\partial\sigma(u)}{\partial t},\ \lambda(u)>dt. \tag{37}$$

Remarking that $\Lambda(u)\ge0$ for all $u\in R$, we can drop the second summand of the right hand side of (36) and let $h\to0$. Thus we see that for almost all τ

$$\int_\Omega\Lambda(u(\tau,x))\ dx\ge\int_0^\tau<\frac{\partial\sigma(u)}{\partial t},\ \lambda(u)>dt. \tag{38}$$

The inequalities (37), (38) imply inclusion (29) and formula (30). $\square$

In following lemma we consider two different values of δ and we denote $\sigma_i = \sigma_{\delta_i}, \ i = 1,2$. Let $F \in (\mathbb{R}^2 \to \mathbb{R}^1)$ be a continuously differentiable function with piecewise continuous derivatives of second order. Denote

$$F_i(z_1, z_2) = \frac{\partial F(z_1, z_2)}{\partial z_i}, \ i = 1, 2.$$

Lemma 2 *Let $u_i \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(Q), \ i = 1,2,$ be functions satisfying (12) and (13) with $\sigma(u) = \sigma_i(u_i)$, $\frac{\partial \sigma_i(u_i)}{\partial t} \in L^2(0,t; [\overset{\circ}{W}{}^{1,2}(\Omega)]^*)$. Suppose that condition ρ), (7) and (8) hold. Then for almost all $\tau \in (0,T)$*

$$\begin{aligned}
&\sum_{i=1}^{2} \int_0^\tau < \frac{\partial \sigma_i(u_i)}{\partial t}, F_i(\sigma_1(u_1), \sigma_2(u_2)) - F_i(\sigma_1(f), \sigma_2(f)) > dt \\
&= \int_\Omega \big[F(\sigma_1(u_1(\tau,x)), \sigma_2(u_2(\tau,x))) - F(\sigma_1(g), \sigma_2(g))\big] \, dx \\
&+ \sum_{i=1}^{2} \int_0^\tau \int_\Omega \big[\sigma_i(u_i) - \sigma_i(g)\big] \frac{\partial}{\partial t} F_i(\sigma_1(f), \sigma_2(f)) \, dx \, dt \\
&- \sum_{i=1}^{2} \int_\Omega \big[\sigma_i(u_i(\tau,x)) - \sigma_i(g(x))\big] F_i(\sigma_1(f(\tau,x)), \sigma_2(f(\tau,x))) \, dx.
\end{aligned} \tag{39}$$

Proof. Evidently (39) holds for $\sigma_i(u_i) \in C^1(0,T; W^{1,2}(\Omega))$. Our assumptions imply $\sigma_i(u_i) \in L^2(0,T; W^{1,2}(\Omega))$. Hence, by Lemma 1.12, Chapter 4 of monograph [4], we can choose sequences $v_j^{(i)}(t,x) \in C^1(0,T, W^{1,2}(\Omega))$ converging to $\sigma_i(u_i)$ in $L^2(0,T; W^{1,2}(\Omega))$ as $j \to \infty$ such that $\frac{\partial v_j^{(i)}}{\partial t} \to \sigma_i(u_i)$ in $L^2(0,T, [\overset{\circ}{W}{}^{1,2}(\Omega)]^*)$. The sequences $\{v_j^{(i)}\}$ can be defined by Steklov averaging of $\sigma_i(u_i) \in L^\infty(Q)$ and can be consequently supposed to be bounded in $L^\infty(Q)$. Thus, replacing $\sigma_i(u_i)$ by $v_j^{(i)}$ in (39) and afterwards taking the limit $j \to \infty$, we arrive at the assertion. $\square$

Next we formulate also particular cases of Lemma 2 which will be used in next Sections.

Corollary 1 *Assume conditions ρ), (7), (8). Let $\lambda \in (\mathbb{R}^1 \to \mathbb{R}^1)$ be a function with piecewise continuous derivative λ' such that $\lambda(0) = 0$. Then for arbitrary functions $u \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(Q)$ satisfying conditions (10), (12), (13),*

$$\begin{aligned}
&\int_0^\tau < \frac{\partial \sigma_\delta(u)}{\partial t}, \lambda(u) - \lambda(f) > \ dt = \int_\Omega \big\{\Lambda_\delta(u(\tau,x)) - \Lambda_\delta(g(x))\big\} \, dx \\
&+ \int_0^\tau \int_\Omega \big[\sigma_\delta(u) - \sigma_\delta(g)\big] \frac{\partial \lambda(f)}{\partial t} \, dx \, dt - \int_\Omega \big[\sigma_\delta(u(\tau,x)) - \sigma_\delta(g(x))\big] \lambda(f(\tau,x)) \, dx,
\end{aligned} \tag{40}$$

where the function $\Lambda_\delta(u)$ is defined by equality (26).

Proof. Equality (40) follows from (39) with $F(z_1, z_2) = \Lambda_\delta(\sigma_\delta^{-1}(z_1))$, $u_1 = u$, $\sigma_1 = \sigma_\delta$, $u_2 = 1$, $F_1(z_1, z_2) = \frac{\partial \Lambda_\delta(\sigma_\delta^{-1}(z_1))}{\partial z_1} = \lambda(\sigma_\delta^{-1}(z_1))$. □

Corollary 2 *Suppose that the functions u_i, σ_i, $i = 1, 2$, satisfy the conditions of Lemma 2. Let $\lambda_i \in (\mathbb{R}^1 \to \mathbb{R}^1)$, $i = 1, 2$, be functions with piecewise continuous derivatives λ_i' and $\lambda_i(0) = 0$. Then for almost all τ*

$$\begin{aligned}
\int_0^\tau \Big\{ &< \frac{\partial \sigma_1(u_1)}{\partial t}, \lambda_1(u_1)\Lambda_2(u_2) - \lambda_1(f)\Lambda_2(f) > \\
&+ < \frac{\partial \sigma_2(u_2)}{\partial t}, \lambda_2(u_2)\Lambda_1(u_2) - \lambda_2(f)\Lambda_1(f) > \Big\} dt \\
&= \int_\Omega \Big\{ \Lambda_1(u_1(\tau, x))\Lambda_2(u_2(\tau, x)) - \Lambda_1(g)\Lambda_2(g) \Big\}\, dx \\
&+ \int_0^\tau \int_\Omega \Big\{ \big[\sigma_1(u_1) - \sigma_1(g)\big] \frac{\partial}{\partial t}\{\lambda_1(f)\Lambda_2(f)\} \\
&+ \big[\sigma_2(u_2) - \sigma_2(g)\big] \frac{\partial}{\partial t}\{\lambda_2(f)\Lambda_1(f)\} \Big\}\, dx\, dt \\
&- \int_\Omega \Big\{ \big[\sigma_1(u_1(\tau, x)) - \sigma_1(g(x))\big] \lambda_1(f(\tau, x))\Lambda_2(f(\tau, x)) \\
&+ \big[\sigma_2(u_2(\tau, x)) - \sigma_2(g(x))\big] \lambda_2(f(\tau, x))\Lambda_1(f(\tau, x)) \Big\}\, dx,
\end{aligned} \tag{41}$$

where

$$\Lambda_i(u) = \int_0^u \sigma_i'(s)\lambda_i(s)\, ds.$$

Proof. Equality (41) follows from (39) with $F(z_1, z_2) = \Lambda_1(\sigma_1^{-1}(z_1))\Lambda_2(\sigma_2^{-1}(z_2))$,

$$F_1(z_1, z_2) = \frac{\partial \Lambda_1(\sigma_1^{-1}(z_1))}{\partial z_1} \cdot \Lambda_2(\sigma_2^{-1}(z_2)) = \lambda_1(\sigma_1^{-1}(z_1))\Lambda_2(\sigma_2^{-1}(z_2)),$$

$$F_2(z_1, z_2) = \Lambda_1(\sigma_1^{-1}(z_1))\lambda_2(\sigma_2^{-1}(z_2)).$$

□

4 A priori estimates of solutions

Denote

$$Q_\pm(m) = \{(t, x) \in Q : \pm[u(t, x) - m] > 0\},$$

where $m \geq m_0$ and the number m_0 is defined by (28). We shall estimate only the norm of $\left|\frac{\partial u}{\partial x}\right|$ in $L^2(Q_+(m))$. A corresponding estimate in $L^2(Q_-(m))$ can be proved analogously.
By condition $\rho)$ a positive number m_1 exists such that

$$\pm\, \rho'(u) < 0 \quad \text{for} \quad \pm\, u > m_1 \quad \text{if} \quad \alpha_\pm = 0. \tag{42}$$

Let us consider firstly the case $\alpha_+ = 0$.
We introduce following notations

$$v_k(t,x) = [v(t,x)]_k = \min\{v(t,x), k\},\ k \in \mathbb{R}^1,$$
$$v_+(t,x) = [v(t,x)]_+ = \max\{v(t,x), 0\}$$

for arbitrary functions v defined on Q and we shall denote by M_i, C_i constants depending only on the same parameters as constants M_0, M_1 in Theorems 1, 2.

Lemma 3 *Suppose that $\alpha_+ = 0$. Let the conditions of Theorem 1 be satisfied. Then for $m \geq m_0 + m_1$*

$$\underset{t \in (0,T)}{ess\ \sup} \int_\Omega \left[u(t,x) - m\right]_+^2 dx + \int_{Q_+(m)} \left|\frac{\partial u(t,x)}{\partial x}\right|^2 dx\ dt \leq M_2. \tag{43}$$

Proof. In view of Remark 3 we can put the test function

$$\varphi = \frac{1}{\rho_\delta(u_k)}[u_k - m]_+ \quad \text{with} \quad k > m \geq m_0 + m_1 \tag{44}$$

into the integral identity

$$\begin{gathered} \int_0^\tau < \frac{\partial \sigma_\delta(u)}{\partial t}, \varphi > dt + \int_0^\tau \int_\Omega \Big\{ \sum_{i=1}^n \rho_\delta(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big) \frac{\partial \varphi}{\partial x_i} \\ + a\Big(t,x,u,\frac{\partial u}{\partial x}\Big)\varphi \Big\}\, dx\ dt = 0,\ \tau \in (0,T). \end{gathered} \tag{45}$$

Here the first term can be rewritten using Lemma 1 as

$$\int_0^\tau < \frac{\partial \sigma_\delta(u)}{\partial t}, \frac{1}{\rho_\delta(u_k)}[u_k - m]_+ > dt = \int_\Omega \Lambda^{(1)}_{\delta,k}(u(\tau,x))\, dx, \tag{46}$$

where

$$\Lambda^{(1)}_{\delta,k}(u) = \begin{cases} 0 & \text{for} \quad u \leq m, \\ \frac{(u-m)^2}{2} & \text{for} \quad m < u < k, \\ \frac{(k-m)^2}{2} + \frac{(k-m)}{\rho_\delta(k)} \int_k^u \rho_\delta(s)\, ds & \text{for} \quad u \geq k. \end{cases} \tag{47}$$

Using (42), we can estimate the second term in (45),

$$\int_0^\tau \int_\Omega \sum_{i=1}^n \rho_\delta(u) b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} dxdt \geq \int_0^\tau \int_{\{m<u_t<k\}} \sum_{i=1}^n b_i\Big(t,x,\frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x_i} dxdt, \tag{48}$$

where $\{m < u_t < k\} = \{x \in \Omega : m < u(t,x) < k\}$. Further, condition iii) implies

$$\begin{aligned} a\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\varphi &= \frac{a_0(x,t,u,\frac{\partial u}{\partial x})}{\rho_\delta(u)} \cdot \frac{\rho_\delta(u)}{\rho_\delta(u_k)}[u_k - m]_+ \; + a_1\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\varphi \\ &\geq \Big[\frac{a_0(x,t,0,\frac{\partial u}{\partial x})}{\rho_\delta(0)} + \frac{a_1(x,t,u,\frac{\partial u}{\partial x})}{\rho_\delta(u)}\Big]\frac{\rho_\delta(u)}{\rho_\delta(u_k)}[u_k - m]_+ \\ &\geq -c_1\Big\{\Big|\frac{\partial u}{\partial x}\Big| + |u| + 1 + \alpha(t,x)\Big\}[u-m]_+. \end{aligned} \tag{49}$$

Because of ii)$_1$ we obtain by (45) – (49)

$$\begin{aligned} &\int_{\{m<u_\tau<k\}} [u(\tau,x) - m]^2 \, dx + \int_0^\tau \int_{\{m<u_t<k\}} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt \\ &\leq c_2 \int_0^\tau \int_\Omega \Big\{\Big|\frac{\partial u}{\partial x}\Big| + |u| + \alpha + 1\Big\}[u-m]_+ \, dx\, dt. \end{aligned} \tag{50}$$

Passing to the limit as $k \to \infty$ and applying the monotone convergence theorem, we obtain from (50)

$$\begin{aligned} &\int_\Omega [u(\tau,x) - m]_+^2 \, dx + \int_0^\tau \int_{\{u_t>m\}} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt \\ &\leq c_2 \int_0^\tau \int_\Omega \Big\{\Big|\frac{\partial u}{\partial x}\Big| + |u| + \alpha + 1\Big\}[u-m]_+ \, dx\, dt\ , \end{aligned}$$

an by Cauchy's inequality

$$\int_\Omega [u(\tau,x) - m]_+^2 \, dx + \int_0^\tau \int_{\{u_t>m\}} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt \leq c_3\Big\{1 + \int_0^\tau \int_\Omega [u(t,x) - m]_+^2 \, dx\, dt\Big\}.$$

Hence the asserted inequality (43) follows from Gronwall's lemma. □

Now we turn to the proof of an analogous result for the alternative case $\alpha_+ > 0$.

Lemma 4 *Suppose that $\alpha_+ > 0$ for $u \in \mathbb{R}^1$. Let the conditions of Theorem 1 be satisfied. Then for arbitrary numbers $r \geq 0$ there exist constants $M(r)$ depending only on known parameters and r such that each solution of the problem (20), (2), (3) satisfies*

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\{u_t>m_0\}} \sigma_\delta^{1+2r}(u(t,x))\, dx + \int_{Q_+(m_0)} \Big|\frac{\partial}{\partial x}\sigma_\delta^{\frac{1}{2}+r}(u)\Big|^2 dx\, dt \leq M(r), \tag{51}$$

with m_0 defined by (28).

Proof. Condition $\rho)$ together with $\alpha_+ > 0$ imply $\rho'(u) \geq 0$ for $u \in \mathbb{R}^1$. By Remark 3 we can put the test function

$$\varphi = \left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right]_+, \; k > m_0, \; r > 0, \tag{52}$$

into the integral identity (45). Evaluating the first term of (45) with φ specified by (52), we get by Lemma 1

$$\int_0^\tau < \frac{\partial \sigma_\delta(u)}{\partial t}, \left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right]_+ > dt = \int_\Omega \Lambda_{\delta,k}^{(2)}(u(\tau,x))\, dx, \tag{53}$$

where

$$\Lambda_{\delta,k}^{(2)}(u) = \begin{cases} 0 & \text{if } u \leq m_0, \\ \frac{\left[\sigma_\delta^{2r+1}(u) - \sigma_\delta^{2r+1}(m_0)\right]}{2r+1} - \sigma_\delta^{2r}(m_0)\left[\sigma_\delta(u) - \sigma_\delta(m_0)\right] & \text{if } m_0 < u \leq k, \\ \Lambda_{\delta,k}^{(2)}(k) + \left[\sigma_\delta^{2r}(k) - \sigma_\delta^{2r}(m_0)\right]\left[\sigma_\delta(u) - \sigma_\delta(k)\right] & \text{if } u > k. \end{cases} \tag{54}$$

Using ii)$_1$, we estimate the second term in (45) with φ defined by (52):

$$\begin{aligned} &\int_0^\tau \int_\Omega \sum_{i=1}^n \rho_\delta(u) b_i\Big(t, x, \frac{\partial u}{\partial x}\Big) \frac{\partial}{\partial x_i}\left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right]_+ dx\, dt \\ &\geq \frac{c_4 r}{(r+1)^2} \int_0^\tau \int_{\{m_0 < u_t < k\}} \left|\frac{\partial}{\partial x}\sigma_\delta^{r+\frac{1}{2}}(u)\right|^2 dx dt. \end{aligned} \tag{55}$$

The next estimate follows from condition iii):

$$\begin{aligned} &a\Big(t, x, u, \frac{\partial u}{\partial x}\Big)\left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right]_+ \\ &\geq -c_5\Big\{\rho(u)\Big|\frac{\partial u}{\partial x}\Big| + \sigma(u) + \alpha\Big\}\left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right]_+. \end{aligned} \tag{56}$$

We obtain from (45) with φ defined by (52) and from (53) – (56)

$$\begin{aligned} &\int_{\{m_0 < u_\tau < k\}} \sigma_\delta^{2r+1}(u(\tau,x))\, dx + \frac{r}{r+1}\int_0^\tau \int_{\{m_0 < u_t < k\}} \left|\frac{\partial}{\partial x}\sigma_\delta^{r+\frac{1}{2}}(u)\right|^2 dx\, dt \\ &\leq c_6(r+1)\Big\{\sigma_\delta^{2r+1}(m_0) \text{ meas } Q \\ &+ \int_0^\tau \int_{\{u_t > m_0\}} \Big[\rho(u)\Big|\frac{\partial u}{\partial x}\Big| + \sigma(u) + \alpha\Big]\left[\sigma_\delta^{2r}(u_k) - \sigma_\delta^{2r}(m_0)\right] dx\, dt\Big\}. \end{aligned} \tag{57}$$

We shall prove firstly that the right hand side of (57) is uniformly bounded with respect to $k \in [m_0, \infty)$ if r is small enough. In order to check this we start proving that

$$\sigma_\delta(u) \in L^{1+\frac{2}{n}}(Q_+(m_0)). \tag{58}$$

Indeed, from Lemma 1 with $\lambda(u) = \Big[ln\frac{\sigma_\delta(u)}{\sigma_\delta(m_0)}\Big]_+$ we have

$$\int_\Omega \Lambda_\delta(u(\tau,x))\, dx = \int_0^\tau < \frac{\partial \sigma_\delta(u)}{\partial t}, \Big[ln\frac{\sigma_\delta(u)}{\sigma_\delta(m_0)}\Big]_+ > \, dt, \tag{59}$$

where

$$\Lambda_\delta(u) \geq c_7\big[\sigma_\delta(u) - \sigma_\delta(m_0+1)\big] \quad \text{for} \quad u \geq m_0+1. \tag{60}$$

Note that by condition $\rho)$ for $u > 0$

$$\int_u^{u+1} \frac{\rho'(s)}{\rho(s)}\, ds \leq \int_u^{u+1} \frac{\rho'(0)}{\rho(0)}\, ds = \frac{\rho'(0)}{\rho(0)}.$$

Consequently, we obtain

$$\rho(u+1) \leq \rho(u) \exp\Big\{\frac{\rho'(0)}{\rho(0)}\Big\}.$$

This inequality implies for $u > 1$

$$\frac{\rho_\delta(u)}{\sigma_\delta(u)} \leq \frac{\delta + \rho(u)}{\delta + \int_{u-1}^u \rho(s) ds} \leq \frac{\delta+\rho(u)}{\delta + \rho(u-1)} \leq \max\Big\{1, \exp\frac{\rho'(0)}{\rho(0)}\Big\}. \tag{61}$$

Using the last inequality and the assumption $u \in L^2(0,T;W^{1,2}(\Omega))$, we get

$$\Big[ln\frac{\sigma_\delta(u)}{\sigma_\delta(m_0)}\Big]_+ \in L^2(0,T;W^{1,2}(\Omega)). \tag{62}$$

From (59), (60), (62) and (10) we obtain

$$\sigma_\delta(u) \in L^\infty(0,T;L^1(\Omega)). \tag{63}$$

By Hölder's inequality, we have with $q = \frac{n}{n-1}$

$$\begin{aligned} &\int_0^T \Big(\int_\Omega \Big|\frac{\partial \sigma_\delta(u)}{\partial x}\Big|^q \, dx\Big)^{\frac{1}{q}} \, dt \\ &\leq \Big\{\int_0^T \Big(\int_\Omega \big[\rho_\delta(u)\big]^{\frac{n}{n-2}} \, dx\Big)^{\frac{n-2}{n}} \, dt\Big\}^{\frac12} \cdot \Big\{\int_Q \rho_\delta(u)\Big|\frac{\partial u}{\partial x}\Big|^2 \, dx\, dt\Big\}^{\frac12}, \end{aligned} \tag{64}$$

where the right hand side is finite because of the inequalities (9), (18) for solutions to equation (20). Using (64), (63) and Sobolev's embedding theorem, we obtain

$$\begin{aligned} &\int_Q \big[\sigma_\delta(u) - \sigma(m_0)\big]_+^{1+\frac{2}{n}} \, dx\, dt \\ &\leq \int_0^T \Big\{\int_\Omega \sigma_\delta(u)\, dx\Big\}^{\frac{2}{n}} \cdot \Big\{\int_\Omega \big[\sigma_\delta(u) - \sigma(m_0)\big]_+^{\frac{n}{n-2}} \, dx\Big\}^{\frac{n-2}{n}} \, dt \\ &\leq c_8 \Big\{ \operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \sigma_\delta(u(t,x))\, dx\Big\}^{\frac{2}{n}} \cdot \int_0^T \Big\{\int_\Omega \Big|\frac{\partial \sigma_\delta(u)}{\partial x}\Big|^q \, dx\Big\}^{\frac{1}{q}} \, dt\ , \end{aligned}$$

that completes the proof of the desired inclusion (58).
From (58), (61) and assumption (9) we can estimate the right hand side of (57) by a constant independent of k for $r = r_1 = \frac{1}{2n}$. Passing to the limit as $k \to \infty$ in (57) with $r = r_1$ and using the monotone convergence theorem, we get

$$\underset{t \in (0,T)}{ess\ \sup} \int_\Omega \left[\sigma_\delta^{r_1+\frac{1}{2}}(u(t,x)) - \sigma_\delta^{r_1+\frac{1}{2}}(m_0)\right]_+^2 dx + \int_{Q_+(m_0)} \left|\frac{\partial}{\partial x}\sigma_\delta^{r_1+\frac{1}{2}}(u)\right|^2 dxdt \le c_9.$$

Hence the embedding $V^2(Q) \to L^{\frac{2(n+2)}{n}}(Q)$ (comp. (22) and [11]) implies

$$\sigma_\delta(u) \in L^{(1+\frac{1}{n})\frac{n+2}{n}}(Q_+(m_0)).$$

Now we can continue the previous discussion choosing $r = r_2 = r_1 \cdot \frac{n+1}{n} + \frac{1}{2n}$. Iterating the described process, we complete the proof of Lemma 4. □

Corollary 3 *If the conditions of Lemma 4 hold, then*

$$\underset{t \in (0,T)}{ess\ \sup} \int_\Omega \left[u(t,x) - m_0\right]_+^2 dx + \int_{Q_+(m_0)} \left|\frac{\partial u(t,x)}{\partial x}\right|^2 dx\, dt \le M_3.$$

Proof. Under the conditions of Lemma 4 we have $\sigma(u) \ge \rho(0)u$. Thus we get the assertion from (51) with $r = \frac{1}{2}$. □

Further we shall estimate $||u||_{L^\infty(Q_\pm(m))}$ separately for the sets $Q_+(m)$ and $Q_-(m)$. As in the previous lemmas, we can restrict us to $Q_+(m)$ in the two cases $\alpha_+ = 0$, $\alpha_+ > 0$. □

Lemma 5 *Suppose that $\alpha_+ = 0$. Let the conditions of Theorem 2 be satisfied. Then there exists a constant M_4 such that each solution of problem (20), (2), (3) satisfies*

$$ess \sup \left\{|u(t,x)| : (t,x) \in Q_+(m)\right\} \le M_4 \quad \textit{with} \quad m = m_0 + m_1. \tag{65}$$

Proof. We substitute in identity (45) the test function

$$\varphi = \frac{1}{\rho_\delta(u_k)}\left[u_k - m\right]_+^{r+1}, \ r \ge 0, \qquad k > m = m_0 + m_1. \tag{66}$$

Analogously to the proof of inequality (50) we obtain

$$\frac{1}{r+2}\int_{\{m<u_\tau<k\}} \left[u(\tau,x) - m\right]^{r+2} dx + (r+1)\int_0^\tau \int_{\{m<u_t<k\}} \left[u - m\right]^r \left|\frac{\partial u}{\partial x}\right|^2 dxdt$$
$$\le c_9 \int_0^\tau \int_\Omega \left\{\left|\frac{\partial u}{\partial x}\right| + |u| + \alpha + 1\right\}\left[u_k - m\right]_+^{r+1} dx\, dt. \tag{67}$$

From Corollary 3 and the embedding $V^2(Q) \to L^{\frac{2(n+2)}{n}}(Q)$ we have

$$[u(t,x)-m]_+ \in L^{\frac{2(n+2)}{n}}(Q).$$

Repeating the discussion from the end of the proof of Lemma 4, we verify that the right hand side of inequality (67) can be estimated by a constant independent of k for arbitrary r. Thus we can take the limit $k \to \infty$ to get

$$\begin{aligned}&\frac{1}{r+2}\int_\Omega \big[u(\tau,x)-m\big]_+^{r+2}\,dx + (r+1)\int_{Q_+(m)} \big[u-m\big]^r \Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt \\ &\le c_{10}\int_0^\tau\int_\Omega \big[u-m\big]_+^{r+1}([u-m]_+ + \alpha + 1)\,dx\,dt\,.\end{aligned} \tag{68}$$

From (68) we obtain estimate (65) by Moser iteration [13, 2]. □

Lemma 6 *Assume $\rho'(u) \ge 0$ for $u \in \mathbb{R}^1$ and that the conditions of Theorem 2 are satisfied. Then each solution of problem (20), (2), (3) satisfies*

$$\operatorname{ess\,sup}\big\{|u(t,x)| : (t,x) \in Q_+(m_0)\big\} \le M_5. \tag{69}$$

Proof. We specify the test function in the integral identity(45) by

$$\varphi = \big[\sigma_\delta(u_k) - \sigma_\delta(m_0)\big]_+^{2r},\ k > m_0,\ r > \frac{1}{2}.$$

Analogously to the proof of inequality (57) we obtain

$$\begin{aligned}&\int_{\{m_0<u_\tau<k\}} \big[\sigma_\delta(u(\tau,x)) - \sigma_\delta(m_0)\big]^{2r+1}\,dx \\ &+ \int_0^\tau\int_{\{m_0<u_t<k\}} \Big|\frac{\partial}{\partial x}\big[\sigma_\delta(u)-\sigma_\delta(m_0)\big]^{r+\frac12}\Big|^2\,dx\,dt \\ &\le c_{11} r\int_0^\tau\int_{\{u_t>m_0\}} \Big[\rho(u)\Big|\frac{\partial u}{\partial x}\Big| + \sigma(u) + \alpha\Big]\big[\sigma_\delta(u_k)-\sigma_\delta(m_0)\big]^{2r}\,dx\,dt.\end{aligned} \tag{70}$$

Using Lemma 4, we can pass to the limit in (70) as $k \to \infty$ to get

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega \big[\sigma_\delta(u(\tau,x)) - \sigma_\delta(m_0)\big]_+^{2r+1}dx + \int_Q \Big|\frac{\partial}{\partial x}\big[\sigma_\delta(u)-\sigma_\delta(m_0)\big]_+^{r+\frac12}\Big|^2 dxdt \\ &\le c_{12} r\int_Q \big[\sigma_\delta(u)-\sigma_\delta(m_0)\big]_+^{2r}\big\{\big[\sigma_\delta(u)-\sigma_\delta(m_0)\big]_+ + \alpha + 1\big\}\,dx\,dt\end{aligned}$$

for arbitrary $r > \frac{1}{2}$. Now Moser iteration leads us to the boundedness of the function $\sigma_\delta(u)$ on $Q_+(m_0)$ and consequently to the boundedness of u on this set. Lemma 6 is proved. □

Proof of Theorem 2. Estimate (23) follows immediately from (65), (69) and analogous estimates on $Q_-(m)$ that can be proved by repeating the arguments of the proofs of the Lemmas 5 and 6. $\square$

Proof of Theorem 1. Taking into account the proved boundedness of solutions to problem (20), (2), (3), we need only an estimate of the second integral in (22). For this purpose we choose in (45) the test function

$$\varphi = u - f. \tag{71}$$

Using conditions (7), ii)$_1$ and iii) and the estimate (23), we obtain immediately

$$\begin{aligned} &\int_Q \Big\{ \sum_{i=1}^n \rho_\delta(u) b_i\Big(t, x, \frac{\partial u}{\partial x}\Big) \frac{\partial (u-f)}{\partial x_i} \\ &+ a\Big(t, x, u, \frac{\partial u}{\partial x}\Big)(u-f)\Big\} \, dx \, dt \ge c_{13} \int_Q \int \Big|\frac{\partial u}{\partial x}\Big|^2 \, dx \, dt - c_{14}. \end{aligned} \tag{72}$$

For estimating the first term in (45) with φ defined by (71) we apply Corollary 1 to get

$$\begin{aligned} \int_0^T < \frac{\partial \sigma_\delta(u)}{\partial t}, u - f > dt = \int_\Omega \{\Lambda_\delta(u, (T, x)) - \Lambda_\delta(g(x))\} \, dx \\ + \int_Q [\sigma_\delta(u) - \sigma_\delta(g)] \frac{\partial f}{\partial t} \, dx \, dt \; - \int_\Omega [\sigma_\delta(u(T,x)) - \sigma_\delta(g(x))] f(T,x) \, dx, \end{aligned}$$

where

$$\Lambda_\delta(u) = \int_0^u \rho_\delta(s) s \, ds \ge 0.$$

By the last inequality, the assumptions (7), (8) and estimate (23), we obtain

$$\int_0^T < \frac{\partial \sigma_\delta(u)}{\partial t}, u - f > dt \ge -c_{15}. \tag{73}$$

Now (22) follows from the integral identity (45) with $\varphi = u - f$ and the inequalities (72), (73). $\square$

5 Proof of Theorem 3

We know that at least one of the numbers α_-, α_+ defined by (14) is zero. Thus we can modify the functions ρ and a in the following way: If $\alpha_+ = \infty$, we define

$$\rho^*(u) = \rho(\min[u, M_0]), \;\; a^*(t, x, u, \xi) = a(t, x, \min[u, M_0], \xi), \tag{74}$$

where M_0 is the constant from Theorem 2. If $\alpha_- = \infty$, we define

$$\rho^*(u) = \rho(\max[u, -M_0]), \;\; a^*(t, x, u, \xi) = a(t, x, \max[u, -M_0], \xi). \tag{75}$$

These new functions ρ^*, a^* satisfy the conditions ρ), i), iii) with the same parameters as ρ, a and with $\sigma^*(u) = \int_0^u \rho^*(s)\, ds$.
Now we consider for $\delta \in [0,1]$ the initial boundary value problem for the equation

$$\frac{\partial \sigma_\delta^*(u)}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big[\rho_\delta^*(u) b_i\Big(t, x, \frac{\partial u}{\partial x}\Big)\Big] + a^*\Big(t, x, u, \frac{\partial u}{\partial x}\Big) = 0,\ (t,x) \in Q \tag{76}$$

with the conditions (2), (3). By Theorems 1, 2 arbitrary solutions u of the problem (76), (2), (3) satisfy the a priori estimates

$$ess\ \sup\{|u(t,x)| : (t,x) \in Q\} \le M_0, \quad \int_Q \left|\frac{\partial u(t,x)}{\partial x}\right|^2 dx\ dt \le M_1 \tag{77}$$

with the constants M_0, M_1 from the inequalities (23), (22). From (74), (75) and the equality $\rho_\delta^*(u) = \delta + \rho^*(u)$ we see that solutions of problem (76), (2), (3) with $\delta = 0$ are automatically solutions of problem (1) – (3). We shall consider firstly the solvability of problem (76), (2), (3) with $\delta > 0$ and let than $\delta \to 0$.

We don't want to go into details of proving solvability of problem (76), (2), (3) with $\delta > 0$. That could be done via Euler's backward time discretization. Such approach was used in [1], [5]. We remark only that solvability of the arising elliptic problems can be proved by using degree theory for operators of class (S_+) [16].

Let $u_\delta(x,t)$, $\delta > 0$, be a solution of problem (76), (2), (3) and consider the limit $\delta \to 0$. From the integral identity for u_δ and the inequalities (77) we obtain immediately

$$\left\|\frac{\partial \sigma_\delta(u_\delta)}{\partial t}\right\|_{L^2(0,T,[\overset{\circ}{W}{}^{1,2}(\Omega)]^*)} \le M_6. \tag{78}$$

The estimates (77) – (78) and Theorem 5.1, Chapter 1 [12] imply compactness of the set $\{u_\delta : \delta \in (0,1]\}$ in $L^2(Q)$. Hence we can choose a sequence $\delta_j \to 0$ such that the corresponding sequence $\{u_j\} = \{u_{\delta_j}\}$ converges to a function u_∞ in following sense

$$u_j(t,x) \to u_\infty(t,x) \ \text{ in } L^2(Q), \tag{79}$$

$$\frac{\partial u_j(t,x)}{\partial x} \rightharpoonup \frac{\partial u_\infty(t,x)}{\partial t} \ \text{ in } [L^2(Q)]^n, \tag{80}$$

$$\frac{\partial \sigma_{\delta_j}(u_j)}{\partial x} \rightharpoonup \frac{\partial \sigma(u_\infty)}{\partial t} \ \text{ in } L^2(0,T;[\overset{\circ}{W}{}^{1,2}(\Omega)]^*), \tag{81}$$

where $\to$ ($\rightharpoonup$) denotes strong (weak) convergence in respective spaces. We shall prove now strong convergence of the sequence $\frac{\partial u_j(t,x)}{\partial x}$ in $[L^2(Q)]^n$. Note that u_j satisfies the integral identy (45) with $\delta = \delta_j$. Defining

$$\rho_j(u) = \rho_{\delta_j}(u),\ \sigma_j(u) = \sigma_{\delta_j}(u), \tag{82}$$

we choose the test functions

$$\varphi(t,x) = \frac{1}{\rho_j(u_j(t,x))}\,[\sigma_j(u_j(t,x)) - \sigma_i(u_i(t,x))], \tag{83}$$

$$\varphi(t,x) = u_j(t,x) - u_i(t,x) \tag{84}$$

in (43) with $\delta = \delta_j$ and $\delta = \delta_i$ respectively. Taking the difference of the two resulting equalities, we get

$$\begin{aligned}
&\int_0^T \Big\{ < \frac{\partial \sigma_j(u_j)}{\partial t},\ \frac{1}{\rho_j(u_j)}[\sigma_j(u_j) - \sigma_i(u_i)] > - < \frac{\partial \sigma_i(u_i)}{\partial t},\ u_j - u_i > \Big\}\, dt \\
&+ \sum_{k=1}^n \int_Q \Big\{ \rho_j(u_j) b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big) \frac{\partial}{\partial x_k}\Big[\frac{1}{\rho_j(u_j)}[\sigma_j(u_j) - \sigma_i(u_i)]\Big] \\
&- \rho_i(u_i) b_k\Big(t,x,\frac{\partial u_i}{\partial x}\Big)\frac{\partial}{\partial x_k}(u_j - u_i)\Big\}\, dx\, dt \\
&+ \int_Q \Big\{ a\Big(t,x,u_j,\frac{\partial u_j}{\partial x}\Big)\frac{[\sigma_j(u_j) - \sigma_i(u_i)]}{\rho_j(u_j)} - a\Big(t,x,u_i,\frac{\partial u_i}{\partial x}\Big)[u_j - u_i]\Big\} dxdt = 0.
\end{aligned} \tag{85}$$

We analyze the behaviour of each summand in (85) as $i, j \to \infty$. Using condition iii), the estimates (77) and the strong convergence in (79), we obtain immediately that the last integral in (85) tends to zero. Now we rewrite the first integral of (83):

$$\begin{aligned}
&\int_0^T \Big\{ < \frac{\partial \sigma_j(u_j)}{\partial t},\ \frac{1}{\rho_j(u_j)} \cdot \sigma_j(u_j) > + < \frac{\partial \sigma_i(u_i)}{\partial t},\ u_i > \Big\}\, dt \\
&- \int_0^T \Big\{ < \frac{\partial \sigma_j(u_j)}{\partial t},\ \frac{1}{\rho_j(u_j)}\sigma_i(u_i) > + < \frac{\partial \sigma_i(u_i)}{\partial t},\ u_j > \Big\}\, dt.
\end{aligned} \tag{86}$$

Applying Corollary 1 resp. Corollary 2 to the first resp. second integral in (86), we find

$$\begin{aligned}
&\int_0^T \Big\{ < \frac{\partial \sigma_j(u_j)}{\partial t},\ \frac{1}{\rho_j(u_j)}[\sigma_j(u_j) - \sigma_i(u_i)] > - < \frac{\partial \sigma_i(u_i)}{\partial t},\ u_j - u_i > \Big\}\, dt \\
&= \int_\Omega \Big\{ \int_0^{u_j(T,x)} \sigma_j(s)\, ds + \int_0^{u_i(T,x)} \rho_i(s) s\, ds - u_j(T,x)\sigma_i(u_i(T,x))\Big\} dx \\
&- \frac{1}{2}(\delta_j - \delta_i)\int_\Omega g^2(x) dx = \int_\Omega \Big\{ \int_{u_i(T,x)}^{u_j(T,x)} [u_j(T,x) - s]\, \rho(s)\, ds \\
&+ \delta_i \int_{u_i(T,x)}^{u_j(T,x)} [u_j(T,x) - s]\, ds + \frac{1}{2}(\delta_j - \delta_i)[u_j^2(T,x) - g^2(x)]\Big\} dx \\
&\geq \frac{1}{2}(\delta_j - \delta_i)\int_\Omega \{u_j^2(T,x) - g^2(x)\}\, dx.
\end{aligned} \tag{87}$$

As to the second summand in (85), we note that

$$-\frac{\rho'(u)}{\rho(u)}\int_v^u \rho(s)\, ds \geq -\int_v^u \frac{\rho'(s)}{\rho(s)}\rho(s)\, ds = \rho(v) - \rho(u), \quad \forall u, v \in \mathbb{R}^1, \tag{88}$$

holds by condition $\rho)$ and that

$$\sum_{i=1}^{n} b_i(t,x,\xi)\xi_i \geq 0 \quad \text{for} \quad (t,x) \in Q, \quad \xi \in \mathbb{R}^n, \tag{89}$$

holds by condition ii)$_1^*$. Using these estimates, we find

$$\sum_{k=1}^{n} \rho_j(u_j) b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big) \frac{\partial}{\partial x_k}\Big[\frac{1}{\rho_j(u_j)}[\sigma_j(u_j) - \sigma_i(u_i)]\Big]\Big\}$$

$$= \sum_{k=1}^{n} b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big)\Big\{\Big[\rho_j(u_j)\frac{\partial u_j}{\partial x_k} - \rho_i(u_i)\frac{\partial u_i}{\partial x_k}\Big]$$

$$- \frac{\partial u_j}{\partial x_k} \cdot \frac{\rho(u_j)}{\rho_j(u_j)} \frac{\rho'(u_j)}{\rho(u_j)}[\sigma(u_j) - \sigma(u_i) + \delta_j u_j - \delta_i u_i]\Big\}$$

$$\geq \sum_{k=1}^{n} b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big)\Big\{\Big[\rho_j(u_j)\frac{\partial u_j}{\partial x_k} - \rho_i(u_i)\frac{\partial u_i}{\partial x_k}\Big]$$

$$+ \frac{\partial u_j}{\partial x_k} \cdot \frac{\rho(u_j)}{\rho_j(u_j)}\Big[\rho(u_i) - \rho(u_j) - \frac{\rho'(u_j)}{\rho(u_j)}(\delta_j u_j - \delta_i u_i)\Big]\Big\} =$$

$$= \sum_{k=1}^{n} b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big)\Big\{\rho_i(u_i)\Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_i}{\partial x_k}\Big)$$

$$+ \frac{\partial u_j}{\partial x_k} \cdot \Big[\rho_j(u_j) - \rho_i(u_i) - \frac{\rho(u_j)}{\rho_j(u_j)}\Big(\rho(u_j) - \rho(u_i) - \frac{\rho'(u_j)}{\rho(u_j)}(\delta_j u_j - \delta_i u_i)\Big)\Big]\Big\}.$$

From this, (77) and condition ii)$_1^*$ we obtain

$$\sum_{k=1}^{n} \int_Q\int \Big\{\rho_j(u_j) b_k\Big(t,x,\frac{\partial u_j}{\partial x}\Big)\frac{\partial}{\partial x_k}\Big[\frac{1}{\rho_j(u_j)}[\sigma(u_j) - \sigma(u_i)]\Big]$$

$$- \rho_i(u_i) b_k\Big(t,x,\frac{\partial u_i}{\partial x}\Big)\frac{\partial}{\partial x_k}(u_j - u_i)\Big\}\, dx\, dt \tag{90}$$

$$\geq \nu_1 \min_{|u|\leq M_0} \rho(u) \int_Q \Big|\frac{\partial}{\partial x}(u_j - u_i)\Big|^2 dx\, dt - c_{15}(\delta_i + \delta_j).$$

Now, using (85), (87), (90) and the fact that the last term in (85) converges to zero, we get

$$\frac{\partial u_j(t,x)}{\partial x} \to \frac{\partial u_\infty(t,x)}{\partial x} \text{ in } [L^2(Q)]^n. \tag{91}$$

The convergences (80), (81), (91) allow to let $\delta = \delta_j \to 0$ in (45). Thus we see that u_∞ satisfies the integral identity (11). We can pass also to the limit as $j \to \infty$ in

$$\int_0^\tau < \frac{\partial \sigma_{\delta_j}(u_j)}{\partial t}, \varphi > dt + \int_0^\tau \int_\Omega \big[\sigma_j(u_j) - \sigma_j(g)\big]\frac{\partial \varphi}{\partial t}\, dx\, dt = 0,$$

with φ satisfying the same condition as in (13). Hence we get that u_∞ fulfils the initial condition in the sense of Definition 1. $\square$

6 Proof of Theorem 4

Theorem 3 guarantees the existence of a bounded solution to the problem (1) - (3). For proving uniqueness let us assume two solutions $u_j \in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q)$ to exist with $\frac{\partial \sigma(u_j)}{\partial t} \in L^2(0,T;[\overset{\circ}{W}{}^{1,2}(\Omega)]^*)$, $j=1,2$. We will show that $u_1 = u_2$. By Theorems 1, 2 we have

$$\left\| \frac{\partial u_j}{\partial x} \right\|^2_{L^2(Q)} \le M_1, \ \|u_j\|_{L^\infty(Q)} \le M_0, \ j = 1,2. \tag{92}$$

Denote $v = u_2 - u_1$ and suppose contradictorily that $v \neq 0$. It is sufficient to prove that the positive part $v_+(t,x) = \max\{v(t,x), 0\}$ of v vanishes.

We substitute in the integral identity (11) with $u = u_2$ the test function

$$\varphi = \frac{1}{\rho(u_2)}[\sigma(u_2) - \sigma(u_1)]_+. \tag{93}$$

Thus we obtain

$$\int_0^\tau < \frac{\partial \sigma(u_2)}{\partial t}, \frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]_+ > dt$$

$$+ \int_{Q_\tau^+} \Big\{ \sum_{i=1}^n b_i\Big(t,x,\frac{\partial u_2}{\partial x}\Big)\Big[\frac{\partial u_2}{\partial x_i}\rho(u_2) - \rho(u_1)\frac{\partial u_1}{\partial x_i} \ - \frac{\rho'(u_2)}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]\frac{\partial u_2}{\partial x_i}\Big]$$

$$+ a\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)\frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]\Big\} \, dx \, dt = 0,$$

where $Q_\tau^+ = \{(t,x) : 0 < t < \tau, \ x \in \Omega, \ v(t,x) > 0\}$.
Additionally we substitute in (11) with $u = u_1$ the test function $\varphi = v_+$. This yields

$$\int_0^\tau < \frac{\partial \sigma(u_1)}{\partial t}, v_+ > dt + \int_{Q_\tau^+} \Big\{ \sum_{i=1}^n \rho(u_1) b_i\Big(t,x,\frac{\partial u_1}{\partial x}\Big)\frac{\partial (u_2-u_1)}{\partial x_i}$$

$$+ a\Big(t,x,u_1,\frac{\partial u_1}{\partial x}\Big)(u_2-u_1)\Big\} \, dx \, dt = 0.$$

Taking the difference of the latter equations, we get

$$\int_0^\tau \Big\{ < \frac{\partial \sigma(u_2)}{\partial t}, \frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]_+ > - < \frac{\partial \sigma(u_1)}{\partial t}, [u_2-u_1]_+ > \Big\}\, dt$$

$$+ \int_{Q_\tau^+} \Big\{ \rho(u_1) \sum_{i=1}^n \Big[b_i\Big(t,x,\frac{\partial u_2}{\partial x}\Big) - b_i\Big(t,x,\frac{\partial u_1}{\partial x}\Big)\Big] \frac{\partial}{\partial x_i}(u_2-u_1)$$

$$+ \sum_{i=1}^n b_i\Big(t,x,\frac{\partial u_2}{\partial x}\Big) \frac{\partial u_2}{\partial x_i} \Big[\rho(u_2)-\rho(u_1) - \frac{\rho'(u_2)}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]\Big]$$

$$+ a\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big) \frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)] - a\Big(t,x,u_1,\frac{\partial u_1}{\partial x}\Big)(u_2-u_1)\Big\}\, dx\, dt = 0. \tag{94}$$

Let us evaluate (94) term by term. We start with the first integral and want to apply Lemma 2 with respect to the function

$$F(z_1,z_2) = \int_{\sigma^{-1}(z_1)}^{\sigma^{-1}(z_2)} \big[\sigma^{-1}(z_2) - s\big]_+ \rho(s)\, ds, \quad F_i(z_1,z_2) = \frac{\partial F(z_1,z_2)}{\partial z_i}, \quad i=1,2.$$

It is simple to check that

$$F_1(z_1,z_2) = -\big[\sigma^{-1}(z_2) - \sigma^{-1}(z_1)\big]_+, \ F_2(z_1,z_2) = \frac{[z_2-z_1]_+}{\rho(\sigma^{-1}(z_2))}, \quad z_1, z_2 \in \mathbb{R}^1.$$

Since $F(\sigma(g),\sigma(g)) = 0$, $F(\sigma(f),\sigma(f)) = 0$, Lemma 2 yields

$$\int_0^\tau \Big\{ < \frac{\partial \sigma(u_2)}{\partial t}, \frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]_+ > - < \frac{\partial \sigma(u_1)}{\partial t}, [u_2-u_1]_+ > \Big\}\, dt$$

$$= \int_\Omega F(\sigma(u_1(\tau,x)), \sigma(u_2(\tau,x)))\, dx \tag{95}$$

$$= \int_\Omega \Big\{ \int_{u_1(\tau,x)}^{u_2(\tau,x)} \big[u_2(\tau,x) - s\big]_+ \rho(s)\, ds \Big\}\, dx \ \geq \ \frac{1}{2} \min_{|u|\leq M_0} \rho(u) \ \int_\Omega v_+^2\, dx.$$

Now we turn to estimate summands in (94) involving functions b_i. Applying condition ii)$_1^*$ and the inequalities (88), (89), we get

$$\int_{Q_\tau^+}\int \Big\{ \rho(u_1) \sum_{i=1}^n \Big[b_i\Big(t,x,\frac{\partial u_2}{\partial x}\Big) - b_i\Big(t,x,\frac{\partial u_1}{\partial x}\Big)\Big] \frac{\partial (u_2-u_1)}{\partial x_i}$$

$$+ \sum_{i=1}^n b_i\Big(t,x,\frac{\partial u_2}{\partial x}\Big) \frac{\partial u_2}{\partial x_i} \Big[\rho(u_2)-\rho(u_1) - \frac{\rho'(u_2)}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)]\Big\}\, dx\, dt \tag{96}$$

$$\geq \nu_1 \min_{|u|\leq M_0} \rho(u) \cdot \int_{Q_\tau^+} \Big|\frac{\partial}{\partial x}(u_2-u_1)\Big|^2\, dx\, dt.$$

Further we have to estimate terms in (94) involving the function a. Let us firstly explain, how the estimate

$$
\begin{aligned}
&a_0\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)\frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)] - a_0\Big(t,x,u_1,\frac{\partial u_1}{\partial x}\Big)(u_2-u_1) \geq \\
&\geq -c_{15}\Big\{\Big[c_1(t,x)+\Big|\frac{\partial u_2}{\partial x}\Big|^{\frac{2}{p}}\Big](u_2-u_1)+c_2(t,x)\Big|\frac{\partial(u_2-u_1)}{\partial x}\Big|\Big\}(u_2-u_1)
\end{aligned}
\tag{97}
$$

for $(t,x)\in Q_\tau^+$ can be derived. Since the proof of (97) in the case iv)$_1$ is simple, we restrict us to the case of condition iv)$_2$. Subdivide Q_τ^+ into three subsets $Q_\tau^+ = Q_\tau^+(0)\cup Q_\tau^+(1)\cup Q_\tau^+(2)$, where

$$
Q_\tau^+(k)=\Big\{(t,x)\in Q_\tau^+ : (-1)^k\int_{u_1(t,x)}^{u_2(t,x)}[\rho(s)-\rho(u_1(t,x))]\,ds>0\Big\}\ ,k=1,2,
$$

and $Q_\tau^+(0)=Q_\tau^+\setminus\{Q_\tau^+(1)\cup Q_\tau^+(2)\}$. We shall prove (97) for example in the case $(t,x)\in Q_\tau^+(1)$. The other cases can be considered analogously.
For $(t,x)\in Q_\tau^+(1)$ the inequality $\rho'(u_2(t,x))<0$ holds by condition ρ). Thus the conditions iii) and iv)$_2$ imply the desired estimate (97) in following way:

$$
\begin{aligned}
&a_0\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)\frac{1}{\rho(u_2)}[\sigma(u_2)-\sigma(u_1)] - a_0\Big(t,x,u_1,\frac{\partial u_1}{\partial x}\Big)(u_2-u_1) \geq \\
&\geq a_0\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)\frac{1}{\rho(u_2)}\big[\sigma(u_2)-\sigma(u_1)\big] - a_0\Big(t,x,u_2,\frac{\partial u_1}{\partial x}\Big)\frac{\rho(u_1)}{\rho(u_2)}(u_2-u_1) \\
&= sign\ \rho'(u_2(t,x))\frac{a_0(t,x,u_2,\frac{\partial u_2}{\partial x})}{\rho(u_2)}\Big|\int_{u_1(t,x)}^{u_2(t,x)}\big[\rho(s)-\rho(u_1(t,x))\big]\,ds\Big| \\
&+\Big[a_0\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)-a_0\Big(t,x,u_2,\frac{\partial u_1}{\partial x}\Big)\Big]\frac{\rho(u_1)}{\rho(u_2)}(u_2-u_1) \\
&\geq -c_{15}\Big\{\Big[c_1(t,x)+\Big|\frac{\partial u_2}{\partial x}\Big|^{\frac{2}{p}}\Big](u_2-u_1)+c_2(t,x)\Big|\frac{\partial(u_2-u_1)}{\partial x}\Big|\Big\}(u_2-u_1).
\end{aligned}
$$

On the other hand condition iv) yields for $(t,x)\in Q_\tau^+$

$$
\begin{aligned}
&\Big|a_1\Big(t,x,u_2,\frac{\partial u_2}{\partial x}\Big)\frac{1}{\rho(u_2)}\big[\sigma(u_2)-\sigma(u_1)\big] - a_1\Big(t,x,u_1,\frac{\partial u_1}{\partial x}\Big)(u_2-u_1)\Big| \\
&\leq c_{16}\Big\{\Big[c_1(t,x)+\Big|\frac{\partial u_2}{\partial x}\Big|^{\frac{2}{p}}\Big](u_2-u_1)+c_2(t,x)\Big|\frac{\partial(u_2-u_1)}{\partial x}\Big|\Big\}(u_2-u_1).
\end{aligned}
\tag{98}
$$

Putting together equalities (94) and the estimates (95)-(98), we get

$$
\begin{aligned}
&\int_\Omega v_+^2(\tau,x)\,dx+\int_0^\tau\int_\Omega\Big|\frac{\partial v_+}{\partial x}\Big|^2\,dx\,dt \\
&\leq c_{17}\int_0^\tau\int_\Omega\Big\{\Big[c_1(t,x)+\Big|\frac{\partial u_1}{\partial x}\Big|^{\frac{2}{p}}+\Big|\frac{\partial u_2}{\partial x}\Big|^{\frac{2}{p}}\Big]v_+^2+c_2(t,x)\Big|\frac{\partial v}{\partial x}\Big|\cdot v_+\Big\}\,dx.
\end{aligned}
$$

Thus, using the inequalities of Cauchy and Hölder together with the conditions on $c_1(t,x)$, $c_2(t,x)$ and the first inequality in (92), we get

$$\begin{aligned} &\underset{\tau \in (0,\ominus)}{ess\ \sup} \int_\Omega v_+^2(\tau,x)\,dx + \int_{Q_\ominus} \left|\frac{\partial v_+}{\partial x}\right|^2 dx\,dt \\ &\le c_{18}\Big\{\int_{Q_\ominus} v_+^{2p'}\,dx\,dt\Big\}^{\frac{1}{p'}} \quad \text{, with} \quad p' = \frac{p}{p-1} < \frac{n+2}{n}, \end{aligned} \tag{99}$$

for an arbitrary $\ominus \in (0,T)$, $Q_\ominus = \{(t,x) : 0 < t < \ominus,\ x \in \Omega\}$.
Estimating the right hand side of (99) by Hölder's inequality and the embedding $V^2(Q) \to L^{\frac{2(n+2)}{n}}(Q)$ and setting $p_1 = n+2-p'n$, we find

$$\Big\{\int_{Q_\ominus} v_+^{2p'}\,dx\,dt\Big\}^{\frac{1}{p'}} \le \Big\{\int_{Q_\ominus} v_+^2\,dx\,dt\Big\}^{\frac{p_1}{2p'}} \cdot \Big\{\int_{Q_\ominus} v_+^{\frac{2(n+2)}{n}}\,dx\,dt\Big\}^{\frac{1}{p'}-\frac{p_1}{2p'}}$$

$$\le c_{19}\Big\{\int_{Q_\ominus} v_+^2 dxdt\Big\}^{\frac{p_1}{2p'}}\Big\{\underset{\tau \in (0,\ominus)}{ess\sup} \int_\Omega v_+^2(\tau,x)dx + \int_{Q_\ominus}\left|\frac{\partial v_+}{\partial x}\right|^2 dxdt\Big\}^{\frac{(n+2)(2-p_1)}{2np'}}.$$

with a constant c_{19} independent of $\ominus$. Thus (99) implies

$$\int_\Omega v_+^2(\ominus,x)\,dx \le c_{20}\int_0^\ominus \int_\Omega v_+^2(t,x)\,dx\,dt$$

for arbitrary $\ominus \in (0,T)$ and a constant c_{20} independent of $\ominus$. Finally, Gronwall's lemma yields $v_+(t,x) = 0$ and finishes the proof of Theorem 4. □

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